$\mathbb{Q}$-Fano Threefolds of Large Fano Index, I

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Abstract. We study $\mathbb{Q}$-Fano threefolds of large Fano index. In particular, we prove that the maximum possible Fano index is attained only by the weighted projective space $\mathbb{P}(3,4,5,7)$.

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1. Introduction

The Fano index of a smooth Fano variety X is the maximal integer $\mathrm{q}(X)$ that divides the anti-canonical class in the Picard group $\mathrm{Pic}(X)$ [IP99]. It is well-known [KO73] that $\mathrm{q}(X) \le \dim X + 1$. Moreover, $\mathrm{q}(X) = \dim X + 1$ if and only if X is a projective space and $\mathrm{q}(X) = \dim X$ if and only if X is a quadric hypersurface. In this paper we consider generalizations of Fano index for the case of singular Fanos admitting terminal singularities.
A normal projective variety X is called *Fano* if some positive multiple $-nK_X$ of its anti-canonical Weil divisor is Cartier and ample. Such X is called a $\mathbb{Q}$-*Fano variety* if it has only terminal $\mathbb{Q}$-factorial singularities and its Picard number is one. This class of Fano varieties is important because they appear naturally in the Minimal Model Program.
For a singular Fano variety X the Fano index can be defined in different ways. For example, we can define

$$\begin{aligned} \mathrm{qW}(X) &:= \max\{q \mid -K_X \sim qA, \quad A \text{ is a Weil } \mathbb{Q}\text{-Cartier divisor}\}, \\ \mathrm{q}\mathbb{Q}(X) &:= \max\{q \mid -K_X \sim_{\mathbb{Q}} qA, \quad A \text{ is a Weil } \mathbb{Q}\text{-Cartier divisor}\}. \end{aligned}$$

If X has at worst log terminal singularities, then the Picard group $\mathrm{Pic}(X)$ and Weil divisor class group $\mathrm{Cl}(X)$ are finitely generated and $\mathrm{Pic}(X)$ is torsion free (see e.g. [IP99, §2.1]). Moreover, the numerical equivalence of $\mathbb{Q}$-Cartier

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divisors coincides with $\mathbb{Q}$-linear one. This implies, in particular, that the Fano indices $\mathrm{qW}(X)$ and $\mathrm{q}\mathbb{Q}(X)$ defined above are positive integers. If X is smooth, these numbers coincide with the Fano index $\mathrm{q}(X)$ defined above. Note also that $\mathrm{q}\mathbb{Q}(X) = \mathrm{qW}(X)$ if the group $\mathrm{Cl}(X)$ is torsion free.

Theorem 1.1 ([Suz04]). *Let X be a $\mathbb{Q}$-Fano threefold. Then* $\mathrm{qW}(X) \in \{1, \dots, 11, 13, 17, 19\}$. *All these values, except possibly for* $\mathrm{qW}(X) = 10$, *occur. Moreover, if* $\mathrm{qW}(X) = 19$, *then the types of non-Gorenstein points and Hilbert series of X coincide with that of* $\mathbb{P}(3, 4, 5, 7)$.

It can be easily shown (see proof of Proposition 3.6) that the index $\mathrm{q}\mathbb{Q}(X)$ takes values in the same set $\{1, \dots, 11, 13, 17, 19\}$. Thus one can expect that $\mathbb{P}(3, 4, 5, 7)$ is the only example of $\mathbb{Q}$-Fano threefolds with $\mathrm{q}\mathbb{Q}(X) = 19$. In general, we expect that Fano varieties with extremal properties (maximal degree, maximal Fano index, etc.) are quasihomogeneous with respect to an action of some connected algebraic group. This is supported, for example, by the following facts:

Theorem 1.2 ([Pro05], [Pro07]).
(i) *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein. Then $-K_X^3 \le 125/2$ and the equality holds if and only if X is isomorphic to the weighted projective space* $\mathbb{P}(1^3, 2)$.
(ii) *Let X be a Fano threefold with canonical Gorenstein singularities. Then $-K_X^3 \le 72$ and the equality holds if and only if X is isomorphic to* $\mathbb{P}(1^3, 3)$ *or* $\mathbb{P}(1^2, 6, 4)$.

The following proposition is well-known (see, e.g., [BB92]). It is an easy exercise for experts in toric geometry.

Proposition 1.3. *Let X be a toric $\mathbb{Q}$-Fano 3-fold. Then X is isomorphic to either* $\mathbb{P}^3$, $\mathbb{P}^3/\boldsymbol{\mu}_5(1, 2, 3, 4)$, *or one of the following weighted projective spaces:*

$$\mathbb{P}(1^3, 2),\ \ \mathbb{P}(1^2, 2, 3),\ \ \mathbb{P}(1, 2, 3, 5),\ \ \mathbb{P}(1, 3, 4, 5),\ \ \mathbb{P}(2, 3, 5, 7),\ \ \mathbb{P}(3, 4, 5, 7).$$

We characterize the weighted projective spaces above in terms of Fano index. The following is the main result of this paper.

Theorem 1.4. *Let X be a $\mathbb{Q}$-Fano threefold. Then* $\mathrm{q}\mathbb{Q}(X) \in \{1, \dots, 11, 13, 17, 19\}$.

(i) *If* $\mathrm{q}\mathbb{Q}(X) = 19$, *then* $X \simeq \mathbb{P}(3, 4, 5, 7)$.
(ii) *If* $\mathrm{q}\mathbb{Q}(X) = 17$, *then* $X \simeq \mathbb{P}(2, 3, 5, 7)$.
(iii) *If* $\mathrm{q}\mathbb{Q}(X) = 13$ *and* $\dim |-K_X| > 5$, *then* $X \simeq \mathbb{P}(1, 3, 4, 5)$.
(iv) *If* $\mathrm{q}\mathbb{Q}(X) = 11$ *and* $\dim |-K_X| > 10$, *then* $X \simeq \mathbb{P}(1, 2, 3, 5)$.
(v) $\mathrm{q}\mathbb{Q}(X) \ne 10$.
(vi) *If* $\mathrm{q}\mathbb{Q}(X) \ge 7$ *and there are two effective Weil divisors* $A \ne A_1$ *such that* $-K_X \sim_{\mathbb{Q}} \mathrm{q}\mathbb{Q}(X)A \sim_{\mathbb{Q}} \mathrm{q}\mathbb{Q}(X)A_1$, *then* $X \simeq \mathbb{P}(1^2, 2, 3)$.
(vii) *If* $\mathrm{qW}(X) = 5$ *and* $\dim |-\frac{1}{5}K_X| > 1$, *then* $X \simeq \mathbb{P}(1^3, 2)$.

Note that in cases (iii) and (iv) assumptions about $|-K_X|$ are really needed. Indeed, there are examples of non-toric $\mathbb{Q}$-Fano threefolds with $\mathrm{q}\mathbb{Q}(X) = 13$ and 11.

Example 1.5 ([BS07], see also Proposition 3.6). Let $X = X_d \subset \mathbb{P}(a_1, \dots, a_5)$ be a general hypersurface of degree d. Assume that X is a $\mathbb{Q}$-Fano with $\mathrm{q}\mathbb{Q}(X) \geq 10$ and such that $\mathscr{O}_{\mathbb{P}}(1)|_X$ is a primitive element of $\mathrm{Cl}(X)$, then X is one of the following:

(i) $X = X_{12} \subset \mathbb{P}(1, 4, 5, 6, 7)$, $\mathrm{q}\mathbb{Q}(X) = 11$, $\dim |-K_X| = 10$;
(ii) $X = X_{10} \subset \mathbb{P}(2, 3, 4, 5, 7)$, $\mathrm{q}\mathbb{Q}(X) = 11$, $\dim |-K_X| = 8$;
(iii) $X \simeq X_{12} \subset \mathbb{P}(3, 4, 5, 6, 7)$, $\mathrm{q}\mathbb{Q}(X) = 13$, $\dim |-K_X| = 5$.

In the proof we follow the use some techniques developed in our previous paper [Pro07]. By Proposition 1.3 it is sufficient to show that our $\mathbb{Q}$-Fano X is toric. First, as in [Suz04], we apply the orbifold Riemann-Roch formula to find all the possibilities for the numerical invariants of X. In all cases there is some special element $S \in |-K_X|$ having four irreducible components. This S should be a toric boundary, if X is toric. Further, we use birational transformations like Fano-Iskovskikh "double projection" [IP99] (see [Ale94] for the $\mathbb{Q}$-Fano version). Typically the resulting variety is a Fano-Mori fiber space having "simpler" structure. In particular, its Fano index is large if this variety is a $\mathbb{Q}$-Fano. By using properties of our "double projection" we can show that the pair (X, S) is log canonical (LC). Then, in principle, the assertion follows by Shokurov's toric conjecture [McK01]. We prefer to propose an alternative, more explicit proof. In fact, the image of X under "double projection" is a toric variety and the inverse map preserves the toric structure. In the last section we describe Sarkisov links between toric $\mathbb{Q}$-Fanos that start with blow ups of *singular* points.

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2. Preliminaries, the orbifold Riemann-Roch formula and its applications

Notation. Throughout this paper, we work over the complex number field $\mathbb{C}$. We employ the following standard notation:

$\sim$ denotes linear equivalence;

$\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence.

Let E be a rank one discrete valuation of the function field $\mathbb{C}(X)$ and let D be a $\mathbb{Q}$-Cartier divisor on X. $a(E, D)$ denotes the discrepancy of E with respect to a boundary D. Let $f \colon \tilde{X} \to X$ be a birational morphism such that E appears as a prime divisor on $\tilde{X}$. Then $\mathrm{ord}_E(D)$ denotes the coefficient of E in f^*D.

2.1. The orbifold Riemann-Roch formula [Rei87]. Let X be a threefold with terminal singularities and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Let $\mathbf{B} = \{(r_P, b_P)\}$ be the basket of singular points of X [Mor85a], [Rei87]. Here each pair (r_P, b_P) correspond to a point $P \in \mathbf{B}$ of type $\frac{1}{r_P}(1, -1, b_P)$. For brevity, describing a basket we will list just indices of singularities, i.e., we will write $\mathbf{B} = \{r_P\}$ instead of $\mathbf{B} = \{(r_P, b_P)\}$. In the above situation, the Riemann-Roch formula has the following form

$$\begin{aligned}(2.2)\quad \chi(D) = \frac{1}{12} D \cdot (D - K_X) \cdot (2D - K_X) + \\ + \frac{1}{12} D \cdot c_2 + \sum_{P \in \mathbf{B}} c_P(D) + \chi(\mathscr{O}_X),\end{aligned}$$

where

$$c_P(D) = -i_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_P - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

Clearly, computing $c_P(D)$, we always may assume that $1 \le b_P \le r_P/2$.

2.3. Now let X be a Fano threefold with terminal singularities, let $q := \mathrm{q}\mathbb{Q}(X)$, and let A be an ample Weil $\mathbb{Q}$-Cartier divisor on X such that $-K_X \sim_{\mathbb{Q}} qA$. By (2.2) we have

$$(2.4)\qquad \chi(tA) = \chi(\mathscr{O}_X) + \frac{t(q+t)(q+2t)}{12} A^3 + \frac{tA \cdot c_2}{12} + \sum_{P \in \mathbf{B}} c_P(tA),$$

where $\chi(\mathscr{O}_X) = 1$ and

$$c_P(tA) = -i_{P,t} \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_{P,t} - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

If $q > 2$, then $\chi(-A) = 0$. Using this equality we obtain (see [Suz04])

$$(2.5)\qquad A^3 = \frac{12}{(q-1)(q-2)} \left(1 - \frac{A \cdot c_2}{12} + \sum_{P \in B} c_P(-A) \right).$$

In the above notation, applying (2.2), Serre duality and Kawamata-Viehweg vanishing to $D = K_X$, we get the following important equality (see, e.g., [Rei87]):

$$(2.6)\qquad 24 = -K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right).$$

Theorem 2.7 ([Kaw92a], [KMMT00]). *In the above notation,*

$$(2.8)\qquad -K_X \cdot c_2(X) \ge 0, \qquad \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) \le 24.$$

Proposition 2.9. *Let X be a Fano threefold with terminal singularities and let Ξ be an n-torsion element in the Weil divisor class group. Let $\mathbf{B}^{\Xi}$ be the collection of points $P \in \mathbf{B}$ where Ξ is not Cartier. Then*

$$(2.10)\qquad 2 = \sum_{P \in \mathbf{B}^{\Xi}} \frac{\overline{b_P i_{\Xi,P}}\, \left(r_P - \overline{b_P i_{\Xi,P}} \right)}{2 r_P}.$$

where $i_{\Xi,P}$ is taken so that $\Xi \sim i_{\Xi,P}K_X$ near $P \in \mathbf{B}$ and $\overline{}$ is the residue mod r_P. *Assume furthermore that n is prime. Then*

(i) $n \in \{2,3,5,7\}$.
(ii) *If $n=7$, then* $\mathbf{B}^{\Xi} = (7,7,7)$.[†]
(iii) *If $n=5$, then* $\mathbf{B}^{\Xi} = (5,5,5,5)$, $(10,5,5)$, *or* $(10,10)$.
(iv) *If $n=3$, then* $\sum_{P\in\mathbf{B}^{\Xi}} r_P = 18$.
(v) *If $n=2$, then* $\sum_{P\in\mathbf{B}^{\Xi}} r_P = 16$.

Proof. By Kawamata-Viehweg vanishing theorem, Riemann-Roch (2.2), and Serre duality we have $\chi(\mathscr{O}_X)=1$,

$$\begin{aligned} 0 &= \chi(\Xi) && = 1 + \textstyle\sum_P c_P(\Xi), \\ 0 &= \chi(K_X+\Xi) && = 1 + \tfrac{1}{12}K_X\cdot c_2(X) + \textstyle\sum_{P\in\mathbf{B}} c_P(K_X+\Xi). \end{aligned}$$

Subtracting we get

$$0 = -\frac{1}{12}K_X\cdot c_2(X) + \sum_{P\in\mathbf{B}} \left(c_P(\Xi) - c_P(K_X+\Xi)\right).$$

Since $ni_{\Xi,P} \equiv 0 \mod r_P$,

$$0 = -\frac{1}{12}K_X\cdot c_2(X) + \frac{1}{12}\sum_{P\in\mathbf{B}}\left(r_P - \frac{1}{r_P}\right) - \sum_{P\in\mathbf{B}} \frac{\overline{b_P i_{\Xi,P}}\ \left(r_P - \overline{b_P i_{\Xi,P}}\right)}{2r_P}.$$

This proves (2.10).

Now assume that n is prime. If $P \in \mathbf{B}^{\Xi}$, then $n \mid r_P$. Write $r_P = nr'_P$. Since $r_P \mid ni_P$, $i_P = r'_P i'_P$, where $n \nmid i'_P$. Let $\overline{(\quad)_n}$ be the residue mod n. Then

$$2 = \sum_{P\in\mathbf{B}^{\Xi}} \frac{\overline{b_P i'_{\Xi,P} r'}\ \left(nr'_P - \overline{b_P i'_{\Xi,P} r'_P}\right)}{2nr'_P} = \frac{r'_P\overline{(b_P i'_{\Xi,P})_n}\ \left(n - \overline{(b_P i'_{\Xi,P})_n}\right)}{2n}.$$

Therefore,

$$4n^2 = \sum_{P\in\mathbf{B}^{\Xi}} r_P\overline{(b_P i'_{\Xi,P})_n}\ \left(n - \overline{(b_P i'_{\Xi,P})_n}\right).$$

Denote $\xi_P := \overline{(b_P i'_{\Xi,P})_n}$. Then $0<\xi_P<n$, $\gcd(n,\xi_P)=1$, and

$$4n = \sum_{P\in\mathbf{B}^{\Xi}} r'_P\xi_P\left(n-\xi_P\right) \ge \frac{n^2}{4}\sum_{P\in\mathbf{B}^{\Xi}} r'_P, \qquad 16 \ge n\sum_{P\in\mathbf{B}^{\Xi}} r'_P.$$

If $n\ge 11$, then $\sum r'_P = 1$, $n \mid r'_P$, and $r_P \ge n^2 \ge 121$, a contradiction. Therefore, $n \le 7$. Consider the case $n=7$. Then $\xi_P\left(n-\xi_P\right) = 6$, 10, or 12. The only solution is $\mathbf{B}^{\Xi} = (7,7,7)$. The case $n=5$ is considered similarly. If $n=3$, then $\xi_P\left(n-\xi_P\right)=3$ and $\sum r_P = 3\sum r'_P = 18$. Similarly, if $n=2$, then $\xi_P\left(n-\xi_P\right)=1$ and $\sum r_P = 2\sum r'_P = 16$. This finishes the proof. □

[†]More delicate computations show that this case does not occur. (We do not need this.)

3. Computations with Riemann-Roch on $\mathbb{Q}$-Fano threefolds of large Fano index

LEMMA 3.1 (see [Suz04]). *Let X be a Fano threefold with terminal singularities with $q := \mathrm{qW}(X)$, let $A := -\frac{1}{q}K_X$, and let r be the Gorenstein index of X. Then*

(i) *r and q are coprime;*
(ii) *rA^3 is an integer.*

LEMMA 3.2. *Let X be a Fano threefold with terminal singularities.*

(i) *If $-K_X \sim qL$ for some Weil divisor L, then q divides* $\mathrm{qW}(X)$.
(ii) *If $-K_X \sim_{\mathbb{Q}} qL$ for some Weil divisor L, then q divides* $\mathrm{q}\mathbb{Q}(X)$.
(iii) $\mathrm{qW}(X)$ *divides* $\mathrm{q}\mathbb{Q}(X)$.
(iv) *Let $q := \mathrm{q}\mathbb{Q}(X)$ and let $K_X + qA \sim_{\mathbb{Q}} 0$. If the order of $K_X + qA$ in the group* $\mathrm{Cl}(X)$ *is prime to q, then* $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$.

Proof. To prove (i) write $-K_X \sim \mathrm{qW}(X)A$ and let $d = \gcd(\mathrm{qW}(X), q)$. Then $d = u\,\mathrm{qW}(X) + vq$ for some $u, v \in \mathbb{Z}$. Hence, $dA = u\,\mathrm{qW}(X)A + vqA \sim quL + qvA = q(uL + vA)$. Since A is a primitive element of $\mathrm{Cl}(X)$, $q = d$ and $q \mid \mathrm{qW}(X)$.
(ii) can proved similarly and (iii) is a consequence of (ii).
To show (iv) assume that $\Xi := K_X + qA$ is of order n. By our condition $qu + nv = 1$, where $u, v \in \mathbb{Z}$. Put $A' := A - u\Xi$. Then $qA' = qA - qu\Xi = qA - \Xi \sim -K_X$. Hence, $q = \mathrm{qW}(X)$ by (i) and (iii). □

LEMMA 3.3. *Let X be a Fano threefold with terminal singularities.*

(i) $\mathrm{q}\mathbb{Q}(X) \in \{1, \ldots, 11, 13, 17, 19\}$.
(ii) *If* $\mathrm{q}\mathbb{Q}(X) \geq 5$, *then* $-K_X^3 \leq 125/2$.

Proof. Denote $q := \mathrm{q}\mathbb{Q}(X)$ and write, as usual, $-K_X \sim_{\mathbb{Q}} qA$. Thus $n(K_X + qA) \sim 0$ for some positive integer n. The element $K_X + qA$ defines a cyclic étale in codimension one cover $\pi\colon X' \to X$ so that X' is a Fano threefold with terminal singularities and $K_{X'} + qA' \sim 0$, where $A' := \pi^*A$. Let $\sigma\colon X'' \to X'$ be a $\mathbb{Q}$-factorialization. (If X' is $\mathbb{Q}$-factorial, we take $X'' = X'$). Run K-MMP on X'': $\psi\colon X'' \dashrightarrow \bar{X}$. At the end we get a Mori-Fano fiber space $\bar{X} \to Z$. Let $A'' := \sigma^{-1}(A')$ and $\bar{A} := \psi_* A''$. Then $-K_{\bar{X}} \sim q\bar{A}$. If $\dim Z > 0$, then for a general fiber F of $\bar{X}/Z$, we have $-K_F \sim q\bar{A}|_F$. This is impossible because $q > 3$. Thus $\dim Z = 0$ and $\bar{X}$ is a $\mathbb{Q}$-Fano.
(i) By Lemma 3.2 the number q divides $\mathrm{qW}(\bar{X})$. On the other hand, by Theorem 1.1 we have $\mathrm{qW}(\bar{X}) \in \{1, \ldots, 11, 13, 17, 19\}$. This proves (i).
To show (ii) we note that $-K_{\bar{X}}^3 \geq -K_{X''}^3 = -K_{X'}^3 \geq -K_{X''}^3$. Here the first inequality holds because for Fanos (with at worst log terminal singularities) the number $-\frac{1}{6}K^3$ is nothing but the leading term in the asymptotic Riemann-Roch and $\dim|-tK_{X''}| \leq \dim|-tK_{\bar{X}}|$. Now the assertion of (ii) follows from Theorem 1.2. □

From Lemmas 3.2 and 3.3 we have

Corollary 3.4. *Let X be a Fano threefold with terminal singularities.*

(i) *If $-K_X \sim qL$ for some Weil divisor L and $q \ge 5$, then $q = \mathrm{qW}(X)$.*

(ii) *If $-K_X \sim_{\mathbb{Q}} qL$ for some Weil divisor L and $q \ge 5$, then $q = \mathrm{q}\mathbb{Q}(X)$.*

Lemma 3.5 (cf. [Suz04]). *Let X be a Fano threefold with terminal singularities and let $q := \mathrm{qW}(X)$. Assume that $\mathrm{qW}(X) \ge 8$. Then one of the following holds:*

$q = 8$, $\mathbf{B} = (3^2, 5)$, $(3^2, 5, 9)$, $(3, 5, 11)$, $(3, 7)$, $(3, 9)$, $(5, 7)$, $(7, 11)$, $(7, 13)$, (11),
$q = 9$, $\mathbf{B} = (2, 4, 5)$, $(2^3, 5, 7)$, $(2, 5, 13)$,
$q = 10$, $\mathbf{B} = (7, 11)$,
$q = 11$, $\mathbf{B} = (2, 3, 5)$, $(2, 5, 7)$, $(2^2, 3, 4, 7)$,
$q = 13$, $\mathbf{B} = (3, 4, 5)$, $(2, 3^2, 5, 7)$,
$q = 17$, $\mathbf{B} = (2, 3, 5, 7)$,
$q = 19$, $\mathbf{B} = (3, 4, 5, 7)$.

In all cases the group $\mathrm{Cl}(X)$ is torsion free.

Proof. We use a computer program written in PARI [PARI] ‡. Below is the description of our algorithm.
Step 1. By Theorem 2.7 we have $\sum_{P \in \mathbf{B}} (1 - 1/r_P) \le 24$. Hence there is only a finite (but very huge) number of possibilities for the basket $\mathbf{B} = \{[r_P, b_P]\}$. In each case we know $-K_X \cdot c_2(X)$ from (2.6). Let $r := \mathrm{lcm}(\{r_P\})$ be the Gorenstein index of X.
Step 2. By Lemma 3.3 $\mathrm{q}\mathbb{Q}(X) \in \{8, \dots, 11, 13, 17, 19\}$. Moreover, the condition $\gcd(q, r) = 1$ (see Lemma 3.1) eliminates some possibilities.
Step 3. In each case we compute A^3 and $-K_X^3 = q^3 A^3$ by formula (2.5). Here, for $D = -A$, the number i_P is uniquely determined by $q i_P \equiv b_P \mod r_P$ and $0 \le i_P < r_P$. Further, we check the condition $rA^3 \in \mathbb{Z}$ (Lemma 3.1) and the inequality $-K_X^3 \le 125/2$ (Lemma 3.3).
Step 4. Finally, by the Kawamata-Viehweg vanishing theorem we have $\chi(tA) = h^0(tA)$ for $-q < t$. We compute $\chi(tA)$ by using (2.4) and check conditions $\chi(tA) = 0$ for $-q < t < 0$ and $\chi(tA) \ge 0$ for $t > 0$.
At the end we get our list. To prove the last assertion assume that $\mathrm{Cl}(X)$ contains an n-torsion element Ξ. Clearly, we also may assume that n is prime. By Proposition 2.9 we have $\sum_{n | r_i} r_i \ge 16$. Moreover, $\sum_{n | r_i} r_i \ge 18$ if $n = 3$. This does not hold in any case of our list. □

Proposition 3.6. *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}\mathbb{Q}(X) \ge 9$. Let $q := \mathrm{q}\mathbb{Q}(X)$ and let $-K_X \sim_{\mathbb{Q}} qA$. Then the group $\mathrm{Cl}(X)$ is torsion free, $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$, and one of the following holds:*

‡The PARI code is available at `http://mech.math.msu.su/department/algebra/staff/prokhorov/q-fano`.

q	$\mathbf{B}$	A^3	$\dim\lvert kA\rvert$							
			$\lvert A\rvert$	$\lvert 2A\rvert$	$\lvert 3A\rvert$	$\lvert 4A\rvert$	$\lvert 5A\rvert$	$\lvert 6A\rvert$	$\lvert 7A\rvert$	$\lvert -K\rvert$
9	$(2,4,5)$	$\frac{1}{20}$	0	1	2	4	6	8	11	19
9	$(2,2,2,5,7)$	$\frac{1}{70}$	-1	0	0	1	1	2	3	5
10	$(7,11)$	$\frac{2}{77}$	-1	0	1	1	3	4	6	13
11	$(2,3,5)$	$\frac{1}{30}$	0	1	2	3	5	7	9	23
11	$(2,5,7)$	$\frac{1}{70}$	0	0	0	1	2	3	4	10
11	$(2,2,3,4,7)$	$\frac{1}{84}$	-1	0	0	1	1	2	3	8
13	$(3,4,5)$	$\frac{1}{60}$	0	0	1	2	3	4	5	19
13	$(2,3,3,5,7)$	$\frac{1}{210}$	-1	-1	0	0	0	1	1	5
17	$(2,3,5,7)$	$\frac{1}{210}$	-1	0	0	0	1	1	2	12
19	$(3,4,5,7)$	$\frac{1}{420}$	-1	-1	0	0	0	0	1	8

Proof. First we claim that $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$. Assume the converse. Then, as in the proof of Lemma 3.3, the class of $K_X + qA$ is a non-trivial n-torsion element in $\mathrm{Cl}(X)$ defining a global cover $\pi\colon X' \to X$. We have $K_{X'} + qA' \sim 0$, where $A' = \pi^*A$. Hence X' is such as in Lemma 3.5 and by Corollary 3.5 we have $\mathrm{Cl}(X') \simeq \mathbb{Z}\cdot A'$ and $\mathrm{qW}(X') = \mathrm{q}\mathbb{Q}(X') \geq q$. The Galois group $\boldsymbol{\mu}_n$ acts naturally on X'. Consider, for example, the case $q = 11$ and $\mathbf{B}_{X'} = (2,3,5)$ (all other cases are similar). Then X' has three cyclic quotient singularities whose indices are 2, 3, and 5. These points must be $\boldsymbol{\mu}_n$-invariant. Hence the variety X has cyclic quotient singularities of indices $2n$, $3n$, and $5n$. By Lemma 3.2 we have $\gcd(q,n) \neq 1$. In particular, $n \geq 11$. This contradicts (2.8). Therefore, $\mathrm{qW}(X) = \mathrm{q}\mathbb{Q}(X)$ and so X is such as in Lemma 3.5.
Now we have to exclude only the case $q = 9$, $\mathbf{B} = (2,5,13)$. But in this case by (2.6) and (2.5) we have $A^3 = 9/130$ and $-K_X\cdot c_2 = 621/130$. On the other hand, by Kawamata-Bogomolov's bounds [Kaw92a] we have $2673/130 = (4q^2-3q)A^3 \leq 4K_X\cdot c_2 = 1242/65$ [Suz04, Proposition 2.2]. The contradiction shows that this case is impossible. Finally, the values of A^3 and dimensions of $\lvert kA\rvert$ are computed by using (2.5) and (2.4). □

Corollary 3.7. *Let X be a $\mathbb{Q}$-Fano threefold satisfying assumptions of* (i)-(v) *of Theorem* 1.4. *Then X has only cyclic quotient singularities.*

Proof. Indeed, in these cases the indices of points in the basket $\mathbf{B}$ are distinct numbers and moreover $\mathbf{B}$ contains no pairs of points of indices 2 and 4. Then the assertion follows by [Mor85a], or [Rei87] □

Corollary 3.8. *Let X be a $\mathbb{Q}$-Fano threefold with* $\mathrm{q}\mathbb{Q}(X) \geq 9$. *Then* $\dim\lvert A\rvert \leq 0$.

Computer computations similar to that in Lemma 3.5 allow us to prove the following.

Lemma 3.9. *Let X be a Fano threefold with terminal singularities, let $q := \mathrm{qW}(X)$, and let $A := -\frac{1}{q}K_X$.*

(i) *If $q \geq 5$ and $\dim |A| > 1$, then $q = 5$, $\mathbf{B} = (2)$, and $A^3 = 1/2$.*
(ii) *If $q \geq 7$ and $\dim |A| > 0$, then $q = 7$, $\mathbf{B} = (2, 3)$, $A^3 = 1/6$.*

3.10. Proof of (vi) and (vii) of Theorem 1.4. (vii) Apply Lemma 3.9. Then the result is well-known: in fact, $2A$ is Cartier and by Riemann-Roch $\dim |2A| = 6 = \dim X + 3$. Hence X is a variety of Δ-genus zero [Fuj75], i.e., a variety of minimal degree. Then $X \simeq \mathbb{P}(1^3, 2)$.
(vi) Put $q := \mathrm{q}\mathbb{Q}(X)$, $\Xi := K_X + qA$, and $\Xi_1 := A - A_1$. By our assumption $n\Xi \sim n\Xi_1 \sim 0$ for some integer n. If either $\Xi \not\sim 0$ or $\Xi_1 \not\sim 0$, then elements Ξ and Ξ' define an étale in codimension one finite cover $\pi \colon X' \to X$ such that $K_{X'} + qA' \sim 0$ and $A' \sim A'_1$, where $A' := \pi^*A$ and $A'_1 := \pi^*A_1$. If $\Xi \sim \Xi_1 \sim 0$, we put $X' = X$. In both cases, the following inequalities hold: $\mathrm{qW}(X') \geq 7$ and $\dim |A'| \geq 1$. By Lemma 3.9 we have $\mathbf{B}(X') = (2, 3)$ and $\mathrm{q}\mathbb{Q}(X') = \mathrm{qW}(X') = 7$. Note that the Gorenstein index of X' is strictly less than $\mathrm{qW}(X')$. In this case, $X' \simeq \mathbb{P}(1^2, 2, 3)$ according to [San96]. [§] Now it is sufficient to show that π is an isomorphism. Assume the converse. By our construction, there is an action of a cyclic group $\boldsymbol{\mu}_p \subset \mathrm{Gal}(X'/X)$, p is prime, such that π is decomposed as $\pi \colon X' \to X'/\boldsymbol{\mu}_p \to X$. Here $X'/\boldsymbol{\mu}_p$ is a $\mathbb{Q}$-Fano threefold and there is a torsion element of $\mathrm{Cl}(X'/\boldsymbol{\mu}_p)$ which is not Cartier exactly at points where $X' \to X'/\boldsymbol{\mu}_p$ is not étale. There are exactly four such points and two of them are points of indices 2 and 3. Thus the basket of $X'/\boldsymbol{\mu}_p$ consists of points of indices p, p, $2p$, and $3p$. This contradicts Proposition 2.9.

Lemma 3.11. *Let X be a $\mathbb{Q}$-Fano threefold with $q := \mathrm{q}\mathbb{Q}(X)$. If there are three effective different Weil divisors A, A_1, A_2 such that $-K_X \sim_{\mathbb{Q}} qA \sim_{\mathbb{Q}} qA_1 \sim_{\mathbb{Q}} qA_2$ and $A \not\sim A_1$, then $q \leq 5$.*

Proof. Assume that $q \geq 6$. As in 3.10 consider a cover $\pi \colon X' \to X$. Thus on X' we have $A' \sim A'_1 \sim A'_2$ and $-K_{X'} \sim qA'$. Moreover, $\dim |A'| = 1$ according to Lemma 3.9. In this case, the action of $\mathrm{Gal}(X'/X)$ on the pencil $|A'|$ is trivial (because there are three invariant members A', A'_1, and A'_2). But then $A \sim A_1 \sim A_2$, a contradiction. □

4. Birational construction

4.1. Let X be a $\mathbb{Q}$-Fano threefold and let A be the ample Weil divisor that generates the group $\mathrm{Cl}(X)/\sim_{\mathbb{Q}}$. Thus we have $-K_X \sim_{\mathbb{Q}} qA$. Let $\mathscr{M}$ be a mobile linear system without fixed components and let $c := \mathrm{ct}(X, \mathscr{M})$ be the canonical threshold of $(X, \mathscr{M})$. So the pair $(X, c\mathscr{M})$ is canonical but not terminal. Assume that $-(K_X + c\mathscr{M})$ is ample.

Recall that, for any point $P \in X$, the class of K_X is a generator of the local Weil divisor class group $\mathrm{Cl}(X, P)$.

[§] The result also can be easily proved by using birational transformations similar to that in §4.

LEMMA 4.2. *Let $P \in X$ be a point of index $r > 1$. Assume that $\mathscr{M} \sim -mK_X$ near P, where $0 < m < r$. Then $c \le 1/m$.*

Proof. According to [Kaw92b] there is an exceptional divisor Γ over P of discrepancy $a(\Gamma) = 1/r$. Let $\varphi\colon Y \to X$ be a resolution. Clearly, Γ is a prime divisor on Y. Write

$$K_Y = \varphi^* K_X + \frac{1}{r}\Gamma + \sum \delta_i \Gamma_i, \quad \mathscr{M}_Y = \varphi^*\mathscr{M} - \operatorname{ord}_\Gamma(\mathscr{M})\Gamma - \operatorname{ord}_{\Gamma_i}(\mathscr{M})\Gamma_i,$$

where $\mathscr{M}_Y$ is the birational transform of $\mathscr{M}$ and Γ_i are other φ-exceptional divisors. Then

$$K_Y + c\mathscr{M}_Y = \varphi^*(K_X + c\mathscr{M}) + (1/r - c\operatorname{ord}_\Gamma(\mathscr{M}))\Gamma + \dots$$

and so $1/r - c\operatorname{ord}_\Gamma(\mathscr{M}) \ge 0$. On the other hand, $\operatorname{ord}_\Gamma(\mathscr{M}) \equiv m/r \mod \mathbb{Z}$ (because $mK_X + \mathscr{M} \sim 0$ near P). Hence, $\operatorname{ord}_\Gamma(\mathscr{M}) \ge m/r$ and $c \le 1/m$. $\square$

4.3. In the construction below we follow [Ale94]. Let $f\colon \tilde{X} \to X$ be a $K + c\mathscr{M}$-crepant blowup such that $\tilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities:

$$K_{\tilde{X}} + c\tilde{\mathscr{M}} = f^*(K_X + c\mathscr{M}). \tag{4.4}$$

As in [Ale94], we run $K + c\mathscr{M}$-MMP on $\tilde{X}$. We get the following diagram (Sarkisov link of type I or II)

$$\begin{tikzcd} & \tilde{X} \arrow[r, dashed] \arrow[dl, "f"'] & \bar{X} \arrow[dr, "g"] & \\ X & & & \hat{X} \end{tikzcd} \tag{4.5}$$

where the varieties $\tilde{X}$ and $\bar{X}$ have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\tilde{X}) = \rho(\bar{X}) = 2$, f is a Mori extremal divisorial contraction, $\tilde{X} \dashrightarrow \bar{X}$ is a sequence of log flips, and g is a Mori extremal contraction (either divisorial or fiber type). Thus one of the following possibilities holds:

a) $\dim \hat{X} = 1$ and g is a $\mathbb{Q}$-del Pezzo fibration;
b) $\dim \hat{X} = 2$ and g is a $\mathbb{Q}$-conic bundle; or
c) $\dim \hat{X} = 3$, g is a divisorial contraction, and $\hat{X}$ is a $\mathbb{Q}$-Fano threefold. In this case, denote $\hat{q} := \mathrm{q}\mathbb{Q}(\hat{X})$.

Let E be the f-exceptional divisor. In all what follows, for a divisor D on X, let $\tilde{D}$ and $\bar{D}$ denote strict birational transforms of D on $\tilde{X}$ and $\bar{X}$, respectively. If g is birational, we put $\hat{D} := g_*\bar{D}$.

CLAIM 4.6 ([Ale94]). *If the map g of (4.5) is birational, then $\bar{E}$ is not an exceptional divisor. If g is of fiber type, then $\bar{E}$ is not composed of fibers.*

Proof. Assume the converse. If g is birational, this implies that the map $g \circ \chi \circ f^{-1}\colon X \dashrightarrow \hat{X}$ is an isomorphism in codimension 1. Since both X and $\hat{X}$ are Fano threefolds, this implies that $g \circ \chi \circ f^{-1}$ is in fact an isomorphism. On the other hand, the number of $K + c\mathscr{M}$-crepant divisors on $\hat{X}$ is less than that on X, a contradiction. If $\dim \hat{X} \le 2$, then $\bar{E}$ is a pull-back of an ample

Weil divisor on $\hat{X}$. But then $n\bar{E}$ is a movable divisor for some $n > 0$. This contradicts exceptionality of E. □

4.7. Notation. If $|kA| \neq \varnothing$, let $S_k \in |kA|$ be a general member. Write

$$
\begin{array}{rcl}
K_{\tilde{X}} & = & f^*K_X + \alpha E, \\
\tilde{S}_k & = & f^*S_k - \beta_k E, \\
\tilde{\mathscr{M}} & = & f^*\mathscr{M} - \beta_0 E.
\end{array}
\tag{4.8}
$$

Then

$$
c = \alpha/\beta_0. \tag{4.9}
$$

Remark 4.10. If $\alpha < 1$, then $a(E, |-K_X|) < 1$. On the other hand, $0 = K_X + |-K_X|$ is Cartier. Hence, $a(E, |-K_X|) \le 0$ and $K_{\tilde{X}} + f_*^{-1}|-K_X|$ is linearly equivalent to a non-positive multiple of E. Therefore, $f_*^{-1}|-K_X| \subset |-K_{\tilde{X}}|$ and so

$$
\dim|-K_{\bar{X}}| = \dim|-K_{\tilde{X}}| \ge \dim|-K_X|.
$$

In our situation X has only cyclic quotient singularities (see Corollary 3.7). So, the following result is very important.

Theorem 4.11 ([Kaw96]). *Let $(Y \ni P)$ be a terminal cyclic quotient singularity of type $\frac{1}{r}(1, a, r-a)$, let $f\colon \tilde{Y} \to Y$ be a Mori divisorial contraction, and let E be the exceptional divisor. Then $f(E) = P$, f is the weighted blowup with weights $(1, a, r-a)$ and the discrepancy of E is $a(E) = 1/r$.*

We call this f the *Kawamata blowup* of P.

4.12. Notation. Assume that g is birational. Let $\bar{F}$ be the g-exceptional divisor and let $\tilde{F}$ and F be its proper transforms on $\tilde{X}$ and X, respectively. Let n be the maximal integer dividing the class of $\bar{F}$ in $\mathrm{Cl}(\bar{X})$. Let Θ be an ample Weil divisor on $\hat{X}$ that generates $\mathrm{Cl}(\hat{X})/\sim_{\mathbb{Q}}$. Write

$$
\hat{S}_k \sim_{\mathbb{Q}} s_k\Theta \quad \text{and} \quad \hat{E} \sim_{\mathbb{Q}} e\Theta,
$$

where s_k, $e \in \mathbb{Z}$, $s_k \ge 0$, $e \ge 1$. Note that $s_k = 0$ if and only if $\bar{S}_k$ is contracted by g.

Lemma 4.13. *In the above notation assume that the group $\mathrm{Cl}(X)$ is torsion free. Write $F \sim dA$, where $d \in \mathbb{Z}$, $d \ge 1$. Then $\mathrm{Cl}(\hat{X}) \simeq \mathbb{Z} \oplus \mathbb{Z}_n$ and $d = ne$.*

Proof. Write $\bar{F} \sim n\bar{G}$, where $\bar{G}$ is an integral Weil divisor. Then $\bar{E} \sim e\bar{\Theta} + k\bar{G}$ for some $k \in \mathbb{Z}$ and $\mathrm{Cl}(\hat{X}) \simeq \mathrm{Cl}(\bar{X})/\bar{F}\mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}_n$. We have

$$
\mathbb{Z}_d \simeq \mathrm{Cl}(X)/\langle F \rangle \simeq \mathrm{Cl}(\bar{X})/\langle \bar{E},\, \bar{F} \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}/\langle e\bar{\Theta} + k\bar{G},\, n\bar{G} \rangle.
$$

Since the last group is of order ne, we have $d = ne$. □

From now until the end of this section we consider the case where $\hat{X}$ is a surface.

Lemma 4.14. *Assume that $\hat{X}$ is a surface. Then $\hat{X}$ is a del Pezzo surface with Du Val singularities of type A_n. The linear system $|-K_{\hat{X}}|$ is base point free. If moreover the group $\mathrm{Cl}(X)$ is torsion free, then so is $\mathrm{Cl}(\hat{X})$ and there are only the following possibilities:*

(i) $K_{\hat{X}}^2 = 9$, $\hat{X} \simeq \mathbb{P}^2$;

(ii) $K_{\hat{X}}^2 = 8$, $\hat{X} \simeq \mathbb{P}(1^2, 2)$;

(iii) $K_{\hat{X}}^2 = 6$, $\hat{X} \simeq \mathbb{P}(1, 2, 3)$;

(iv) $K_{\hat{X}}^2 = 5$, *$\hat{X}$ has a unique singular point, point of type A_4.*

Proof. By the main result of [MP08b] the surface $\hat{X}$ has only Du Val singularities of type A_n. Since $\rho(\hat{X}) = 1$ and $\hat{X}$ is uniruled, $-K_{\hat{X}}$ is ample. Further, since both $\bar{X}$ and $\hat{X}$ have only isolated singularities and $\mathrm{Pic}(\bar{X}/\hat{X}) \simeq \mathbb{Z}$, there is a well-defined injective map $g^*\colon \mathrm{Cl}(\hat{X}) \to \mathrm{Cl}(\bar{X})$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free whenever $\mathrm{Cl}(X)$ is. The remaining part follows from the classification of del Pezzo surfaces with Du Val singularities (see, e.g., [MZ88]). □

Lemma 4.15. *Let $\varphi\colon Y \to Z$ be a $\mathbb{Q}$-conic bundle (we assume that Y is $\mathbb{Q}$-factorial and $\rho(Y/Z) = 1$). Suppose that there are two prime divisors D_1 and D_2 such that $\varphi(D_i) = Z$, the log divisor $K_Y + D_1 + D_2$ is φ-linearly trivial and canonical. Suppose furthermore that Z is singular and let $o \in Z$ be a singular point. Then $o \in Z$ is of type A_{r-1} for some $r \geq 2$ and there is a Sarkisov link*

$$\begin{tikzcd}
& \tilde{Y} \arrow[rr, dashed, "\chi"] \arrow[dl, "\sigma"'] & & \bar{Y} \arrow[dr, "\bar{\varphi}"] & \\
Y \arrow[drr, "\varphi"] & & & & \bar{Z} \arrow[dll, "\delta"'] \\
& & Z & &
\end{tikzcd}$$

where σ is the Kawamata blowup of a cyclic quotient singularity $\frac{1}{r}(1, a, r-a)$ over o, χ is a sequence of flips, $\bar{\varphi}$ is a $\mathbb{Q}$-conic bundle with $\rho(\bar{Y}/\bar{Z}) = 1$, and δ is a crepant contraction of an irreducible curve to o. Moreover, if $\bar{D}_i$ is the proper transform of D_i on $\bar{Y}$, then the divisor $K_{\bar{Y}} + \bar{D}_1 + \bar{D}_2$ is linearly trivial over Z and canonical.

Proof. Regard Y/Z as an algebraic germ over o. Since D_i are generically sections, the fibration φ has no discriminant curve. By [MP08c] the central fiber $C := \varphi^{-1}(o)_{\mathrm{red}}$ is irreducible and by the main result of [MP08b] Y/Z is toroidal, that is, it is analytically isomorphic to a toric contraction:

$$Y \simeq (\mathbb{C}^2 \times \mathbb{P}^1)/\boldsymbol{\mu}_r(a, r-a, 1)$$

for some r, $a \in \mathbb{Z}$ with $r \geq 2$ and $\gcd(a, r) = 1$. Here the map $Y \to Z$ is the projection to $Z \simeq \mathbb{C}^2/\boldsymbol{\mu}_r(a, r-a)$. In particular, Y has exactly two singular points and these points are cyclic quotients of types $\frac{1}{r}(1, a, r-a)$ and

$\frac{1}{r}(-1, a, r-a)$. Since the pair $(Y, D_1 + D_2)$ is canonical, $D_1 \cap D_2 = \varnothing$. On the other hand, the divisor $D_1 + D_2 \sim -K_Y$ must contain all points on indices > 1. Hence $\operatorname{Sing}(Y) = (D_1 + D_2) \cap C$. Further, the divisors D_i are quotients of two disjointed sections of $\mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ by $\boldsymbol{\mu}_r$. Therefore, $D_i \cdot C = 1/r$. Now consider the Kawamata blowup $\sigma : \tilde{Y} \to Y$ of $C \cap D_1$. Let E be the exceptional divisor and let $\tilde{D}_i$ be the proper transform of D_i. Since $K_{\tilde{Y}} = \sigma^* K_Y + \frac{1}{r} E$ and the pair $(Y, D_1 + D_2)$ is canonical, we have

$$K_{\tilde{Y}} + \tilde{D}_1 + \tilde{D}_2 = \sigma^*(K_Y + D_1 + D_2).$$

It is easy to check locally that the proper transform $\tilde{C}$ of the central fiber C does not meet $\tilde{D}_1$. Moreover, $\tilde{C} \cap E$ is a smooth point of $\tilde{Y}$ and E. Thus we have $\tilde{D}_1 \cdot \tilde{C} = 0$, $E \cdot \tilde{C} = 1$, and $\tilde{D}_2 \cdot \tilde{C} = D_2 \cdot C = 1/r$. Hence, $K_{\tilde{Y}} \cdot \tilde{C} = -1/r$. Since the set-theoretical fiber over o in $\tilde{Y}$ coincides with $E \cup \tilde{C}$, the divisor $-K_{\tilde{Y}}$ is ample over Z and $\tilde{C}$ generates a (flipping) extremal ray R. Run the MMP over Z in this direction, i.e., starting with R. Assume that we end up with a divisorial contraction $\bar{\varphi} \colon \bar{Y} \to \bar{Z}$. Then $\bar{\varphi}$ must contract the proper transform $\bar{E}$ of E. Here $\bar{Z}/Z$ is a Mori conic bundle and the map $Y \dashrightarrow \bar{Z}$ is an isomorphism in codimension one, so it is an isomorphism. Moreover, $\bar{Z}/Z$ has a section, the proper transforms of D_i. Hence the fibration $\bar{Z}/Z$ is toroidal over o. Consider Shokurov's difficulty [Sho85]

$$\mathrm{d}(W) := \#\{\text{exceptional divisors of discrepancy} < 1\}.$$

Then $\mathrm{d}(Y) = \mathrm{d}(\bar{Z}) = 2(r-1)$. On the other hand,

$$\mathrm{d}(\bar{Z}) - 1 \le \mathrm{d}(\bar{Y}) < \mathrm{d}(\tilde{Y}) = r - 1 + a - 1 + r - a - 1 = 2r - 3$$

(because the map $\tilde{Y} \dashrightarrow \bar{Y}$ is not an isomorphism). The contradiction shows that our MMP ends up with a $\mathbb{Q}$-conic bundle. Clearly, the divisor $K_{\bar{Y}} + \bar{D}_1 + \bar{D}_2$ is linearly trivial and canonical. By [MP08b] the surface $\bar{Z}$ has at worst Du Val singularities of type A. Hence the morphism δ is crepant [Mor85b]. □

Corollary 4.16. *In the above notation assume that $\bar{Y}$ is a toric variety. Then so is Y.*

Corollary 4.17. *Notation as in Lemma* 4.15. *Assume that the base surface Z is toric. Then so is Y.*

Proof. Induction by the number e of crepant divisors of Z. If $e = 0$, then Y is smooth and $Y \simeq \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a decomposable rank-2 vector bundle on Z. □

Proposition 4.18. *In notation of* 4.3, *let $\hat{X}$ be a surface. Let $\Gamma \in |-K_{\hat{X}}|$ and let $G := g^{-1}(\Gamma)$. Suppose that there are two prime divisors D_1 and D_2 such that $g(D_i) = \hat{X}$ and $K_{\hat{X}} + D_1 + D_2 + G \sim 0$. Then the pair $(\bar{X}, D_1 + D_2)$ is canonical. If furthermore the surface $\hat{X}$ is toric, then so are $\bar{X}$ and X.*

Proof. Clearly, we may replace Γ with a general member of $|-K_{\hat{X}}|$. Note that G is an elliptic ruled surface and $K_G + D_1|_G + D_2|_G \sim 0$. Hence divisors

$D_1|_G$ and $D_2|_G$ are disjointed sections. This shows that $D_1 \cap D_2$ is either empty or consists of fibers. Assume that $D_1 \cap D_2 \neq \varnothing$. We can take Γ so that $G \cap D_1 \cap D_2 = \varnothing$. By adjunction $-K_{D_1} \sim \bar{G}|_{D_1} + D_2|_{D_1}$. Since D_1 is a rational surface (birational to $\hat{X}$), the divisor $\bar{G}|_{D_1} + D_2|_{D_1}$ must be connected, a contradiction. Thus, $D_1 \cap D_2 = \varnothing$.
Therefore both divisors D_1 and D_2 contain no fibers and so $D_1 \simeq D_2 \simeq \hat{X}$. Then the pair $(\bar{X}, D_1 + D_2)$ is PLT by the Inversion of Adjunction. Since $K_{\bar{X}} + D_1 + D_2$ is Cartier, this pair must be canonical. The second assertion follows by Corollary 4.17. □

5. Case $\mathrm{q}\mathbb{Q}(X) = 10$

Consider the case $\mathrm{q}\mathbb{Q}(X) = 10$. We assume that a $\mathbb{Q}$-Fano threefold with $\mathrm{q}\mathbb{Q}(X) = 10$ exists and get a contradiction applying Construction (4.5).
By Proposition 3.6 the group $\mathrm{Cl}(X)$ is torsion free and $\mathbf{B} = (7, 11)$. Recall also that

$$|A| = \varnothing, \quad \dim |2A| = 0, \quad \text{and} \quad \dim |3A| = 1. \tag{5.1}$$

For $r = 7$ and 11, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |3A|$. By (5.1) there exist a (unique) irreducible divisor $S_2 \in |-2K_X|$ and $\mathscr{M}$ is a pencil without fixed components. Let $S_3 \in \mathscr{M} = |3A|$ be a general member.
Apply Construction (4.5). Notations of 4.3 and 4.7 will be used freely. Near P_{11} we have $A \sim -10K_X$, so $\mathscr{M} \sim -8K_X$. By Lemma 4.2 we get $c \le 1/8$. In particular, the pair $(X, \mathscr{M})$ is not canonical. For some a_1, $a_2 \in \mathbb{Z}$ we can write

$$\begin{aligned} K_{\tilde{X}} + 5\tilde{S}_2 &= f^*(K_X + 5S_2) - a_1 E &&\sim -a_1 E, \\ K_{\tilde{X}} + 2\tilde{S}_2 + 2\tilde{S}_3 &= f^*(K_X + 2S_2 + 2S_3) - a_2 E &&\sim -a_2 E. \end{aligned}$$

Therefore,

$$\begin{aligned} K_{\bar{X}} + 5\bar{S}_2 + a_1\bar{E} &\sim 0, \\ K_{\bar{X}} + 2\bar{S}_2 + 2\bar{S}_3 + a_2\bar{E} &\sim 0, \end{aligned} \tag{5.2}$$

where $\dim |S_2| = 0$ and $\dim |S_3| = 1$. Using (4.8) we obtain

$$\begin{aligned} 5\beta_2 &= a_1 + \alpha, \\ 2\beta_2 + 2\beta_3 &= a_2 + \alpha. \end{aligned} \tag{5.3}$$

Since $S_3 \in \mathscr{M}$ is a general member, by (4.9) we have $c = \alpha/\beta_3 \le 1/8$, so

$$8\alpha \le \beta_3 \quad \text{and} \quad a_2 \ge 15\alpha + 2\beta_2. \tag{5.4}$$

5.5. First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are non-negative integers. In particular, $a_2 \ge 15$. From (5.2) and (5.4) we obtain that g is birational. Indeed, otherwise restricting the second relation of (5.2) to a general fiber V we get that $-K_V$ is divisible by some number $a' \ge a_2 \ge 15$. This is impossible because V is either $\mathbb{P}^1$ or a smooth del Pezzo surface.

Thus g is birational and $\hat{X}$ is a $\mathbb{Q}$-Fano. Again from (5.2) and (5.4) in notation of 4.12 we get

$$-K_{\hat{X}} \sim 2\hat{S}_2 + 2\hat{S}_3 + a_2\hat{E} \sim_{\mathbb{Q}} (2s_2 + 2s_3 + a_2e)\Theta,$$

where $s_2,\ s_3 \geq 0$, $e \geq 1$, and $a_2 \geq 15$. This immediately gives us $\mathrm{q}\mathbb{Q}(\hat{X}) \geq 15$ and $e = 1$, that is, $\hat{E} \sim_{\mathbb{Q}} \Theta$. By Proposition 3.6 the group $\mathrm{Cl}(\hat{X})$ is torsion free. In particular, $\hat{E} \sim \Theta$ and $|\Theta| \neq \varnothing$. On the other hand, again by Proposition 3.6 we have $|\Theta| = \varnothing$, a contradiction.

5.6. Therefore $f(E)$ is a non-Gorenstein point P_r of index $r = 7$ or 11. By Theorem 4.11 f is Kawamata blowup and $\alpha = 1/r$. Near P_r we can write $A \sim -l_rK_X$, where $l_r \in \mathbb{Z}$ and $10l_r \equiv 1 \mod r$. Then $S_k + kl_rK_X$ is Cartier near P_r. Therefore, $\beta_k \equiv kl_r\alpha \mod \mathbb{Z}$ and so $\beta_k = kl_r/r + m_k$, where $m_k = m_{k,r} \in \mathbb{Z}$. Explicitly, we have the following values of α, β_k, and a_k:

r	α	β_2	β_3	a_1	a_2
7	$\frac{1}{7}$	$\frac{3}{7} + m_2$	$\frac{1}{7} + m_3$	$2 + 5m_2$	$1 + 2m_2 + 2m_3$
11	$\frac{1}{11}$	$\frac{9}{11} + m_2$	$\frac{8}{11} + m_3$	$4 + 5m_2$	$3 + 2m_2 + 2m_3$

Claim 5.7. *If $r = 7$, then $m_3 \geq 1$.*

Proof. Follows from $c = \alpha/\beta_3 \leq 1/8$. □

If g is not birational, then, as above, restricting relations (5.2) to a general fiber V we get

$$-K_V \sim 5\bar{S}_2|_V + a_1\bar{E}|_V \sim 2\bar{S}_2|_V + 2\bar{S}_3|_V + a_2\bar{E}|_V,$$

where $E|_V \neq 0$ and $S_2|_V$, $S_3|_V$, and $E|_V$ are proportional to $-K_V$ (because $\rho(\bar{X}/\hat{X}) = 1$). Since V is either $\mathbb{P}^1$, or a smooth del Pezzo surface, $S_2|_V = 0$ and $a_i \leq 3$. So, $r = 7$. By the above claim and computations in the table we have $a_2 = 3$, $m_1 = 1$, and $m_2 = 0$. Hence, $a_1 = 2$. But then

$$-K_V \sim 2\bar{E}|_V \sim 2\bar{S}_3|_V + 3\bar{E}|_V,$$

a contradiction.

Thus g is birational. Below we will use notation of 4.12. Since $\bar{S}_3$ is moveable, $s_3 \geq 1$. Put

$$u := s_2 + em_2, \qquad v := s_3 + em_3.$$

5.8. Case: $r = 11$. Since $\mathrm{Cl}(X)/\!\sim_{\mathbb{Q}} \simeq \mathbb{Z}\cdot\Theta$, pushing down (5.2) to $\hat{X}$ we obtain the following relations

$$\begin{aligned} \hat{q} &= 5s_2 + (4 + 5m_2)e &&= 5u + 4e, \\ \hat{q} &= 2s_2 + 2s_3 + (3 + 2m_2 + 2m_3)e &&= 2u + 2v + 3e. \end{aligned} \tag{5.9}$$

Assume that $u = 0$. Then $\hat{q} = 4e$. The only solution of (5.9) with $\hat{q}$ allowed by Proposition 3.6 is the following: $\hat{q} = 8$, $v = 1$, $e = 2$. Hence, $s_2 = 0$ and $s_3 = 1$. In particular, $\dim|\Theta| \geq \dim|S_3| = 1$. On the other hand, by Lemma 4.13 the

group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 3.9 the divisor Θ is not moveable, a contradiction.
Therefore, $u \geq 1$. By the first relation in (5.9) $\hat{q} \geq 9$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free (Proposition 3.6). Then by Lemma 4.13 we have $F \sim eA$. Since $|A| = \varnothing$, $e \geq 2$. Again by (5.9) $\hat{q} \geq 13$ and e is odd. Thus, $e = 3$, $u = 1$, and $\hat{q} = 17$. Further, $s_3 + em_3 = v = 3$ and $s_3 = 3$ (because $\bar{S}_3$ is moveable). By Proposition 3.6 we have $1 = \dim |S_3| \leq \dim |3\Theta| = 0$, a contradiction.

5.10. Case: $r = 7$. Recall that $m_3 \geq 1$ by Claim 5.7. As in 5.8 write

$$
\begin{array}{lccccc}
(5.11) & \hat{q} & = & 5s_2 + (2 + 5m_2)e & = & 5u + 2e, \\
& \hat{q} & = & 2s_2 + 2s_3 + (1 + 2m_2 + 2m_3)e & = & 2u + 2v + e.
\end{array}
$$

Hence, $v = s_3 + em_3 \geq 1 + e$.
If $u = 0$, then $\hat{q} = 2e = 2v + e$, $e = 2v$, and $\hat{q} = 4v \geq 4(1 + e) = 4(1 + 2v)$, a contradiction. If $u = 2$, then $\hat{q}$ is even ≥ 12. Again we have a contradiction.
Assume that $u \geq 3$. Using the first relation in (5.11) and Proposition 3.6 we get successively

$$u = 3, \quad \hat{q} \geq 17, \quad |\Theta| = \varnothing, \quad e \geq 2, \quad \hat{q} \geq 19, \quad |2\Theta| = \varnothing, \quad e \geq 3,$$

and so $\hat{q} \geq 21$, a contradiction.
Therefore, $u = 1$. Then $\hat{q} = 5 + 2e = 2 + 2v + e$ and $2v = 3 + e = 2v \geq 2 + 2e$. So, $e = 1$, $v = 2$, $\hat{q} = 7$. Since $m_3 \geq 1$, $s_3 = v - em_3 = 1$. Hence, $\hat{S}_3 \sim_{\mathbb{Q}} \Theta$. Since $\dim |\hat{S}_3| \geq 1$, by (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. In particular, the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.13 the divisor F generates the group $\mathrm{Cl}(X)$. This contradicts $|A| = \varnothing$.
The last contradiction finishes the proof of (v) of Theorem 1.4.

6. Case $\mathrm{q}\mathbb{Q}(X) = 11$ and $\dim |-K_X| \geq 11$

In this section we consider the case $\mathrm{q}\mathbb{Q}(X) = 11$ and $\dim |-K_X| \geq 11$. By Proposition 3.6 the group $\mathrm{Cl}(X)$ is torsion free and $\mathbf{B} = (2, 3, 5)$. Recall that

$$\dim |A| = 0, \quad \dim |2A| = 1, \quad \text{and} \quad \dim |3A| = 2.$$

It is easy to see that, for $m = 1$, 2, and 3, general members $S_m \in |-mK_X|$ are irreducible. For $r = 2$, 3, 5, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |2A|$. By the above, $\mathscr{M}$ is a pencil without fixed components. Apply Construction (4.5). Near P_5 we have $A \sim -K_X$ and $\mathscr{M} \sim -2K_X$. By Lemma 4.2 we get $c \leq 1/2$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 6.1. *In the above notation, f is the Kawamata blowup of P_5 and $\hat{X}$ is a del Pezzo surface with Du Val singularities with $K_{\hat{X}}^2 = 5$ or 6. Moreover, for $k = 1$, 2 and 3, the image $C_k := g(\bar{S}_k)$ is a curve on $\hat{X}$ with $-K_{\hat{X}} \cdot C_k = k$.*

Proof. Similar to (5.2)-(5.3) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$
\begin{array}{lrcl}
& K_{\bar{X}} + 11\bar{S}_1 + a_1\bar{E} & \sim & 0, \\
(6.2) & K_{\bar{X}} + \bar{S}_1 + 5\bar{S}_2 + a_2\bar{E} & \sim & 0, \\
& K_{\bar{X}} + 2\bar{S}_1 + 3\bar{S}_3 + a_3\bar{E} & \sim & 0,
\end{array}
$$

$$
\begin{array}{lrcl}
(6.3) & 11\beta_1 & = & a_1+\alpha, \\
& \beta_1+5\beta_2 & = & a_2+\alpha, \\
& 2\beta_1+3\beta_3 & = & a_3+\alpha.
\end{array}
$$

Since $S_2 \in \mathscr{M}$ is a general member, by (4.9) we have $c = \alpha/\beta_2 \le 1/2$, so $2\alpha \le \beta_2$ and $a_2 \ge 9\alpha + \beta_1$. Since $2S_1 \sim S_2$, we have $2\beta_1 \ge \beta_2$. Thus $\beta_1 \ge \alpha$ and a_1, $a_2 \ge 10\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers, so a_1, $a_2 \ge 10$. Restricting (6.2) to a general fiber of g we obtain that g is birational. Moreover, in notation of 4.12 we have $\hat{q} \ge 15$, the group $\operatorname{Cl}(\hat{X})$ is torsion free, and $\hat{E} \sim \Theta$. In particular, $|\Theta| \ne \varnothing$. This contradicts Proposition 3.6.

6.4. Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 2, 3$ or 5. As in 5.6 we have the following values of β_k and a_k:

r	β_1	β_2	β_3	a_1	a_2	a_3
2	$\frac{1}{2}+m_1$	m_2	$\frac{1}{2}+m_3$	$5+11m_1$	m_1+5m_2	$2+2m_1+3m_3$
3	$\frac{2}{3}+m_1$	$\frac{1}{3}+m_2$	m_3	$7+11m_1$	$1+m_1+5m_2$	$1+2m_1+3m_3$
5	$\frac{1}{5}+m_1$	$\frac{2}{5}+m_2$	$\frac{3}{5}+m_3$	$2+11m_1$	$2+m_1+5m_2$	$2+2m_1+3m_3$

Claim 6.5. *If $r = 2$ or 3, then $m_2 \ge 1$.*

Proof. Follows from $1/2 \ge c = \alpha/\beta_2 = 1/r\beta_2$. □

Assume that g is birational. By Proposition 3.6 and Remark 4.10 we have $\dim|-K_{\hat{X}}| \ge |-K_X| = 23$. So, in notation of 4.12, $\hat{q} \le 11$. If $\bar{S}_1$ is not contracted, then by the first relation in (6.2) we have $\hat{q} \ge 11 + a_1 \ge 13$, a contradiction. Therefore the divisor $\bar{S}_1$ is contracted. By Lemma 4.13 the group $\operatorname{Cl}(\hat{X})$ is torsion free and $\hat{E} \sim \Theta$. Hence, $\hat{q} = a_1 \le 7$, $m_1 = 0$, and $r \ne 5$. But then $m_2 \ge 1$ (see Claim 6.5) and $a_2 \ge 5$. This contradicts the second relation in (6.2).

Therefore g is of fiber type. Restricting (6.2) to a general fiber we get $a_i \le 3$. Thus, $r = 5$ and $a_1 = a_2 = a_3 = 2$. Moreover, divisors $\bar{S}_1$, $\bar{S}_2$, and $\bar{S}_3$ are g-vertical. Since $\bar{S}_3$ is irreducible and $\dim|\bar{S}_3| = 2$, the variety $\hat{X}$ cannot be a curve. Therefore $\hat{X}$ is a surface and the images $g(\bar{S}_1)$, $g(\bar{S}_2)$, and $g(\bar{S}_3)$ are curves. Since $\dim|\bar{S}_1| = 0$, we have $\dim|g(\bar{S}_1)| = 0$. Hence, $K_{\hat{X}}^2 \le 6$ and $g(\bar{S}_1)$ is a line on $\hat{X}$. By Lemma 4.14 there are only two possibilities: $\hat{X} \simeq \mathbb{P}(1,2,3)$ and $\hat{X}$ is an A_4-del Pezzo surface. □

6.6. Consider the case where $\hat{X}$ is an A_4-del Pezzo surface. Assume that $\bar{S}_6$ is g-vertical. By Riemann-Roch for Weil divisors on surfaces with Du Val singularities [Rei87] we have $\dim|\bar{S}_6| = \dim|g(\bar{S}_6)| = 6$. On the other hand, $\dim|\bar{S}_6| = \dim|S_6| = 7$, a contradiction. Thus $g(\bar{S}_5) = \hat{X}$. Since $K_X + S_5 + S_6 \sim 0$,

$$K_{\bar{X}} + \bar{S}_5 + \bar{S}_6 + \bar{E} \sim 0.$$

Therefore $\bar{S}_6$ and $\bar{E}$ are sections of g. By Proposition 4.18 the pair $(\bar{X}, \bar{S}_6 + \bar{E})$ is canonical. Now since $\bar{S}_5$ is nef, the map $\bar{X} \dashrightarrow \tilde{X}$ is a composition of steps of the $K_{\bar{X}} + \bar{S}_6 + \bar{E}$-MMP. Hence the pair $(\tilde{X}, \tilde{S}_6 + E)$ is also canonical. In particular, $\tilde{S}_6 \cap E = \varnothing$ and so $P_5 = f(E) \notin S_6$, a contradiction.

6.7. Now consider the case $\hat{X} \simeq \mathbb{P}(1,2,3)$. As above, if $g(\bar{S}_5)$ is a curve, then $\dim |g(\bar{S}_5)| = 5$ and $g(\bar{S}_5) \sim 5g(\bar{S}_1)$. On the other hand, $g(\bar{S}_5) \sim -\frac{5}{6}K_{\hat{X}}$. But then $\dim |g(\bar{S}_5)| = 4$, a contradiction. Therefore, $g(\bar{S}_5) = \hat{X}$. Similar to (6.2) we have $K_{\bar{X}} + 2\bar{S}_5 + \bar{S}_1 + a_4\bar{E} \sim 0$. This shows that $a_4 = 0$ and $\bar{S}_5$ is a section of g. Thus we can write $K_{\bar{X}} + \bar{S}_5 + G + \bar{E} \sim 0$, where G is a g-trivial Weil divisor, i.e., $G = g^*\Gamma$ for some Weil divisor Γ. Pushing down this equality to X we get $G \sim 6\bar{S}_1$, i.e., $\Gamma \in |-K_{\hat{X}}|$. By Proposition 4.18 varieties $\bar{X}$ and X are toric. This proves (iv) of Theorem 1.4.

7. Case $q\mathbb{Q}(X) = 13$ and $\dim|-K_X| \geq 6$

In this section we consider the case $q\mathbb{Q}(X) = 13$ and $\dim|-K_X| \geq 6$. By Proposition 3.6 $\mathbf{B} = (3,4,5)$. Recall that

$$\dim |A| = \dim |2A| = 0,\ \dim |3A| = 1,\ \dim |4A| = 2, \text{ and } \dim |5A| = 3.$$

Therefore, for $m = 1$, 3, 4, and 5, general members $S_m \in |-mK_X|$ are irreducible. For $r = 3$, 4, 5, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |4A|$. Since $1 = \dim|3A| > \dim \mathscr{M} = 2$, the linear system $\mathscr{M}$ has no fixed components. Apply Construction (4.5). Near P_5 we have $A \sim -2K_X$ and $\mathscr{M} \sim -3K_X$. By Lemma 4.2 we get $c \leq 1/3$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 7.1. *In the above notation, f is the Kawamata blowup of P_5, g is birational, it contracts $\bar{S}_1$, and $\hat{X} \simeq \mathbb{P}(1^3, 2)$. Moreover, in notation of* 4.12 *we have $\hat{S}_3 \sim \hat{S}_4 \sim \hat{E} \sim \Theta$ and $\hat{S}_5 \sim 2\Theta$.*

Proof. Similar to (5.2)-(5.3) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$\begin{array}{lrcl} (7.2) & K_{\bar{X}} + 13\bar{S}_1 + a_1\bar{E} & \sim & 0, \\ & K_{\bar{X}} + \bar{S}_1 + 4\bar{S}_3 + a_2\bar{E} & \sim & 0, \\ & K_{\bar{X}} + \bar{S}_1 + 3\bar{S}_4 + a_3\bar{E} & \sim & 0, \end{array}$$

$$\begin{array}{lrcl} (7.3) & 13\beta_1 & = & a_1 + \alpha, \\ & \beta_1 + 4\beta_3 & = & a_2 + \alpha, \\ & \beta_1 + 3\beta_4 & = & a_3 + \alpha. \end{array}$$

Since $S_4 \in \mathscr{M}$ is a general member, by (4.9) we have $c = \alpha/\beta_4 \leq 1/3$, $3\alpha \leq \beta_4$ and $a_3 \geq 8\alpha + \beta_1$. Since $4S_1 \sim S_4$, we have $4\beta_1 \geq \beta_4$. Thus $\beta_1 \geq \alpha$ and $a_1 \geq 12\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. In particular, $a_1 \geq 12$. From the first relation in (7.2) we obtain that g is birational. Moreover, in notation of 4.12

we have $\hat{q} \geq 13$ and $\hat{E} \sim \Theta$. In particular, $|\Theta| \neq \varnothing$. By Proposition 3.6 we have $\hat{q} = 13$, $a_1 = 13$, $\bar{S}_1$ is contracted, and $\alpha = 1$. This contradicts (7.3).
Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 3, 4$ or 5. By Theorem 4.11 $\alpha = 1/r$. Similar to 5.6 we have (here $m_k \in \mathbb{Z}_{\geq 0}$)

r	β_1	β_3	β_4	β_5	a_1	a_2	a_3
3	$\frac{1}{3} + m_1$	m_3	$\frac{1}{3} + m_4$	$\frac{2}{3} + m_5$	$4 + 13m_1$	$m_1 + 4m_3$	$1 + m_1 + 3m_4$
4	$\frac{1}{4} + m_1$	$\frac{3}{4} + m_3$	m_4	$\frac{1}{4} + m_5$	$3 + 13m_1$	$3 + m_1 + 4m_3$	$m_1 + 3m_4$
5	$\frac{2}{5} + m_1$	$\frac{1}{5} + m_3$	$\frac{3}{5} + m_4$	m_5	$5 + 13m_1$	$1 + m_1 + 4m_3$	$2 + m_1 + 3m_4$

Claim 7.4. *If $r = 3$ or 4, then $m_4 \geq 1$.*

Proof. Follows from $1/3 \geq c = \alpha/\beta_4 = 1/r\beta_4$. □

If g is not birational, then $a_1 = 3$, $r = 4$, $m_4 \geq 1$, and $a_3 \geq 3$. In this case, $a_2 = a_3 = 3$, g is a generically $\mathbb{P}^2$-bundle, and divisors $\bar{S}_1$, $\bar{S}_3$, $\bar{S}_4$ are g-vertical. Since $\dim|\bar{S}_4| > 1$ and the divisor $\bar{S}_4$ is irreducible, we have a contradiction. Therefore g is birational. Below we will use notation of 4.12.
By Proposition 3.6 we have $\dim|-K_{\hat{X}}| \geq |-K_X| = 19$ and $\hat{q} \leq 13$. From the first relation in (7.2) we see that $\bar{S}_1$ is contracted. By Lemma 4.13 the group $\mathrm{Cl}(\hat{X})$ is torsion free and $\hat{E} \sim \Theta$. Moreover, $m_1 = 0$ (because $13m_1 < a_1 e = \hat{q} \leq 13$). Thus $\hat{q} = a_1 = 4$, 3, and 5 in cases $r = 3$, 4, and 5, respectively.
In cases $r = 3$ and 4 we have $\hat{q} \geq 3 + a_3 \geq 6$, a contradiction. Therefore, $r = 5$, $\hat{q} = 5$, and $s_3 = s_4 = 1$. Since $\dim|\Theta| \geq 1$, by (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^3, 2)$. Since $\dim|S_5| = 3$ and $\dim|\Theta| = 2$, $s_5 \geq 2$. Similar to (7.2)-(7.3) we have $K_{\bar{X}} + \bar{S}_3 + 2\bar{S}_5 + a_4\bar{E} \sim 0$, $2s_5 + a_4 = 4$, and $a_4 = \beta_3 + 2\beta_5 - \alpha = m_3 + 2m_5$. Thus, $s_5 = 2$ and $a_4 = \beta_5 = 0$, i.e., $P_5 \notin S_5$. □

Lemma 7.5. (i) *$S_1 \cap S_3$ is a reduced irreducible curve.*
(ii) *$S_1 \cap S_3 \cap S_4 = \{P_5\}$.*

Proof. (i) Recall that $A^3 = 1/60$ by Proposition 3.6. Write $S_1 \cap S_3 = C + \Gamma$, where C is a reduced irreducible curve passing through P_5 and Γ is an effective 1-cycle. Suppose, $\Gamma \neq 0$. Then $1/4 = S_1 \cdot S_3 \cdot S_5 > S_5 \cdot C$. Since $P_5 \notin S_5$, $C \not\subset S_5$ and $S_5 \cdot C \geq 1/4$, a contradiction. Hence, $S_1 \cap S_3 = C$.
(ii) Assume that $S_1 \cap S_3 \cap S_4 \ni P \neq P_5$. Since $1/5 = S_1 \cdot S_3 \cdot S_4 = S_4 \cdot C$ and $P, P_5 \in S_4 \cap C$, we have $C \subset S_4$. If there is a component $C' \neq C$ of $S_1 \cap S_4$ not contained in S_5, then, as above, $1/3 = S_1 \cdot S_4 \cdot S_5 \geq S_5 \cdot C + S_5 \cdot C' \geq 1/2$, a contradiction. Thus we can write $S_1 \cap S_4 = C + \Gamma$, where Γ is an effective 1-cycle with $\operatorname{Supp}\Gamma \subset S_5$. In particular, $P_5 \notin \Gamma$. The divisor $12A$ is Cartier at P_3 and P_4. We get

$$\frac{1}{5} = 12A^3 = 12A \cdot S_1 \cdot (S_4 - S_3) = 12A \cdot \Gamma \in \mathbb{Z},$$

a contradiction. □

Lemma 7.6. *Let X be a $\mathbb{Q}$-Fano threefold and $D = D_1 + \cdots + D_4$ be a divisor on X, where D_i are irreducible components. Let $P \in X$ be a cyclic quotient singularity of index r. Assume that $K_X + D \sim_{\mathbb{Q}} 0$, $P \notin D_4$, $D_1 \cap D_2 \cap D_3 = \{P\}$, and $D_1 \cdot D_2 \cdot D_3 = 1/r$. Then the pair (X, D) is LC.*

Proof. Let $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ be the index-one cover. For $k = 1, 2, 3$, let $D_k^\sharp$ be the preimage of D_k and let $D^\sharp := D_1^\sharp + D_2^\sharp + D_3^\sharp$. By our assumptions $D_1^\sharp \cap D_2^\sharp \cap D_3^\sharp = \{P^\sharp\}$. Since $D_1 \cdot D_2 \cdot D_3 = 1/r$, locally near $P^\sharp$ we have $D_1^\sharp \cdot D_2^\sharp \cdot D_3^\sharp = 1$. Hence $D^\sharp$ is a simple normal crossing divisor (near $P^\sharp$). In particular, $(X^\sharp, D^\sharp)$ is LC near $P^\sharp$ and so is (X, D) near P.
Thus the pair (X, D) is LC in some neighborhood $U \ni P$. Since $D_1 \cap D_2 \cap D_3 = \{P\}$, P is a center of LC singularities for (X, D). Let H be a general hyperplane section through P. Write $\lambda D_4 \sim_{\mathbb{Q}} H$, where $\lambda > 0$. If (X, D) is not LC in $X \setminus U$, then the locus of log canonical singularities of the pair $(X, D + \epsilon H - (\lambda\epsilon + \delta) D_4)$ is not connected for $0 < \delta \ll \epsilon \ll 1$. This contradicts Connectedness Lemma [Sho92], [Kol92]. Therefore the pair (X, D) is LC. □

7.7. Proof of (iii) of Theorem 1.4. By Lemma 7.6 the pair $(X, S_1 + S_3 + S_4 + S_5)$ is LC. Since $K_X + S_1 + S_3 + S_4 + S_5 \sim 0$, it is easy to see that $a(E, S_1 + S_3 + S_4 + S_5) = -1$. Thus $K_{\tilde{X}} + \tilde{S}_1 + \tilde{S}_3 + \tilde{S}_4 + \tilde{S}_5 = f^*(K_X + S_1 + S_3 + S_4 + S_5) \sim 0$. Therefore the pairs $(\bar{X}, \bar{S}_1 + \bar{S}_3 + \bar{S}_4 + \bar{S}_5 + \bar{E})$ and $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E})$ are also LC. It follows from Proposition 7.1 and its proof that $\hat{X} \simeq \mathbb{P}(1^3, 2)$, $\hat{E} \sim \hat{S}_3 \sim \hat{S}_4 \sim \Theta$, and $\hat{S}_5 \sim 2\Theta$. We claim that $\hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E}$ is a toric boundary (for a suitable choice of coordinates in $\mathbb{P}(1^3, 2)$). Let $(x_1 : x_1' : x_1'' : x_2)$ be homogeneous coordinates in $\mathbb{P}(1^3, 2)$. Clearly, we may assume that $\hat{E} = \{x_1 = 0\}$, $\hat{S}_3 = \{x_1' = 0\}$, and $\hat{S}_4 = \{\alpha x_1 + \alpha' x_1' + \alpha'' x_1'' = 0\}$ for some constants α, α', α''. Since $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{E})$ is LC, $\alpha'' \neq 0$ and after a coordinate change we may assume that $\hat{S}_4 = \{x_1'' = 0\}$. Further, the surface $\hat{S}_5$ is given by the equation $\beta x_2 + \psi(x_1, x_1', x_1'') = 0$, where β is a constant and ψ is a quadratic form. If $\beta = 0$, then $\hat{S}_3 \cap \hat{S}_4 \cap \hat{E} \cap \hat{S}_5 \neq \varnothing$ and the pair $(\hat{X}, \hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E})$ cannot be LC. Thus $\beta \neq 0$ and after a coordinate change we may assume that $\hat{S}_5 = \{x_2 = 0\}$. Therefore $\hat{S}_3 + \hat{S}_4 + \hat{S}_5 + \hat{E}$ is a toric boundary. Then by Lemma 7.8 below the varieties $\bar{X}$, $\tilde{X}$, and X are toric. This proves (iii) of Theorem 1.4.

Lemma 7.8 (see, e.g., [McK01, 3.4]). *Let V be a toric variety and let Δ be the toric (reduced) boundary. Then every valuation ν with discrepancy -1 with respect to $K_V + \Delta$ is toric, that is, there is a birational toric morphism $\tilde{V} \to V$ such that ν corresponds to an exceptional divisor.*

8. Case $\mathrm{q}\mathbb{Q}(X) = 17$

Consider the case $\mathrm{q}\mathbb{Q}(X) = 17$. By Proposition 3.6 $\mathbf{B} = (2, 3, 5, 7)$ and $|A| = \varnothing$, $\dim |2A| = \dim |3A| = \dim |4A| = 0$, $\dim |5A| = \dim |6A| = 1$, $\dim |7A| = 2$. Therefore, for $m = 2$, 3, 5, and 7 general members $S_m \in |-mK_X|$ are irreducible. For $r = 2$, 3, 5, 7, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |5A|$ and apply Construction (4.5). Near P_7 we have $A \sim -5K_X$ and $\mathscr{M} \sim -4K_X$. By Lemma 4.2 we get $c \le 1/4$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 8.1. *In the above notation, f is the Kawamata blowup of P_7, g is birational, it contracts $\bar{S}_2$, and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. Moreover, in notation of* 4.12 *we have $\hat{S}_3 \sim \hat{S}_5 \sim \Theta$, $\hat{E} \sim 2\Theta$, and $\hat{S}_7 \sim 3\Theta$.*

Proof. Similar to (5.2)-(5.3) we have for some $a_1, a_2, a_3 \in \mathbb{Z}$:

$$\begin{array}{rcl} K_{\bar{X}} + 7\bar{S}_2 + \bar{S}_3 + a_1\bar{E} & \sim & 0, \\ K_{\bar{X}} + \bar{S}_2 + 5\bar{S}_3 + a_2\bar{E} & \sim & 0, \\ K_{\bar{X}} + \bar{S}_2 + 3\bar{S}_5 + a_3\bar{E} & \sim & 0, \end{array} \tag{8.2}$$

$$\begin{array}{rcl} 7\beta_2 + \beta_3 & = & a_1 + \alpha, \\ \beta_2 + 5\beta_3 & = & a_2 + \alpha, \\ \beta_2 + 3\beta_5 & = & a_3 + \alpha. \end{array} \tag{8.3}$$

Since $S_5 \in \mathscr{M}$ is a general member, by (4.9) we have $c = \alpha/\beta_5 \le 1/4$, so $4\alpha \le \beta_5$ and $a_3 \ge 11\alpha + \beta_2$. Since $S_2 + S_3 \sim S_5$, we have $\beta_2 + \beta_3 \ge \beta_5 \ge 4\alpha$. Hence, $a_1 \ge 6\beta_2 + 3\alpha$ and $a_2 \ge 4\beta_3 + 3\alpha$.

First we consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. In particular, $a_3 \ge 11$ and by the third relation in (8.2) we obtain that g is birational. Moreover, in notation of 4.12 we have $\hat{q} \ge 11$. In particular, the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $\hat{E} \ge 2\Theta$. Hence, $\hat{q} \ge 2a_3 \ge 22$, a contradiction.

Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 2, 3, 5$ or 7. Similar to 5.6 we have $\alpha = 1/r$ and

r	β_2	β_3	β_5	β_7	a_1	a_2	a_3
2	m_2	$\frac{1}{2} + m_3$	$\frac{1}{2} + m_5$	$\frac{1}{2} + m_7$	$7m_2 + m_3$	$2 + m_2 + 5m_3$	$1 + m_2 + 3m_5$
3	$\frac{1}{3} + m_2$	m_3	$\frac{1}{3} + m_5$	$\frac{2}{3} + m_7$	$2 + 7m_2 + m_3$	$m_2 + 5m_3$	$1 + m_2 + 3m_5$
5	$\frac{1}{5} + m_2$	$\frac{4}{5} + m_3$	m_5	$\frac{1}{5} + m_7$	$2 + 7m_2 + m_3$	$4 + m_2 + 5m_3$	$m_2 + 3m_5$
7	$\frac{3}{7} + m_2$	$\frac{1}{7} + m_3$	$\frac{4}{7} + m_5$	m_7	$3 + 7m_2 + m_3$	$1 + m_2 + 5m_3$	$2 + m_2 + 3m_5$

Claim 8.4. (i) *If $r = 2$, then $m_5 \ge 2$ and $m_2 + m_3 \ge 2$.*
(ii) *If $r = 3$, then $m_5 \ge 1$ and $m_2 + m_3 \ge 1$.*
(iii) *If $r = 5$, then $m_5 \ge 1$.*

Proof. Note that $1/4 \ge c = \alpha/\beta_5 = 1/r\beta_5$ and $r\beta_5 \ge 4$. This gives us inequalities for m_5. The inequalities for $m_2 + m_3$ follows from $\beta_2 + \beta_3 \ge \beta_5$. □

From this we have $\min(a_1, a_2, a_3) \ge 3$. Moreover, the equality $\min(a_1, a_2, a_3) = 3$ holds only if $r = 7$. Therefore the contraction g can be of fiber type only if $a_1 = 3$, $r = 7$, $m_2 = m_3 = 0$, $\min(a_1, a_2, a_3) = 3$, $r = 7$, $m_2 = m_3 = m_5 = 0$, $a_3 = 2$, and $a_2 = 1$. Then g is a del Pezzo fibration of degree 9 and by the first relation in (8.2) divisors $\hat{S}_2$ and $\hat{S}_3$ are g-vertical. But then $a_2 = 3$, a contradiction. From now on we assume that g is birational. Thus we use notation of 4.12 as usual.

Since $\bar{S}_5$ is moveable, it is not contracted. Therefore, $s_5 \geq 1$. By (8.2) we have

$$\begin{aligned}\hat{q} &= 7s_2 + s_3 + a_1 e,\\ \hat{q} &= s_2 + 5s_3 + a_2 e,\\ \hat{q} &= s_2 + 3s_5 + a_3 e.\end{aligned}$$

Put

$$u := s_2 + em_2, \qquad v := s_3 + em_3, \qquad w := s_5 + em_5.$$

8.5. Case: $r = 2$. Then $a_3 \geq 7$ and $\hat{q} \geq 3s_5 + a_3 \geq 10$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free. So, $e \geq 2$ and $\hat{q} \geq 3s_5 + 2a_3 \geq 17$. In this case, $|\Theta| = \varnothing$. Therefore, $s_5 \geq 2$ and $\hat{q} \geq 3s_5 + 2a_3 \geq 20$, a contradiction.

8.6. Case: $r = 3$. Then

$$\begin{aligned}\hat{q} &= 7s_2 + s_3 + (2 + 7m_2 + m_3)e &&= 7u + v + 2e,\\ \hat{q} &= s_2 + 5s_3 + (m_2 + 5m_3)e &&= u + 5v,\\ \hat{q} &= s_2 + 3s_5 + (1 + m_2 + 3m_5)e &&= u + 3w + e.\end{aligned}$$

Assume that $u > 0$. Then $\hat{q} \geq 9$. Hence the group $\mathrm{Cl}(\hat{X})$ is torsion free and $e \geq 2$. Since $\dim |S_5| = 1$ and $\dim |\Theta| \leq 0$, we have $s_5 \geq 2$. Since $m_5 \geq 1$ (see Claim 8.4), we have $w \geq 4$ and $\hat{q} > 13$. In this case, $s_5 \geq 5$, a contradiction.

Therefore, $u = 0$, $m_2 = 0$, $s_3 \neq 0$, $m_3 \geq 1$, and $v \geq 2$. So, $\hat{q} = 5v \geq 10$. Then we get a contradiction by (v) of Theorem 1.4.

8.7. Case: $r = 5$. Then

$$\begin{aligned}\hat{q} &= 7s_2 + s_3 + (2 + 7m_2 + m_3)e &&= 7u + v + 2e,\\ \hat{q} &= s_2 + 5s_3 + (4 + m_2 + 5m_3)e &&= u + 5v + 4e,\\ \hat{q} &= s_2 + 3s_5 + (m_2 + 3m_5)e &&= u + 3w.\end{aligned}$$

From the first two relations we have $3u = 2v + e$ and $1 \leq u \leq 2$. Further, $\hat{q} - 4u = 3(v + e)$, so $\hat{q} \equiv u \mod 3$.

If $u = 2$, then e is even and $\hat{q} = 14 + v + 2e \geq 18$. So, $\hat{q} = 19$, a contradiction. Thus $u = 1$, $3 = 2v + e$, and $\hat{q} = 7 + v + 2e \geq 9$. By (v) of Theorem 1.4 $\hat{q}$ is odd. Hence, v is even, $e = 3$, $v = 0$, $\hat{q} = 13$. In this case, $s_5 + 3m_5 = w = 4$. By Claim 8.4 $m_5 = s_5 = 1$. Note that the group $\mathrm{Cl}(\hat{X})$ is torsion free and $s_2 = 1$. Thus $\dim |\Theta| > 0$. This contradicts Proposition 3.6.

8.8. Case: $r = 7$. Then

$$\begin{aligned}\hat{q} &= 7s_2 + s_3 + (3 + 7m_2 + m_3)e &&= 7u + v + 3e,\\ \hat{q} &= s_2 + 5s_3 + (1 + m_2 + 5m_3)e &&= u + 5v + e,\\ \hat{q} &= s_2 + 3s_5 + (2 + m_2 + 3m_5)e &&= u + 3w + 2e.\end{aligned}$$

Assume that $u > 0$. Then $\hat{q} \geq 10$, the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $e \geq 2$, $\hat{q} \geq 13$, $u = 1$. From the first two relations we get $\hat{q} + 2 = 7v$. Hence, $v = 3$, $\hat{q} = 19$, $e = 3$, and $s_2 = 0$. This contradicts the equality $1 = u = s_2 + em_2$.

Therefore, $u = 0$ and $s_2 = m_2 = 0$. From the first two relations we get $\hat{q} = 7v$. Thus, $\hat{q} = 7$, $v = 1$, $e = 2$, $w = 1$, $m_3 = m_5 = 0$, and $s_3 = s_5 = 1$. By Lemma 4.13 the group $\mathrm{Cl}(\hat{X})$ is torsion free and so $\dim|\Theta| \geq \dim|\bar{S}_5| > 0$. From (vi) of Theorem 1.4 we have $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. In particular, $\dim|\Theta| = 1$. Further, similar to (8.2) we have

$$
\begin{aligned}
&K_{\bar{X}} + \bar{S}_3 + 2\bar{S}_7 + a_4\bar{E} \sim 0,\\
&\beta_3 + 2\beta_7 = a_4 + \alpha.
\end{aligned}
$$

This gives us $a_4 = 2\beta_7$ and $s_7 + a_4 = 3$. Since $\dim|S_7| = 2$, $s_7 > 1$, $s_7 = 3$, $\hat{S}_7 \sim 3\Theta$, $a_4 = 0$, and $\beta_7 = 0$, i.e., $P_7 \notin S_7$.

□

Lemma 8.9. (i) *$S_2 \cap S_3$ is a reduced irreducible curve.*
(ii) *$S_2 \cap S_3 \cap S_5 = \{P_7\}$.*

Proof. (i) Similar to the proof of (i) of Lemma 7.5.
(ii) Put $C := S_3 \cap S_4$. Assume that $S_2 \cap S_3 \cap S_5 \ni P \neq P_7$. Since $1/7 = S_2 \cdot S_3 \cdot S_5 = S_5 \cdot C$ and P, $P_7 \in S_5 \cap C$, we have $C \subset S_5$. If there is a component $C' \neq C$ of $S_2 \cap S_5$ not contained in S_7, then, as above, $7/15 = S_2 \cdot S_7 \cdot S_7 \geq S_7 \cdot C + S_7 \cdot C' \geq 2/5$, a contradiction. Thus we can write $S_2 \cap S_5 = C + \Gamma$, where Γ is an effective 1-cycle with $\mathrm{Supp}\,\Gamma \subset S_7$. In particular, $P_7 \notin \Gamma$. The divisor $30A$ is Cartier at P_2, P_3, and P_5. We get

$$
\frac{120}{210} = 120A^3 = 30A \cdot S_2 \cdot (S_5 - S_3) = 30A \cdot \Gamma \in \mathbb{Z},
$$

a contradiction. □

Now the proof of (ii) of Theorem 1.4 can be finished similar to 7.7: the pair $(\hat{X}, \hat{S}_3 + \hat{S}_5 + \hat{E} + \hat{S}_7)$ is LC and the corresponding discrepancy of $\bar{S}_2$ is equal to -1.

9. Case $\mathrm{q}\mathbb{Q}(X) = 19$

Consider the case $\mathrm{q}\mathbb{Q}(X) = 19$. By Proposition 3.6 $\mathbf{B} = (3, 4, 5, 7)$ and $|A| = \varnothing$, $|2A| = \varnothing$, $\dim|3A| = \dim|4A| = \dim|5A| = \dim|6A| = 0$, $\dim|7A| = 1$. Therefore, for $m = 3$, 4, 5, and 7 general members $S_m \in |-mK_X|$ are irreducible. For $r = 3$, 4, 5, 7, let P_r be a (unique) point of index r. In notation of §4, take $\mathscr{M} := |7A| = |S_7|$ and apply Construction (4.5). Near P_5 we have $A \sim -4K_X$ and $\mathscr{M} \sim -3K_X$. By Lemma 4.2 we get $c \leq 1/3$. In particular, the pair $(X, \mathscr{M})$ is not canonical.

Proposition 9.1. *In the above notation, f is the Kawamata blowup of P_5, g is birational, it contracts $\bar{S}_3$, and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$. Moreover, in notation of* 4.12 *we have $\hat{S}_4 \sim \hat{S}_7 \sim \Theta$, $\hat{E} \sim 3\Theta$, and $\hat{S}_5 \sim 2\Theta$.*

Proof. Similar to (5.2)-(5.3) we have for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}$:

$$\begin{array}{lrcl} & K_{\bar{X}} + 5\bar{S}_3 + \bar{S}_4 + a_1\bar{E} & \sim & 0, \\ & K_{\bar{X}} + \bar{S}_3 + 4\bar{S}_4 + a_2\bar{E} & \sim & 0, \\ (9.2) & K_{\bar{X}} + \bar{S}_4 + 3\bar{S}_5 + a_3\bar{E} & \sim & 0, \\ & K_{\bar{X}} + \bar{S}_5 + 2\bar{S}_7 + a_4\bar{E} & \sim & 0, \end{array}$$

$$\begin{array}{lrcl} & 5\beta_3 + \beta_4 & = & a_1 + \alpha, \\ & \beta_3 + 4\beta_4 & = & a_2 + \alpha, \\ (9.3) & \beta_4 + 3\beta_5 & = & a_3 + \alpha, \\ & \beta_5 + 2\beta_7 & = & a_4 + \alpha. \end{array}$$

Remark 9.4. Since $S_7 \in \mathscr{M}$ is a general member, by (4.9) we have $c = \alpha/\beta_7 \le 1/3$, so $3\alpha \le \beta_7$ and $a_4 \ge 5\alpha + \beta_5$. Further, $S_3 + S_4 \sim S_7$. Thus, $\beta_3 + \beta_4 \ge \beta_7 \ge 3\alpha$, $a_1 \ge 4\beta_3 + 2\alpha$, and $a_2 \ge 3\beta_4 + 2\alpha$.

Assume that $\hat{X}$ is a surface. Then $\hat{X}$ is such as in Lemma 4.14. From the first and second relations in (9.2) we obtain that S_3 and S_4 are g-vertical. Since $\dim|\bar{S}_k| = 0$, $\dim|g(\bar{S}_k)| = 0$, $k = 3, 4$. Hence, $K_{\hat{X}}^2 \le 6$ and the curves $g(\bar{S}_k)$ are in fact lines on $\hat{X}$. In particular, $g(\bar{S}_3) \sim g(\bar{S}_4)$. This implies $\bar{S}_3 \sim \bar{S}_4$ and $S_3 \sim S_4$, a contradiction.
Now assume that $\hat{X}$ is a curve and let G be a general fiber of g. Clearly, divisors $\bar{S}_3$ and $\bar{S}_4$ are g-vertical. If the divisor $\bar{S}_5$ is also g-vertical, then $k_3\bar{S}_3 \sim k_4\bar{S}_4 \sim k_5\bar{S}_5 \sim G$, where the k_i are the multiplicities of corresponding fibers. Considering proper transforms on X we get $3k_3 = 4k_4 = 5k_5$ and so $k_3 = 20k$, $k_4 = 14k$, $k_5 = 12k$ for some $k \in \mathbb{Z}$. This contradicts the main result of [MP08a]. Therefore the divisor $\bar{S}_5$ is g-horizontal. In this case, the degree of the general fiber is 9. As above we have $k_3\bar{S}_3 \sim k_4\bar{S}_4 \sim G$, $3k_3 = 4k_4$. So, $k_3 = 4k$, $k_4 = 3k$, $k \in \mathbb{Z}$. Again by [MP08a] g has no fibers of multiplicity divisible by 4.
From now on we assume that g is birational. Then in notation of 4.12,

$$\hat{q} = 5s_3 + s_4 + a_1 e = s_3 + 4s_4 + a_2 e = s_4 + 3s_5 + a_3 e. \tag{9.5}$$

Consider the case where $f(E)$ is either a curve or a Gorenstein point on X. Then α and β_k are integers. By Remark 9.4

$$a_1 + a_2 = 5(\beta_3 + \beta_4) + \beta_3 - 2\alpha \ge 13\alpha \ge 13.$$

On the other hand, from (9.5) we obtain $2\hat{q} \ge 6s_3 + 5s_4 + 13 \ge 18$. So, $\hat{q} \ge 9$ (both $\bar{S}_3$ and $\bar{S}_4$ cannot be contracted). In this case, the group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 4.13 we have $\hat{E} \ge 3\Theta$. Since $a_4 \ge 5$, we have $\hat{E} \sim 3\Theta$, $\hat{q} \ge 15$, and $\bar{S}_3$ is contracted. In this situation, $|\Theta| = \varnothing$, so s_5, $s_7 \ge 2$. This contradicts the fourth relation in (9.2).

Therefore $P := f(E)$ is a non-Gorenstein point of index $r = 3, 4, 5$ or 7. Similar to 5.6 we have $\alpha = 1/r$ and

r	β_3	β_4	β_5	β_7	a_1	a_2	a_3
3	m_3	$\frac{1}{3}+m_4$	$\frac{2}{3}+m_5$	$\frac{1}{3}+m_7$	$5m_3+m_4$	$1+m_3+4m_4$	$2+m_4+3m_5$
4	$\frac{1}{4}+m_3$	m_4	$\frac{3}{4}+m_5$	$\frac{1}{4}+m_7$	$1+5m_3+m_4$	m_3+4m_4	$2+m_4+3m_5$
5	$\frac{2}{5}+m_3$	$\frac{1}{5}+m_4$	m_5	$\frac{3}{5}+m_7$	$2+5m_3+m_4$	$1+m_3+4m_4$	m_4+3m_5
7	$\frac{2}{7}+m_3$	$\frac{5}{7}+m_4$	$\frac{1}{7}+m_5$	m_7	$2+5m_3+m_4$	$3+m_3+4m_4$	$1+m_4+3m_5$

Claim 9.6. (i) *If $r = 3$ or 4, then $m_7 \geq 1$ and $m_3 + m_4 \geq 1$.*
(ii) *If $r = 7$, then $m_7 \geq 1$.*

Proof. To get inequalities for m_7 we use $1/3 \geq c = \alpha/\beta_7 = 1/r\beta_7$, $r\beta_7 \geq 3$. The inequalities for $m_3 + m_4$ follows from $\beta_3 + \beta_4 \geq \beta_7$. □

Thus, in all cases $a_1, a_2 \geq 1$. Put

$$u := s_3 + em_3, \qquad v := s_4 + em_4, \qquad w := s_5 + em_5.$$

9.7. Case: $r = 3$. Then $u + v > e(m_3 + m_4) \geq e$ by Claim 9.6. Further,

$$\begin{aligned} \hat{q} &= 5s_3 + s_4 + (5m_3 + m_4)e &&= 5u + v, \\ \hat{q} &= s_3 + 4s_4 + (1 + m_3 + 4m_4)e &&= u + 4v + e, \\ \hat{q} &= s_4 + 3s_5 + (2 + m_4 + 3m_5)e &&= v + 3w + 2e. \end{aligned}$$

If $u = 0$, then $v = \hat{q} = e + 4v$, a contradiction.
Assume that $u \geq 2$. Then $\hat{q} \geq 10$, $u \leq 3$, the group $\mathrm{Cl}(\hat{X})$ is torsion free and by Lemma 4.13 we have $e \geq 3$. If $u = 2$, then $v \geq 2$, $\hat{q} \geq 13$, $v = \hat{q} - 10$, and $e \leq \hat{q} - 2 - 4v \leq 2$, a contradiction. If $u = 3$, then $v = 2$, $e = 6$, $\hat{q} = 17$, and $m_3 = m_4 = 0$. This contradicts Claim 9.6.
Therefore, $u = 1$. Then $v = \hat{q} - 5$, $19 = e + 3\hat{q}$, and $\hat{q} \leq 6$. We get only one solution: $\hat{q} = 6$, $u = v = w = e = 1$. Recall that $m_3 + m_4 \geq 1$ by Claim 9.6. Hence either $s_3 = 0$ and $\hat{S}_4 \sim_{\mathbb{Q}} \hat{S}_5 \sim_{\mathbb{Q}} \hat{E} \sim_{\mathbb{Q}} \Theta$ or $s_4 = 0$ and $\hat{S}_3 \sim_{\mathbb{Q}} \hat{S}_5 \sim_{\mathbb{Q}} \hat{E} \sim_{\mathbb{Q}} \Theta$. In both cases $\hat{S}_5 \not\sim \hat{E}$ (otherwise $\bar{S}_5 \sim \bar{E} + l\bar{F}$ for some $l \in \mathbb{Z}$ and so $S_5 \sim lF$, a contradiction). Then we get a contradiction by Lemma 3.11.

9.8. Case: $r = 4$. As in the previous case, $u + v > e$ and

$$\begin{aligned} \hat{q} &= 5s_3 + s_4 + (1 + 5m_3 + m_4)e &&= 5u + v + e, \\ \hat{q} &= s_3 + 4s_4 + (m_3 + 4m_4)e &&= u + 4v. \end{aligned}$$

If u is even, then so is $\hat{q}$. Hence, $\hat{q} \leq 10$. From the first relation we have $u = 0$, $\hat{q} = 4v$, and $e = 3v$. This contradicts $u + v > e$. Therefore u is odd.
Assume that $u = 1$. Then $\hat{q} = 5 + v + e = 1 + 4v$ and $e = 3v - 4$. Since $u + v > e$, there is only one possibility: $v = e = 2$, $\hat{q} = 9$. Then the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.13 we have $F \in |2A| \neq \varnothing$, a contradiction.
Finally, assume $u \geq 3$. Then $u = 3$ and $\hat{q} = 15 + v + e = 3 + 4v \geq 16$. Thus, $\hat{q} = 19$, $v = 4$, and $e = 0$, a contradiction.

9.9. CASE: $r = 7$. Then

$$\begin{array}{rclcl} \hat{q} & = & 5s_3 + s_4 + (2 + 5m_3 + m_4)e & = & 5u + v + 2e, \\ \hat{q} & = & s_3 + 4s_4 + (3 + m_3 + 4m_4)e & = & u + 4v + 3e, \\ \hat{q} & = & s_4 + 3s_5 + (1 + m_4 + 3m_5)e & = & v + 3w + e. \end{array}$$

In this case, $u = (3v + e)/4 > 0$. Assume that $u \geq 2$. Then $\hat{q} \geq 13$ and the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.13 we have $e \geq 3$. Further, $u = 2$, and $\hat{q} \geq 17$. We get $m_3 = 0$, $s_3 = 2$, $e \geq 4$, $\hat{q} = 19$, $e = 4$, and $v = 1$. This contradicts the last relation.
Therefore, $u = 1$. Then $3v + e = 4$. Assume that $e = 4$. Then $v = 0$, $\hat{q} = 13$, $w = 3$, $s_4 = 0$, $s_3 = 1$, and $m_4 = m_3 = 0$. Since $\dim|\Theta| = \dim|2\Theta| = 0$, we have $s_5 \geq 3$. Recall that $m_7 \geq 1$ by Claim 9.6. Hence, $\beta_7 \geq 1$ and $a_4 = 2\beta_7 \geq 2$. This contradicts the fourth relation in (9.2).
Therefore, $e < 4$. In this case, $e = 1$, $v = 1$, and $\hat{q} = 8$. Then $\hat{E} \sim_{\mathbb{Q}} \Theta$ and either $\hat{S}_3 \sim_{\mathbb{Q}} \Theta$ or $\hat{S}_4 \sim_{\mathbb{Q}} \Theta$ (because $u = v = 1$). This contradicts (vi) of Theorem 1.4.

9.10. CASE: $r = 5$. From (9.2) we obtain

$$\begin{array}{rclcl} \hat{q} & = & 5s_3 + s_4 + (2 + 5m_3 + m_4)e & = & 5u + v + 2e, \\ \hat{q} & = & s_3 + 4s_4 + (1 + m_3 + 4m_4)e & = & u + 4v + e, \\ \hat{q} & = & s_4 + 3s_5 + (m_4 + 3m_5)e & = & v + 3w. \end{array} \tag{9.11}$$

Then $e = 3v - 4u$. If $u \geq 2$, then $e = 3v - 4u \leq 3v - 6$, and so $v \geq 3$. Hence, $\hat{q} \geq 15$ and the group $\mathrm{Cl}(\hat{X})$ is torsion free. By Lemma 4.13 we have $e \geq 3$. So $\hat{q} = 19$, $e = 3$, $s_3 = 0$, and $2 = u = em_3 \geq 3$, a contradiction.
Assume that $u = 1$, then $e = 3v - 4$ and $v \geq 2$. Further, $\hat{q} = 7v - 3 = v + 3w \leq 19$. We get $\hat{q} = 11$ and $e = 2$. This contradicts Lemma 4.13.
Therefore, $u = 0$. Then $e = 3v$, $\hat{q} = 7v = 7$, $v = 1$, $e = 3$, and $w = 2$. By Lemma 4.13 the group $\mathrm{Cl}(\hat{X})$ is torsion free. Thus $s_3 = 0$, i.e., $\bar{S}_3$ is contracted, $s_4 = 1$, $s_5 = 2$, and $m_5 = \beta_5 = 0$. This means, in particular, that $P_5 \notin S_5$. From the fourth relation in (9.2) we get $a_4 = 1$ and $s_7 = 1$. In particular, $\dim|\Theta| > 0$ and $\hat{X} \simeq \mathbb{P}(1^2, 2, 3)$ by (vi) of Theorem 1.4.

□

LEMMA 9.12. (i) *$S_3 \cap S_4$ is a reduced irreducible curve.*
(ii) *$S_3 \cap S_4 \cap S_7 = \{P_5\}$.*

Proof. (i) Similar to the proof of (i) of Lemma 7.5.
(ii) Put $C := S_3 \cap S_4$. Assume that $S_3 \cap S_4 \cap S_7 \ni P \neq P_5$. Since $1/5 = S_3 \cdot S_4 \cdot S_7 = S_7 \cdot C$ and P, $P_5 \in S_7 \cap C$, we have $C \subset S_7$. If there is a component $C' \neq C$ of $S_3 \cap S_7$ not contained in S_5, then, as above, $1/4 = S_3 \cdot S_7 \cdot S_5 \geq S_5 \cdot C + S_5 \cdot C' \geq 2/7$, a contradiction. Thus we can write $S_3 \cap S_7 = C + \Gamma$, where Γ is an effective 1-cycle with $\mathrm{Supp}\,\Gamma \subset S_5$. In particular, $P_5 \notin S_5$. The divisor $84A$ is Cartier at P_3, P_4, and P_7. We get

$$\frac{9}{5} = 84A \cdot S_3 \cdot (S_7 - S_4) = 84A \cdot \Gamma \in \mathbb{Z},$$

a contradiction. □

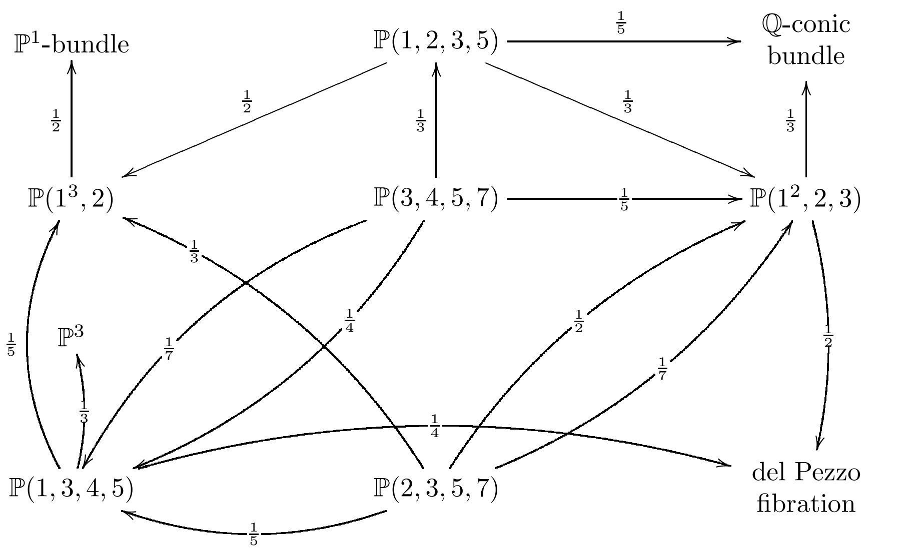
Now the proof of (i) of Theorem 1.4 can be finished similar to 7.7.

10. Toric Sarkisov links

Proposition 10.1. *Let X be a toric $\mathbb{Q}$-Fano threefold and let $P \in X$ be a cyclic quotient singularity of index r. Let $f\colon \tilde{X} \to X$ be the Kawamata blowup of $P \in X$. Then a general member of $|-K_X|$ is a normal surface having at worst Du Val singularities. The linear system $|-K_X|$ has only isolated base points. In particular, $-K_{\tilde{X}}$ is nef and big. The map $f\colon \tilde{X} \to X$ can be completed by a toric Sarkisov link* (*cf.* (4.5)).

Proof. This can be shown by explicit computations in all cases of Proposition 1.3. Consider, for example, the case $X = \mathbb{P}(3,4,5,7)$. Let x_3, x_4, x_5, x_7 be quasi-homogeneous coordinates in $\mathbb{P}(3,4,5,7)$. A section $S \in |-K_X|$ is given by a quasi-homogeneous polynomial of degree 19. By taking this polynomial as a general linear combination of $x_3^5x_4$, $x_3^3x_5^2$, $x_3^4x_7$, $x_4x_5^3$, $x_4^3x_7$, $x_5x_7^2$ we see that the base locus of $|-K_X|$ is the union of four coordinate points and the surface S has only quotient singularities. Since K_S is Cartier, the singularities of S are Du Val. Further, we can write $K_{\tilde{X}} + \tilde{S} = f^*(K_X + S) \sim 0$, where $\tilde{S}$ is the proper transform of S. Hence, $\tilde{S} \in |-K_{\tilde{S}}|$ and the linear system $|-K_{\tilde{X}}|$ has only isolated base points outside of $f^{-1}(P)$. In particular, $-K_{\tilde{X}}$ is nef. It is easy to check that $-K_{\tilde{X}}^3 > 0$, i.e., $-K_{\tilde{X}}$ is big. Recall that $\rho(\tilde{X}) = 2$. So, the Mori cone $\mathrm{NE}(\tilde{X})$ has exactly two extremal rays, say R_1 and R_2. Let R_1 is generated by f-exceptional curves. If $-K_{\tilde{X}}$ is ample, we run the MMP starting from R_2. Otherwise we make a flop in R_2 and run the MMP. Clearly, we obtain Sarkisov link (4.5). □

Explicitly, for weighted projective spaces from Proposition 1.3, we have the following diagram of Sarkisov links. Here an arrow $X_1 \xrightarrow{\frac{1}{r}} X_2$ indicates that there is a Sarkisov link described above that starts from the Kawamata blowup of a cyclic quotient singularity of index $r > 1$ on X_1 and the target variety is X_2.



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