Symplectic Involutions of $K3$ Surfaces Act Trivially on CH_0

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Abstract. A symplectic involution on a $K3$ surface is an involution which preserves the holomorphic 2-form. We prove that such a symplectic involution acts as the identity on the CH_0 group of the $K3$ surface, as predicted by Bloch's conjecture.

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1 Introduction

For a smooth complex projective variety X, Mumford has shown in [9] that the triviality of the Chow group $CH_0(X)$, i.e. $CH_0(X)_{hom} = 0$, implies the vanishing of holomorphic forms of positive degree on X. An immediate generalization is the fact that a 0-correspondence $\Gamma \in CH^d(Y \times X)$, with $d = \dim X$, which induces the 0-map $\Gamma_* : CH_0(Y)_{hom} \to CH_0(X)_{hom}$ has the property that the maps $\Gamma^* : H^{i,0}(X) \to H^{i,0}(Y)$ vanish for $i > 0$.
Bloch's conjecture is a sort of converse to the above statement, but it needs the introduction of a certain filtration on CH_0 groups of smooth projective varieties. The beginning of this conjectural filtration is

$$F^0CH_0(X) = CH_0(X),\ F^1CH_0(X) = CH_0(X)_{hom}, \tag{1}$$
$$F^2CH_0(X) = CH_0(X)_{alb} := \mathrm{Ker}\,(\mathrm{alb}_X : CH_0(X)_{hom} \to \mathrm{Alb}(X)).$$

As the filtration is supposed to satisfy $F^kCH^0(X) = 0$ for $k > \dim X$, we find that for surfaces, the filtration is fully determined by (1).
Bloch's conjecture for correspondences with values in surfaces is then the following:

Conjecture 1.1 *Let S be a smooth projective surface, and let X be a smooth projective variety, $\Gamma \in CH^2(X \times S)$ be a correspondence such that the maps $\Gamma^* : H^{i,0}(S) \to H^{i,0}(X)$ vanish for $i > 0$. Then*

$$\Gamma_* : CH_0(X)_{alb} \to CH_0(S)_{alb}$$

vanishes.

This question can be addressed in particular in the case of self-correspondences associated to finite group actions on surfaces. A particular case of the conjecture above is the following:

Conjecture 1.2 *Let G be a finite group acting on a smooth projective complex surface S with $q = 0$. Let $\chi : G \to \{1, -1\}$ be a character. Assume that $H^{2,0}(S)^\chi = 0$. Then $CH_0(S)^\chi_{hom} = 0$.*

Here

$$H^{2,0}(S)^\chi := \{\omega \in H^{2,0}(S),\, g^*\omega = \chi(g)\omega,\, \forall g \in G\},$$

$$CH_0(S)^\chi_{hom} := \{z \in CH_0(S)_{hom},\, g^*z = \chi(g)z,\, \forall g \in G\}.$$

This is indeed the particular case of the conjecture 1.1 applied to the 0-correspondence

$$\pi_\chi := \sum_{g \in G} \chi(g)\Gamma_g \in CH^2(S \times S),$$

where $\Gamma_g \subset S \times S$ is the graph of g.

Conjecture 1.2 is proved in [13] in the situation where S is the zero set of a transverse section of a G-invariant vector bundle on any variety X with trivial Chow groups (that is $CH^*(X)_{hom} \otimes \mathbb{Q} = 0$), under the assumption that E has many G-invariant sections. This generalizes our previous work in [12], where the case of the Godeaux action of $\mathbb{Z}/5\mathbb{Z}$ on the CH_0 group of invariant quintic surfaces was solved. This also covers the case (already considered in [12]) of the action of the involution i on $\mathbb{P}^3$ acting with two -1 eigenvectors and two $+1$ eigenvectors on homogeneous coordinates, if we take for S a quartic surface defined by an i-invariant equation and we look at the antiinvariant part of $CH_0(S)$.

In the paper [5], Huybrechts proved that a derived autoequivalence of a $K3$ surface S acting as the identity on $H^*(S, \mathbb{Z})$ acts as the identity on $CH_0(S)$. The next situation to consider is that of a symplectic finite order automorphism g of a $K3$ surface S. Thus g is by definition an automorphism of S such that $g^*\omega = \omega$, where ω is the holomorphic 2-form on S. Such a g acts trivially on $H^{2,0}(S)$ so it has trivial action on the transcendental lattice of S, so the difference

$$g^* - Id \in \operatorname{Aut} H^*(S, \mathbb{Z})$$

is, at least over $\mathbb{Q}$, induced by the cohomology class of a cycle of the form $\sum_i \alpha_i C_i \otimes C'_i$. where C_i, C'_i are curves on S and α_i are rational coefficients. It seems that if one could take the α_i to be integers, the above mentioned result

of Huybrechts would apply to show that g_* is the identity on $CH_0(S)$. Still the problem remains open for these symplectic automorphisms and was explicitly raised by Huybrechts in [7]. In this note, the case of a symplectic involution i acting on a $K3$ surface S is considered. The fact that such symplectic involutions act trivially on $CH_0(S)$ has been proved on one hand in a finite number of cases in [4], [12], [13], and on the other hand (and more significantly), it has been established in [6] for any $K3$ surface with symplectic involution in one of the three series introduced by van Geemen and Sarti [3] (each series contains itself an infinite number of families indexed by an integer d, and the three series differ first of all by the parity of this integer d, and secondly, when d is even, by the structure of the Néron-Severi lattice of the general such surface admitting an invariant line bundle of self-intersection $2d$).
The present paper solves the problem in general :

Theorem 1.3 *Let S be an algebraic $K3$ surface, and let $i : S \to S$ be a symplectic involution. Then i_* acts as the identity on $CH_0(S)$.*

The proof is elementary : It uses the fact that Prym varieties of étale double covers of curves of genus g are of dimension $g-1$. This departure point is the obvious generalization of the starting point of Huybrechts and Kemeny's work [6], who work with elliptic curves and their étale double covers. This observation is applied to the étale double covers of generic smooth ample curves $C \subset S/i$ and allows us to prove in section 3 that the group of i-antiinvariant 0-cycles on S is finite dimensional in the Roitman sense (the definition is recalled in section 2). One then uses a mild generalization (Theorem 2.3 established in section 2) of a fundamental result due to Roitman (cf. [10]) in order to conclude that the group of i-antiinvariant 0-cycles on S is in fact trivial.

2 Finite dimensionality in the sense of Roitman

Let X be a smooth (connected for simplicity) projective variety over $\mathbb{C}$, and let $P \subset CH_0(X)$ be a subgroup.

Definition 2.1 *We will say that P is finite dimensional in the Roitman sense if there exist a (nonnecessarily connected) smooth projective variety W, and a correspondence $\Gamma \subset W \times X$ such that P is contained in the set $\{\Gamma_*(w),\, w \in W\}$.*

Remark 2.2 As P is a subgroup and the cycles $\Gamma_*(w)$ have finitely many possible degrees (depending on the connected component of W to which w belongs), we conclude that if P is finite dimensional in the Roitman sense, all elements of P have degree 0 (so $P \subset CH_0(X)_{hom}$ as X is connected).

The following result is essentially due to Roitman. (It is in fact due to Roitman in the case where $M = X$ and $\operatorname{Im} Z_* = CH_0(X)_{hom}$, see also [14], lecture 5). The proof we give below is slightly different, as it makes use of Proposition 2.4, while Roitman uses only elementary arguments. The proof given here

also has the advantage that it does not need the torsion freeness of the group $\mathrm{Ker}\,(\mathrm{alb}_M : CH_0(M)_{hom} \to \mathrm{Alb}\, M))$.
Let M and X be smooth connected projective varieties with X of dimension d. Let $Z \in CH^d(M \times X)$ be a correspondence.

Theorem 2.3 *Assume that* $\mathrm{Im}\,(Z_* : CH_0(M) \to CH_0(X))$ *is finite dimensional in the Roitman sense. Then the map* $Z_* : CH_0(M)_{hom} \to CH_0(X)$ *factors through the Albanese morphism* $\mathrm{alb}_M : CH_0(M)_{hom} \to \mathrm{Alb}\, M$ *of* M.

Proof. By definition, there exist a smooth projective variety W and a correspondence $\Gamma \subset W \times X$ such that $\mathrm{Im}\, Z_*$ is contained in the set $\{\Gamma_*(w),\, w \in W\}$. Let $C \subset M$ be a curve which is a very general complete intersection of sufficiently ample hypersurfaces $H_i \subset M$. Then by the Lefschetz theorem on hyperplane sections, the Jacobian $J(C)$ maps surjectively to $\mathrm{Alb}(M)$ and the kernel $K(C)$ is connected, hence an abelian variety. We will prove for completeness the following result:

Proposition 2.4 *When the* H_i*'s are sufficiciently ample and very general,* $K(C)$ *is a simple abelian variety.*

We fix now C as above, satisfying the conclusion of Proposition 2.4 and let $j : C \to M$ be the inclusion, which induces the morphism $j_* : J(C) = CH_0(C)_{hom} \to CH_0(M)$. We note that by taking the H_i sufficiently ample, the dimension of $K(C)$ can be made arbitrarily large, so we may assume $\dim K > \dim W$.
Let $R \subset K(C) \times W$ be the following set:

$$R = \{(k, w) \in K(C) \times W,\, Z_*(j_*(k)) = \Gamma_*(w) \text{ in } CH_0(X)\}.$$

It is known (cf. [15, 10.1.1]) that R is a countable union of closed irreducible algebraic subsets R_i of $K(C) \times W$. As $\mathrm{Im}\, Z_*$ is contained in the set $\{\Gamma_*(w),\, w \in W\}$, the union of the images of the first projections $p_{|R_i} : R_i \to K(C)$ is equal to $K(C)$. A countability argument then shows that there exists an i such that

$$pr_{1|R_i} : R_i \to K(C)$$

is dominating. It follows in particular that $\dim R_i \geq \dim K(C) > \dim W$. The fibers of the second projection

$$pr_{2|R_i} : R_i \to W$$

are thus positive dimensional. Let $w \in W$, and $F_w \subset K(C)$ be the fiber over w. Then $F_w \subset K(C)$ is positive dimensional, hence it generates $K(C)$ as a group because $K(C)$ is simple. On the other hand, by definition of R, for any $f \in F_w$, we have $Z_*(j_*(f)) = \Gamma_*(w)$ in $CH_0(X)$, hence the cycle $Z_*(j_*(f))$ is independent of $f \in F_W$. Thus for any 0-cycle z of F_w, we have $Z_*(j_*(z)) = \deg z\, \Gamma_*(w)$ and it follows then from the fact that F_w generates $K(C)$ as a group that $Z_* \circ j_*$ vanishes identically on $K(C)$.

In order to conclude that $Z_* : CH_0(M)_{hom} \to CH_0(X)$ factors through $\mathrm{Alb}\, M$, we now observe the following: For any degree 0 cycle of M, which we write in the form $z_m = z_m^+ - z_m^-$ with $z_m^+ = \sum_{l \leq k} m_l$, $z_m^- = \sum_{k+1 \leq l \leq 2k} m_l$, with $m = (m_1, \ldots, m_{2k}) \in M^{2k}$, we can blow up M at the points m_i, $i = 1, \ldots, 2k$, which gives us $\tau : M' \to M$, with exceptional divisors E_i over the points m_i. We can choose $H \in \mathrm{Pic}\, M$ so that $L = \tau^* H - \sum_i E_i$ is ample on M' and then apply Proposition 2.4 to an adequate multiple of L. We conclude that there is a complete intersection curve $j : C \hookrightarrow M'$ which satisfies the conclusion of Proposition 2.4, that is the kernel of the map

$$j_* : J(C) \to \mathrm{Alb}\, M' = \mathrm{Alb}\, M$$

is a simple abelian variety of high dimension. Note that $\tau(C)$ contains all the points m_i. Assuming now that the 0-cycle $z_m = z_m^+ - z_m^-$ is annihilated by alb_M, any of its lifts z' to M' belongs to $\mathrm{Ker}\, \mathrm{alb}_{M'}$, hence choosing a lift z' supported in C, we conclude that as a 0-cycle of C, z' belongs to $j_*(K(C))$. Applying the previous reasoning to the correspondence $Z' := Z \circ \tau$ between M' and X, we conclude that $Z_*(z_m) = Z'_*(z') = 0$ in $CH_0(X)$.

■

Proof of Proposition 2.4. First of all, we reduce the problem to the case where M is a surface, by replacing M by a smooth complete intersection $T = H_1 \cap \ldots \cap H_{m-2}$ of ample hypersurfaces and recalling that due to the Lefschetz theorem on hyperplane sections [15, 2.3.2], $\mathrm{Alb}\, M = \mathrm{Alb}\, T$. Now we take on T a Lefschetz pencil of very ample curves T_t, $t \in \mathbb{P}^1$. Picard-Lefschetz theory has for consequence (see [15, 3.2.3]) the irreducibility of the monodromy action $\rho : \pi_1(\mathbb{P}^1_{reg}, t_0) \to \mathrm{Aut}\, H^1(T_{t_0}, \mathbb{Q})_{van}$, where

$$H^1(T_{t_0}, \mathbb{Q})_{van} := \mathrm{Ker}\, (H^1(T_{t_0}, \mathbb{Q}) \to H^3(T, \mathbb{Q})).$$

The same proof shows as well the irreducibility of the action of any finite index subgroup $\Gamma \subset \pi_1(\mathbb{P}^1_{reg}, t_0)$.

Assume by contradiction that for the general curve T_t, the abelian variety $K(C_t)$ is not simple. Then there is a finite cover $r : D \to \mathbb{P}^1$, and a proper sub-abelian fibration

$$\mathcal{A} \subset \mathcal{K}_D,$$

where $\mathcal{K}_D \to D_{reg}$ is the pull-back to $D_{reg} := r^{-1}(\mathbb{P}^1_{reg})$ of the family of abelian varieties $K(C_t)$, $t \in \mathbb{P}^1_{reg}$. This sub-abelian fibration (taken up to isogenies) corresponds to a sub-local system $\mathbb{L}$ of the pull-back to D_{reg} of the local system on $\mathbb{P}^1_{reg}$ with fiber $H^1(C_t, \mathbb{Q})_{van}$.

The monodromy action on $\mathbb{P}^1_{reg}$ being irreducible on any finite index subgroup of $\pi_1(\mathbb{P}^1_{reg}, t_0)$, it is irreducible on the image $r_*(\pi_1(D_{reg}, s_0))$, $r(s_0) = t_0$. This contradicts the existence of $\mathbb{L}$.

■

In the next section, we will prove the following:

PROPOSITION 2.5 *Let S be an algebraic $K3$ surface, and let $i : S \to S$ be a symplectic involution. Then the antiinvariant part*

$$CH_0(S)^- = \{z \in CH_0(S),\, i_*(z) = -z\}$$

is finite dimensional in the Roitman sense.

PROOF OF THEOREM 1.3 We apply Theorem 2.3 to the case where $X = S$, $M = S$ and Z is the cycle $\Delta_S - \mathrm{Graph}(i)$. Here Δ_S is the diagonal of S and $\mathrm{Graph}(i)$ is the graph of i. Proposition 2.5 says that $\mathrm{Im}\, Z_*$ is finite dimensional in the Roitman sense and Theorem 2.3 tells us then that $Z_* : CH_0(S)_{hom} \to CH_0(S)_{hom}$ factors through $\mathrm{Alb}\, S = 0$. Hence Z_* vanishes on $CH_0(S)_{hom}$. On the other hand, Z_* is multiplication by 2 on $CH_0(S)^- \subset CH_0(S)_{hom}$ and we thus proved that $CH_0(S)^-$ is a 2-torsion group; as $CH_0(S)$ has no torsion by [11], we conclude that $CH_0(S)^- = 0$. Thus $Z_* = Id$ on $CH_0(S)$.

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3 PROOF OF PROPOSITION 2.5

We start with the following lemma: Let M, X be smooth projective varieties with $\dim X = d$. Let $\Gamma \in CH^d(M \times X)$ be a correspondence. Each point $(m_1, \ldots, m_k) \in M^k$ determines an element $\sum_i m_i \in CH_0(M)$. Hence we get a map

$$\Gamma_* : M^k \to CH_0(X).$$

LEMMA 3.1 *Assume there is a point $m \in M$ such that $\Gamma_*(m) = 0$ in $CH_0(X)$ and for some integer $g > 0$, one has $\Gamma_*(M^{g-1}) = \Gamma_*(M^g)$ as subsets of $CH_0(X)$. Then $\mathrm{Im}\,\Gamma_*$ is finite dimensional in the Roitman sense.*

PROOF. Since $\Gamma_*(M^{g-1}) = \Gamma_*(M^g)$, it is obvious by induction that $\Gamma_*(M^{g-1}) = \Gamma_* M^k$ for any $k \geq g-1$. Any cycle $z \in CH_0(M)$ can be written as $z^+ - z^-$, where z^+ and z^- are effective cycles, of degree k^+, k^-. Up to adding the adequate multiples of m to z^+ and z^-, which does not change $\Gamma_* z$, we may assume that $k^+ = k^- \geq g$. Thus $\Gamma_*(z) = \Gamma_*(z^+) - \Gamma_*(z^-)$, where $\Gamma_*(z^+)$ and $\Gamma_*(z^-)$ belong to $\Gamma_*(M^k) = \Gamma_*(M^{g-1})$. Hence we proved that the correspondence $\Gamma' \in CH^d(M^{2g-2} \times X)$, defined as

$$\Gamma' = \sum_{i \leq g-1} (pr_i, p_X)^* \Gamma - \sum_{g \leq i \leq 2g-2} (pr_i, p_X)^* \Gamma$$

satisfies

$$\mathrm{Im}\,\Gamma_* = \Gamma'_*(M^{2g-2}).$$

According to Definition 2.1, $\mathrm{Im}\,\Gamma_*$ is finite dimensional in the Roitman sense.

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PROOF OF PROPOSITION 2.5. Let S be a $K3$ surface endowed with a symplectic involution i. The quotient surface $\Sigma = S/i$ is a singular $K3$ surface. (By blowing-up its singular points, which correspond to the fixed points of i, it becomes a honest $K3$ surface.) The canonical bundle of Σ (or rather Σ_{reg}) is trivial. Let $L \in \operatorname{Pic}\Sigma$ be very ample, and let $2g-2 = \deg c_1(L)^2$. By triviality of $K_{\Sigma_{reg}}$, g is the genus of the smooth curves in $|L|$. Furthermore, we have $\dim|L| = g$, due to the exact sequence

$$0 \to \mathbb{C} \to H^0(\Sigma, L) \to H^0(C, L_{|C}) = H^0(C, K_C) \to 0,$$

which comes from the similar exact sequence on the desingularization $\widetilde{\Sigma}$ of Σ, which has $H^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}) = 0$.
Note also that for a smooth ample curve $C \subset \Sigma$, the inverse image $\widetilde{C} \subset S$ is smooth, connected, and is an étale double cover of C. (Only the connectedness is to be proved, and this follows from the fact that otherwise each component C_1, C_2 of $\widetilde{C} \subset S$ has positive self-intersection and $C_1 \cdot C_2 = 0$ since $\widetilde{C}$ is smooth. This contradicts the Hodge index theorem.)
Let $\Gamma \in CH^2(S \times S)$ be the correspondence $\Delta_S - \operatorname{Graph}(i)$. We prove now the following, where c_S is the effective 0-cycle of degree 1 introduced in [1]:

CLAIM 3.2 *We have* $\Gamma_*(c_S) = 0$ *and* $\Gamma_*(S^g) = \Gamma_*(S^{g-1})$.

According to Lemma 3.1, this proves Proposition 2.5, since $CH_0(S)^- = \operatorname{Im}\Gamma_*$. (The last fact follows from the fact that Γ_* acts as $-2\,Id$ on $CH_0(S)^-$, which is a divisible group.)

PROOF OF THE CLAIM. The cycle c_S is obviously i-invariant since it is the class of any point of S belonging to a rational curve $D \subset S$, and if $x \in D$ then $i(x) \in i(D)$ also belongs to a rational curve in S.
Let $s = (s_1, \ldots, s_g)$ be a general point of S^g. Then if we denote by σ_i the image of s_i in $\Sigma = S/i$, the g-uple $(\sigma_1, \ldots, \sigma_g)$ is generic in Σ^g and there exists a unique curve $C_s \in |L|$ containing all the σ_i's. The curve C_s being general in $|L|$, it is smooth and thus we have the étale double cover $\widetilde{C_s} \to C_s$, with $\widetilde{C_s} \subset S$ containing the points s_i. Consider the 0-cycle

$$z_s = \sum_l s_l - i(\sum_l s_l) = \Gamma_*(\sum_l s_l) \in CH_0(S).$$

This cycle clearly depends only on the Abel image

$$\mathrm{alb}_{\widetilde{C_s}}(\sum_l s_l - i(\sum_l s_l)),$$

which is an antiinvariant element of $J(\widetilde{C_s})$ or, up to 2-torsion, an element of the Prym variety $P(\widetilde{C_s}/C_s)$ which is a $g-1$-dimensional abelian variety.

In other words, we find that, on a Zariski open set U of S^g, the map

$$S^g \to CH_0(S)^-,\ (s_1,\ldots,s_g) \mapsto z_s,$$

factors through the morphism

$$f : U \to \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C}), (s_1,\ldots,s_g) \mapsto \mathrm{alb}_{\widetilde{C_s}}(s_1+\ldots+s_g - i(s_1) - \ldots - i(s_g)),$$

where $\mathcal{C} \to |L|_0$ is the universal smooth curve over the Zariski open set $|L|_0$ of $|L|$ parameterizing smooth curves, $\widetilde{\mathcal{C}} \to |L|_0$ is the universal family of double covers, and $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C}) \to |L|_0$ is the corresponding Prym fibration.
The total space of the Prym fibration $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$ has dimension $2g-1$, while U has dimension $2g$, so the morphism f has positive dimensional fibers. It follows that for $s \in U$, there is a curve $F_s \subset S^g$ such that the 0-cycle $z_t = \sum_l t_l - i(\sum_l t_l)$ is rationally equivalent to z_s in S for any $(t_1,\ldots,t_g) \in F_s$. Choose an ample curve $D \subset S$ whose irreducible components are rational (the existence of such a curve is well-known and due to Mori-Mukai, cf. [8]). The curve F_s meets the ample divisor $\sum_l pr_l^{-1}(D)$, where $pr_l : S^g \to S$ is the l-th projection. Hence the 0-cycle z_s is rationally equivalent to a 0-cycle of the form $z_t = \sum_l t_l - i(\sum_l t_l)$, where we have $t_{l_0} \in D$ for some l_0. We have seen already that the 0-cycle $t_{l_0} - i(t_{l_0})$ vanishes in $CH_0(S)$ and it follows that z_s is rationally equivalent to the cycle $\sum_{l\neq l_0} t_l - i(\sum_l t_{l\neq l_0})$. Thus $z_s \in \Gamma_*(S^{g-1})$ for $s = (s_1,\ldots,s_g) \in U$.
To conclude the proof, we have to show that the above result is true for any $(s_1,\ldots,s_g) \in S^g$. This follows from the following statement :

Fact 3.3 *Let Y be a connected complex projective variety. Let $U \subset Y$ be the complement of a countable union of proper closed algebraic subsets. Then any 0-cycle z of Y is rationally equivalent in Y to a 0-cycle supported on U.*

A proof of Fact 3.3 is as follows: there exists a curve $C \subset Y$ which is irreducible, contains $\mathrm{Sup}\, z$ and intersects U non-trivially. Then $C \setminus C \cap U$ is countable. It suffices to prove that there exists a 0-cycle z' of C supported on $C \cap U$ which is rationally equivalent to z on C. We may assume that C is smooth by taking normalization if necessary. Then we write $z = z_1 - z_2$ in $\mathrm{Pic}\, C$, where z_1 and z_2 are very ample divisors on C. Since $[z_1|$ and $|z_2|$ are base-point free, there exist members $z_1' \in |z_1|$, $z_2' \in |z_2|$ which avoid the countably many points in $C \setminus C\cap U$, hence are supported on $C \cap U$. Then $z = z_1' - z_2'$ in $\mathrm{Pic}\, C = CH_0(C)$.

We apply this statement to $Y = S^g$ to conclude that the cycles z_s for $s = (s_1,\ldots,s_g) \in U$ fill-in the image $\Gamma_*(S^g)$. We thus conclude that $z_s \in \Gamma_*(S^{g-1})$ for any $s = (s_1,\ldots,s_g) \in S^g$ since we already know the result for $s \in U$. Proposition 2.5 is thus proved.

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