On the Structure of Selmer Groups of Λ-Adic Deformations over p-Adic Lie Extensions

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Abstract. In this paper, we consider the Λ-adic deformations of Galois representations associated to elliptic curves. We prove that the Pontryagin dual of the Selmer group of a Λ-adic deformation over certain p-adic Lie extensions of a number field, that are not necessarily commutative, has no non-zero pseudo-null submodule. We also study the structure of various arithmetic Iwasawa modules associated to such deformations.

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Introduction

Let E be an elliptic curve defined over a number field K. Assume that p is an odd prime such that E has good ordinary reduction at the primes of K lying over p. Let K_∞ be a (not necessarily commutative) p-adic Lie extension of K containing the cyclotomic $\mathbb{Z}_p$-extension K_{cyc} of K with the additional property that the Galois group $\mathrm{Gal}(K_\infty/K)$ is pro-p with no elements of order p. The dual Selmer group $X(E/K_\infty)$ of the elliptic curve over K_∞ of the associated Galois representation has been extensively studied (see [OV], [OV1], [HV]). In recent works, Greenberg proved broad generalizations on the structure of the dual Selmer group for deformations, when the extension K_∞/K is commutative (see [Gr3] and [Gr4]). The aim of this paper is to establish analogues of these results when K_∞/K is a non-commutative p-adic Lie extension. More precisely, we consider R-adic deformations of the Galois representation associated to the elliptic curve; here R is a complete, local Noetherian domain which is finite flat over the Iwasawa algebra $\mathbb{Z}_p[[X]]$. Such deformations give rise to a big Galois representation

$$\rho : G_K \to \mathrm{GL}_2(R).$$

In this context, there are associated Iwasawa modules that are finitely generated over the Iwasawa algebra $R[[G]]$ where $G := \mathrm{Gal}(K_\infty/K)$. We study the ranks of these Iwasawa modules and also analyse the dual Selmer group of the deformation, proving in particular that it has no non-zero pseudo-null $R[[G]]$-submodule. Our investigation was inspired by the article of Greenberg [Gr3] but the methods used are different as Greenberg's methods crucially rely on the Iwasawa algebra $R[[G]]$ being a commutative Noetherian local ring. To prove the corresponding results in this non-commutative setting, we need to use subtle methods from the theory of non-commutative Iwasawa algebras. We then adapt the techniques of Ochi and Venjakob [OV] and Jannsen [J] to our context along with specialisation arguments. We stress that our main result on the non-existence of non-zero pseudo-null submodules is for the strict Selmer group, and in section 6 we compare the strict Selmer group with the Greenberg Selmer group. As an application of our methods, we study the variation of the Iwasawa invariants of the Selmer groups in the Hida family. Analogous results for the cyclotomic $\mathbb{Z}_p$-extension were proved by Emerton-Pollack-Weston [EPW], while Chandrakant [Sh] and Jha [Jh] study non-commutative p-adic Lie extensions, and our results strengthen those of [Sh] and [Jh]. Specifically, it is assumed in [Sh] that the cyclotomic μ-invariant of the dual Selmer group at a specialisation vanishes, whereas we do not make this strong assumption.

The paper consists of eight sections. Section 1 is preliminary in nature while in sections 2 and 3 we formalize the tools and techniques needed to adapt the methods of Ochi and Venjakob [OV] and Jannsen [J]. In sections 4 and 5, we prove the reflexivity of some local Iwasawa modules and establish results on the $R[[G]]$-ranks of certain cohomology modules. In section 6, we prove the main result on the dual Selmer group and in section 7, we discuss the surjectivity of the global to local map defining the Selmer group. In section 8, we apply the main result to study the invariance of various Iwasawa invariants at the specialisations.

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1. Notation and preliminaries

Let $\bar{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$ and for every prime integer l let $\bar{\mathbb{Q}}_l$ be a fixed algebraic closure of $\mathbb{Q}_l$. Fix an embedding $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_l$ for every prime l. Throughout, p will denote an odd prime number. Let $G_{\mathbb{Q}}$ denote the absolute Galois group of $\mathbb{Q}$ and G_l denote the decomposition subgroup above a prime of l given by this embedding. Suppose that E is an elliptic curve defined over $\mathbb{Q}$ such that E has ordinary reduction at p. We denote by T_pE the Tate module of the elliptic curve E. Let

$$\rho_0 : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T_pE)$$

be the associated Galois representation giving the action of $G_{\mathbb{Q}}$ on T_pE. Since E has ordinary reduction at p, there exists a G_p-submodule F^+T_pE of T_pE of $\mathbb{Z}_p$-rank one such that the action of G_p on $F^-T_pE := T_pE/F^+T_pE$ is unramified. In fact, F^-T_pE is canonically identified with the Tate module $T_p\tilde{E}$ of $\tilde{E}$ where $\tilde{E}$ denotes the reduction of E modulo p.

Let Λ be a commutative Noetherian local domain isomorphic to the power series ring $\mathbb{Z}_p[[X]]$. Our aim is to study the structure of Iwasawa modules arising from the Galois cohomology of certain Λ-adic deformations of ρ_0. Specifically, we consider an R-adic representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(R) \tag{1}$$

for a complete Noetherian local domain R which is a finite flat extension of Λ, such that ρ satisfies the following hypotheses:

Hypothesis 1.

(a) There exists a continuous $\mathbb{Z}_p$-algebra homomorphism $\lambda : R \longrightarrow \mathbb{Z}_p$ such that $\lambda \circ \rho = \rho_0$.

(b) ρ is unramified outside the primes p, the infinite prime and the primes dividing the conductor N of E.

(c) Let $\mathcal{T}$ denote the free R-module of rank two on which $G_{\mathbb{Q}}$ acts by ρ. Then there exists an R-submodule $F^+\mathcal{T}$ which is invariant under the action of G_p and such that both $F^+\mathcal{T}$ and $\mathcal{T}/F^+\mathcal{T}$ are free R-modules of rank one. The decomposition group G_p acts on $\mathcal{T}/F^+\mathcal{T}$ via an unramified character η_p.

An important example we have in mind is the Hida deformation. We recall this here briefly. For more details see [Gr1], [Wi], [Hi]. Let $\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^* = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ be the p-th cyclotomic character, where μ_{p-1} denotes the group of $(p-1)$-th roots of unity. Let κ denote the composition of χ with the projection onto $1 + p\mathbb{Z}_p$. Then κ induces an isomorphism $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}) = \Gamma \xrightarrow{\sim} 1 + p\mathbb{Z}_p$, where $\mathbb{Q}_{cyc}$ denotes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\Lambda = \mathbb{Z}_p[[\Gamma]] := \varprojlim \mathbb{Z}_p[\Gamma/p^i\Gamma]$ be the Iwasawa algebra of Γ. Then the ring Λ is a commutative Noetherian local domain isomorphic to the power series ring $\mathbb{Z}_p[[X]]$ in one variable. If E is defined over $\mathbb{Q}$ and the residual Galois representation associated to ρ_0 is absolutely irreducible, then by Hida theory, there exists a complete Noetherian local domain R which is finite flat over Λ, and an R-adic form $\mathcal{F}$ that gives rise to a representation $\tilde{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(R)$ which satisfies all the properties stated in Hypothesis 1. Further, it also satisfies the following two properties:

(i) Let $\epsilon : 1 + p\mathbb{Z}_p \longrightarrow \bar{\mathbb{Q}}_p^*$ be the inclusion. Then for every $k \in \mathbb{Z}_p$ the homomorphism $\epsilon^k : 1 + p\mathbb{Z}_p \longrightarrow \bar{\mathbb{Q}}_p^*$ induces a $\mathbb{Z}_p$-algebra homomorphism $\epsilon^k : \Lambda \longrightarrow \bar{\mathbb{Q}}_p$. Let $k \geq 2$ and let $\phi \in \mathrm{Hom}(R, \bar{\mathbb{Q}}_p)$ be a $\mathbb{Z}_p$-algebra homomorphism such that $\phi|_\Lambda = \epsilon^k$. Then, $\phi \circ \tilde{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$

is equivalent (over $\bar{\mathbb{Q}}_p$) to the $\bar{\mathbb{Q}}_p$-representation corresponding to a weight k cusp form for $\Gamma(Np)$.

(ii) Let $\mathfrak{p}_k$ denote the prime ideal of Λ defined by the kernel of the homomorphism ϵ^k. If $k \in \mathbb{Z}$ and $k \geq 2$, then $\mathfrak{p}_k$ remains unramified in R.

For property (ii) we refer the reader to [EPW, Theorem 2.1.3(3)] (or [GS, Theorem 2.6(a)]). For more details and examples, see the last part of §6. In general R is not equal to Λ. One needs to impose extra conditions to get $R = \Lambda$. In fact, $R = \Lambda$ holds whenever E is a CM elliptic curve whose residual representation is absolutely irreducible. When E is not a CM elliptic curve, then there exists a set of primes of $\mathbb{Q}$ of density one for which this condition is satisfied (for instance see [NP, section 4]).

Throughout this article, R will denote a commutative, complete local Noetherian domain with maximal ideal m and residue field $\mathbb{F}$, which is a finite field extension of $\mathbb{Z}/p\mathbb{Z}$. In Sections 4 and 8, we shall assume that R is a regular local ring to prove our results. Our main results in Section 6 and Section 7 are however valid for more general rings, namely Noetherian local domains of Krull dimension two. For a profinite group G and a commutative, Noetherian ring S, we denote by $S[[G]]$ the completed group algebra $\varprojlim_U S[G/U]$ where U varies over open normal subgroups of G, and the inverse limit is taken with respect to the natural maps. All modules considered over the ring $S[[G]]$ will be as left modules. For a compact (resp. discrete) $S[[G]]$-module M, we denote by $\widehat{M}$ the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M which is a discrete (resp. compact) $S[[G]]$-module. Let $\mathcal{D}$ be a discrete R-module with a continuous action of G such that the dual $\widehat{\mathcal{D}}$ is a free R-module of rank r. For a finitely generated $R[[G]]$-module M, we define

$$M[\mathcal{D}] := \widehat{\mathrm{Hom}_R(M, \mathcal{D})} = M \otimes_R \widehat{\mathcal{D}},$$

where G acts diagonally on the tensor product.

2. Fox Lyndon and The Big Diagram

Let K be a number field. Fix an R-adic representation

$$\sigma : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(R),$$

and let $\mathcal{T}$ be the free R-module of rank r corresponding to this representation. We denote by $\mathcal{D}$ be the discrete $R[[\mathrm{Gal}(\bar{K}/K)]]$-module defined by $\mathcal{T} \otimes \widehat{R}$. Throughout, we assume that the action of $\mathrm{Gal}(\bar{K}/K)$ on $\mathcal{D}$ is unramified outside a finite set of primes of K. We denote by $S(K)$ a finite set of primes of K containing each prime above p, every archimedean prime and every prime whose inertia group acts non-trivially on $\mathcal{D}$. Let $S_f(K)$ denote the set of all finite primes in $S(K)$ and put $S_p(K) = \{v \in S_f(K) : v|p\}$. By K_S, we shall mean the maximal $S(K)$-ramified extension of K. Let $K(\mathcal{D})$ be the extension of K defined by the kernel of σ and let K_1 be the extension of K defined by the kernel of the residual representation of σ.

Lemma 2.1. *The extension $K(\mathcal{D})$ is contained in K_S and $K(\mathcal{D})/K_1$ is pro-p.*

Proof. The first assertion is clear. Let K_n be the extension defined by the kernel of the representation $\sigma_n : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(R/m^n))$ and let $G_n = \mathrm{Gal}(K(\mathcal{D})/K_n)$. We have $\mathrm{Gal}(K(\mathcal{D})/K_1) = \varprojlim_{n\geq 1} G_1/G_n$. Let GL_r^n be the subgroup of GL_r consisting of matrices $(a_{i,j})$ such that $a_{i,j} \in m^n$ if $i \neq j$ and $a_{i,j} \in 1 + m^n$ if $i = j$. Then

$$\frac{G_n}{G_{n+1}} \hookrightarrow \frac{\mathrm{GL}_r^n}{\mathrm{GL}_r^{n+1}} \cong \mathrm{M}_r\left[\frac{m^n}{m^{n+1}}\right],$$

where $\mathrm{M}_r[\frac{m^n}{m^{n+1}}]$ is the corresponding additive subgroup of the matrix ring $M_r[\frac{R}{m^{n+1}}]$. This shows that each G_n/G_{n+1} is a p-group. Therefore $\mathrm{Gal}(K(\mathcal{D})/K_1)$ is pro-p. $\square$

Let Ω be the maximal S-ramified p-extension of $K(\mathcal{D})$ (resp. K_1). The extension Ω/K is Galois. Let K_∞ be a p-adic Lie extension of K such that (i) $K_\infty \subset \Omega$, (ii) the Galois group $\mathrm{Gal}(K_\infty/K)$ is a pro-p, p-adic Lie group with no elements of order p and (iii) K_∞ contains the cyclotomic $\mathbb{Z}_p$-extension K_{cyc} of K; here $\mathrm{Gal}(K_\infty/K)$ is not assumed to commutative. Such an extension is called an *admissible* extension of K. Put

$$\mathrm{Gal}(\Omega/K) = \mathcal{G}, \quad \mathrm{Gal}(K_S/K) = \mathcal{G}_S, \quad \mathrm{Gal}(\Omega/K_\infty) = \mathcal{H},$$

$$\mathrm{Gal}(K_S/K_\infty) = \mathcal{H}_S \text{ and } \mathrm{Gal}(K_\infty/K) = G.$$

As $\mathcal{D}$ is fixed now, for a compact $R[[\mathcal{G}]]$-module M, we will denote $M[\mathcal{D}]$ by $M^{\#}$. Then the profinite group $\mathcal{G}$ is a pro-C-group which is topologically finitely generated such that $cd_p(\mathcal{G}) \leq 2$ (see [OV]); here C is a class of finite groups closed under taking subgroups, homomorphic images and group extensions. Let $\mathcal{G}$ be generated by d elements. Then there is the following commutative diagram with exact rows and columns

(2)
$$\begin{tikzcd}
& 1 & 1 & & \\
1 \arrow[r] & \mathcal{H} \arrow[u] \arrow[r] & \mathcal{G} \arrow[u] \arrow[r] & G \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{R} \arrow[u] \arrow[r] & F_d \arrow[u] \arrow[r] & G \arrow[u, equal] \arrow[r] & 1 \\
& \mathcal{N} \arrow[u] \arrow[r, equal] & \mathcal{N} \arrow[u] & & \\
& 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

where F_d is a free pro-C-group of rank d. We fix the following notation.
$X = X_{\mathcal{H},\mathcal{D}} := H^1\widehat{(\mathcal{H},\mathcal{D})}$
$Y = Y_{\mathcal{H},\mathcal{D}} := (\mathrm{I}_{\mathcal{G}}^{\#})_{\mathcal{H}}$
$J = J_{\mathcal{H},\mathcal{D}} := \ker(R[[\mathcal{G}]]_{\mathcal{H}}^{\#} \longrightarrow (\widehat{\mathcal{D}})_{\mathcal{H}})$.
Note that there is an isomorphism $H^1\widehat{(\mathcal{H}_S,\mathcal{D})} \simeq H^1\widehat{(\mathcal{H},\mathcal{D})}$.

Lemma 2.2. *We have a commutative diagram with exact rows and columns as below*

$$\begin{tikzcd}
 & & & 0 & 0 & \\
 & & & J \arrow[u] \arrow[r, equal] & J \arrow[u] & \\
0 \arrow[r] & H^2\widehat{(\mathcal{H}_S,\mathcal{D})} \arrow[r] & H_1(\mathcal{N},R)_{\mathcal{H}}^{\#} \arrow[r] & R[[G]]^{dr} \arrow[u] \arrow[r] & Y \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & H^2\widehat{(\mathcal{H}_S,\mathcal{D})} \arrow[u] \arrow[r] & H_1(\mathcal{N},\widehat{\mathcal{D}})_{\mathcal{H}} \arrow[u] \arrow[r] & H_1(\mathcal{R},\widehat{\mathcal{D}}) \arrow[u] \arrow[r] & H^1\widehat{(\mathcal{H}_S,\mathcal{D})} \arrow[u] \arrow[r] & 0 \\
 & & & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd}$$

Also $H_1(\mathcal{N},R)^{\#}$ is a projective $R[[\mathcal{G}]]$-module and $H_1(\mathcal{N},R)_{\mathcal{H}}^{\#}$ is a projective $R[[G]]$-module.

□

There is a similar lemma when the number field K is replaced by a finite extension of $\mathbb{Q}_l$, l a prime number. In this case, in place of K_S, we take an algebraic closure $\bar{K}$ of K. We also have the following two lemmas.

Lemma 2.3. *The $R[[\mathcal{G}]]$-module $R[[\mathcal{G}]][\mathcal{D}]$ is free of rank r and therefore if M is a projective $R[[\mathcal{G}]]$-module, then $M[\mathcal{D}]$ is also a projective $R[[\mathcal{G}]]$-module.*

Lemma 2.4. *There is a canonical exact sequence*

$$0 \longrightarrow H_1(\mathcal{N},R) \longrightarrow R[[\mathcal{G}]]^d \longrightarrow R[[\mathcal{G}]] \longrightarrow R \longrightarrow 0.$$

The proof of the above statements are entirely analogous to the proof of [OV, Lemma 4.5, Lemma 4.2] and [NSW, Theorem 5.6.6] respectively.

3. A spectral sequence

Let S be a not necessarily commutative ring. For a finitely generated S-module M, we will denote by $\mathrm{E}_S^i(M)$ the (right) S-module $\mathrm{Ext}_S^i(M,S)$ for any integer $i \geq 0$ and set $\mathrm{E}_S^i(M) = 0$ for $i < 0$. However, we shall always consider the case when S is a completed group ring and hence $\mathrm{E}_S^i(M)$ is endowed with a natural left module structure (see [J, Section 1, page 175]). The subscript S from the notation will be dropped whenever there is no ambiguity. We also write M^+ for $\mathrm{E}^0(M)$.

Definition 1. *If $M \neq 0$ is an S-module, then $j(M) := min\{i | \mathrm{E}^i(M) \neq 0\}$ is called the grade of M.*

Next, we recall the definition of the "transpose functor". For a finitely presented S-module M, take any projective resolution $P_1 \to P_0 \to M \to 0$ of M by finitely generated projective modules P_0 and P_1. Then DM is defined as the cokernel of the induced map $(P_0)^+ \to (P_1)^+$. This is well defined up to homotopy. We will also need the following exact sequence (see [J]).

$$0 \to \mathrm{E}^1(DM) \to M \xrightarrow{\phi_M} M^{++} \to \mathrm{E}^2(DM) \to 0.$$

A finitely generated S-module M is said to be *reflexive* if ϕ_M is an isomorphism and M is said to be *torsion free* (resp. *torsion*) if ϕ_M is injective (resp. ϕ_M is zero). If S is a Noetherian domain then the torsion elements of M form a submodule which coincides with $\mathrm{E}^1(DM)$. We say that a finitely generated S-module M is *pseudo-null* if grade $j(M) \geq 2$.

In this section, we prove various results which are generalizations of analogous results proved in [NSW, (Section 5.4, 5.6)] and [J]. Again, the proofs are not substantially different but we include them for the sake of completeness, checking that the appropriate modifications are valid in this general setting.

Let $\mathcal{G}$ be any profinite group and M be a finitely generated $R[[\mathcal{G}]]$-module. Then

$$\begin{aligned} M^+ &= \mathrm{Hom}_{R[[\mathcal{G}]]}(M, R[[\mathcal{G}]]) \\ &\cong \varprojlim_{U \lhd_{open} \mathcal{G}} \mathrm{Hom}_{R[\mathcal{G}/U]}(M, R[\mathcal{G}/U]) \\ &\cong \varprojlim_{U} \mathrm{Hom}_{R[\mathcal{G}/U]}(M_U, R[\mathcal{G}/U]) \\ &\cong \varprojlim_{U} \mathrm{Hom}_{R}(M_U, R) \qquad \text{(limit being taken via the norm map)} \\ &\cong \varprojlim_{i} \varprojlim_{U} \mathrm{Hom}_{R}(M_U, R/m^i) \end{aligned}$$

The third isomorphism is given by

$$\mathrm{Hom}_R(M_U, R) \longrightarrow \mathrm{Hom}_{R[\mathcal{G}/U]}(M_U, R[\mathcal{G}/U])$$

$$f \mapsto (x \mapsto \sum_{\sigma \in \mathcal{G}/U} f(\sigma^{-1}x)\sigma).$$

The above calculation shows that

$$\widehat{M^+} \cong \varinjlim_{i} \varinjlim_{U} \mathrm{Hom}_R(M_U, R/m^i)'$$

where N' denotes $\mathrm{Hom}_{\mathbb{Z}_p}(N, \mathbb{Q}_p/\mathbb{Z}_p)$ for an abelian group N. Note that $\mathrm{Hom}_R(M_U, R/m^i)$ is a discrete (in fact finite) abelian group. Therefore we have $\mathrm{Hom}(M_U, R/m^i)' \cong \mathrm{Hom}(\widehat{M_U, R}/m^i)$ (see also [J, section 2]). For an open normal subgroup $U \subset \mathcal{G}$

$$\begin{aligned}&\mathrm{Hom}_R(M_U, R/m^i)'\\ &\cong \mathrm{Hom}_R(R \otimes_{R[[U]]} M, R/m^i)'\\ &\cong \mathrm{Hom}_{R[[U]]}(R, \mathrm{Hom}_R(M, R/m^i))'\\ &\cong (\mathrm{Hom}_R(M, R/m^i)^U)'.\end{aligned}$$

Here, $\mathrm{Hom}_R(M, R/m^i)$ is a discrete $\mathcal{G}$-module because R/m^i is a discrete $\mathcal{G}$-module. The proof of this fact follows from [Br, Lemma 3.4]. Therefore $\widehat{M^+}$ can be written as a composition of the following two covariant functors:

$$M \xrightarrow{LC_p} (\mathrm{Hom}_R\widehat{(M, R}/m^i)) = (M \otimes_R \widehat{R/m^i})$$

and

$$(M_i) \xrightarrow{\varinjlim D_0(\widehat{-})} \varinjlim_{i,U}((\widehat{M_i})^U)',$$

where $D_0(\widehat{-}) := \varinjlim_U H^0(U, \widehat{-})'$ is the functor defined by Tate [S, Appendix 1] whose rth derived functor is $D_r(\widehat{-}) := \varinjlim_U H^r(U, \widehat{-})'$. Let P be a finitely generated projective $R[[\mathcal{G}]]$-module. Since $\mathrm{Hom}_R(P, R/m^i)$ is a direct summand of a coinduced $R[[\mathcal{G}]]$-module, it is cohomologically trivial. Therefore $\mathrm{Hom}_R\widehat{(P, R}/m^i)$ is a $D_0(\widehat{-})$ -acyclic module for every $i \geq 1$. This shows that the first functor takes projective objects to acyclic objects of the second functor. Therefore if $R[[\mathcal{G}]]$ is a Noetherian ring, then we have the convergence of the following spectral sequence of homological type.

Theorem 3.1. *Let $\mathcal{G}$ be any profinite group such that $R[[\mathcal{G}]]$ is a Noetherian ring and let M be a finitely generated $R[[\mathcal{G}]]$-module. Then,*

$$\varinjlim_i D_r(L^s\widehat{C_p(M)}) \Rightarrow \widehat{\mathrm{E}^{r+s}(M)}$$

where $L^sC_p(M) = \mathrm{Tor}^s_R(M, \widehat{R/m^i}) \cong \mathrm{Ext}^s_R\widehat{(M, R}/m^i)$.

Corollary 3.2. *Let $\mathcal{D}$ be a discrete $R[[\mathcal{G}]]$-module such that $\widehat{\mathcal{D}}$ is a finitely generated free R-module. Then, as U varies over the open normal subgroups of $\mathcal{G}$, for every $r \geq 0$ we have*

$$\varinjlim_i \varinjlim_U H^r(U, \mathrm{Hom}_R(\widehat{\mathcal{D}}, R/m^i))' \cong \widehat{\mathrm{E}^r(\widehat{\mathcal{D}})}.$$

Proof. Since $\widehat{\mathcal{D}}$ is free R-module $\mathrm{Ext}^s_R(\widehat{\mathcal{D}}, R/m^i) = 0$ for all $s \geq 1$. Therefore the spectral sequence degenerates and the result follows. □

Remark 3.3. In the proof of Theorem 3.1 we have assumed that the ring $R[[\mathcal{G}]]$ is Noetherian. Without this assumption, the category of finitely generated $R[[\mathcal{G}]]$-modules need not be an abelian category and therefore we can not use the spectral sequence arguments to prove the theorem. Nevertheless, Corollary 3.2 can be proved without the assumption that the ring $R[[\mathcal{G}]]$ is Noetherian. Indeed, since $\widehat{\mathcal{D}}$ is a finitely generated free R-module, it follows from Lemma 2.3 that $\widehat{\mathcal{D}}$ has a resolution by finitely generated projective (in fact free) $R[[\mathcal{G}]]-$modules. For any finitely generated projective module P, the module $\mathrm{Hom}_R(P, R/m^i)$ is cohomologically trivial as explained in the proof of Theorem 3.1. Therefore if $P_\bullet \longrightarrow \widehat{\mathcal{D}} \longrightarrow 0$ is a resolution of $\widehat{\mathcal{D}}$ by finitely generated projective $R[[\mathcal{G}]]$-modules, then $0 \longrightarrow \mathrm{Hom}(\widehat{\mathcal{D}}, R/m^i) \longrightarrow \mathrm{Hom}(P_\bullet, R/m^i)$ is a resolution of the discrete $\mathcal{G}$-module $\mathrm{Hom}(\widehat{\mathcal{D}}, R/m^i)$ by cohomologically trivial $\mathcal{G}$-modules. Thus we have,

$$\begin{aligned}
\widehat{\mathrm{E}^r(\widehat{\mathcal{D}})} &= H^r(\widehat{P_\bullet^+}) \\
&= H^r(\varinjlim_i \varinjlim_U (\mathrm{Hom}_R(P_\bullet, R/m^i)^U)') \\
&= \varinjlim_i \varinjlim_U H^r(\mathrm{Hom}_R(P_\bullet, R/m^i)^U)' \\
&= \varinjlim_i \varinjlim_U H^r(U, \mathrm{Hom}_R(\widehat{\mathcal{D}}, R/m^i))'
\end{aligned}$$

Let K be a finite field extension of $\mathbb{Q}$ or a finite field extension of $\mathbb{Q}_l$, l a prime integer. We follow the notation of the previous section.

Lemma 3.4. *The $R[[\mathcal{G}]]$-module $H_1(\mathcal{N}, R)$ is finitely generated.*

Proof. Let U be an open normal pro-p subgroup of $\mathcal{G}$. Then it suffices to show that $H_1(\mathcal{N}, R)$ is a finitely generated $R[[U]]$-module. Since R is a complete Noetherian local domain with finite residue field of characteristic p, there exists an integer $n \geq 0$ such that R is a finitely generated module over the power series ring $\mathbb{Z}_p[[T_1, \cdots, T_n]]$. Thus we may assume that $R = \mathbb{Z}_p[[T_1, \cdots, T_n]]$. Using induction on the number of variables n, and Nakayama's lemma, it is enough to prove the lemma when $R = \mathbb{Z}_p$. Now the assertion of the lemma follows from [J, Theorem 5.1(c)]. □

Proposition 3.5. *Let $Z_{\mathcal{D}} = Z_{\mathcal{D}, K_\infty}$ denote the Pontryagin dual of $(\varinjlim D_2(\mathrm{Hom}_R(\mathcal{D}, R/m^i)))^{\mathcal{H}}$ Then,*

(i) $Z_{\mathcal{D}} = DY = \mathrm{E}^2_{R[[\mathcal{G}]]}(\widehat{\mathcal{D}})_{\mathcal{H}}$.

(ii) $Z^+_{\mathcal{D}} = H^2(\widehat{\mathcal{H}_S, \mathcal{D}})$.

Proof. From Lemma 3.4 and the proof of [NSW, Proposition 5.6.8] we have, $DY = \mathrm{E}^2_{R[[\mathcal{G}]]}(\widehat{\mathcal{D}})_{\mathcal{H}}$. From Corollary 3.2 we have, $Z_{\mathcal{D}} = \mathrm{E}^2_{R[[\mathcal{G}]]}(\widehat{\mathcal{D}})_{\mathcal{H}}$. This proves the first assertion. From Lemma 2.2 we have the following exact sequence

$$0 \longrightarrow H^2(\widehat{\mathcal{H}_S, \mathcal{D}}) \longrightarrow H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}} \longrightarrow R[[G]]^{dr} \longrightarrow Y \longrightarrow 0,$$

where the two terms in the middle are projective $R[[G]]$-modules. The second assertion follows from the proof of [J, Lemma 4.6] and the above exact sequence. □

4. Grade and Reflexivity

For a compact (resp. discrete) $R[[\mathcal{G}]]$-module M, let M^* denote the discrete (resp. compact) $R[[\mathcal{G}]]$-module $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mu_{p^\infty})$ where μ_{p^∞} is the group of all p-power roots of unity. Throughout this section, we assume that R is a regular local domain with residue field $\mathbb{F}_p$. Then $R[[G]]$ is an Auslander regular ring (see [V, Theorem 3.30] and [E, Theorem 4.3]). In fact, using [Wa, Lemma 2.6] it can be shown that if R is of mixed characteristic, then $R[[G]]$ is a complete Auslander regular local domain with residue field $\mathbb{F}_p$. More generally, if G is a uniform pro-p group and R is a complete regular local domain of characteristic p, then $R[[G]]$ is a complete Auslander regular local domain with finite residue field (see for instance [V] for the definition of uniform group).

Lemma 4.1. *Suppose that K is a finite extension of $\mathbb{Q}_l$, where l is a prime number. Then, $Z_{\mathcal{D}}$ is isomorphic to the Pontryagin dual of* $\mathrm{Hom}_R(\widehat{\mathcal{D}}, R)^{*\mathcal{H}}$ *and in particular, is a finitely generated R-module.*

Proof. We have,

$$\begin{aligned}\widehat{Z}_{\mathcal{D}} &= (\varinjlim_i D_2(\mathrm{Hom}(\widehat{\mathcal{D}}, R/m^i)))^{\mathcal{H}} = (\varinjlim_i \varinjlim_U H^2(U, \mathrm{Hom}(\widehat{\mathcal{D}}, R/m^i))')^{\mathcal{H}} \\ &= (\varinjlim_i \varinjlim_U (\mathrm{Hom}(\widehat{\mathcal{D}}, R/m^i)^*)^U)^{\mathcal{H}} \quad (\text{ by local duality}) \\ &= (\varinjlim_i \mathrm{Hom}_R(\widehat{\mathcal{D}}, R/m^i)^*)^{\mathcal{H}} \cong (\mathrm{Hom}_R(\widehat{\mathcal{D}}, R)^*)^{\mathcal{H}}.\end{aligned}$$

□

For an $R[[G]]$-module M and an ideal I of R, put $M[I] = \{x \in M : rx = 0 \ \ \forall r \in I\}$. We prove the following algebraic lemma which will be needed later in this section .

Lemma 4.2. *Let M be a finitely generated $R[[G]]$-module and assume that $\lambda \in R$ such that the following holds:*

(i) The grade of $M/\lambda M$ as an $(R/\lambda)[[G]]$-module is $> k$.
(ii) The grade of $M[\lambda]$ as an $(R/\lambda)[[G]]$-module is $> k-1$.

Then the grade of M as an $R[[G]]$-module is $> k$.

Proof. For any ring S and $\lambda \in S$ a central element, we have the following spectral sequence

$$\mathrm{Ext}^p_{S/\lambda}(M/\lambda, \mathrm{Ext}^q_s(S/\lambda, S)) \Rightarrow \mathrm{Ext}^{p+q}_S(M/\lambda, S)$$

[W, Exercise 5.6.3]. If λ is a non-zero divisor of S then we have an exact sequence

$$0 \to S \xrightarrow{\lambda} S \to S/\lambda \to 0.$$

Thus we have,

$$\begin{aligned}\mathrm{Ext}^i(S/\lambda, S) &\cong S/\lambda \text{ for } i = 1 \\ &= 0 \text{ for } i \neq 1\end{aligned}$$

Therefore the spectral sequence collapses to give $\mathrm{Ext}^{p-1}_{S/\lambda}(M/\lambda, S/\lambda)$ $\cong \mathrm{Ext}^{p}_{S}(M/\lambda, S)$. Now, we take $S = R[[G]]$. Then by assumption we have $\mathrm{Ext}^{p}_{R[[G]]}(M/\lambda, R[[G]]) = 0$ for all $p \leq k+1$. From the short exact sequence

$$0 \to M/M[\lambda] \xrightarrow{\lambda} M \to M/\lambda M \to 0,$$

for all $i \leq k$ we have an isomorphism

$$\mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]]) \xrightarrow{\lambda} \mathrm{Ext}^{i}_{R[[G]]}(M/M[\lambda], R[[G]]).$$

We also have,

$$\begin{aligned}\mathrm{Ext}^{i+1}_{R[[G]]}(M[\lambda], R[[G]]) &= \mathrm{Ext}^{i+1}_{R[[G]]}(M[\lambda]/\lambda, R[[G]]) \\ &= \mathrm{Ext}^{i}_{R/\lambda[[G]]}(M[\lambda]/\lambda, R/\lambda[[G]]) \\ &= \mathrm{Ext}^{i}_{R/\lambda[[G]]}(M[\lambda], R/\lambda[[G]]) = 0 \text{ for all } i \leq k-1.\end{aligned}$$

Therefore the short exact sequence

$$0 \longrightarrow M[\lambda] \longrightarrow M \longrightarrow M/M[\lambda] \longrightarrow 0,$$

gives an isomorphism

$$\mathrm{Ext}^{i}_{R[[G]]}(M/M[\lambda], R[[G]]) \xrightarrow{\sim} \mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]]) \text{ for all } i \leq k.$$

Thus, we have shown that for all $i \leq k$, we have isomorphisms

$$\mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]]) \overset{\lambda}{\tilde{\to}} \mathrm{Ext}^{i}_{R[[G]]}(M/M[\lambda], R[[G]]) \tilde{\to} \mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]]).$$

Therefore $\mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]]) = \lambda \mathrm{Ext}^{i}_{R[[G]]}(M, R[[G]])$ for all $i \leq k$. Now the lemma follows using Nakayama lemma. $\square$

Proposition 4.3. *Let K be a finite extension of $\mathbb{Q}_l$ and suppose that K_∞/K is a p-adic Lie extension of K with $cd_p(G) = k$, where $G = \mathrm{Gal}(K_\infty/K)$. Then the grade of DY as an $R[[G]]$-module is greater than or equal to k .*

Proof. Since $Z_{\mathcal{D}}$ is a finitely generated R-module, $\mathrm{Ext}^{s}_{R}(Z_{\mathcal{D}}, R/m_i)$ is finite for all $s \geq 0$. Being a pro-p, p-adic Lie group, $G = \mathrm{Gal}(K_\infty/K)$ is a Poincaré p-group [La]. Therefore $D_r(\mathrm{Ext}^{s}_{R}(Z_{\mathcal{D}}, R/m^i)) = 0$ for all $r \neq k$ (see [S, Appendix 1, Theorem 3]). From Theorem 3.1, we conclude that $\widehat{\mathrm{E}^n(Z_{\mathcal{D}})} \cong \varinjlim_i D_k(\mathrm{Ext}^{n-k}_{R}(Z_{\mathcal{D}}, R/m^i))$ for all $n \geq k$ and is zero for all $n < k$. Therefore the grade of $DY = Z_{\mathcal{D}}$ is $\geq k$. $\square$

Corollary 4.4. *If $cd_pG \geq 2$ then both Y and X are torsion free, and if $cd_pG \geq 3$ then both Y and X are reflexive.*

Proof. The assertion for Y follows from Proposition 4.3. Since X is a submodule of Y, it follows that X is also torsion free. Consider the right most vertical sequence in Lemma 2.2. It follows by an application of snake lemma that $\mathrm{E}^2(DY)$ is isomorphic to a submodule of J (see [OV, Lemma 5.4]). Since J is torsion free and $\mathrm{E}^2(DY)$ is pseudo-null, $\mathrm{E}^2(DY)$ must be zero. Hence X is a reflexive $R[[G]]$-module. $\square$

Proposition 4.5. *Let K be a finite extension of $\mathbb{Q}_l$ and K_∞ be a p-adic Lie extension of K such that $cd_p\mathrm{Gal}(K_\infty/K) \geq 2$. Suppose that there exists a prime ideal P of R such that R/P is normal, and a finite extension of $\mathbb{Z}_p$ with the property that $Z_{\mathcal{D}}/PZ_D$ is finite. Then the grade of $Z_{\mathcal{D}} \geq 3$ as an $R[[G]]$-module.*

Proof. Since R/P is normal and finite over $\mathbb{Z}_p$, the ring R/P is regular local. Therefore P is generated by a regular sequence. We shall prove the proposition by induction on the number of generators of P. We shall repeatedly use the fact (see Lemma 4.1) that Z_D is finitely generated as an R-module. First, suppose that P is generated by one element λ. Since $Z_{\mathcal{D}}/\lambda$ is finite, the grade of $Z_{\mathcal{D}}/\lambda$ as an $R/\lambda[[G]]$-module is ≥ 3 [J, Corollary 2.6]. Further, as $Z_{\mathcal{D}}[\lambda]$ is a finitely generated R/λ-module, it follows from [*loc.cit.*], that grade of $Z_{\mathcal{D}}[\lambda]$ is ≥ 2. It then follows from Lemma 4.2 that grade $Z_{\mathcal{D}} \geq 3$. It is easily seen using Lemma 4.1 that for an element $\lambda \in R$, we have $Z_{\mathcal{D}}/\lambda \cong Z_{\mathcal{D}[\lambda]}$. Now choose λ to be an element which is part of a regular sequence of P. Then by induction hypothesis, the grade of $Z_{\mathcal{D}[\lambda]}$ as an $R/\lambda[[G]]$-module is ≥ 3. Since $Z_{\mathcal{D}}[\lambda]$ is a finitely generated R-module, $\mathrm{Ext}^s_R(Z_{\mathcal{D}}[\lambda], R/m^i)$ is finite for each i and s. Thus by an argument similar to Proposition 4.3, and using the fact the dimension of G is ≥ 2, it can be seen that $\mathrm{E}^n(Z_{\mathcal{D}}[\lambda]) = 0$ for all $n \leq 1$. Thus grade $Z_{\mathcal{D}}[\lambda] \geq 2$. Now the proposition follows using Lemma 4.2. □

Corollary 4.6. *Under the assumptions of Proposition 4.5, both Y and X are reflexive.* □

5. Coranks of Galois cohomology modules

In this section, we compute the ranks of certain Galois cohomology modules. We assume that $R = \Lambda$, where Λ is isomorphic to the power series ring $\mathbb{Z}_p[[X_1, \cdots, X_n]]$ in n variables with $n \geq 1$. More generally, if R is a finite flat extension of Λ, and in addition, is a regular local domain, then the rank of a finitely generated $R[[G]]$-module M can be computed in terms of the rank of M over $\Lambda[[G]]$ and the rank of R over Λ. Note that if G is a pro-p, p-adic Lie group such that G has no p-torsion, then by the remark of the previous section, $R[[G]]$ is a complete local Noetherian domain which satisfies the Auslander regular condition. The non-zero elements of this ring form an Ore set and we denote the skew field of fractions of $R[[G]]$ by $\mathcal{Q}$. The rank of any module over $\mathcal{Q}$ is well defined. If M is a finitely generated compact $R[[G]]$-module, then we define the *rank of* M, denoted by $\mathrm{rank}(M)$, to be the rank of $\mathcal{Q} \otimes_{R[[G]]} M$ over $\mathcal{Q}$. We prove the following theorem which generalizes [OV1, Theorem 3.2] to Λ-adic representations.

Theorem 5.1. *Let K be a number field such that $K(\mathcal{D})/K$ is pro-p and let K_∞ be a pro-p, p-adic Lie extension of K. Assume that $G = \mathrm{Gal}(K_\infty/K)$ has no p-torsion. Then $\mathrm{rank}(H^1\widehat{(\mathcal{H}_S, \mathcal{D})}) = r(K).r + \mathrm{rank}(H^2\widehat{(\mathcal{H}_S, \mathcal{D})})$, where r denotes the corank of $\mathcal{D}$ as an R-module and $r(K)$ denotes the number of complex places of K.*

Proof. From the right vertical exact sequence in Lemma 2.2 we have,

$$\operatorname{rank}(H^1(\widehat{K_S/K_\infty}, \mathcal{D})) = \operatorname{rank}(Y) - \operatorname{rank}(J) \tag{3}$$

$$\operatorname{rank}(J) = \operatorname{rank}(R[[\mathcal{G}]]^{\#})_{\mathcal{H}} \tag{4}$$

$$\operatorname{rank}(Y) = \operatorname{rank}(R[[G]]^{dr}) - \operatorname{rank}(H_1(\mathcal{N}, R)^{\#})_{\mathcal{H}} + \operatorname{rank}(H_2(\mathcal{H}, \widehat{\mathcal{D}})). \tag{5}$$

Let P be prime ideal of R such that $R/P \cong \mathbb{Z}_p$. Then, since R and $\mathbb{Z}_p$ are regular local, P is generated by a regular sequence. Let $\lambda \in P$ be a generator of a height one prime ideal (λ) of R. From the exact sequence

$$0 \longrightarrow R \xrightarrow{\lambda} R \longrightarrow R/\lambda \longrightarrow 0,$$

we get

$$0 \longrightarrow H_1(\mathcal{N}, R) \xrightarrow{\lambda} H_1(\mathcal{N}, R) \longrightarrow H_1(\mathcal{N}, R/\lambda) \longrightarrow 0.$$

The above sequence is exact because $\mathcal{N}$ has cohomological dimension ≤ 1 and $\mathcal{N}$ acts on R trivially. This in turn gives the following exact sequence.

$$H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}} \xrightarrow{\lambda} H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}} \longrightarrow H_1(\mathcal{N}, R/\lambda)^{\#}_{\mathcal{H}} \longrightarrow 0.$$

Therefore

$$\frac{H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}}{\lambda H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}} \cong H_1(\mathcal{N}, R/\lambda)^{\#}_{\mathcal{H}}$$

Since $H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}$ is $R[[G]]$-projective, and $R[[G]]$ is local, the module $H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}$ is in fact free. Therefore the rank of $H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}$ as an $R[[G]]$-module is the same as the rank of $H_1(\mathcal{N}, R/\lambda)^{\#}_{\mathcal{H}}$ as an $(R/\lambda)[[G]]$-module. Choosing a height one prime ideal P, and applying this argument successively, we get that the rank of $H_1(\mathcal{N}, R/P)^{\#}_{\mathcal{H}}$ as an $R/P[[G]]$-module is the same as the rank of $H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}$ as an $R[[G]]$-module. Now it follows from [OV1, Theorem 3.2] that the rank of $H_1(\mathcal{N}, R)^{\#}_{\mathcal{H}}$ as an $R[[G]]$-module is $r(d - r(K) - 1)$. Recall that d denotes the rank of the free pro-C-group F_d defined in (2). Therefore from equation (5), we see that the rank of Y is $r(r(K) + 1)$. From equation (4) and Lemma 2.3 the rank of J is r. Substituting the rank of Y and the rank of J in equation (3), we get $\operatorname{rank}(H_1(K_S/K_\infty, \widehat{\mathcal{D}})) = r(K)r + \operatorname{rank}(H_2(K_S/K_\infty, \widehat{\mathcal{D}}))$. $\square$

Theorem 5.2. *Let K be a finite extension of $\mathbb{Q}_l$, l a prime number. Suppose that K_∞ is a pro-p, p-adic Lie extension such that* $\operatorname{Gal}(K_\infty/K)$ *has no p torsion. Then* $\operatorname{rank}(H^1(\widehat{K_\infty}, \mathcal{D})) = r[K : \mathbb{Q}_p]$ *if $l = p$ and is 0 if $l \neq p$.*

Proof. The proof of this theorem is analogous to the global case. Notice that $H^2(K_\infty, \mathcal{D}) = 0$ because $cd_p \operatorname{Gal}(\bar{K}/K_\infty) \leq 1$. $\square$

We have the following more general theorem, for which we need not assume that $K(\mathcal{D})/K$ is pro-p.

Theorem 5.3. *Let K be number field and let K_∞ be a p-adic Lie extension of K. Then*

$$\begin{aligned} & rk_{R[[G]]}(H^1(\widehat{K_\infty, \mathcal{D}})) - \mathrm{rank}_{R[[G]]}(H^2(\widehat{K_\infty, \mathcal{D}})) \\ &= (r_1(K) + r_2(K))r - \sum_{v\ real} \dim_{\mathbb{F}_p}(\mathcal{D}[m])^+, \end{aligned}$$

where $(-)^+$ denotes the invariant part with respect to the complex conjugation and $\mathcal{D}[m] = \{x \in \mathcal{D} | ax = 0 \ \forall\ a \in m\}$.

Proof. We shall prove the theorem for $R = \mathbb{Z}_p[[X]]$. A proof for higher dimensional power series rings can be given using a similar method. Using the Hochschild-Serre spectral sequence and following the proof of [HV, 7.4] we obtain

$$\begin{aligned} & \sum_{i \geq 0} (-1)^{i+1} \mathrm{corank}_{R[[G]]} H^i(K_S/K_\infty, \mathcal{D}) \\ &= \sum_{i,j \geq} (-1)^{i+j+1} \mathrm{corank}_R H^j(G, H^i(K_S/K_\infty, \mathcal{D})) \\ &= \sum_{n \geq 0} (-1)^{n+1} \mathrm{corank}_R H^n(K_S/K, \mathcal{D}) \end{aligned}$$

By [Gr3, 2.1, 3.5], we can choose a height one prime ideal λ such that $R/\lambda \cong \mathbb{Z}_p$ and $\mathrm{corank}_R(H^n(K_S/K, \mathcal{D})) = \mathrm{corank}_{\mathbb{Z}_p}(H^n(K_S/K, \mathcal{D}[\lambda]))$. Therefore we have,

$$\begin{aligned} & \sum_{i \geq 0} (-1)^{i+1} \mathrm{corank}_{R[[G]]} H^i(K_S/K_\infty, \mathcal{D}) \\ &= \sum_{n \geq 0} (-1)^{n+1} \mathrm{corank}_{\mathbb{Z}_p} H^n(K_S/K, \mathcal{D}[\lambda]) \\ &= \sum_{n \geq 0} (-1)^{n+1} \dim_{\mathbb{F}} H^n(K_S/K, \mathcal{D}[m]) \\ &= (r_1(K) + r_2(K))r - \sum_{v\ real} \dim_{\mathbb{F}_p}(\mathcal{D}[m])^+ \end{aligned}$$

For the last equality see for example [NSW, 8.6.4] or [HV, 7.4]. The $R[[G]]$-rank of $H^0(K_S/K_\infty, \mathcal{D})$ is zero and $H^i(K_S/K_\infty, \mathcal{D}) = 0$ for all $i \geq 3$. Hence the theorem follows. □

6. Selmer groups

In this section, we study the Selmer groups of R-adic representations over p-adic Lie extensions of a number field. We will mainly consider two particular types of admissible extensions, namely the False Tate extensions [HV] and certain higher dimensional p-adic Lie extensions of K. We continue with the notation of sections 1 and 2. From now on Λ will denote the power series ring $\mathbb{Z}_p[[X]]$ in one variable. Fix an R-adic representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(R)$$

as in section 1 and let $\mathcal{T}$ be the underlying $R[[G_{\mathbb{Q}}]]$-module. We assume that ρ satisfies Hypothesis 1. In this section, we shall assume that R is a complete Noetherian local domain which is a finite flat extension of Λ. Let K be a number field and suppose that K_∞ is a pro-p, p-adic Lie extension of K such that $G = \mathrm{Gal}(K_\infty/K)$ does not have any p-torsion. As before, let $S(K)$ be the set of primes of K containing the primes over p, the primes where ρ is ramified and the infinite primes. Let S_f denote the set of finite primes of $S(K)$ and S_p denote the primes in $S(K)$ lying over p. Consider the discrete $R[[G_{\mathbb{Q}}]]$-module defined by

$$\mathcal{A} = \mathcal{T} \otimes_R \widehat{R}$$

where $\widehat{R} = \mathrm{Hom}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual of R with trivial action of $G_{\mathbb{Q}}$. Let $\mathrm{F}^+\mathcal{A}$ denote the submodule $F^+\mathcal{T} \otimes_R \widehat{R}$ and $F^-\mathcal{A}$ denote the quotient $\mathcal{A}/F^+\mathcal{A} \cong \mathcal{T}/F^+\mathcal{T} \otimes_R \widehat{R}$ of $\mathcal{A}$. Put $A = T_pE \otimes \mathbb{Q}_p/\mathbb{Z}_p$. By Hypothesis 1, there exists a prime ideal P with $P = \ker \lambda$ such that $\mathcal{T}/P$ corresponds to T_pE. Then $A \cong \mathcal{A}[P]$ [Gr3, section 1]. Recall that the *Selmer group* of $\mathcal{A}$ over K_∞ is defined by

$$\mathrm{Sel}(\mathcal{A}/K_\infty) = \mathrm{Ker}(H^1(K_S/K_\infty, \mathcal{A}) \xrightarrow{\gamma_\infty} \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_\infty))$$

where

$$J_v(\mathcal{A}, K_\infty) = \begin{cases} \prod_{w|v} H^1(K_{\infty,w}, \mathcal{A}/F^+\mathcal{A}) & \text{if } v|p \\ \prod_{w|v} H^1(K_{\infty,w}, \mathcal{A}) & \text{if } v \nmid p \end{cases}$$

We remark that dimension theory for Auslander regular rings (see for example [V]) enables one to prove results on the existence of non-trivial pseudo-null submodules for a finitely generated module over an Auslander regular domain. But when R is not regular local, the ring $R[[G]]$ is not necessarily Auslander regular. The following algebraic lemma is therefore useful in the study of pseudo-null submodules. This may be well known to experts, but we give a proof of this as it is important for establishing results in a more general context.

Lemma 6.1. *Let M be a finitely generated $R[[G]]$-module. Then M is a pseudo-null $R[[G]]$-module if and only if M is a pseudo-null $\Lambda[[G]]$-module.*

Proof. We shall show that for a finitely generated $R[[G]]$-module M, $j_{R[[G]]}(M) = j_{\Lambda[[G]]}(M)$. Let r denote the rank of R over Λ. We have

$$\begin{aligned} R \otimes_\Lambda \mathrm{Ext}^i_{\Lambda[[G]]}(M, \Lambda[[G]]) &\cong \mathrm{Ext}^i_{R[[G]]}(R \otimes_\Lambda M, R[[G]]) \\ &\cong \mathrm{Ext}^i_{R[[G]]}(M^r, R[[G]]) \\ &\cong \mathrm{Ext}^i_{R[[G]]}(M, R[[G]])^r. \end{aligned}$$

Note that $R \otimes_\Lambda \mathrm{Ext}^i_{\Lambda[[G]]}(M, \Lambda[[G]]) = 0$ if and only $\mathrm{Ext}^i_{\Lambda[[G]]}(M, \Lambda[[G]]) = 0$. Therefore $\mathrm{Ext}^i_{\Lambda[[G]]}(M, \Lambda[[G]]) = 0$ if and only if $\mathrm{Ext}^i_{R[[G]]}(M, R[[G]]) = 0$. This proves the assertion. $\square$

Our next result is a generalization of the results proved by Greenberg and Hachimori-Venjakob. Greenberg has considered the case when G is commutative and $\mathcal{A}$ is a cofinitely generated module over R [Gr3, Theorem 1]. Hachimori and Venjakob consider the case of an elliptic curve with G not necessarily commutative [OV, Theorem 4.7].

Theorem 6.2. *If $H^2(K_S/K_\infty, \mathcal{A}) = 0$, then $H^1(\widehat{K_S/K_\infty}, \mathcal{A})$ has no non-zero pseudo-null $R[[G]]$-submodule.*

Proof. From Lemma 2.2 and the proof of [OV, Theorem 4.6], we have $H^1(\widehat{K_S/K_\infty}, \mathcal{A})$ does not have any non-trivial pseudo-null $\Lambda[[G]]$-submodule. Note that here we have used that $\Lambda[[G]]$ is an Auslander regular domain. Now the theorem follows from Lemma 6.1. □

Lemma 6.3. *If a finitely generated Λ-module M is torsion, then there exist infinitely many height one prime ideals $\mathfrak{p}$ of Λ such that $M/\mathfrak{p}$ is finite.*

Proof. By the structure theorem for finitely generated torsion modules over Λ, any prime ideal $\mathfrak{p}$ of height one that does not divide the characteristic ideal of M has the desired property.

□

Let us denote $\mathcal{A}/F^+\mathcal{A}$ by $\mathcal{B}$. For a prime v of K and w be a prime of K_∞ lying over v, put $\mathcal{H}_w = \mathrm{Gal}(\bar{K_v}/K_{\infty,w})$. Let G_w (resp. T_w) denote the decomposition group (resp. the inertia group) of w in G. To prove the non-existence of pseudo-null submodules of the dual Selmer group, we need to prove the following result related to the module $Z_{\mathcal{B},K_{\infty,w}}$ which we considered in Section 3. Recall that the decomposition group G_p at p acts on $\mathcal{T}/F^+\mathcal{T}$ via an unramified character η. Let Fr_p denote the Frobenius at p.

Lemma 6.4. *Suppose that $\dim G_w = \dim T_w$ for all primes w of K_∞ lying above primes $v \in S_p(K)$. If $\eta(\mathrm{Fr}_p)$ is not of finite order, then $Z_{\mathcal{B},K_{\infty,w}}$ is a finitely generated torsion Λ-module.*

Proof. From Lemma 4.1, we have $\widehat{Z_{\mathcal{B},K_{\infty,w}}} \cong (\mathrm{Hom}_R(\widehat{B}, R)^*)^{\mathcal{H}_w}$, where $\mathcal{H}_w$ denotes the absolute Galois group of $K_{\infty,w}$. Since K_∞ contains K_{cyc}, $\mathcal{H}_w$ acts trivially on μ_{p^∞}, thus we have an isomorphism of $\mathcal{H}_w$-modules $\mathrm{Hom}_R(\widehat{B}, R)^* \cong \widehat{\mathrm{Hom}_R(\widehat{B}, R)}$. Since G_w and T_w have the same dimension, there exists a finite extension F_v of K_v in $K_{\infty,w}$ such that $K_{\infty,w}$ is totally ramified over F_v. Since $\mathcal{B}$ is unramified at p, we have that

$$\widehat{\mathrm{Hom}_R(\widehat{B}, R)}^{\mathcal{H}_w} = \widehat{\mathrm{Hom}_R(\widehat{B}, R)}^{G_{F_v}},$$

where G_{F_v} denotes the absolute Galois group of F_v. Now taking the Pontryagin dual, we get that

$$Z_{\mathcal{B},K_{\infty,w}} \cong \mathrm{Hom}_R(\widehat{B}, R)_{G_{F_v}} \cong \frac{\mathrm{Hom}_R(\widehat{B}, R)}{(\mathrm{Fr_v} - 1)\mathrm{Hom}_R(\widehat{B}, R)},$$

where $\mathrm{Fr_v}$ denotes the Frobenius at v in G_{F_v}. By assumption, $\eta(\mathrm{Fr_p})$ is not of finite order. Therefore $\eta(\mathrm{Fr_v})$ is not the identity element. Since $\mathrm{Hom}_R(\widehat{B}, R)$ is a free R-module of rank one and R is domain, it follows that $Z_{\mathcal{B},K_{\infty,w}}$ is a torsion R-module, and hence a torsion Λ-module. $\square$

Lemma 6.5. *Suppose that the assumptions of Lemma 6.4 hold. Then for every prime w of K_∞ lying above a prime $v \in S_p(K)$ such that* $\dim G_w = 2$, *we have that $J_v(\mathcal{A}, K_\infty)$ is a coreflexive $\Lambda[[G]]$-module.*

Proof. Choose a prime w of K_∞ such that $w|v$ for some $v \in S_p(K)$. Then we know that the $\Lambda[[G]]$-module $J_v(\mathcal{A}, K_\infty)$ can also be written as $\mathrm{Coind}_G^{G_w}(H^1(K_{\infty,w}, \mathcal{B}))$. Therefore $J_v\widehat{(\mathcal{A}, K_\infty)} = \mathrm{Ind}_G^{G_w}(H_1(K_{\infty,w}, \widehat{\mathcal{B}}))$. Since $\Lambda[[G]]$ is flat over $\Lambda[[G_w]]$, it follows that

$$\mathrm{E}^0_{\Lambda[[G]]}E^0_{\Lambda[[G]]}(\mathrm{Ind}_G^{G_w}(H_1(K_{\infty,w}, \widehat{\mathcal{B}})) \cong \mathrm{Ind}_G^{G_w}\mathrm{E}^0_{\Lambda[[G_w]]}E^0_{\Lambda[[G_w]]}(H_1(K_{\infty,w}, \widehat{\mathcal{B}}).$$

From Lemma 6.4 and 6.3 we conclude that there exists a height one prime ideal $\mathfrak{p}$ of Λ such that $\Lambda/\mathfrak{p}$ is a finite extension of $\mathbb{Z}_p$ and $Z_{\mathcal{B},K_{\infty,w}}/\mathfrak{p}$ is finite. Now it follows from Corollary 6.7 that $H_1(K_{\infty,w}, \widehat{\mathcal{B}})$ is a reflexive $\Lambda[[G_w]]$-module. Therefore

$$\mathrm{E}^0_{\Lambda[[G_w]]}E^0_{\Lambda[[G_w]]}(H_1(K_{\infty,w}, \widehat{\mathcal{B}}) \cong H_1(K_{\infty,w}, \widehat{\mathcal{B}}).$$

This shows that

$$\mathrm{E}^0_{\Lambda[[G]]}E^0_{\Lambda[[G]]}(\mathrm{Ind}_G^{G_w}(H_1(K_{\infty,w}, \widehat{\mathcal{B}})) \cong \mathrm{Ind}_G^{G_w}(H_1(K_{\infty,w}, \widehat{\mathcal{B}})),$$

thereby proving the assertion of the proposition. $\square$

Remark 6.6. *If $\eta(\mathrm{Fr}_p)$ is of finite order, then there exists a finite extension F' of $\mathbb{Q}$ and a prime v of F' lying above p such that η factors through F'_v. Thus if we take $F' = K$, then the image of Frobenius in G_F acts by multiplication by one. In this case we see that $Z_{\mathcal{B},K_{\infty,w}}$ is free of rank one. Thus the above proposition does not hold and the local Galois cohomology groups fail to be coreflexive.*

We mention that if P is a prime ideal of R such that $T_P := \mathcal{T}/P$ is a lattice of the Galois representation associated to a Hecke eigenform f_P of weight $k \geq 3$ with ordinary reduction at p, then it is known that the action of Frobenius in G_p on the unramified quotient of T_P is given by multiplication by an element whose complex norm is not one. If T_P is the Tate module T_pE of E and E has good ordinary reduction at p, then the same assertion holds. From this, it follows that if $\mathcal{T}$ is the Hida deformation of E as discussed in Section 1, then $\eta(\mathrm{Fr}_p)$ is not of finite order, and hence the assumptions of Lemma 6.5 are satisfied.

Proposition 6.7. *Suppose that for every prime w of K_∞ lying over a prime $v|p$ of K,* $\dim G_w \geq 3$. *Then $J_v(\mathcal{A}, K_\infty)$ is a coreflexive $\Lambda[[G]]$-module.*

Proof. From our assumption and Corollary 4.4, it follows that for every prime w of K_∞ lying above a prime $v|p$ in K, the $\Lambda[[G_w]]$-module $H_1(\bar{K_w}/K_{\infty,w}, \widehat{\mathcal{A}})$

is reflexive. It now follows from the proof of Proposition 6.5 that $J_v(\mathcal{A}, K_\infty)$ is a coreflexive $\Lambda[[G]]$-module for all $v|p$. $\square$

PROPOSITION 6.8. *Let v be a prime of K such that $v \nmid p$ and w be a prime of K_∞ such that $w|v$. If $\dim G_w \geq 2$ then we have $H^1(K_{\infty,w}, \mathcal{A}) = 0$.*

Proof. From Corollary 4.4 and the assumption of the proposition, it follows that $H^1(K_{\infty,w}, \mathcal{A})$ is cotorsion free as a $\Lambda[[G]]$-module. But Theorem 5.2 implies that $H^1(K_{\infty,w}, \mathcal{A})$ is a cotorsion module. This forces $H^1(K_{\infty,w}, \mathcal{A}) = 0$. $\square$

LEMMA 6.9. *Consider the following exact sequence of $\Lambda[[G]]$-modules:*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

Suppose V does not have any non-trivial pseudo-null $\Lambda[[G]]$-submodule and U is a reflexive $\Lambda[[G]]$-submodule. Then W does not contain any non-trivial pseudo-null submodule.

Proof. See [HO, Lemma 3.5]. $\square$

We finally state the following important theorem for R-adic representations which generalises a similar theorem proved in [HV, Theorem 2.6(ii)] for p-adic representations.

THEOREM 6.10. *Suppose that the following assumptions hold:*

(i) For every prime w of K_∞ lying above a prime $v \in S_p(K)$, $\dim G_w = \dim T_w = 2$ and $\eta(\mathrm{Fr}_p)$ is not of finite order.

(ii) For every prime w of K_∞ lying above a prime $v \in S(K) \setminus S_p(K)$, $\dim G_w \geq 2$.

(iii) $H^2(K_S/K_\infty, \mathcal{A}) = 0$.

If the global to local map γ_∞ in the exact sequence defining the Selmer group is surjective, then $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ has no non-trivial $R[[G]]$ pseudo-null submodule.

Proof. From Proposition 6.8 and the assumption in the theorem, we have

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_\infty) \longrightarrow H^1(K_S/K_\infty, \mathcal{A}) \xrightarrow{\gamma_\infty} \bigoplus_{v|p} J_v(\mathcal{A}, K_\infty) \longrightarrow 0$$

Taking Pontryagin duals, we get the exact sequence

$$0 \longrightarrow \bigoplus_{v|p} J_v\widehat{(\mathcal{A}, K_\infty)} \longrightarrow H^1(\widehat{K_S/K_\infty}, \mathcal{A}) \longrightarrow \mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)} \longrightarrow 0.$$

By Proposition 6.5 the first term in the above sequence is a reflexive $\Lambda[[G]]$-module and the middle term has no non-trivial pseudo-null submodule (Proposition 6.2). From Lemma 6.9 we have $\mathrm{Sel}(\mathcal{A}/K_\infty)$ does not contain any non-trivial pseudo-null $\Lambda[[G]]$-submodule. Thus, the theorem follows on applying Lemma 6.1. $\square$

We mention that the assumptions (i) and (ii) of the theorem hold when $K = K(\mu_p)$ and K_∞ is a False Tate extension $K(\mu_{p^\infty}, m^{1/p^\infty})$ of K where m is an integer prime to p such that m is divisible by the conductor N of E, and is also p-power free.

We now study an other type of Selmer group defined by Greenberg in [Gr] for cyclotomic extensions. Let $v \in S_p(K)$ and w be a prime of K_∞ such that $w|v$. We write $I_{\infty,w}$ for the inertia subgroup of the Galois group $G(\overline{K_{w,\infty}}/K_{w,\infty})$. The Selmer group $\mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty)$ of $\mathcal{A}$ over K_∞ is defined as

$$\mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty) = \mathrm{Ker}(H^1(K_S/K_\infty, \mathcal{A}) \longrightarrow \bigoplus_{v \in S(K)} J_v^{Gr}(\mathcal{A}, K_\infty))$$

where

$$J_v^{Gr}(\mathcal{A}, K_\infty) = \begin{cases} \prod_{w|v} H^1(I_{\infty,w}, \mathcal{A}/F^+\mathcal{A}) & \text{if } v|p \\ \prod_{w|v} H^1(K_{\infty,w}, \mathcal{A}) & \text{if } v \nmid p \end{cases}$$

We shall show that if Hypothesis 1 and assumption (i) of Theorem 6.10 hold, then $\mathrm{Sel}(\mathcal{A}/K_\infty) = \mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty)$. This will show that the $\widehat{\mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty)}$ has no non-trivial pseudo-null submodule. Let $v|p$ be a prime of K, and $w|v$ be a prime of K_∞. Then from the Hochschild-Serre spectral sequence, we have

$$0 \longrightarrow H^1(\mathfrak{H}_w, (\mathcal{A}/F^+\mathcal{A})^{I_{\infty,w}}) \longrightarrow H^1(\mathcal{H}_w, \mathcal{A}/F^+\mathcal{A}) \longrightarrow H^1(I_{\infty,w}, \mathcal{A}/F^+\mathcal{A})^{\mathcal{H}_w}$$

where, $\mathfrak{H}_w = \mathrm{Gal}(\overline{K_{w,\infty}}/K_{w,\infty})/I_{\infty,w} \hookrightarrow \mathrm{Gal}(\overline{K_w}/K_w)/I_w \cong \widehat{\mathbb{Z}}$, and is hence topologically cyclic. We shall show that the first term in the above exact sequence vanishes. It follows from Hypothesis 1(c) that $I_{\infty,w}$ acts trivially on $\mathcal{A}/F^+\mathcal{A}$. Thus we need to show that $H^1(\mathfrak{H}_w, \mathcal{A}/F^+\mathcal{A}) = 0$. We claim that $\mathfrak{H}_w$ acts non-trivially on $\mathcal{A}/F^+\mathcal{A}$. Suppose this is not true. This implies that $\mathcal{H}_w = \mathrm{Gal}(\overline{K_{w,\infty}}/K_{w,\infty})$ acts trivially on $\mathcal{A}/F^+\mathcal{A}$, and hence on $(\mathrm{Hom}_R(\widehat{B}, R)^*$, where $\mathcal{B} = \mathcal{A}/F^+\mathcal{A}$. Now by an argument similar to Lemma 6.4, it follows that there exists a finite extension F_v of K_v inside $K_{\infty,w}$ such that the absolute Galois group G_{F_v} of F_v acts trivially on $\mathcal{B}$. But this contradicts assumption (i) of Theorem 6.10. Therefore $\mathfrak{H}_w$ act non-trivially on $\mathcal{B}$. Now using the fact that $\mathfrak{H}_w$ is procyclic we conclude that $H^1(\mathfrak{H}_w, \mathcal{B}) \cong \mathcal{B}/(s-1)\mathcal{B}$ where, s is the generator of $\mathfrak{H}_w$. Since s does not act trivially and $\mathcal{B}$ is cofree, we see that $(s-1)\mathcal{B} = \mathcal{B}$. Therefore $H^1(\mathfrak{H}_w, \mathcal{B}) = H^1(\mathfrak{H}_w, (\mathcal{A}/F^+\mathcal{A})^{I_{\infty,w}}) = 0$. We consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}(\mathcal{A}/K_\infty) \arrow[r] \arrow[d] & H^1(K_S/K_\infty, \mathcal{A}) \arrow[r] \arrow[d] & \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_\infty)) \arrow[d, "f"] \\
0 \arrow[r] & \mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty) \arrow[r] & H^1(K_S/K_\infty, \mathcal{A}) \arrow[r] & \bigoplus_{v \in S(K)} J_v^{Gr}(\mathcal{A}, K_\infty).
\end{tikzcd}$$

The kernel of the last vertical map is product of $H^1(\mathfrak{H}_w, (\mathcal{A}/F^+\mathcal{A})^{I_{\infty,w}})$, where w varies over set of primes of K_∞ lying above p. But we have already shown that $H^1(\mathfrak{H}_w, (\mathcal{A}/F^+\mathcal{A})^{I_{\infty,w}}) = 0$. Now it follows from the snake lemma that $\mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty) \cong \mathrm{Sel}(\mathcal{A}/K_\infty)$. Thus we have have the following Corollary to Theorem 6.10.

COROLLARY 6.11. *Suppose that the assumptions of Theorem 6.10 hold. Then* $\widehat{\mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty)}$ *does not have any non-trivial pseudo-null* $R[[G]]$*-submodule.*

□

Suppose that E has split multiplicative reduction at p and consider the Selmer group $\mathrm{Sel}(A/K_\infty)$ associated to the Galois representation of E. Then the module $H^1(\mathfrak{H}_w, A/F^+A)$ is a cofinitely generated non-trivial $\mathbb{Z}_p$-module. This is true because on the unramified quotient A/F^+A, the Frobenius acts trivially. Let M denote the product of $H^1(\mathfrak{H}_w, A/F^+A)$, where w varies over primes of K_∞ lying above p. If dim G = dim G_w = 2 for every prime $v|p$, then there are finitely many primes of K_∞ above each prime v of K as the index of decomposition groups are finite. Therefore M is cofinitely generated as a $\mathbb{Z}_p$-module. Now using snake lemma we see that there exists an exact sequence

$$0 \longrightarrow \mathrm{Sel}(A/K_\infty) \longrightarrow \mathrm{Sel}^{Gr}(\mathcal{A}/K_\infty) \longrightarrow M \longrightarrow 0.$$

Therefore the Pontryagin dual of $\mathrm{Sel}^{Gr}(A/K_\infty)$-contains a non-trivial $\mathbb{Z}_p[[G]]$-submodule which is also a finitely generated $\mathbb{Z}_p$-module and therefore a pseudo-null $\mathbb{Z}_p[[G]]$-module. However, we remark that this does not rule out the existence of non-trivial pseudo-null $\mathbb{Z}_p[[G]]$-submodules for $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ as the local Galois cohomology groups above p fail to be reflexive (cf. Lemma 6.5).

In the previous section, we considered the Selmer group of $\mathcal{A}$ over p-adic Lie extensions satisfying condition (i) of Theorem 6.10. Now we consider the Selmer group over certain higher dimensional p-adic Lie extensions for which this condition is not needed.

THEOREM 6.12. *Suppose that for every prime w of K_∞ lying above a primes $v \in S_p(K)$,* $\dim G_w \geq 3$. *Suppose further that assumptions* (ii) *and* (iii) *of Theorem 6.10 hold. If the global to local map γ_∞ defining the Selmer group* $\mathrm{Sel}(\mathcal{A}/K_\infty)$ *is surjective, then* $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ *has no non-trivial* $R[[G]]$ *pseudo-null submodule.*

Proof. We notice from Proposition 6.7 that for every prime v of K lying above a prime p, the $\Lambda[[G]]$-module $J_v(\mathcal{A}, K_\infty)$ is coreflexive. The rest of the argument is analogous to Theorem 6.10 and the result follows. □

Let K be a number field such that $K = K(E_p)$ and K_∞ be a p-adic Lie pro-p extension of K that contains $K(E_{p^\infty})$. Put $G = \mathrm{Gal}(K_\infty/K)$. If E has no complex multiplication, then it is well known that, the Galois group of K_∞/K is a pro-p, p-adic Lie group without p-torsion and is of dimension at least 4. Further, if E has good reduction at the primes of K dividing p, then for every prime $v|p$, the dimension of G_v is greater than equal to 3 ([S1, IV-43], see also [CH, Lemma 5.1]). If $v \nmid p$ and E does not have good reduction at v, then E has potentially multiplicative reduction at v by a result of Serre-Tate [ST, 2,Corollary 2] and thus the dimension of G_v is ≥ 2. Let $S(K) = \{v|p\} \cup \{v|v \nmid p \text{ and E has bad reduction at } v\} \cup \{v|\infty\}$. Suppose that

$K_\infty \subset K_S$. Then all the assumptions of Theorem 6.12 on the dimension of the p-adic Lie group G are satisfied.
We remark that if E has split multiplicative reduction at a prime $v|p$ then the dimension of G_w may not be equal to 3 for a prime $w|v$. But if we consider the Hida deformation of E as discussed in Section 1, then for every $k \equiv 2 \mod p-1$ the modular form corresponding to the specialisation of weight k will be the ordinary p-stabilization of a modular form of level prime to p and trivial nebentype. If we take K_∞ to be a pro-p, p-adic Lie extension whose Galois group has no p-torsion and contains the trivializing extension for the Galois representation of any such modular form, then one always expect that $\dim G_w \geq 3$ for every prime w of K_∞ lying above p. Nevertheless, under certain assumptions it is proved in [Ek] that for all but finitely many such specialisations of Hida deformation this is indeed true. Thus for all those p-adic Lie extensions, the same result holds.

So far we have considered the case when R is a local domain. Now suppose that R is a complete semi-local reduced Noetherian ring and there exists a representation ρ satisfying Hypothesis 1(b) and 1(c). Let $\mathcal{T}$ denote the associated G_K-module. Then in this case, we can once again define the discrete $R[[G_K]]$-module $\mathcal{A} := \mathcal{T} \otimes_R \widehat{R}$. One may wonder about the existence of non-trivial pseudo-null $R[[G]]$-submodules for the corresponding dual Selmer group. We discuss this question now. In this case, R splits into a direct product of local rings R_m, where m varies over the set of maximal ideals of R. Then $\mathcal{T}$ splits as a direct sum of $\mathcal{T}_m$. Consider the discrete $R[[G]]$-module $\mathcal{A}_m$ defined by $\mathcal{T}_m \otimes_R \widehat{R} \cong \mathcal{T}_m \otimes_{R_m} R_m \otimes_R \widehat{R}$. Then $\mathcal{A}$ splits into a direct sum of $\mathcal{A}_m$. Note that $R_m \otimes_R \widehat{R} \cong \widehat{\mathrm{Hom}_R(R_m, R)} \cong \widehat{\mathrm{Hom}_{R_m}(R_m, R_m)}$. Therefore $\mathcal{A}_m \cong \mathcal{T}_m \otimes_{R_m} \widehat{R_m}$.

Proposition 6.13. *Suppose that for every maximal ideal m of R, $\widehat{\mathrm{Sel}(\mathcal{A}_m/K_\infty)}$ does not have any non-trivial, pseudo-null $R_m[[G]]$-submodule. Then $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ does not have any non-trivial, pseudo-null $R[[G]]$-submodule.*

Proof. Since $\mathcal{A}$ splits into a direct sum of $\mathcal{A}_m$ as Galois modules, $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ splits into a direct sum of $\widehat{\mathrm{Sel}(\mathcal{A}_m/K_\infty)}$ as a module over $R[[G]]$, where each component is an $R_m[[G]]$-module and the action is component wise. Now by assumption, each component $\widehat{\mathrm{Sel}(\mathcal{A}_m/K_\infty)}$ has no non-trivial, pseudo-null $R_m[[G]]$-submodule and therefore for every maximal ideal m of R, $\widehat{\mathrm{Sel}(\mathcal{A}_m/K_\infty)}$ does not have any non-trivial, pseudo-null $\Lambda[[G]]$-submodule. This shows that $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ does not have any non-trivial, pseudo-null $\Lambda[[G]]$-submodule. Therefore $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ does not have any non-trivial, pseudo-null $R[[G]]$-submodule. $\square$

We mention that a local factor R_m of R in the above proposition may not be a domain. But our result on the non-existence of non-trivial, pseudo-null submodules of the dual Selmer group associated to a Galois representation over the local ring R assumes that it is a domain. In fact, we have used that R is a domain only in the proof of Lemma 6.4. Now we generalize the above result to a more general context. Suppose that R is a reduced, complete local Noetherian ring and let $\mathcal{T}$ be the rank 2 R-module associated to the Galois representation satisfying Hypothesis 1.

LEMMA 6.14. *Suppose that condition one of 6.10(i) is satisfied for the Galois module $T/\mathfrak{a}$ for every minimal prime ideal $\mathfrak{a}$ of R. Then $Z_{\mathcal{B},K_{\infty,w}}$ is a finitely generated torsion Λ-module.*

Proof. First recall that $\mathcal{B}$ denotes the Galois module $F^-\mathcal{T}\otimes\widehat{R}$ unramified at p, where $F^-\mathcal{T}\cong\mathcal{T}/F^+\mathcal{T}$. Since R is a reduced ring, the natural map from R to $\prod_{\mathfrak{a}} R/\mathfrak{a}$ is an injection. Let X denote the cokernel of this map. Then $X_{\mathfrak{a}}=0$ for every minimal prime ideal $\mathfrak{a}$ of R, where $X_{\mathfrak{a}}$ denote the localisation of X at $\mathfrak{a}$. This implies that the height of a prime ideal of R in the support of X is at least 1. Therefore the zero ideal of Λ is not in the support of X, and hence X is a torsion Λ-module. Now consider the following exact sequence

$$0\longrightarrow \mathrm{Hom}_R(\widehat{\mathcal{B}},R)\longrightarrow \oplus_{\mathfrak{a}}\,\mathrm{Hom}_R(\widehat{\mathcal{B}},R)/\mathfrak{a}\longrightarrow X\otimes\mathrm{Hom}_R(\widehat{\mathcal{B}},R)\longrightarrow 0.$$

Since $\mathcal{B}$ is a cofree R-module we have $\mathrm{Hom}_R(\widehat{\mathcal{B}},R)/\mathfrak{a}\cong\mathrm{Hom}_{R/\mathfrak{a}}(\widehat{\mathcal{B}}/\mathfrak{a},R/\mathfrak{a})$ for every minimal prime ideal $\mathfrak{a}$ of R. The last term in the above exact sequence is a torsion Λ-module. We conclude that $\mathrm{Hom}_R(\widehat{\mathcal{B}},R)_{G_{F_v}}$ is torsion Λ-module if and only if $\mathrm{Hom}_{R/\mathfrak{a}}(\widehat{\mathcal{B}}/\mathfrak{a},R/\mathfrak{a})_{G_{F_v}}$ is a torsion Λ-module for every minimal ideal $\mathfrak{a}$ of R. Here F_v is the finite extension of $\mathbb{Q}_p$ considered in the proof of Lemma 6.4. Under the hypothesis of the lemma, it follows from the proof of Lemma 6.4 that $\mathrm{Hom}_{R/\mathfrak{a}}(\widehat{\mathcal{B}}/\mathfrak{a},R/\mathfrak{a})_{G_{F_v}}$ is a torsion Λ-module for every minimal prime ideal $\mathfrak{a}$ of R. This shows that $Z_{\mathcal{B},K_{\infty,w}}\cong\mathrm{Hom}_R(\widehat{\mathcal{B}},R)_{G_{F_v}}$ is a torsion Λ-module. $\square$

In the remaining part of this section, we shall discuss the vanishing of the second cohomology group $H^2(K_S/K_\infty,\mathcal{A})$. This is the analogue of the classical Weak Leopoldt conjecture. Recall that this vanishing was a necessary assumption for Theorems 6.10 and 6.12. The proof of this vanishing will depend on certain ramification behaviour of prime ideals of Λ in R. We start by fixing some notation. Let $\mathfrak{S}$ denote a set of height one prime ideals of Λ which remain unramified in R and such that for every $\mathfrak{p}\in\mathfrak{S}$, $\Lambda/\mathfrak{p}$ is a finite extension of $\mathbb{Z}_p$.

LEMMA 6.15. *Let $\mathfrak{p}\in\mathfrak{S}$ be a height one prime ideal of Λ. Then the kernel and cokernel of the map*

$$R/\mathfrak{p}\longrightarrow\oplus_{P|\mathfrak{p}}R/P$$

are finite.

Proof. If $\mathfrak{p}$ is unramified in R, then for every minimal prime ideal P of $R/\mathfrak{p}$ we have that $R_P/\mathfrak{p}_P\cong R_P/P_P$. This shows that the kernel and cokernel of the above map has support consisting of only maximal ideals m of R. Therefore it must be finite. $\square$

Proposition 6.16. *Let $\mathfrak{p} \in \mathfrak{S}$ be a height one prime ideal of Λ. Suppose that for every height one prime ideal P of R lying above $\mathfrak{p}$ we have that $H^2(K_S/K_\infty, \mathcal{A}[P]) = 0$. Then $H^2(K_S/K_\infty, \mathcal{A}) = 0$.*

Proof. Let $\mathfrak{p}$ be a height one prime ideal of Λ such that it remains unramified in R. Now from Lemma 6.15, the kernel and cokernel of the map from

$$\widehat{\mathcal{A}}/\mathfrak{p} \longrightarrow \oplus_{P|\mathfrak{p}} \widehat{\mathcal{A}} \otimes R/P$$

are finite. Therefore the kernel and cokernel of the map

$$\oplus_{P|\mathfrak{p}} \mathcal{A}[P] \longrightarrow \mathcal{A}[\mathfrak{p}].$$

are finite. This shows that the kernel and cokernel of the map

$$\oplus_{P|\mathfrak{p}} H^2(K_S/K_\infty, \mathcal{A}[P]) \longrightarrow H^2(K_S/K_\infty, \mathcal{A}[\mathfrak{p}])$$

are $\mathbb{Z}_p$-torsion. Since we have assumed that $H^2(K_S/K_\infty, \mathcal{A}[P]) = 0$, we get that $H^2(K_S/K_\infty, \mathcal{A}[\mathfrak{p}])$ is torsion as a $\mathbb{Z}_p[[G]]$-module. But Proposition 3.5 says that $H^2(K_S/K_\infty, \mathcal{A}[\mathfrak{p}])$ is a coreflexive $\mathbb{Z}_p[[G]]$-module. Therefore it must be zero. Since $\mathfrak{p}$ is a principal ideal and R is flat over Λ and hence torsion free, we have a surjective map from $H^2(K_S/K_\infty, \mathcal{A}[\mathfrak{p}])$ to $H^2(K_S/K_\infty, \mathcal{A})[\mathfrak{p}]$. Therefore $H^2(K_S/K_\infty, \mathcal{A})[\mathfrak{p}] = 0$, and by Nakayama's lemma it follows that $H^2(K_S/K_\infty, \mathcal{A}) = 0$. This proves the proposition. □

Corollary 6.17. *Suppose that K is an abelian extension of $\mathbb{Q}$ and $\mathrm{Gal}(K_\infty/K_{cyc})$ is a p-adic Lie extension of dimension ≤ 1. Let $\mathfrak{p} \in \mathfrak{S}$ be a prime ideal of Λ. If for every height one prime ideal P of R lying above $\mathfrak{p}$, the Galois module $\mathcal{T}/P\mathcal{T}$ is a lattice corresponding to the Galois representation of a Hecke eigenform of tame level N, then $H^2(K_S/K_\infty, \mathcal{A}) = 0$.*

Proof. Let P be a height one prime ideal of R lying above $\mathfrak{p}$. If K is an abelian extension of $\mathbb{Q}$, then it is a deep result of Kato that the group $H^2(K_S/K_{cyc}, \mathcal{A}[P]) = 0$. It then follows from an argument entirely analogous to [HV, Remark 2.2] that $H^2(K_S/K_\infty, \mathcal{A}[P]) = 0$. Now the Corollary follows from Proposition 6.16. □

We mention that if K_∞ is a False Tate extension of K, then $\mathrm{Gal}(K_\infty/K_{cyc}) \simeq \mathbb{Z}_p$ and hence has p-cohomological dimension 1. Thus the assumption of the above corollary holds. The next result proves a similar assertion for a p-adic Lie extension of larger dimension. For every height one prime ideal P of R, let $K(\mathcal{A}[P])$ denote the trivialising extension of $\mathcal{A}[P]$.

Corollary 6.18. *Let $\mathfrak{p} \in \mathfrak{S}$ and $K = K(E[p])$. Assume that for every height one prime ideal P of R lying above $\mathfrak{p}$, the Galois module $\mathcal{T}/P\mathcal{T}$ is a lattice corresponding to the Galois representation of a Hecke eigenform of tame level N. Further assume that K_∞ is a Galois extension of K such that the group $\mathrm{Gal}(K_\infty/K)$ is a pro-p, p-adic Lie extension and without p-torsion. If $K(\mathcal{A}[P]) \subset K_\infty$ for every prime P lying above $\mathfrak{p}$, then $H^2(K_S/K_\infty, \mathcal{A}) = 0$.*

Proof. Let P be a prime of R such that P lies above $\mathfrak{p}$. Since R/P is a finite $\mathbb{Z}_p$-algebra, $\mathcal{T}/P$ is isomorphic to a finitely generated free $\mathbb{Z}_p$-module with the trivial action of $\mathrm{Gal}(K_S/K_\infty)$. Thus $\mathcal{A}[P]$ is a finite direct sum of $\mathbb{Q}_p/\mathbb{Z}_p$ as a module over $\mathrm{Gal}(K_S/K_\infty)$. Now it follows from the proof of [OV, Corollary 4.8] that $H^2(K_S/K_\infty, \mathcal{A}[P]) = 0$, and the assertion follows from Proposition 6.16. □

We remark that if we consider the extension $K_\infty = K(\mathcal{A}[P])$ for one particular prime P of R lying above $\mathfrak{p}$ in the previous corollary, then we cannot conclude using Proposition 6.16 that $H^2(K_S/K_\infty, \mathcal{A}) = 0$, as we do not know at present that, $H^2(K_S/K_\infty, \mathcal{A}[Q]) = 0$ for a prime $Q \neq P$ of R lying above $\mathfrak{p}$. But if we assume that P is principal prime ideal, then $H^2(K_S/K_\infty, \mathcal{A}[P])$ surjects onto $H^2(K_S/K_\infty, \mathcal{A})[P]$. In that case, using Nakayama's lemma, we may conclude that $H^2(K_S/K_\infty, \mathcal{A}) = 0$.

We end this section by recalling various rings considered in [EPW] over which there exists a Galois representation satisfying Hypothesis 1. Let $\mathbf{T}_N$ denote the universal ordinary Hecke algebra of tame level N and $\mathbf{T}_N^{new}$ denote the new quotient of $\mathbf{T}_N$. Then it is known that $\mathbf{T}_N^{new}$ is a finite, reduced, torsion free Λ-algebra. The algebra $\mathbf{T}_N^{new}$ splits as a product of local rings $(\mathbf{T}_N^{new})_m$ where m varies over the maximal ideals of $\mathbf{T}_N^{new}$ and the subscript m denotes the localization at m. Then it is known that each such local factor is a finite flat extension of Λ([Hi, Section 9]).

If the residual representation associated to E at p is irreducible, then there exists a maximal ideal m of $\mathbf{T}_N^{new}$ and a rank two free module $\mathcal{T}_m$ over $(\mathbf{T}_N^{new})_m$ satisfying Hypothesis 1(see [EPW, Proposition 2.2.7]). We mention that the ring $(\mathbf{T}_N^{new})_m$ need not be domain. But one can replace this ring by $(\mathbf{T}_N^{new})_m/\mathfrak{a} \cong \mathbf{T}_N^{new}/\mathfrak{a}$ for a minimal prime ideal $\mathfrak{a} \subset m$ of $(\mathbf{T}_N^{new})_m$ depending on E. For every integer $k \geq 2$ we had considered a height one prime ideal $\mathfrak{p}_k$ of Λ in section 1. Then it is known that the rings $(\mathbf{T}_N^{new})_m$ and $\mathbf{T}_N^{new}/\mathfrak{a}$ are unramified over every such prime ideal $\mathfrak{p}_k$. This follows from the fact that $((\mathbf{T}_N^{new})_m)_{\mathfrak{p}_k} \cong (\mathbf{T}_N^{new}/\mathfrak{a})_{\mathfrak{p}_k} \cong (\mathbf{T}_N^{new})_{\mathfrak{p}_k}$ and $\mathfrak{p}_k$ is unramified in $\mathbf{T}_N^{new}$ for each $k \geq 2$ ([EPW, Theorem 2.1.3]). Here the subscript $\mathfrak{p}_k$ denotes the localization of the corresponding ring at $\mathfrak{p}_k$.

The Galois module $\mathcal{T}_m/\mathfrak{a}$ satisfies all the properties of Hypothesis 1, except that if $\mathbf{T}_N^{new}/\mathfrak{a}$ is not integrally closed, then it is not known whether this local ring is a flat extension of Λ (we thank the referee for kindly pointing this out to us). To remedy this problem, one can again replace this ring by various larger rings in its field of fractions. Let $\widehat{\mathbf{T}}$ denote the integral closure of $(\mathbf{T}_N^{new})_m/\mathfrak{a}$ in its field of fractions. This new ring is a finite flat extension of Λ (see [EPW, Proposition 2.2.4]).

For every integer $k \geq 2$, let $\widehat{\mathbf{T}}_{\mathfrak{p}_k}$ denote the localization of $\widehat{\mathbf{T}}$ at $\mathfrak{p}_k$. Then the natural map from $(\mathbf{T}_N^{new}/\mathfrak{a})_{\mathfrak{p}_k}$ to $\widehat{\mathbf{T}}_{\mathfrak{p}_k}$ is an isomorphism and $\mathfrak{p}_k$ remains unramified in $\widehat{\mathbf{T}}$ (see the proof of [EPW, Proposition 2.2.4]). Now we have a rank two free module $\widehat{\mathcal{T}}$ over $\widehat{\mathbf{T}}$ defined as $\mathcal{T}_m/\mathfrak{a} \otimes \widehat{\mathbf{T}}$. This Galois module

satisfies all the assumptions of Hypothesis 1(see also [EPW, Remark 2.2.3] and [Gr2, Section 3.4]). In fact condition (b) and (c) of Hypothesis 1 are satisfied by construction. Since the natural map from $(\mathbf{T}_N^{new}/\mathfrak{a})_{\mathfrak{p}_k}$ to $\widehat{\mathbf{T}}_{\mathfrak{p}_k}$ is an isomorphism for every height one prime ideal $\mathcal{P}_k$ of $\mathbf{T}_N^{new}/\mathfrak{a}$ lying above $\mathfrak{p}_k$, we have $\mathcal{T}_m/\mathfrak{a}\otimes \widehat{\mathbf{T}}_{\mathcal{P}_k} \cong (\mathcal{T}_m)_{\mathcal{P}_k}$. Thus $\widehat{\mathcal{T}}_{\mathcal{P}_k}/\mathcal{P}_k$ is equivalent to the Galois representation of a Hecke eigenform of weight k of tame level N. In particular $\widehat{\mathcal{T}}_{\mathcal{P}_2}/\mathcal{P}_2 \cong T_pE\otimes\mathbb{Q}_p$ for a height one prime ideal $\mathcal{P}_2$ lying over $\mathfrak{p}_2$. Therefore T_pE is a $\mathbb{Z}_p$-lattice of the Galois representation $\widehat{\mathcal{T}}_{\mathcal{P}_2}/\mathcal{P}_2$. Now using the assumption that the residual representation associated to E at p is irreducible, it can be shown that $\widehat{\mathcal{T}}/\mathcal{P}_2$ is equivalent to T_pE. Thus all the conditions of Hypothesis 1 are satisfied.
We mention that there is yet another ring, namely the reflexive hull of $(\mathbf{T}_N^{new}/\mathfrak{a})$ in its field of fractions over which there exists a Galois representation satisfying Hypothesis 1. If $\tilde{\mathbf{T}}$ denotes the reflexive hull of $\mathbf{T}_N^{new}/\mathfrak{a}$, then we have $\tilde{\mathbf{T}}_{\mathfrak{p}_k} \cong \widehat{\mathbf{T}}_{\mathfrak{p}_k}$ for every $k\geq 2$ ([Hi, Corollary 1.4]). Therefore $\tilde{\mathbf{T}}$ is unramified at $\mathfrak{p}_k$.

7. Surjectivity of global to local map

In this section, we discuss the surjectivity of the global to local map defining the Selmer group. We continue to assume that K_∞ is a pro-p, p-adic Lie extension of a number field K such that $\mathrm{Gal}(K_\infty/K)$ has no p-torsion.
Let F be a finite extension of K such that $K\subset F\subset K_\infty$ and let $S(F)$ denote the set of primes of F that lie above the set of primes in $S(K)$. Then for every $i\geq 0$, the module $\text{Ш}^i(F,\mathcal{A})$ [Gr3] is defined by

$$\text{Ш}^i(F,\mathcal{A}) = \mathrm{Ker}(H^i(F_S/F,\mathcal{A}) \longrightarrow \bigoplus_{v\in S(F)} H^i(F_v,\mathcal{A})).$$

The compact version of the above cohomology groups with continuous cocycles taking values in $\mathcal{T}^* =: \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{A},\mu_{p^\infty})$ are defined by

$$\text{Ш}^i(F,\mathcal{T}^*) = \mathrm{Ker}(H^i(F_S/F,\mathcal{T}^*) \longrightarrow \bigoplus_{v\in S(F)} H^i(F_v,\mathcal{T}^*)).$$

Let w be a prime of F such that $w|v$ for some $v\in S_p(K)$. Then put

$$L_{w,F} = \mathrm{Image}(H^1(F_w,F^+\mathcal{A}) \longrightarrow H^1(F_w,\mathcal{A}))$$
$$Q_{w,F} = H^1(F_w,\mathcal{A})/L_{w,F}$$

If w is a prime of F such that $w\nmid p$ then we take $L_{w,F}=0$ and $Q_{w,F} = H^1(F_w,\mathcal{A})$. Let $L^*_{w,F}$ denote the orthogonal complement of $L_{w,F}$ in $H^1(F_w,\mathcal{T}^*)$ under local duality. The compact version of the Selmer group $\mathrm{Sel}(\mathcal{T}^*/F)$ with respect to these local conditions is defined as

$$\mathrm{Sel}(\mathcal{T}^*/F) = \mathrm{Ker}(H^1(F_S/F,\mathcal{T}^*) \longrightarrow \bigoplus_{v\in S(K)} J_v(\mathcal{T}^*,F))$$

where $J_v(\mathcal{T}^*,F) = \prod_{w|v} H^1(F_w,\mathcal{T}^*)/L^*_{w,F}$.

Notice that $L^*_{w,F}$ and $H^1(F_w,\mathcal{T}^*)/L^*_{w,F}$ are compact R-modules isomorphic to the Pontryagin dual of $Q_{w,F}$ and $L_{w,F}$ respectively. The map $H^1(F,\mathcal{T}^*) \longrightarrow \widehat{Q_{w,F}}$ is induced by the restriction map $G_{F_w} \longrightarrow G(F_S/F)$ (see [Gr2, section 2]). The compact $R[[G]]$-module $\mathcal{R}(\mathcal{T}^*, K_\infty)$ is defined as $\varprojlim_F \mathrm{Sel}(\mathcal{T}^*/F)$, where F varies over finite Galois extensions of K contained in K_∞.

LEMMA 7.1. *Suppose that $H^2(K_S/K_\infty,\mathcal{A}) = 0$ and that assumption (ii) of Theorem 6.10 hold. Then the compact $\Lambda[[G]]$-module $\mathcal{R}(\mathcal{T}^*, K_\infty)$ is torsion free.*

Proof. From the global duality theorem, for every finite extension F of K, we have $\widehat{Ш^2(F,\mathcal{A})} \cong Ш^i(F,\mathcal{T}^*)$ [NSW, 8.6.8]. Therefore $Ш^2(K_\infty,\mathcal{A}) \cong \varprojlim_F Ш^1(F,\mathcal{T}^*)$, where $Ш^2(K_\infty,\mathcal{A}) = \varinjlim_F Ш^2(F,\mathcal{A})$. Since $Ш^2(K_\infty,\mathcal{A}) \subset H^2(K_S/K_\infty,\mathcal{A})$ and $H^2(K_S/K_\infty,\mathcal{A}) = 0$ by assumption of the lemma, we get $\varinjlim_F Ш^2(F,\mathcal{A}) = 0$. This shows that

$$\varprojlim_F H^1(F_S/F,\mathcal{T}^*) \longrightarrow \varprojlim_F \bigoplus_{v\in S(F)} H^1(F_v,\mathcal{T}^*)$$

is an injective map. On the other hand, from local duality and Corollary 4.4, it follows that $\varprojlim_F \bigoplus_{v\in S(F)} H^1(F_v,\mathcal{T}^*)$ is a torsion free $R[[G]]$-module. Therefore $\varprojlim_F H^1(F_S/F,\mathcal{T}^*)$ is a torsion free $R[[G]]$-module. As $\mathcal{R}(\mathcal{T}^*, K_\infty)$ injects into $\varprojlim_F H^1(F_S/F,\mathcal{T}^*)$, the assertion is true. □

THEOREM 7.2. *Suppose that Hypothesis 1 and assumption (ii) of Theorem 6.10 hold. If $H^2(K_S/K_\infty,\mathcal{A}) = 0$ then the sequence*

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_\infty) \longrightarrow H^1(K_S/K_\infty,\mathcal{A}) \longrightarrow \bigoplus_{v\in S(K)} J_v(\mathcal{A},K_\infty) \longrightarrow 0$$

is exact if and only if $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ is a torsion $R[[G]]$-module.

Proof. The Poitou-Tate exact sequence gives an exact sequence

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_\infty) \longrightarrow H^1(K_S/K_\infty,\mathcal{A}) \longrightarrow \bigoplus_{v\in S(K)} J_v(\mathcal{A},K_\infty)$$

$$\longrightarrow \widehat{\mathcal{R}(\mathcal{T}^*, K_\infty)} \longrightarrow \varinjlim_F \widehat{Ш^1(F,\mathcal{T}^*)} \longrightarrow 0.$$

As shown in the proof of Lemma 7.1, the last term in the above exact sequence is zero. Let s denote the rank of R as a Λ-module. We denote by m_Λ and m_R the maximal ideal of Λ and R respectively. From Theorem 5.3 we have

$$\mathrm{corank}_{\Lambda[[G]]}(H^1(K_S/K_\infty,\mathcal{A})) = 2s(r_1(K)+r_2(K)) + \oplus_v \dim_{\Lambda/m_\Lambda}(\mathcal{D}[m_\Lambda])^+$$

where $r_1(K)$ (resp. $r_2(K)$) denotes the number of real (resp. complex) embeddings of K, v varies over the set of all real primes of K and superscript $(-)^+$ denotes the invariant part with respect to complex conjugation. Note that Hypothesis 1(a) implies that $R/m_R \cong \Lambda/m_\Lambda$. From this we conclude that $\dim_{\Lambda/m_\Lambda}(\mathcal{D}[m_\Lambda])^+ = s \times \dim_{R/m_R}(\mathcal{D}[m_R])^+$. Again by Hypothesis 1, we have

$\mathcal{D}[m_R] \cong E[p]$. Now it is well known that the action of complex conjugation on $E[p]$ has two eigenvalues namely $+1$ and -1. Therefore $\dim_{R/m_R}(\mathcal{D}[m_R])^+ = 1$. Thus we have

$$\mathrm{corank}_{\Lambda[[G]]}(H^1(K_S/K_\infty, \mathcal{A})) = 2s(r_1(K) + r_2(K)) + sr_1(K) = s[K : \mathbb{Q}].$$

Since the module $\mathcal{A}/F^+\mathcal{A}$ is a cofree rank one R-module, it follows from Theorem 5.2 that the Λ-corank of the third term in the above exact sequence is equal to $s[K : \mathbb{Q}]$. Thus the $R[[G]]$-coranks of the middle term and the last term are equal. This shows that $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ is torsion if and only if $\mathcal{R}(\mathcal{T}^*, K_\infty)$ is torsion. But the previous lemma asserts that $\mathcal{R}(\mathcal{T}^*, K_\infty)$ is torsion free. Hence $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ is torsion if and only if $\mathcal{R}(\mathcal{T}^*, K_\infty)$ is zero. This completes the proof of the theorem. □

We remark that the dual Selmer group of the p-adic representation associated to an ordinary modular form is conjectured to be torsion as $\mathbb{Z}_p[[G]]$-module, and it is hence not unreasonable to expect that the same holds in this general situation. Thus it is natural to ask whether $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ is $R[[G]]$-torsion whenever a specialisation $\mathrm{Sel}(\widehat{\mathcal{A}[P]/K_\infty})$ is torsion as an $R/P[[G]]$-module. We show that this is indeed true in the following special case.

Proposition 7.3. *Suppose that R is a finite flat extension of $\mathbb{Z}_p[[X]]$ and let $\mathfrak{p}$ be a height one prime ideal of Λ such that $R/\mathfrak{p}$ is finite extension of $\mathbb{Z}_p$. If the ideal $\mathfrak{p}$ remains unramified in R and $\mathrm{Sel}(\widehat{\mathcal{A}[P]/K_\infty})$ is a torsion $\mathbb{Z}_p[[G]]$-module for every height one prime ideal P of R lying above $\mathfrak{p}$, then $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ is a torsion $\Lambda[[G]]$-module.*

Proof. Using an argument similar to Proposition 6.16 we see that the kernel and cokernel of the map

$$\oplus_{P|\mathfrak{p}} H^1(K_S/K_\infty, \mathcal{A}[P]) \longrightarrow H^2(K_S/K_\infty, \mathcal{A}[\mathfrak{p}])$$

are $\mathbb{Z}_p$-cotorsion. A similar assertion holds for local Galois cohomology modules. Now using the snake lemma, we conclude that the kernel and cokernel of the map

$$\oplus_{P|\mathfrak{p}} \mathrm{Sel}(\mathcal{A}[P]/K_\infty) \longrightarrow \mathrm{Sel}(\mathcal{A}[\mathfrak{p}]/K_\infty)$$

are $\mathbb{Z}_p$-cotorsion modules. Therefore $\mathrm{Sel}(\mathcal{A}[P]/K_\infty)$ is a $\mathbb{Z}_p[[G]]$-cotorsion module for every P if and only if $\mathrm{Sel}(\mathcal{A}[\mathfrak{p}]/K_\infty)$ is $\mathbb{Z}_p[[G]]$-cotorsion. Thus by assumption we get that $\mathrm{Sel}(\mathcal{A}[\mathfrak{p}]/K_\infty)$ is $\mathbb{Z}_p[[G]]$-cotorsion. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}(\mathcal{A}[\mathfrak{p}]/K_\infty) \arrow[r] \arrow[d, "\alpha"] & H^1(K_S/K_\infty, \mathcal{A}[\mathfrak{p}]) \arrow[r, "f"] \arrow[d, "\beta"] & \bigoplus_{v \in S(K)} J_v(\mathcal{A}[\mathfrak{p}], K_\infty)) \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathrm{Sel}(\mathcal{A}/K_\infty)[\mathfrak{p}] \arrow[r] & H^1(K_S/K_\infty, \mathcal{A})[\mathfrak{p}] \arrow[r, "g"] & \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_\infty)[\mathfrak{p}].
\end{tikzcd}$$

Using an argument similar to [CS, Theorem 4.2], it can be shown that the kernel of γ is a cofinitely generated $\mathbb{Z}_p[[H]]$-module, where $H := \mathrm{Gal}(K_\infty/K_{cyc})$, and hence $\mathbb{Z}_p[[G]]$-torsion. Therefore from the snake lemma we have that kernel and cokernel of α are also cofinitely generated $\mathbb{Z}_p[[G]]$-module. This shows that kernel and cokernel of α are cotorsion $\mathbb{Z}_p[[G]]$-modules. Therefore $\mathrm{Sel}(\mathcal{A}/K_\infty)[\mathfrak{p}]$ must be a cotorsion $\mathbb{Z}_p[[G]]$-module. This implies that the grade of $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}[\mathfrak{p}]$ as a $\mathbb{Z}_p[[G]]$-module is greater than equal to 1. Now $R/\mathfrak{p}[[G]]$ is a free $\mathbb{Z}_p[[G]]$-module. Therefore the grade of $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}[\mathfrak{p}]$ is also greater than or equal to 1 (see Lemma 6.1). Now from Lemma 4.2 it follows that the grade of $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ is also greater then equal to 1. Therefore $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$ must be a torsion $\Lambda[[G]]$-module.

□

We state another result which can be useful in proving the surjectivity of the global to local map defining the Selmer group for certain p-adic Lie extensions of dimension 2.

THEOREM 7.4. *Suppose that K_∞/K_{cyc} is a pro-p, p-adic Lie extension of dimension 1. Assume further that $H^2(K_S/K_\infty, \mathcal{A}) = 0$. If the sequence*

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_{cyc}) \longrightarrow H^1(K_S/K_{cyc}, \mathcal{A}) \xrightarrow{\gamma_{cyc}} \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_{cyc}) \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_\infty) \longrightarrow H^1(K_S/K_\infty, \mathcal{A}) \xrightarrow{\gamma_\infty} \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_\infty) \longrightarrow 0$$

is also exact.

Proof. The argument is as in [CSS]. Let $H = \mathrm{Gal}(K_\infty/K_{cyc})$. One first uses the Hochschild-Serre spectral sequence and the vanishing of $H^2(K_S/K_\infty, \mathcal{A})$ to conclude that $H^1(H, H^1(K_S/K_\infty, \mathcal{A})) = 0$ (see [CSS, Lemma 2.4]). Simlarly, arguing as in [CSS, Lemma 2.5], it is easily seen that $H^1(H, \mathrm{Sel}(\mathcal{A}/K_\infty)) = 0$. We then obtain the exact sequence (see [CSS, Lemma 2.3])

$$0 \longrightarrow \mathrm{Sel}(\mathcal{A}/K_\infty)^H \longrightarrow H^1(K_S/K_\infty, \mathcal{A})^H \xrightarrow{\gamma_\infty} \bigoplus_{v \in S(K)} J_v(\mathcal{A}, K_\infty)^H \longrightarrow 0,$$

which implies that $coker(\gamma_\infty)^H = 0$. Thus from Nakayama's Lemma $coker(\gamma_\infty) = 0$, and hence γ_∞ is surjective. □

8. IWASAWA INVARIANTS

In this section, we discuss the vanishing of the R-torsion submodule of the dual Selmer group $\mathrm{Sel}\widehat{(\mathcal{A}/K_\infty)}$. We also discuss the variation of the Iwasawa μ-invariant attached to the dual Selmer groups of the specialisations of the Hida deformations considered in section 1. For the definition of the μ-invariant of a finitely generated module M over an Iwasawa algebra, the reader is referred to [H, section 1]. Thus R is a Noetherian local domain which is finite, flat over $\Lambda \simeq \mathbb{Z}_p[[X]]$. Recall that in the case of a Hida deformation, for every integer $k \geq 2$, we have the specialisation maps which we denote by ϵ^k, and the prime ideal $\mathfrak{p}_k$ of Λ corresponding to the kernel of the specialisation ϵ^k. Let $\mathcal{P}_k$ be a height one prime ideal of R lying above $\mathfrak{p}_k$ and π_k denote a generator of the maximal ideal of $R/\mathcal{P}_k$. (We mention that there can be more then one such prime ideal but for ease of notation, we ignore the ambiguity, and denote any such prime ideal by the same notation $\mathcal{P}_k$, and choose one of them). Let T_k be the finitely generated $R/\mathcal{P}_k$-module $\mathcal{T}/\mathcal{P}_k$ and A_k denote the cofinitely generated discrete $R/\mathcal{P}_k$-module $T_k \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for every $k \geq 2$. The definition of the Selmer groups $\mathrm{Sel}(A_k/K_\infty)$ for A_k, attached to the specialisations ϵ^k is analogous to the definition of the Selmer group $\mathrm{Sel}(\mathcal{A}/K_\infty)$ (see [CS] for details). For simplicity, we let $X(\mathcal{T}/K_\infty)$ (resp. $X(T_k/K_\infty)$) denote the module $\widehat{\mathrm{Sel}(\mathcal{A}/K_\infty)}$ (resp. $\widehat{\mathrm{Sel}(A_k/K_\infty)}$). We denote by μ_k the μ-invariant attached to $X(T_k/K_\infty)$. Let $X(\mathcal{T}/K_\infty)(R)$ (resp. $X(T_k/K_\infty)(p)$) denote the R-torsion submodule of $X(\mathcal{T}/K_\infty)$, (resp. the p-primary part of $X(\mathcal{T}/K_\infty)$) considered as an $R[[G]]$-submodule. Put $Y(\mathcal{T}/K_\infty) = X(\mathcal{T}/K_\infty)/X(\mathcal{T}/K_\infty)(p)$, $Y(T_k/K_\infty) = X(T_k/K_\infty)/X(T_k/K_\infty)(p)$, and $H = \mathrm{Gal}(K_\infty/K_{cyc})$. First we prove the following lemma which we shall need in this section.

Lemma 8.1. *Suppose that M is a finitely generated $R[[G]]$-module such that it does not contain any non-trivial pseudo-null submodule. Then, for all but finitely many height one prime ideals $\mathcal{P}$ of R, we have $M[\mathcal{P}] = 0$.*

Proof. Since M does not contain any non-trivial pseudo-null $R[[G]]$-submodule, from Lemma 6.1, it follows that M does not contain any non-trivial pseudo-null $\Lambda[[G]]$-submodule. Now from the structure theory for finitely generated $\Lambda[[G]]$-modules M, it follows that M injects into a finitely generated $\Lambda[[G]]$-module N such that for all but finitely many height one prime ideals $\mathfrak{p}$ of Λ, we have $N[\mathfrak{p}] = 0$ (see [H1, Theorem 2.5]). Now, if we take a height one prime ideal $\mathcal{P}$ of R such that its restriction to Λ does not annihilate any element of M, then $M[\mathcal{P}] = 0$. □

Proposition 8.2. *Assume that there exists an integer $k \geq 2$ such that (i) $X(\mathcal{T}/K_\infty)$ has no $\mathcal{P}_k$ torsion, (ii) $\mu_k = 0$. If the assumptions of Theorem 6.10 (resp. 6.12) hold, then $X(\mathcal{T}/K_\infty)(R) = 0$.*

Proof. The assertion follows from Theorem 6.10 (resp. 6.12) and [CS, Theorem 6.3]. □

In the following proposition, we prove a result on the variation of the μ-invariant with the specialisations. Results of this kind have been proven for $K_\infty = \mathbb{Q}_{cyc}$ in [EPW]. We prove this for more general p-adic Lie extensions of K. For this proposition, we only need to assume Hypothesis 1 and that G is a uniform pro-p, p-adic Lie group without p-torsion. In the next proposition, we need the Selmer group associated to the residual representation $\bar{\rho}$ defined by the action of Galois group of the R/m-module $\mathcal{T}/m$. This is defined as follows:

$$\mathrm{Sel}(\mathcal{A}[m]/K_\infty) = \mathrm{Ker}(H^1(K_S/K_\infty, \mathcal{A}[m]) \longrightarrow \bigoplus_{v \in S(K)} J_v(\mathcal{A}[m], K_\infty)).$$

The filtration on $\mathcal{T}$ induces a filtration on $\mathcal{T}/m$. The local terms $J_v(\mathcal{A}[m], K_\infty))$ are defined using this induced filtration on $\mathcal{T}/m$. Note that the set of primes at which $\mathcal{A}[m]$ is ramified can be smaller than the set $S(K)$ that we are considering. But we shall use this larger set to define this Selmer group as it is more suitable for comparing the μ-invariant of specialisations. First we state the following well known result (cf. [V, Remark 3.33]).

Lemma 8.3. *Let $\mathcal{O}$ be the ring of integers of a finite field extension of $\mathbb{Q}_p$. Let M be a finitely generated torsion $\mathcal{O}[[G]]$-module. Then the μ-invariant of M vanishes if and only if the $\mathbb{F}[[G]]$-rank of M/π is zero. Here π is a generator of the maximal ideal of $\mathcal{O}$ and $\mathbb{F}$ denotes the residue field of $\mathcal{O}$.* □

Theorem 8.4. *Assume that $X(T_k/K_\infty)$ is a torsion $R/\mathcal{P}_k[[G]]$-module for all $k \geq 2$. If there exists a $k \geq 2$ such that $\mu_k = 0$, then $\mu_{k'} = 0$ for every $k' \geq 2$.*

Proof. For every $k \geq 2$, we have a homomorphism

$$X(T_k/K_\infty)/\pi_k \xrightarrow{f_k} X((T_k/\pi_k)/K_\infty)$$

where $X((T_k/\pi_k)/K_\infty)$ is the Pontryagin dual of the Selmer group defined for the the residual representation T_k/π_k. Now, it can be shown using a method similar to ([CS, Theorem 4.2]) that the kernel and cokernel of f_k is a finitely generated $\mathbb{F}[[H]]$-module. Therefore the kernel and cokernel of f_k is a torsion $\mathbb{F}[[G]]$-module and the $\mathbb{F}[[G]]$-rank of $X(T_k/K_\infty)/\pi_k$ and $X((T_k/\pi_k)/K_\infty)$ are the same. Let $k \geq 2$ be an integer such that $\mu_k = 0$. Then from Lemma 8.3, it follows that the $\mathbb{F}[[G]]$-rank of $X((T_k/\pi_k)/K_\infty)$ is zero. Let $k' \geq 2$ be any integer. Since $X((T_k/\pi_k)/K_\infty) \cong X((T_k/\pi_{k'})/K_\infty)$ we see that the $\mathbb{F}[[G]]$-rank of $X(T_{k'}/K_\infty)/\pi_{k'}$ is also zero. Again using Lemma 8.3 we conclude that $\mu_{k'}$ vanishes. □

From now on we shall assume in this section that R is a regular local ring.

Corollary 8.5. *Suppose that the hypotheses of Theorem 6.10 (resp. 6.12) hold. If $\mu_k = 0$ for some $k \geq 2$, then $X(\mathcal{T}/K_\infty)(R) = 0$. In particular $X(\mathcal{T}/K_\infty)(p) = 0$.*

Proof. Since $X(\mathcal{T}/K_\infty)$ does not contain any non-trivial pseudo-null $R[[G]]$-submodule, we have $X(\mathcal{T}/K_\infty)[\mathcal{P}_k] = 0$ for all but finitely many height one prime ideals $\mathcal{P}_k$, by Lemma 8.1. From Theorem 8.4, we have $\mu_{k'} = 0$ for all $k' \geq 2$. Therefore we can choose a height one prime ideal $\mathcal{P}_{k'}$ such that $\mu_{k'} = 0$ and $X(\mathcal{T}/K_\infty)[\mathcal{P}_{k'}] = 0$. Now the corollary follows from Proposition 8.2. □

Let $\mathcal{O}$ be the ring of integers of a finite field extension of $\mathbb{Q}_p$ and π be a generator of the maximal ideal of $\mathcal{O}$. Then a torsion $\mathcal{O}[[G]]$-module M is said to be in the category $\mathfrak{M}_H(G)$ if $M/M(\pi)$ is a finitely generated $\mathcal{O}[[H]]$-module.

Proposition 8.6. *Assume that $Y(\mathcal{T}/K_\infty)$ is a finitely generated $R[[H]]$-module. Then $X(T_k/K_\infty)$ is in $\mathfrak{M}_H(G)$ for all $k \geq 2$. Conversely, if there exists a $k \geq 2$ such that μ_k vanishes and $X(T_k/K_\infty)$ does not have any non-trivial pseudo-null $R/\mathcal{P}_k[[G]]$-submodule, then $X(T_k/K_\infty)$ belongs to $\mathfrak{M}_H(G)$ if and only if $X(\mathcal{T}/K_\infty)$ is a finitely generated $R[[H]]$-module.*

Proof. Consider the following exact sequence,

$$\begin{tikzcd}
X(\mathcal{T}/K_\infty)/\mathcal{P}_k \arrow[r] \arrow[d] & Y(\mathcal{T}/K_\infty)/\mathcal{P}_k \arrow[r] \arrow[d] & 0 \\
X(T_k/K_\infty) \arrow[r] & Y(T_k/K_\infty) \arrow[r] & 0.
\end{tikzcd}$$

Since $Y(\mathcal{T}/K_\infty)$ is a finitely generated $R[[H]]$-module, it follows from the above diagram that $Y(T_k/K_\infty)$ is a finitely generated $R/\mathcal{P}_k[[H]]$-module if and only if the cokernel of the second vertical map is a finitely generated $R/\mathcal{P}_k[[H]]$-module. But the cokernel of the first vertical map is a finitely generated $R/\mathcal{P}_k$ -module [CS, Theorem 4.2]. Therefore the cokernel of the second vertical map is finitely generated as a $R/\mathcal{P}_k$-module, and hence as an $R/\mathcal{P}_k[[H]]$-module. This proves the first assertion. For the proof of the converse assertion, we notice that the hypothesis implies that $X(T_k/K_\infty)$ is a finitely generated $R/\mathcal{P}_k[[H]]$-module. Now the assertion follows from [CS, Proposition 5.4]. $\square$

We remark that in the above proposition, the assumption that the dual Selmer group associated to some specialisation does not contain any non-trivial pseudo-null submodule holds for the specialisation satisfying Hypothesis 1. It is also expected in general that the Pontryagin dual of the corresponding dual Selmer group is in $\mathfrak{M}_H(G)$. Thus it seems probable that $Y(\mathcal{T}/K_\infty)$ is a finitely generated $R[[H]]$-module. Now consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}(A_k/K_\infty) \arrow[r] \arrow[d, "\alpha_k"] & H^1(K_S/K_\infty, A_k) \arrow[r] \arrow[d, "\beta_k"] & \bigoplus_{v\in S(K)} J_v(A_k, K_\infty) \arrow[d, "\oplus_v \gamma_{k,v}"] \\
0 \arrow[r] & \mathrm{Sel}(\mathcal{A}/K_\infty)[\mathcal{P}_k] \arrow[r] & H^1(K_S/K_\infty, \mathcal{A})[\mathcal{P}_k] \arrow[r] & \bigoplus_{v\in S(K)} J_v(\mathcal{A}, K_\infty)[\mathcal{P}_k].
\end{tikzcd}$$

In [CS] it is shown that the cokernel of α_k is a cofinitely generated $R/\mathcal{P}_k[[H]]$-module. But in the following lemma we shall show that under the hypothesis of Theorem 6.10 (resp. 6.12), it is infact $R/\mathcal{P}_k[[H]]$-cotorsion.

Lemma 8.7. *If the assumptions of Theorem 6.10(resp. 6.12) hold, then the cokernel of α_k is a cotorsion $R/\mathcal{P}_k[[H]]$-module.*

Proof. By the snake lemma, it is enough to show that the kernel of $\gamma_{k,v}$ is a cotorsion $R/\mathcal{P}_k[[H]]$-module for each v. If the hypothesis of Theorem 6.10 (resp. 6.12) holds, then for each prime v of K_∞, the dimension of the decomposition group G_v is ≥ 2. Fix a finite prime v of K_∞. Let H_v denote the decomposition subgroup of H corresponding to v. Since the dimension of G_v/H_v is equal to one for each v, the dimension of H_v must be greater than or equal to one for each v. Let $\mathcal{A}(K_{v,\infty})$ denote the group $H^0(K_{v,\infty}, \mathcal{A})$. We shall first show that $\mathrm{Coind}_H^{H_v}(\mathcal{A}(K_{v,\infty})/\mathcal{P}_k)$ is a cotorsion $R/\mathcal{P}_k[[H]]$-module. The Pontryagin dual of $\mathrm{Coind}_H^{H_v}(\mathcal{A}(K_{v,\infty})/\mathcal{P}_k)$ is isomorphic to $\mathrm{Ind}_H^{H_v}(\widehat{\mathcal{A}(K_{v,\infty})}[\mathcal{P}_k])$. Since $\widehat{\mathcal{A}(K_{v,\infty})}[\mathcal{P}_k]$ is a finitely generated $R/\mathcal{P}_k$-module and dimension of H_v is greater than or equal to 1, $\mathcal{A}(\widehat{K_{v,\infty}})/\mathcal{P}_k$ must be a torsion $R/\mathcal{P}_k[[H_v]]$-module. Now it follows from [V, Proposition 2.7(i)] that induced module of a torsion module is torsion. Therefore $\mathrm{Coind}_H^{H_v}(\mathcal{A}(K_{v,\infty})/\mathcal{P}_k)$ is a cotorsion $R/\mathcal{P}_k[[H]]$-module. Note that the kernel of $\gamma_{k,v}$ is equal to $\bigoplus_w \mathrm{Coind}_H^{H_w}(\mathcal{A}(K_{w,\infty})/\mathcal{P}_k)$, where w varies over the finite primes of K_{cyc} such that $w|v$. Since each $\mathrm{Coind}_H^{H_w}(\mathcal{A}(K_{w,\infty}))$ is a cotorsion $R/\mathcal{P}_k[[H]]$-module and there are finitely many primes of K_{cyc} over each prime v of K, the kernel of $\gamma_{k,v}$ is a cotorsion $R/\mathcal{P}_k[[H]]$-module for each v. □

THEOREM 8.8. *Suppose that $Y(\mathcal{T}/K_\infty)$ is a finitely generated $R[[H]]$-module. If the assumptions of Theorem 6.10 (resp. 6.12) hold, then the $R/\mathcal{P}_k[[H]]$-rank of $Y(T_k/K_\infty)$ is same as the $R[[H]]$-rank of $Y(\mathcal{T}/K_\infty)$ for all $k \geq 2$.*

Proof. Since $X(\mathcal{T}/K_\infty)(p)$ is a finitely generated $R[[G]]$-module, there exists a positive integer n such that $p^nX(\mathcal{T}/K_\infty)(p) = 0$. Thus we can view $Y(\mathcal{T}/K_\infty)$ as the submodule $p^nX(\mathcal{T}/K_\infty)$ of $X(\mathcal{T}/K_\infty)$ for such a large n. Therefore $Y(\mathcal{T}/K_\infty)$ does not contain any non-trivial pseudo-null $R[[G]]$-submodule. We shall show that $Y(\mathcal{T}/K_\infty)$ is $R[[H]]$-torsion free. Indeed, since R is assumed to be regular local, the ring $R[[G]]$ is Auslander regular, and standard dimension theory in this context yields that a finitely generated $R[[H]]$-module is pseudo-null as an $R[[G]]$-module, if and only if it is $R[[H]]$-torsion. Therefore by our assumption, $Y(\mathcal{T}/K_\infty)$ is $R[[H]]$-torsion free. From this, we conclude that the $R[[H]]$-rank of $Y(\mathcal{T}/K_\infty)$ is same as the $R/\mathcal{P}_k$-rank of $Y(\mathcal{T}/K_\infty)/\mathcal{P}_k$ for all $k \geq 2$ (cf [Jh, Lemma 5]). Now consider the following commutative diagram.

$$\begin{tikzcd}
 & X(\mathcal{T}/K_\infty)(p)/\mathcal{P}_k \arrow[r] \arrow[d] & X(\mathcal{T}/K_\infty)/\mathcal{P}_k \arrow[r] \arrow[d] & Y(\mathcal{T}/K_\infty)/\mathcal{P}_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X(T_k/K_\infty)(p) \arrow[r] & X(T_k/K_\infty) \arrow[r] & Y(T_k/K_\infty) \arrow[r] & 0.
\end{tikzcd}$$

The cokernel of the middle vertical map in the above diagram is a finitely generated $R/\mathcal{P}_k$-module [CS, Proposition 5.4] and therefore it is $R/\mathcal{P}_k[[H]]$-torsion . Since $X(T_k/K_\infty)(p)$ is $\mathbb{Z}_p$-torsion and by Lemma 8.7, the kernel of the middle map is $R/\mathcal{P}_k[[H]]$-torsion, it follows from the snake lemma that the kernel of the last vertical map is $R/\mathcal{P}_k[[H]]$-torsion. Therefore, the $R/\mathcal{P}_k[[H]]$-rank of $Y(T_k/K_\infty)$ and $Y(\mathcal{T}/K_\infty)/\mathcal{P}_k$ are the same. Thus it follows that the $R/\mathcal{P}_k[[H]]$-rank of $Y(T_k/K_\infty)$ is same as the $R[[H]]$-rank of $Y(\mathcal{T}/K_\infty)$. □

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