# An Equivariant Main Conjecture in Iwasawa Theory and the Coates-Sinnott Conjecture

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Abstract. We formulate and prove an Equivariant Main Conjecture (EMC) for *all* prime numbers $p$ under the assumptions $\mu = 0$ and the validity of the 2-adic Main Conjecture in Iwasawa theory [47]. This equivariant version coincides with the version, which Ritter and Weiss formulated and proved for odd $p$ under the assumption $\mu = 0$ in [35]. Our proof combines the approach of Ritter and Weiss with ideas and techniques used by Greither and Popescu in [15] in a recent proof of an equivalent formulation of the above EMC under the same assumptions ($p$ odd and $\mu = 0$) as in [35]. As an application of the EMC we prove the Coates-Sinnott Conjecture, again assuming $\mu = 0$ and the 2-adic Main Conjecture.

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## 1 Introduction

One of the most fascinating discoveries in Arithmetic Algebraic Geometry is the still mysterious relationship between certain algebraic and analytic data attached to a given arithmetic object. Classical examples include the Conjecture of Birch and Swinnerton-Dyer, which conjecturally relates the order of vanishing of the $L$-function attached to an elliptic curve at 1 to the rank of the algebraically defined Mordell-Weil group of the curve, and Dirichlets Analytic Class Number Formula, which gives a precise algebraic interpretation of the residue of the zeta-function of a number field at 1. This last connection has been generalised to yield interpretations of special values of zeta-functions at arbitrary negative integers in terms of algebraic $K$-theory and motivic cohomology. One of the main tool to understand the deep relations between algebraic and analytic objects is Iwasawa Theory. In this theory a precise formulation of such a relationship is called the Main Conjecture. We first recall the formulation of the Main Conjecture in Iwasawa theory in the classical form:

Let $p$ be a prime number, let $F$ be a totally real number field, and let $\psi$ be a 1-dimensional $p$-adic Artin character for $F$ with $F_\psi$ totally real, where $F_\psi$ denotes the fixed field of the kernel of $\psi$. Let $F_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $F$. We recall Greenberg's terminology about the different types of the characters $\psi$: $\psi$ is of type S, if $F_\psi \cap F_\infty = F$, and $\psi$ is of type W, if $F_\psi \subseteq F_\infty$. Let $\mathcal{O}_\psi$ denote the ring obtained by adjoining all $\psi$-values to the ring $\mathbb{Z}_p$. Let $F_{\psi,\infty}$ be the cyclotomic $\mathbb{Z}_p$-extension of $F_\psi$ with Galois group $\Gamma$ over $F_\psi$. Throughout we fix a topological generator $\gamma$ of $\Gamma$. We denote by $S$ a finite set of primes of $F$ containing the set $S_p$ of primes above $p$ and the set of the infinite primes, and by $S_f$ the set of finite primes in $S$. Deligne and Ribet [8] and independently Cassou-Noguès [4] showed the existence of a $p$-adic $L$-function for the character $\psi$, which is continuous for $s \in \mathbb{Z}_p \setminus \{1\}$, and even at $s = 1$, if $\psi$ is not trivial. This satisfies the following interpolation property for any integer $n \geq 1$:

$$L_p(1-n,\psi) = L(1-n,\psi\omega^{-n}) \prod_{\mathfrak{p}\in S_p} (1-\psi\omega^{-n}(\mathfrak{p})Nm(\mathfrak{p})^{1-n}).$$

Here $L(1-n,\psi\omega^{-n})$ is the usual Artin $L$-function with respect to the character $\psi\omega^{-n}$, where $\omega : F(\mu_{2p}) \to \mathbb{Z}_p^\times$ is the Teichmüller character. Let $H_\psi \in \mathcal{O}_\psi[T]$ be defined as $\psi(\gamma)(T+1)-1$ if $\psi$ is of type W, and 1 otherwise. Deligne and Ribet showed that there exists a power series $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ so that

$$L_p^S(1-s,\psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s-1)}{H_\psi(\kappa(\gamma)^s-1)}, \tag{1.1}$$

where $L_p^S(1-s,\psi)$ denotes the $p$-adic $L$-function with Euler factors removed at the primes in $S$, and $\kappa$ is the restriction of the cyclotomic character to $\Gamma$. By the Weierstrass Preparation Theorem (cf. §7.1 in [16]) we have the following decomposition:

$$G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} g^*_{\psi,S}(T) u_{\psi,S}(T),$$

where $g^*_{\psi,S}(T)$ is a distinguished polynomial in $\mathcal{O}_\psi[T]$, and $u_{\psi,S}(T)$ is a unit power series in $\mathcal{O}_\psi[[T]]$. This power series represents the analytic object in the Main Conjecture.

For the cyclotomic $\mathbb{Z}_p$-extension $F_{\psi,\infty}$ of $F_\psi$ let $M^S_{\psi,\infty}$ be the maximal abelian pro-$p$-extension of $F_{\psi,\infty}$, which is unramified outside the primes in $S$, with Galois group $\mathfrak{X}^S_\infty := Gal(M^S_{\psi,\infty}/F_{\psi,\infty})$. The pro-$p$-group $\mathfrak{X}^S_\infty$ is equipped with a (torsion) $\mathcal{O}_\psi[[\Gamma]]$-module structure, as well as a $Gal(F_\psi/F)$-action given by inner automorphisms. Serre showed that the completed group ring $\mathcal{O}_\psi[[\Gamma]]$ can be identified with the one variable power series $\mathcal{O}_\psi[[T]]$, by mapping $\gamma - 1$ to $T$. By the Structure Theorem of Iwasawa theory (cf. §13.2 in [46]) for $\mathcal{O}_\psi[[T]]$-modules, the $\psi$-eigenspace

$$\mathfrak{X}^{S,\psi}_\infty := \left\{x \in \mathfrak{X}^S_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi | \sigma(x) = \psi(\sigma)x \quad \text{for all } \sigma \in Gal(F_\psi/F)\right\}$$

of $\mathfrak{X}^S_\infty$ is pseudo-isomorphic, as an $\mathcal{O}_\psi[[T]]$-module, to a unique $\mathcal{O}_\psi[[T]]$-module of the form

$$\bigoplus_{i=1}^{m} \mathcal{O}_\psi[[T]]/\mathfrak{p_i}^{n_i}$$

for $m \geq 1$ and $n_i \geq 1$. Here $\mathfrak{p_i}$ is the ideal generated by either a fixed uniformizer $\pi \in \mathcal{O}_\psi$ or a monic irreducible polynomial in $\mathcal{O}_\psi[T]$. We call the ideal $\prod_{i=1}^m \mathfrak{p_i}^{n_i}$ the characteristic ideal. By the Weierstrass Preparation Theorem (cf. §7.1 in [16]) we can choose a unique generator for the characteristic ideal of the following form

$$F_{\psi,S}(T) = \pi^{\mu(F_{\psi,S})} f^*_{\psi,S}(T),$$

where $f^*_{\psi,S}(T)$ is a distinguished polynomial in $\mathcal{O}_\psi[T]$. The polynomial $F_{\psi,S}(T)$ is called the characteristic polynomial of $\mathfrak{X}^{S,\psi}_\infty$. The classical Main Conjecture in Iwasawa theory is formulated as follows: If the character $\psi$ is of type S, then

$$f^*_{\psi,S}(T) = g^*_{\psi,S}(T).$$

This was proved by Wiles in [47] for any totally real number field $F$ and an odd prime $p$. He also proved the conjecture for the prime 2 and the character $\psi$ provided $F_\psi$ is an abelian extension of $\mathbb{Q}$. He also showed the equality of the $\mu$-invariants $\mu(F_{\psi,S}) = \mu(G_{\psi,S})$ for odd primes $p$ and characters $\psi$. For odd primes $p$, both invariants $\mu(F_{\psi,S})$ and $\mu(G_{\psi,S})$ are known to be zero in the case $F_\psi/\mathbb{Q}$ is abelian (cf. [10]), and are conjectured to be zero in general.

The $\mathcal{O}_\psi[[T]]$-torsion module $\mathfrak{X}^{S,\psi}_\infty$ is of projective dimension at most one and has a principal Fitting ideal generated by the characteristic polynomial

$F_{\psi,S}(T)$ (see for instance Lemma 2.4 in [32]). Therefore another formulation of the Main Conjecture for odd primes reads as follows: If the character $\psi$ is of type S, then

$$Fitt_{\mathcal{O}_\psi[[T]]}(\mathfrak{X}_\infty^{S,\psi}) = (G_{\psi,S}(T)).$$

To obtain a similar formulation of the Main Conjecture in terms of ideals for the prime 2, we replace $\mathfrak{X}_\infty$ by $\mathfrak{X}_\infty^f$, where $\mathfrak{X}_\infty^f := \mathfrak{X}_\infty^{S_f}$ is the Galois group of the abelian pro-$p$-extension of $F_{\psi,\infty}$ unramified outside the primes in $S_f$, over $F_{\psi,\infty}$. For an odd prime $p$ the two $\mathcal{O}_\psi[[T]]$-modules are the same, since infinite primes are unramified in $p$-extensions for $p$ odd. However, for $p = 2$ the two modules are related by Lemma 5.9, which shows that they may differ in their $\mu$-invariants. If we assume that $\mathfrak{X}_\infty^f$ has trivial $\mu$-invariant, then the analogous formulation of the Main Conjecture in terms of ideals for the prime 2 reads as follows:

$$Fitt_{\mathcal{O}_\psi[[T]]}(\mathfrak{X}_\infty^{f,\psi}) = (G^*_{\psi,S}(T)), \tag{1.2}$$

where $G^*_{\psi,S}(T) = g^*_{\psi,S}(T)u_{\psi,S}(T)$. We note that for odd primes $p$ this version is the same as before provided the $\mu$-invariant of $\mathfrak{X}_\infty^f$ vanishes.

As we see, for any character $\psi$ of an abelian extension $E/F$ of totally real number fields, the Main Conjecture gives an equality of ideals over the power series ring $\mathcal{O}_\psi[[T]]$ in the form (1.2) under the assumption that the $\mu$-invariant is vanishing. If we denote by $G$ the Galois group of $E/F$, then the $G$-equivariant formulation over the Iwasawa algebra $\mathbb{Z}_p[G][[T]]$ is the so-called Equivariant Main Conjecture in Iwasawa theory. Ritter and Weiss formulated such a conjecture in [35] for any odd prime $p$, and proved it under the assumption of the vanishing of a certain Iwasawa $\mu$-invariant. To explain this, we need the following set-up:

Let $E/F$ be an abelian extension of totally real number fields with Galois group $G$, and let $E_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $E$. Let $S$ be a finite set of primes of $F$ containing the primes which ramify in $E_\infty$, and infinite primes, and let $S_f$ be the set of finite primes of $S$ as before. Let $M_\infty^S$ denote the maximal abelian pro-$p$-extension of $E_\infty$, unramified outside the primes in $S$, and let $\mathfrak{X}_\infty := \mathfrak{X}_\infty^S = Gal(M_\infty^S/E_\infty)$. We denote by $G_\infty$ the Galois group of $E_\infty/F$, by $H$ the Galois group of $E_\infty/F_\infty$, and by $\mathbb{A}$ the completed group ring $\mathbb{Z}_p[[G_\infty]]$. In this article the assumption $\mu = 0$ refers to the following assumption:

$$\boldsymbol{\mu = 0}: \quad \text{The } \mu\text{-invariant of } \mathfrak{X}_\infty^f := \mathfrak{X}_\infty^{S_f} \text{ is zero, i.e. the } \mathbb{Z}_p\text{-module } \mathfrak{X}_\infty^f \text{ is finitely generated.} \tag{1.3}$$

We note that $\mathfrak{X}_\infty^f$ maps by Galois restriction to the Galois group $Gal(M_{\psi,\infty}^{S_f}/F_{\psi,\infty})$, where $M_{\psi,\infty}^{S_f}$ is the maximal abelian pro-$p$-extension of $F_{\psi,\infty}$ unramified outside the primes in $S_f$, with a finite cokernel for any character $\psi$ of $G$. Here the cokernel is $Gal(E_\infty \cap M_{\psi,\infty}^{S_f}/F_{\psi,\infty})$ which is a quotient of a subgroup of $H$, and whence finite. As a result the assumption $\mu = 0$ implies that the $\mu$-invariant of $Gal(M^{S_f}_{\psi,\infty}/F_{\psi,\infty})$ is zero, and therefore the $\mu$-invariant of $\mathfrak{X}^{f,\psi}_\infty$ vanishes for any (even) character $\psi$ of $G$.

The pro-$p$ group $\mathfrak{X}_\infty$ has a (torsion) $\mathbb{A}$-module structure, whose projective dimension is not necessarily at most one. However to formulate an Equivariant Main Conjecture, similar to the classical Main Conjecture, one needs a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one. Let $d_\infty$ be a non-zero divisor of the augmentation ideal $\Delta G_\infty$ of $\mathbb{A}$, let $c_\infty$ be an invertible element of the total ring of fraction of $\mathbb{A}$ so that $d_\infty = c_\infty((\gamma-1)e + (1-e))$, where $e$ is the idempotent attached to the trivial character of $H$. We denote by $L$ the fixed field of $E/F$ under the action of the $p$-Sylow subgroup of $G$, by $\mathcal{G}$ the Galois group of the maximal algebraic extension $\Omega^S_L$ of $L$ unramified outside the primes in $S$, over $F$, and by $\mathcal{H}$ the Galois group of $\Omega^S_L/E_\infty$. There is a commutative diagram of $\mathbb{A}$-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{A} & = & \mathbb{A} & & \\
 & & & & \downarrow \Psi & & \downarrow \psi & & \\
0 & \to & \mathfrak{X}_\infty & \to & \mathcal{Y}_\infty & \to & \Delta G_\infty & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{X}_\infty & \to & \mathcal{Z}_\infty & \to & z_\infty & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where $\psi$ maps 1 to $d_\infty$, $\Psi$ maps 1 to a pre-image $y_\infty$ of $d_\infty$, and $\mathcal{Y}_\infty = H_0(\mathcal{H}, \Delta\mathcal{G})$. Here $\Delta\mathcal{G}$ denotes the augmentation ideal of $\mathbb{Z}_p[[\mathcal{G}]]$. We will see that this definition of $\mathcal{Y}_\infty$ is the same as the definition of Ritter-Weiss in [35]. The $\mathbb{A}$-torsion module $\mathcal{Z}_\infty$ in the diagram above, whose projective dimension is at most one, shows up as the algebraic object in the Equivariant Main Conjecture of Ritter-Weiss. We note that the construction of $\mathcal{Z}_\infty$ depends on the choice of $d_\infty$. Before stating the algebraic object we remark that there exists a subgroup $\Gamma \leq G_\infty$, topologically generated by $\gamma$, so that $G_\infty = H \times \Gamma$ for the abelian group $G_\infty$. The analytic object is defined as follows:

$$G_S := \sum_{\psi \in \hat{H}} G_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]],$$

where $e_\psi$ is the idempotent attached to the character $\psi$ of $H$, i.e.

$$e_\psi := \frac{1}{\mid H \mid} \sum_{\sigma \in G} \psi(\sigma)\sigma^{-1}.$$

One version of the Equivariant Main Conjecture of Ritter-Weiss [35] for odd primes is as follows (cf. [28], §2, (CPE2)):

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty) = (c_\infty G_S),$$

which was verified under the assumption of the vanishing of the $\mu$-invariant of $\mathfrak{X}_\infty$, i.e. assuming that $\mathfrak{X}_\infty$ is a finitely generated $\mathbb{Z}_p$-module. It is worth mentioning that they have generalized and proved their Equivariant Main Conjecture in the non-commutative case, still assuming the vanishing of the $\mu$-invariant of a certain Iwasawa module (see [36]). In [15], Greither and Popescu have recently formulated and proved an Equivariant Main Conjecture in Iwasawa theory in the abelian case in terms of the Tate module of a certain Iwasawa-theoretic abstract 1-motive again under the assumptions $\mu = 0$ and $p$ odd. More recently, Nickel [29] showed that this formulation is equivalent to the formulation of Ritter-Weiss.

We now describe our Equivariant Main Conjecture for an arbitrary prime $p$. For an abelian extension $E/F$, by applying the algebraic construction of the Equivariant Main Conjecture of Ritter-Weiss to the set $S_f$ of finite primes in $S$, we construct the $\mathbb{A}$-torsion module $\mathcal{Z}^f_\infty$, which is of projective dimension at most one. We show that it satisfies the following exact sequence:

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^\# \to \alpha(\mathcal{Z}^f_\infty) \to \alpha(\mathfrak{X}^f_\infty) \to 0,$$

in which

$$pd_{\mathbb{A}}((\mathbb{A}/d_\infty\mathbb{A})^\#) \leq 1 \quad \text{and} \quad pd_{\mathbb{A}}(\alpha(\mathcal{Z}^f_\infty)) \leq 1.$$

Here $\alpha$ is the adjoint functor in Iwasawa theory with the contravariant action and $M^\#$, for any $\mathbb{A}$-module $M$, denotes the same underlying module but with $g$ acting as $g^{-1}$ for any $g \in G_\infty$. The Equivariant Main Conjecture is then formulated as follows (cf. Conjecture 4.1):

$$Fitt_{\mathbb{A}}(\mathcal{Z}^f_\infty) = (c_\infty G^*_S),$$

where

$$G^*_S := \sum_{\psi \in \hat{H}} G^*_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]].$$

Here we recall that $G^*_{\psi,S}(T) = g^*_{\psi,S}(T)u_{\psi,S}(T)$. In Section 4, we prove that this conjecture follows from the classical Main Conjecture under the assumption $\mu = 0$ by taking advantage of the idea of determinantal ideals used by Greither and Popescu [15].

In the last section we show that the Coates-Sinnott Conjecture follows from the Equivariant Main Conjecture assuming $\mu = 0$ (cf. Theorem 5.10). After some fundamental work of Coates-Sinnott in [6] and more recent results by Ritter-Weiss, Nguyen Quang Do, Burns-Greither, Greither-Popescu et al. the Coates-Sinnott Conjecture is completely known up to powers of 2, assuming $\mu = 0$. However, the 2-primary information was neglected more or less completely due to various technical problems. For example, there was no formulation of an Equivariant Main Conjecture in Iwasawa theory for the prime 2 at the time.

The Coates-Sinnott Conjecture is a generalization of the classical Stickelberger Theorem, which provides elements annihilating the class group of a cyclotomic field, using special values of analytic functions. To make it more precise, let $E/F$ be an abelian extension with Galois group $G$, and let $S$ be a finite set of primes in $F$ containing the primes ramified in $E$ and the infinite primes. Let

$$\Theta^S_{E/F}(s) := \sum_{\chi \in \hat{G}} L^S_{E/F}(s, \chi^{-1}) \cdot e_\chi \in \mathbb{C}[G]$$

be the $S$-incomplete equivariant $L$-function, where $e_\chi$ is the idempotent attached to any character $\chi$ of $G$. Deligne and Ribet [8] and independently Cassou-Noguès [4] proved that

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G]$$

for any integer $n \geq 1$. Stickelberger's Theorem shows that the following analytic object is in the annihilator ideal of the class group $Cl(\mathcal{O}_E)$ of the field $E$ in the case $F = \mathbb{Q}$:

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(1))) \cdot \Theta^S_{E/F}(0) \subseteq Ann_{\mathbb{Z}[G]}(Cl(\mathcal{O}_E)).$$

This setup has been generalized in two directions: First of all one looks at an arbitrary relative abelian extension $E/F$ of number fields. Here the analogue of Stickelberger's theorem (Brumer's Conjecture) is still not completely known. In a different direction one replaces the class group by algebraic $K$-groups or motivic cohomology groups and studies annihilators of these groups as Galois modules for relative abelian extensions. In [6], Coates and Sinnott formulated the relevant conjecture in terms of higher Quillen $K$-groups as

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}[G]}(K_{2n-2}(\mathcal{O}_E))$$

for any integer $n \geq 2$. As a result of the recent work of Voevodsky in [45] the relation between algebraic $K$-theory, étale cohomology for all prime numbers and motivic cohomology is known. This yields the motivic formulation of the Coates-Sinnott Conjecture, which implies the $K$-theoretic version. Moreover, it enables us to study each $p$-primary part of the conjecture separately for any prime number $p$ as follows:

$$Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}_p[G]}(H^2_{ét}(\mathcal{O}'_E, \mathbb{Z}_p(n)))$$

for any integer $n \geq 2$, where $H^2_{ét}(\mathcal{O}'_E, \mathbb{Z}_p(n)) = \varprojlim_m H^2_{ét}(\mathcal{O}'_E, \mu_{p^m}^{\otimes n})$ and $H^*_{ét}(\mathcal{O}'_E, \mu_{p^m}^{\otimes n})$ refers to the étale cohomology of the scheme $Spec(\mathcal{O}_E[1/p])$ with values in the étale sheaf $\mu_{p^m}^{\otimes n}$. In the last section we complete the proof of the Coates-Sinnott Conjecture under the assumption $\mu = 0$ by proving it for the prime 2.

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## 2 ALGEBRAIC CONSTRUCTION

Let $E/F$ be a finite abelian extension of totally real number fields with Galois group $G$, and let $p$ be an arbitrary prime. Let $E_\infty$ (resp. $F_\infty$) be the cyclotomic $\mathbb{Z}_p$-extension of $E$ (resp. $F$). We denote the multiplicative group $Gal(E_\infty/E)$ (resp. $Gal(F_\infty/F)$) by $\Gamma_E$ (resp. $\Gamma_F$). Let $H$ denote the Galois group of the finite abelian extension $E_\infty/F_\infty$. We denote by $G_\infty := Gal(E_\infty/F)$ the Galois group of the *abelian* extension $E_\infty/F$. We let $S$ denote a finite set of primes of $F$ which ramify in $E_\infty$, and the infinite primes. In particular $S$ contains the set $S_p$ of the primes above $p$. The set of finite primes in $S$ is also denoted by $S_f$. We use the same notations for the set of primes above the primes in $S$ and $S_f$, respectively, in any intermediate field of $E_\infty/F$. Since $\Gamma_F$ is topologically generated by one element, the exact sequence

$$0 \to H \to G_\infty \leftrightarrows \Gamma_F \to 0 \tag{2.1}$$

splits. We denote by $\Gamma \leq G_\infty$ the image of $\Gamma_F$, so that $G_\infty \simeq H \times \Gamma$, and by $\Lambda$ the completed group ring $\mathbb{Z}_p[[\Gamma]]$. Let $E'$ be the fixed field of $E_\infty$ under the action of the closed subgroup $\Gamma$. Then $E' \cap F_\infty = F$, $E_\infty = E' \cdot F_\infty$, $Gal(E'/F) \simeq H$ and $E_\infty/E'$ is also a cyclotomic $\mathbb{Z}_p$-extension.

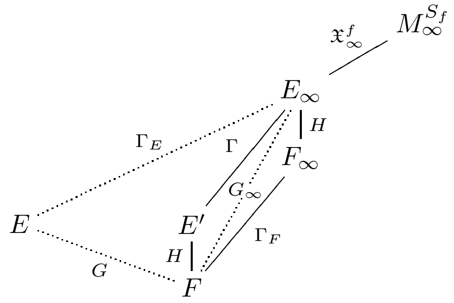
Let $M_\infty^S$ and $M_\infty^{S_f}$ be the maximal abelian pro-$p$-extensions of $E_\infty$ unramified outside the primes in $S$ and $S_f$, respectively. We recall that $\mathfrak{X}_\infty = \mathfrak{X}_\infty^S$ and $\mathfrak{X}_\infty^f := \mathfrak{X}_\infty^{S_f}$ denote the Galois group of the extensions $M_\infty^S/E_\infty$ and $M_\infty^{S_f}/E_\infty$, respectively. Since $E$ is totally real, the $\Lambda$-module $\mathfrak{X}_\infty$ is a torsion module with no non-trivial finite submodule by Propositions 10.3.22 and 10.2.25 in [31]. The $\Lambda$-module $\mathfrak{X}_\infty^f$, which is a quotient of $\mathfrak{X}_\infty$, is also torsion and has no non-trivial finite submodule (cf. [38], §6.4). Finally, we set $\mathbb{A} := \mathbb{Z}_p[[G_\infty]]$ and we freely use the identification

$$\mathbb{A} \simeq \mathbb{Z}_p[H][[T]], \tag{2.2}$$

which is given by mapping the topological generator $\gamma$ of $\Gamma$ to $1 + T$.

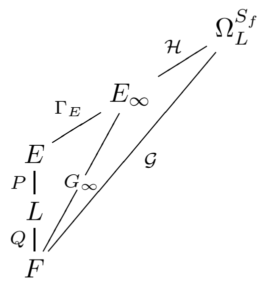
REMARK 2.1. *For an odd prime $p$, infinite primes of $F$ are unramified in a pro-$p$-extension. Hence $M_\infty^S$ and $M_\infty^{S_f}$ coincide and therefore, $\mathfrak{X}_\infty = \mathfrak{X}_\infty^f$ for odd primes.*

The following diagram illustrates the situation:




We recall that the Main Conjecture in Iwasawa theory for a character $\psi$ of $G$ can be written in the form (1.2), assuming $\mu = 0$. Here the Fitting ideal of the (finitely generated) $\mathcal{O}_\psi[[T]]$-torsion module $\mathfrak{X}^{f,\psi}$ is principal, because it has projective dimension at most one, and is generated by the $p$-adic $L$-function associated to $\psi$. Hence to formulate an Equivariant Main Conjecture we construct an appropriate (finitely generated) $\mathbb{A}$-torsion module of projective dimension at most one. The resulting Fitting ideal is then principal, and conjecturally generated by an equivariant $p$-adic $L$-function. This was done by Ritter-Weiss for odd primes in [35]. The strategy of this part is as follows: Since the $\mathbb{A}$-torsion module $\mathfrak{X}^f_\infty$ is not necessarily of projective dimension at most one, we first construct an $\mathbb{A}$-module $\mathcal{Y}^f_\infty$ of projective dimension at most one. Since this module is not necessarily $\mathbb{A}$-torsion, we pass to a quotient $\mathcal{Z}^f_\infty$ of $\mathcal{Y}^f_\infty$, which is then shown to be a (finitely generated) $\mathbb{A}$-torsion module of projective dimension at most one, whose principal Fitting ideal is conjecturally generated by an equivariant $L$-function.

Let $P$ be the $p$-Sylow subgroup of $G$ and let $L$ be the fixed field of $E$ under the action of $P$ with Galois group $Q$ over $F$. Let $\Omega_L^{S_f}$ be the maximal algebraic pro-$p$-extension of $L$, which is unramified outside the primes in $S_f$. We denote by $\mathcal{H}$ the Galois group of $\Omega_L^{S_f}$ over $E_\infty$, and by $\mathcal{G}$ the Galois group of $\Omega_L^{S_f}$ over $F$. The finitely generated group $\mathcal{G}$ has a presentation of the form $\mathcal{G} \simeq \mathcal{F}/\mathcal{W}$, where $\mathcal{F}$ is an appropriate free profinite group of rank $d$ and $\mathcal{W}$ is a relation subgroup of $\mathcal{F}$ of rank $r$. For a certain relation subgroup $\mathcal{R}$ of $\mathcal{F}$ we then have an isomorphism $G_\infty \simeq \mathcal{F}/\mathcal{R}$. The following diagram illustrates the situation:



We apply Proposition 5.6.7 in [31] (see also Lemma 4.3 in [17]) to the profinite groups in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{W} & = & \mathcal{W} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathcal{R} & \to & \mathcal{F} & \to & G_\infty & \to & 1 \\
 & & \downarrow & & \downarrow & & || & & \\
1 & \to & \mathcal{H} & \to & \mathcal{G} & \to & G_\infty & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

and obtain a commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},\mathcal{W}^{ab}) & \to & \mathcal{R}^{ab}(p) & \to & \mathfrak{X}^f_\infty & \to & 0 \\
 & & || & & || & & \downarrow & & \downarrow & & \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},\mathcal{W}^{ab}) & \to & \mathbb{A}^d & \to & \mathcal{Y}^f_\infty & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & \Delta G_\infty & = & \Delta G_\infty & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0, & &
\end{array} \tag{2.3}$$

where $\Delta G_\infty$ denotes the augmentation ideal of $\mathbb{A}$. Here $\mathcal{Y}^f_\infty := H_0(\mathcal{H},\Delta\mathcal{G})$ and $\Delta\mathcal{G}$ and $\Delta G_\infty$ denote the augmentation ideals of $\mathcal{G}$ and $G_\infty$, respectively.

Remark 2.2. *The same construction leads to a similar diagram for an arbitrary intermediate field of $\Omega_L^{S_f}/L$.*

Since the cyclotomic $\mathbb{Z}_p$-extension $E_\infty/E$ satisfies the weak Leopoldt Conjecture by Proposition 10.3.25 in [31], the group $H_2(\mathcal{H},\mathbb{Z}_p)$ in diagram (2.3) vanishes (cf. Proposition 10.3.22 in [31]). Moreover we have the following proposition:

Proposition 2.3. *The $\mathbb{A}$-module $H_0(\mathcal{H},\mathcal{W}^{ab})$ is projective.*

*Proof.* Since $|Q|$ is prime to $p$, we have the equality of cohomological dimensions

$$cd_p(\mathcal{G}) = cd_p(Gal(\Omega_L^{S_f}/L)).$$

The $p$-cohomological dimension of the pro-$p$ group $Gal(\Omega_L^{S_f}/L)$ is at most 2 by Proposition 8.3.17 in [31] for odd primes $p$ (note that infinite primes are unramified in any $p$-extension for $p$ odd), and by Theorem 1 in [38] for $p=2$, i.e.

$$cd_p(\mathcal{G}) \leq 2$$

for any prime $p$. Now Proposition 5.6.7 in [31] completes the proof. □

Let $\chi$ be a $\mathbb{C}_p$-valued character of the group $Q$, let

$$e_\chi := \frac{1}{\mid Q \mid} \sum_{\sigma \in Q} \chi(\sigma)\sigma^{-1}$$

be the idempotent of $Q$ attached to the character $\chi$, and let $\mathbb{A}_\chi := \mathcal{O}_\chi[[G_\infty(p)]]$, where $\mathcal{O}_\chi$ is the ring obtained by adjoining all character values of $\chi$ to $\mathbb{Z}_p$. Since $G_\infty(p) \simeq P \times \Gamma$ is a pro-$p$ group, $\mathbb{A}_\chi$ is a local ring and therefore, $e_\chi H_0(\mathcal{H}, \mathcal{W}^{ab})$ is a free $\mathbb{A}_\chi$-module of rank $r_\chi$:

$$e_\chi H_0(\mathcal{H}, \mathcal{W}^{ab}) \simeq \mathbb{A}_\chi^{r_\chi}. \tag{2.4}$$

Now by applying $e_\chi$ to the exact sequence in the second row of diagram (2.3) we obtain:

$$0 \to \mathbb{A}_\chi^{r_\chi} \to \mathbb{A}_\chi^{d} \to e_\chi \mathcal{Y}_\infty^f \to 0.$$

From the last column of diagram (2.3) we have

$$0 \to e_\chi \mathfrak{X}_\infty^f \to e_\chi \mathcal{Y}_\infty^f \to \mathbb{A}_\chi \to e_\chi \mathbb{Z}_p \to 0.$$

This implies that $e_\chi \mathcal{Y}_\infty^f$ has rank one and as a result $r_\chi = d-1$ for any character $\chi$. Now by taking the direct sum over all characters of $Q$ in equality (2.4) we obtain:

$$H_0(\mathcal{H}, \mathcal{W}^{ab}) \simeq \mathbb{A}^r$$

for $r = d - 1$. Therefore, diagram (2.3) can be rewritten as

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{A}^r & \xrightarrow{f} & \mathcal{R}^{ab}(p) & \to & \mathfrak{X}_\infty^f & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{A}^r & \xrightarrow{f} & \mathbb{A}^{r+1} & \to & \mathcal{Y}_\infty^f & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Delta G_\infty & = & \Delta G_\infty & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{2.5}$$

So far we have constructed the module $\mathcal{Y}_\infty^f$, which fits into the exact sequence

$$0 \to \mathfrak{X}_\infty^f \to \mathcal{Y}_\infty^f \to \Delta G_\infty \to 0. \tag{2.6}$$

The second row in diagram (2.5) implies that the $\mathbb{A}$-module $\mathcal{Y}_\infty^f$ is of projective dimension at most one. To replace it by a torsion $\mathbb{A}$-module we now take a quotient of $\mathcal{Y}_\infty^f$ by a certain submodule as follows:

Let $d_\infty \in \Delta G_\infty$ be a non-zero divisor in the augmentation ideal of $\mathbb{A}$ and let $c_\infty$ be an invertible element in $Q(\mathbb{A})$ such that

$$d_\infty = c_\infty((\gamma - 1)e + (1 - e)), \tag{2.7}$$

where $\gamma$ is the fixed (topological) generator of $\Gamma \leq G_\infty$ and $e = \frac{1}{|H|}\sum_{h\in H} h$ is the idempotent of $\mathbb{Q}_p[H]$ attached to the trivial character of $H$. Here we note that $\gamma - 1$ and $1-e$ generate $\Delta G_\infty \otimes \mathbb{Q}_p$, and that $\gamma - 1$ and $1-e$ can be written in the form (2.7) as follows:

$$\begin{aligned}\gamma - 1 &= (e + (\gamma-1)(1-e))((\gamma-1)e + (1-e)),\\ 1-e &= (1-e)((\gamma-1)e+(1-e)).\end{aligned}$$

Let $y_\infty$ be a pre-image of $d_\infty$ in $\mathcal{Y}^f_\infty$ in diagram (2.5). We have the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbb{A} & = & \mathbb{A} & & \\ & & & & \downarrow \Phi & & \downarrow \phi & & \\ 0 & \to & \mathfrak{X}^f_\infty & \to & \mathcal{Y}^f_\infty & \to & \Delta G_\infty & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \mathfrak{X}^f_\infty & \to & \mathcal{Z}^f_\infty & \to & z^f_\infty & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0, & & \end{array} \tag{2.8}$$

where $\Phi$ and $\phi$ are defined by mapping $1 \in \mathbb{A}$ to $y_\infty$ and to $d_\infty$, respectively, and $\mathcal{Z}^f_\infty$ and $z^f_\infty$ are the quotients of $\mathcal{Y}^f_\infty$ and $\Delta G_\infty$ by the images of $\Phi$ and $\phi$, respectively. We note that the vertical maps are injective since $d_\infty \in \mathbb{A}$ is a non-zero divisor. By a diagram chase in the diagram

$$\begin{array}{ccccccc} & & \mathbb{A} & = & \mathbb{A} & & \\ & & \downarrow \phi & & \downarrow \phi & & \\ 0 & \to & \Delta G_\infty & \to & \mathbb{A} & \to \mathbb{Z}_p \to 0, & \end{array}$$

we obtain:

Lemma 2.4. *The sequence*

$$0 \to z^f_\infty \to \mathbb{A}/d_\infty\mathbb{A} \to \mathbb{Z}_p \to 0 \tag{2.9}$$

*is exact, where the middle term is of projective dimension one and*

$$Fitt_{\mathbb{A}}(\mathbb{A}/d_\infty\mathbb{A}) = (d_\infty).$$

By using the middle column of diagram (2.8) and the first row of diagram (2.5) we also obtain the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{A}^r & \to & \mathbb{A}^{r+1} & \to & \mathbb{A} & \to & 0 \\ & & \downarrow & & \downarrow \Psi & & \downarrow \phi & & \\ 0 & \to & \mathcal{R}^{ab}(p) & \to & \mathbb{A}^{r+1} & \to & \Delta G_\infty & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathfrak{X}^f_\infty & \to & \mathcal{Z}^f_\infty & \to & z^f_\infty & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array} \tag{2.10}$$

which implies the following proposition:

PROPOSITION 2.5. *$\mathcal{Z}_\infty^f$ is a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one:*

$$pd_{\mathbb{A}}(\mathcal{Z}_\infty^f) \leq 1.$$

## 3 ANALYTIC CONSTRUCTION

Again let $E/F$ be an abelian extension of totally real fields, let $S$ be a finite set of primes in $F$ containing the primes above $p$, the primes ramified in $E$ and the infinite primes, and let $S_f$ denote the set of finite primes in $S$. As before, we denote by $E_\infty$ and $F_\infty$ the cyclotomic $\mathbb{Z}_p$-extensions of $E$ and $F$, respectively, by $H$ the Galois group of $E_\infty/F_\infty$, and by $G_\infty \simeq H \times \Gamma$ the Galois group of $E_\infty/F$. We define equivariant versions of $G_{\psi,S}$ and $H_\psi$ as follows (cf. [34], Proposition 5.4): For a character $\psi$ of $G_\infty$, let $G_{\psi,S}(T), H_\psi(T) \in \mathcal{O}_\psi[[T]]$ be the power series defined in (1.1). Let

$$\begin{aligned} G_S &:= \sum_{\psi \in \hat{H}} G_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]] \\ H_S &:= \sum_{\psi \in \hat{H}} H_\psi(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]] \end{aligned} \tag{3.1}$$

be the equivariant versions of $G_{\psi,S}$ and $H_\psi$. For any character $\chi$ of $G_\infty$, they satisfy the following:

$$\chi(G_S) = G_{\chi,S}(0) \quad , \quad \chi(H_S) = H_\chi(0).$$

We recall that for any character $\psi$ of $G$ one has

$$G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} \cdot g^*_{\psi,S}(T) \cdot u_{\psi,S}(T)$$

by the Weierstrass Preparation Theorem, where $\pi$ is a fixed uniformizer in $\mathcal{O}_\psi$, $g^*_{\psi,S}(T) \in \mathcal{O}_\psi[T]$ is a distinguished polynomial, and $u_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ is a unit. The modified equivariant $L$-function $G_S^*$ is now defined as follows:

$$G_S^* := \sum_{\psi \in \hat{H}} G^*_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]], \tag{3.2}$$

where $G^*_{\psi,S}(T) = g^*_{\psi,S}(T) u_{\psi,S}(T)$. The following lemma relates $G_S$ and $G_S^*$, assuming $\mu = 0$ (cf. (1.3)):

LEMMA 3.1. *Under the assumption $\mu = 0$ we have the following equalities:*

1. *$G_S = G_S^*$ for any odd prime $p$.*

2. $G_S = 2^{r_1(F)} G_S^*$ *for* $p = 2$.

*Proof.* Part 1 follows from the result of Wiles [47] that $\mu(G_{\psi,S})$ is the same as the Iwasawa $\mu$-invariant of $\mathfrak{X}_\infty^{f,\psi}$ for all odd primes $p$. Part 2 follows from the fact that $\pi^{\mu(G_{\psi,S})} = 2^{r_1(F)}$ under the assumption $\mu = 0$, where $\pi \in \mathcal{O}_\psi$ is a uniformizer, for any character $\psi$ of $G$ (see [12], pages 82 and 87). $\square$

We note that Lemma 3.1 holds unconditionally for abelian extensions $E$ of $\mathbb{Q}$, since in this case $\mu = 0$ (cf. [10]).

To prove the next lemma we briefly review the definition of a $p$-adic pseudo-measure on a certain Galois group and its relation to the $p$-adic $L$-function. For more properties one can consult [39]. For a commutative profinite group $\mathcal{G}$, $\lambda_S \in Q(\mathbb{Z}_p[[\mathcal{G}]])$ is called a pseudo-measure on $\mathcal{G}$ if $(g-1)\lambda_S$ is a measure, i.e. $(g-1)\lambda_S \in \mathbb{Z}_p[[\mathcal{G}]]$, for any $g \in \mathcal{G}$, where $Q(\mathbb{Z}_p[[\mathcal{G}]])$ denotes the quotient ring of $\mathbb{Z}_p[[\mathcal{G}]]$. Let $\mathfrak{X}$ denote the Galois group of the maximal abelian extension of $F$ unramified outside the primes in $S_f$, over $F$. By a theorem of Deligne and Ribet there is a unique pseudo-measure on $\mathfrak{X}$ denoted by $\lambda_S \in Q(\mathbb{Z}_p[[\mathfrak{X}]])$, which satisfies the following relation for any finite order character $\chi$ of $\mathfrak{X}$:

$$L_{p,S}(1-s,\chi) = <\chi\kappa^s, \lambda_S> .$$

For the definition of this inner product see [39]. Equivalently, if we let $\varepsilon : \mathfrak{X} \to \mathbb{Z}_p$ be the locally constant function defined by $\varepsilon(g) = 1$ if $g$ has image 1 in $H = Gal(E_\infty/F_\infty)$, and zero otherwise, then

$$\zeta_p^S(\varepsilon_h, 1-n) = <\varepsilon_h \rho^n, \lambda_S> .$$

Here $\rho$ is the cyclotomic character, $\varepsilon_h$ is the locally constant function satisfying $\varepsilon_h(x) = \varepsilon(hx)$ and $\zeta_p^S(\varepsilon_h, s)$ is the $S$-incomplete $p$-adic partial zeta function associated to $\varepsilon_h$. The image of $\lambda_S$ under the natural surjection $\pi : \mathfrak{X} \twoheadrightarrow G_\infty$ is a $p$-adic pseudo-measure on $G_\infty$ which is denoted by $\theta_S \in Q(\mathbb{A})$. So if $\hat{\gamma} \in \mathfrak{X}$ denotes a pre-image of $\gamma \in G_\infty$ under the surjection above, then

$$Z_S := \pi((\hat{\gamma}-1)\lambda_S) \in \mathbb{A}.$$

In fact $\theta_S = (\gamma-1)^{-1} Z_S \in Q(\mathbb{A})$. With notations as above we have the following lemma:

LEMMA 3.2. *Let $d_\infty$ be a non-zero divisor in the augmentation ideal $\Delta G_\infty$, and let $c_\infty$ be an invertible element in $Q(\mathbb{A})$ so that $d_\infty = c_\infty((\gamma-1)e + (1-e)) \in \Delta G_\infty$, where $e$ is the idempotent associated to the trivial character of $H$. Then*

$$c_\infty G_S = d_\infty \theta_S \in \mathbb{A}$$

*Proof.* A calculation in Proposition 12 in [35], which works for any character $\chi$ of $G_\infty$ satisfying $\chi(\gamma) = 1$ and for any prime $p$,provides the following equality:

$$\frac{G_{\chi,S}(T)}{T^{<\chi,1>}} = \sum_{h \in H} \chi(h) \frac{Z_S(h,T)}{T}$$

Here $Z_S(h,T)$ is given by the relation $Z_S = \sum_{h\in H} Z_S(h,\gamma-1)h \in \mathbb{A}$. As a result,

$$G_S/H_S = \sum_{\chi\in\hat{H}} \frac{G_{\chi,S}(T)}{T^{<\chi,1>}} e_\chi = \theta_S,$$

for any prime $p$. Since $H_S = (\gamma-1)e+(1-e)$, we obtain $c_\infty G_S = d_\infty \theta_S$. Hence, for the $p$-adic pseudo-measure $\theta_S$ on $G_\infty$, we have $c_\infty G_S = d_\infty \theta_S \in \mathbb{A}$. □

Let $\tilde{E} = E(\zeta_{2p})$ be the field obtained by adjoining a primitive $2p$-th root of unity $\zeta_{2p}$ to $E$, and let $\tilde{E}_\infty := E_\infty(\zeta_{2p}) = E(\mu_{p^\infty})$ be the $\mathbb{Z}_p$-cyclotomic extension of $\tilde{E}$, where $\mu_{p^\infty}$ is the group of all $p$-power roots of unity. We denote by $\tilde{G}_\infty$ the Galois group of $\tilde{E}_\infty/F$. Since $\tilde{E}_\infty$ contains all $p$-power roots of unity, we have the cyclotomic character

$$\rho : \tilde{G}_\infty \to \mathbb{Z}_p^* = Aut(\mu_{p^\infty})$$

of $\tilde{G}_\infty$. We extend the definitions of a Tate twisted module and an inverse module to the following:

- Let $t_n$ be the unique continuous isomorphism of $\mathcal{O}$-algebras

$$t_n : \mathcal{O}[[\tilde{G}_\infty]] \to \mathcal{O}[[\tilde{G}_\infty]], \tag{3.3}$$

which satisfies $t_n(g) = \rho(g)^n \cdot g$ for all $g \in \tilde{G}_\infty$ and $n \in \mathbb{Z}$. For a $\mathcal{O}[[\tilde{G}_\infty]]$-module $M$ let the Tate twisted module $M(n)$ be the same underlying group $M$ with a new $\mathcal{O}[[\tilde{G}_\infty]]$-action given by $\sigma *_n m := t_n(\sigma)m$ for $\sigma \in \mathcal{O}[[\tilde{G}_\infty]]$ and $m \in M$. For even $n$, we note that it is enough to replace $\tilde{E}_\infty$ by its maximal real subfield $\tilde{E}_\infty^+$ to have the isomorphisms $t_n$.

- Let $\iota$ be the unique continuous isomorphism of $\mathcal{O}$-algebras

$$\iota : \mathcal{O}[[G_\infty]] \to \mathcal{O}[[G_\infty]], \tag{3.4}$$

which satisfies $\iota(g) = g^{-1}$ for all $g \in G_\infty$. For a $\mathcal{O}[[G_\infty]]$-module $M$, let the inverse module $M^\#$ be the same underlying group $M$ with a new $\mathcal{O}[[G_\infty]]$-action given by $\sigma * m := \iota(\sigma)m$ for $\sigma \in \mathcal{O}[[G_\infty]]$ and $m \in M$. In the following we mean by the ideal generated by $m^\#$, for any $m \in \mathcal{O}[[G_\infty]]$, the inverse ideal $(m)^\#$ of $(m)$.

Lemma 3.3. *Assume that $E$ is the maximal real subfield of $\tilde{E} = E(\zeta_{2p})$ where $\zeta_{2p}$ is a primitive $2p$-th root of unity. For all even $n$ we have:*

$$(\iota \circ t_n)(G_S) = \sum_{\psi\in\hat{H}} G_{\psi^{-1}\omega^n,S}(u^n(\gamma)^{-1}-1)\cdot e_\psi,$$
$$(\iota \circ t_n)(H_S) = \sum_{\psi\in\hat{H}} H_{\psi^{-1}\omega^n,S}(u^n(\gamma)^{-1}-1)\cdot e_\psi,$$

*where $u = \kappa(\gamma)$.*

*Proof.* Under the assumptions of the lemma we first note that $E_\infty$ is the maximal real subfield of the field $E_\infty(\zeta_{2p})$ which contains all $p$-power roots of unity. So the actions of $t_n$ on $G_S$ and $H_S$ are defined for all even $n$. Now it suffices to observe that $(\iota \circ t_n)(\gamma - 1) = u^n\gamma^{-1} - 1$ and $(\iota \circ t_n)(e_\psi) = e_{\psi^{-1}\omega^n}$. □

This lemma yields the following equality:

$$(\pi \circ \iota \circ t_n)G_S/H_S = \sum_{\chi \in \hat{G}} \frac{G_{\chi^{-1}\omega^n,S}(u^n - 1)}{H_{\chi^{-1}\omega^n,S}(u^n - 1)} \cdot e_\chi,$$

where $\pi : \mathbb{A} \to \mathbb{Z}_p[G]$ is the projection mapping $\gamma - 1$ to zero, and $u = \kappa(\gamma)$. Therefore, we obtain:

COROLLARY 3.4. *If we assume that $E$ is the maximal real subfield of $\tilde{E} = E(\zeta_{2p})$ where $\zeta_{2p}$ is a primitive $2p$-th root of unity, and that $n$ is even, then*

$$(\pi \circ \iota \circ t_n)G_S/H_S = \Theta^S_{E/F}(1 - n)$$

REMARK 3.5. *We note that in the non-dyadic L-functions we have defined, the set $S$ can be replaced by $S_f$, since infinite primes have no influence on the definitions.*

## 4 AN EQUIVARIANT MAIN CONJECTURE IN IWASAWA THEORY

We recall that for the abelian extension $E/F$ of totally real number fields, $E_\infty$ (resp. $F_\infty$) is the cyclotomic $\mathbb{Z}_p$-extension of $E$ (resp. $F$), $H = Gal(E_\infty/F_\infty)$, and $G_\infty = Gal(E_\infty/F)$, which is abelian and hence of the form $G_\infty = H \times \Gamma$ for $\Gamma \simeq \mathbb{Z}_p$. We also recall that $d_\infty = c_\infty((\gamma - 1)e + (1 - e))$ is a non-zero divisor in the augmentation ideal of $\mathbb{A} = \mathbb{Z}_p[[G_\infty]]$ so that $\mathbb{A}/d_\infty\mathbb{A}$ is a finitely generated $\mathbb{Z}_p$-free module, e.g. $d_\infty = \gamma - 1$.

CONJECTURE 4.1. *(The Equivariant Main Conjecture). With notations as above, we have the following equality of ideals in $\mathbb{A}$:*

$$Fitt_{\mathbb{A}}(\mathcal{Z}^f_\infty) = (c_\infty G^*_S)$$

REMARK 4.2. *For any odd prime $p$, the formulation of the Equivariant Main Conjecture 4.1 is equivalent to the formulation of Ritter-Weiss in [35] (cf. [28], §2, (CPE2)). We note that Ritter-Weiss use the translation functor to define $\mathcal{Y}_\infty$, which equals to $\mathcal{Y}_\infty = \frac{\Delta\mathcal{G}}{\Delta(\mathcal{G},\mathcal{H})\Delta\mathcal{G}}$. Here $\Delta(\mathcal{G},\mathcal{H}) = ker(\mathbb{Z}_p[[\mathcal{G}]] \twoheadrightarrow \mathbb{Z}_p[[G_\infty]])$, which is the same as $(\Delta\mathcal{H})\mathbb{Z}_p[[\mathcal{G}]]$ (cf. for example page 275 in [31]). As a result, $\mathcal{Y}_\infty = \frac{\Delta\mathcal{G}}{\Delta\mathcal{H}\Delta\mathcal{G}} = H_0(\mathcal{H}, \Delta\mathcal{G})$, which is the same as the definition used in this article.*

Under the assumption $\mu = 0$ we prove that Conjecture 4.1 follows from the classical Main Conjecture in Iwasawa theory [47]. For some technical reasons, we need to apply the contravariant functors $E^i(-) := Ext^i_{\mathbb{A}}(-, \mathbb{A})$ to $\mathcal{Z}^f_\infty$, for $i = 0, 1$. We will see that $E^1(\mathcal{Z}^f_\infty)$ is a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one, whose Fitting ideal is generated by the modified equivariant $L$-function. First, we recall the definition and some basic properties of Fitting ideals (see [30] for more properties).

For a commutative ring $R$ with identity, the Fitting ideal $Fitt_R(M)$ of a finitely presented $R$-module $M$ is defined as follows: Given a presentation of $M$ as

$$R^a \xrightarrow{h} R^b \to M \to 0,$$

let $A$ be the matrix associated to the map $h$. The (initial) Fitting ideal of $M$ is defined to be the ideal of $R$ generated by all $b$-minors of $A$ if $a \geq b$, and $(0)$ otherwise. Here are some properties:

1. $Fitt_R(M)$ is a finitely generated ideal of $R$ satisfying

$$(Ann_R(M))^b \subseteq Fitt_R(M) \subseteq Ann_R(M),$$

where $Ann_R(M)$ is the annihilator ideal of $M$ and $b$ is an integer so that $M$ can be generated by $b$ elements as a $R$-module.

2. If $M \twoheadrightarrow M'$ is a surjective map of finitely presented $R$-modules, then

$$Fitt_R(M) \subseteq Fitt_R(M').$$

3. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely presented $R$-modules, then

$$Fitt_R(M') \cdot Fitt_R(M'') \subseteq Fitt_R(M).$$

Moreover, we have equality if the exact sequence splits, i.e.

$$Fitt_R(M' \oplus M'') = Fitt_R(M') \cdot Fitt_R(M'').$$

4. If $M \simeq R/\mathfrak{a}$ is a cyclic module, then

$$Fitt_R(M) = Ann_R(M) = \mathfrak{a}.$$

More generally, if we apply the previous property to a direct sum of $n$ cyclic $R$-modules

$$M \simeq R/\mathfrak{a}_1 \oplus R/\mathfrak{a}_2 \oplus \cdots \oplus R/\mathfrak{a}_n,$$

then we obtain

$$Fitt_R(M) = \mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n.$$

5. Let $M$ be a finite $R$-module for a group ring $R$ of a finite abelian group with coefficients in $\mathbb{Z}_p$. If $M$ is cyclic as a $\mathbb{Z}_p$-module, then

$$Fitt_R(M^*) = Ann_R(M^*) \simeq Ann_R(M) = Fitt_R(M),$$

where $M^* = Hom(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual equipped with the covariant action.

We recall that # denotes the inverse action defined in (3.4). In the next lemma we list some general properties of $E^i(M)$ for an $\mathbb{A}$-module $M$. For a proof see propositions 5.4.17, 5.5.6 and corollary 5.5.7 in [31], or [17].

Lemma 4.3. *Let $M$ be an $\mathbb{A}$-module, let $\alpha(M)$ denote the adjoint of $M$ with the contravariant action, and let $M^\vee = Hom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ be the dual with the contravariant $G_\infty$-action. Then*

1. *$E^i(M) = Ext^i_\Lambda(M, \Lambda)$ as $\Lambda$-modules for any $\mathbb{A}$-module $M$ and $i \geq 0$,*
2. *$E^1(M)^\# \simeq \alpha(M)$ as $\Lambda$-modules, provided $M$ is a finitely generated $\Lambda$-torsion module,*
3. *$E^1(M)^\# \simeq M^\vee$ as $\Lambda$-modules, provided $M$ is a $\Lambda$-torsion module with trivial $\mu$-invariant, i.e. $M$ is a finitely generated $\mathbb{Z}_p$-module.*

We list some results obtained by applying the contravariant functors $E^i(-)$ for $i = 1, 2$, to some of the exact sequences arising from diagram (2.8). We first remark that $E^0(-) = Hom_{\mathbb{A}}(-, \mathbb{A})$ is a left exact functor, and that $E^i(\mathbb{A}) = 0$ for $i \geq 1$, since $pd_{\mathbb{A}}(\mathbb{A}) = 0$ (cf. Proposition 5.2.11 in [31]).

Lemma 4.4. *The $\mathbb{A}$-module $E^1(\mathcal{Z}^f_\infty)^\#$ is of projective dimension at most one, and*

$$Fitt_{\mathbb{A}}(E^1(\mathcal{Z}^f_\infty)^\#) = Fitt_{\mathbb{A}}(\mathcal{Z}^f_\infty)^\#.$$

*Proof.* This lemma is a consequence of Proposition 2 in [13] and Lemma 4.3. We give here a direct proof, since we need some of the methods in the proof of Lemma 4.5 below: We first apply $E^i(-)$ to the last column of diagram (2.8). We observe that $Hom_{\mathbb{A}}(z^f_\infty, \mathbb{A})$ is the set of all morphisms in $Hom_{\mathbb{A}}(\Delta G_\infty, \mathbb{A})$, whose restriction to $d_\infty \mathbb{A}$ vanishes. This observation and the choice of $d_\infty \in \Delta G_\infty$ as a non-zero divisor imply that $E^0(z^f_\infty) = 0$. By part 1 of Lemma 4.3 $E^0(\mathfrak{X}^f_\infty)$ is also trivial for the $\Lambda$-torsion module $\mathfrak{X}^f_\infty$. Hence by applying the contravariant functor $E^i(-)$ to the last row of diagram (2.8), we obtain $E^0(\mathcal{Z}^f_\infty) = 0$. On the other hand $E^i(\mathbb{A})$ is trivial for $i \geq 1$ as we mentioned before. Therefore, applying $E^i(-)$ to the middle column of diagram (2.10) leads to the exact sequence

$$0 \to E^0(\mathbb{A})^{r+1} \overset{E^0(\Psi)}{\to} E^0(\mathbb{A})^{r+1} \to E^1(\mathcal{Z}^f_\infty) \to 0.$$

Here $E^0(\Psi)$ is the transpose of $\Psi$. We now apply "#" to the exact sequence above to obtain:

$$0 \to \mathbb{A}^{r+1} \overset{E^0(\Psi)^\#}{\to} \mathbb{A}^{r+1} \to E^1(\mathcal{Z}^f_\infty)^\# \to 0,$$

which shows that the projective dimension of the $\mathbb{A}$-module $E^1(\mathcal{Z}^f_\infty)^\#$ is at most one. To complete the proof it is enough to note that the Fitting ideal of the $\mathbb{A}$-module $E^1(\mathcal{Z}^f_\infty)^\#$ is given by the determinant of $E^0(\Psi)^\#$, whereas the Fitting ideal of $\mathcal{Z}^f_\infty$ is given by the determinant of the map $\Psi$ defined in diagram (2.10). □

Lemma 4.5. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^{\#} \to E^1(z_\infty^f)^{\#} \to 0.$$

*Proof.* We saw in the proof of Lemma 4.4 that $E^0(z_\infty^f)$ is trivial. As a consequence of Proposition 5.2.11 in [31] we obtain that $E^2(\mathbb{Z}_p)$ is trivial as well, since the projective dimension of the $\Lambda$-module $\mathbb{Z}_p$ is one. By applying $E^i(-)$ to the exact sequence (2.9) we therefore obtain the exact sequence

$$0 \to E^1(\mathbb{Z}_p) \to E^1(\mathbb{A}/d_\infty\mathbb{A}) \to E^1(z_\infty^f) \to 0.$$

By part 2 of Lemma 4.3 together with the fact that an elementary module is isomorphic to the inverse module of its adjoint (cf. 1.3 in [16]) we have $E^1(\mathbb{Z}_p)^{\#} \simeq \mathbb{Z}_p$, where $\mathbb{Z}_p$ has the trivial $G_\infty$-action. We also have the isomorphism $E^1(\mathbb{A}/d_\infty\mathbb{A})^{\#} \simeq (\mathbb{A}/d_\infty\mathbb{A})^{\#}$, since $pd_{\mathbb{A}}(\mathbb{A}) = 0$. Therefore, by applying "#" to the exact sequence above, we obtain:

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^{\#} \to E^1(z_\infty^f)^{\#} \to 0.$$

□

Lemma 4.6. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to E^1(z_\infty^f) \to E^1(\mathcal{Z}_\infty^f) \to E^1(\mathfrak{X}_\infty^f) \to 0.$$

*Proof.* First we observe that $E^2(z_\infty^f)$ is trivial by applying $E^i(-)$ to the exact sequence (2.9) and by noting that the projective dimensions of $\mathbb{Z}_p$ and $\mathbb{A}/d_\infty\mathbb{A}$ are both one. Now we apply $E^i(-)$ to the last row of diagram (2.8) to obtain the exact sequence above. We note that the surjectivity of the last map in the diagram follows from the observation that $E^2(z_\infty^f) = 0$, and that the injectivity of the first map in the diagram is a consequence of the observation that $E^0(\mathfrak{X}_\infty^f) = 0$ in the proof of lemma (4.4). □

We combine Lemmas 4.5 and 4.6 to obtain the following theorem:

Theorem 4.7. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^{\#} \to E^1(\mathcal{Z}_\infty^f)^{\#} \to E^1(\mathfrak{X}_\infty^f)^{\#} \to 0,$$

*in which*

$$pd_{\mathbb{A}}((\mathbb{A}/d_\infty\mathbb{A})^{\#}) \leq 1 \quad \textit{and} \quad pd_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^{\#}) \leq 1.$$

The $\Lambda$-module $E^1(\mathfrak{X}_\infty^f)^{\#}$ is isomorphic to the adjoint of $\mathfrak{X}_\infty^f$ by part 2 of Lemma 4.3 and so it is a finitely generated $\mathbb{Z}_p$-free module under the assumption $\mu = 0$. Therefore we obtain from Theorem 4.7 the first statement in the following proposition:

PROPOSITION 4.8. *Let $d_\infty \in \Delta G_\infty$ be a non-zero divisor so that $\mathbb{A}/d_\infty\mathbb{A}$ is a finitely generated $\mathbb{Z}_p$-free module. Then, under the assumption $\mu = 0$, the sequence*

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^\# \to E^1(\mathcal{Z}^f_\infty)^\# \to E^1(\mathfrak{X}^f_\infty)^\# \to 0$$

*is an exact sequence of finitely generated $\mathbb{Z}_p$-free modules. Moreover, if we consider this sequence as an exact sequence of $\mathbb{Z}_p[H]$-modules, then under the assumptions of $\mu = 0$ we have*

$$pd_{\mathbb{Z}_p[H]}((\mathbb{A}/d_\infty\mathbb{A})^\#) = 0 \quad \textit{and} \quad pd_{\mathbb{Z}_p[H]}(E^1(\mathcal{Z}^f_\infty)^\#) = 0.$$

*Proof.* We only need to prove the second part. We first remark that $\mathbb{A}/d_\infty\mathbb{A}$ and $E^1(\mathcal{Z}^f_\infty)$ are both $H$-cohomologically trivial by Proposition 2.2 in [32], since their projective dimensions are at most one as $\mathbb{A}$-modules. By a classical theorem of Nakayama $pd_{\mathbb{Z}_p[H]}(M) = 0$ if and only if $M$ is $\mathbb{Z}_p$-free and $H$-cohomologically trivial ($p$-adic version of Theorem 8 in [40], Chapter IX, §5). Therefore, both modules are of projective dimension zero as $\mathbb{Z}_p[H]$-modules. □

To finish the proof that the Equivariant Main Conjecture follows from the classical Main Conjecture under the assumption $\mu = 0$, we first review the definition of the determinantal ideal, which plays a role similar to that of the characteristic ideal for some $\Lambda$-modules with an extra group action: For a commutative ring $R$ with identity, a finitely generated projective $R$-module $P$ and $f \in End_R(P)$, the determinant of $f$ is defined as

$$det_R(f \mid P) := det_R(f \oplus id_Q \mid P \oplus Q),$$

where $Q$ is a complement of $P$, i.e. $P \oplus Q$ is free. One can check that the definition is independent of $Q$ by using Schanuel's lemma. By the same strategy, since $P \otimes_R R[X]$ is a finitely generated projective $R[X]$-module, the monic polynomial $det_R(X - f \mid P) \in R[X]$ is defined to be

$$det_R(X - f \mid P) := det_{R[X]}(id_P \otimes X - f \otimes 1 \mid P \otimes_R R[X])$$

for any projective $R$-module $P$. One can see that these definitions are well-behaved under base-change, i.e.

$$\begin{aligned} det_R(f \mid P) &= det_{R'}(f \otimes id_{R'} \mid P \otimes R') \\ det_R(X - f \mid P) &= det_{R'}(X - (f \otimes id_{R'}) \mid P \otimes R'), \end{aligned} \tag{4.1}$$

where $R'$ is any commutative $R$-algebra. We have the following general proposition:

PROPOSITION 4.9. *([15], Proposition 7.2). Let $R$ be a commutative, semi-local, compact topological ring and let $\Gamma$ be a pro-cyclic group with topological generator $\gamma$. Let $M$ be a topological $R[[\Gamma]]$-module, which is projective and finitely generated as an $R$-module. Let*

$$F(X) := det_R(X - m_\gamma \mid M),$$

*where $m_\gamma$ is the $R[[\Gamma]]$-module automorphism of $M$ given by multiplication by $\gamma$. Then the following holds:*

1. *$M$ is finitely presented as an $R[[\Gamma]]$-module. If we let $F(\gamma)$ be the image of $F(X)$ via the $R$-algebra morphism $R[X] \to R[[\Gamma]]$ sending $X$ to $\gamma$, we have an equality of $R[[\Gamma]]$-ideals*
$$Fitt_{R[[\Gamma]]}(M) = (F(\gamma)).$$
2. *If we view $M_R^\vee = Hom_R(M, R)$ as a topological $R[[\Gamma]]$-module with the covariant $\Gamma$-action, then*
$$Fitt_{R[[\Gamma]]}(M) = Fitt_{R[[\Gamma]]}(M_R^\vee).$$
3. *If we view $M^\vee = Hom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ as a topological $R[[\Gamma]]$-module with the covariant $\Gamma$-action, where $R = \mathbb{Z}_p[G]$ and $G$ is a finite abelian group, then*
$$Fitt_{R[[\Gamma]]}(M) = Fitt_{R[[\Gamma]]}(M^\vee).$$

By using Proposition 4.9 for the ring $R = \mathbb{Z}_p[H]$ and the finitely generated $R$-modules $M = E^1(\mathcal{Z}_\infty^f)^\#$ and $M = (\mathbb{A}/d_\infty\mathbb{A})^\#$, which are projective by Proposition 4.8, we obtain:

Lemma 4.10. *If $m_\gamma$ denotes the $R[[\Gamma]]$-module automorphism of $M$ given by multiplication by $\gamma$, then*
$$Fitt_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^\#) = (det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\#)),$$
$$Fitt_{\mathbb{A}}((\mathbb{A}/d_\infty\mathbb{A})^\#) = (det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid (\mathbb{A}/d_\infty\mathbb{A})^\#)).$$

Let $\mathcal{O}$ be the ring of integers obtained by adjoining all character values of the characters of $H$ to $\mathbb{Z}_p$, let $\pi$ be a fixed uniformizer in $\mathcal{O}$ and let $Q(\mathcal{O})$ denote the field of fractions of $\mathcal{O}$. We consider $\mathcal{O}$ and $Q(\mathcal{O})$ as $\mathbb{A}$-modules with trivial $G_\infty$-action. We note that for the idempotent $e$ attached to the trivial character of $H$ we have $H_S(T) = T \cdot e + (1-e)$ using the identification (2.2) (see (3.1) for the definition of $H_S(T)$). Therefore, using Lemma 4.10 we obtain the following lemma:

Lemma 4.11. *We have the following equalities of ideals in $Q(\mathcal{O})[H][[\Gamma]]$:*
$$(H_S(T)) = (det_{Q(\mathcal{O})[H]}((T+1) - m_\gamma \mid Q(\mathcal{O}))),$$
$$(d_\infty^\#) = (det_{Q(\mathcal{O})[H]}((T+1) - m_\gamma \mid (Q(\mathcal{O})[H][[\Gamma]]/d_\infty)^\#)).$$

Remark 4.12. *Any character $\chi$ of $H$ can be extended to a $Q(\mathcal{O})[X]$-algebra homomorphism, for a variable $X$, and to a $Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]]$-algebra homomorphism*
$$\chi : Q(\mathcal{O})[H][X] \to Q(\mathcal{O})[X],$$
$$\chi : Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[G_\infty]] \to Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]],$$
*which maps $h \to \chi(h)$ for $h \in H$.*

Lemma 4.13. *We have the following equality of ideals in $\mathcal{O}[[\Gamma]]$ under the assumption $\mu = 0$:*

$$(det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(\mathfrak{X}^f_\infty \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))^\#$$
$$= (det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(E^1(\mathfrak{X}^f_\infty)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))).$$

*Proof.* Since the $R := \mathcal{O}_\psi[[\Gamma]]$-torsion module $M_1 := e_\psi(\mathfrak{X}^f_\infty \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))$ is of projective dimension at most one (cf. for example Lemma 2.4 in [32]), the $R$-torsion module $M_2 := e_\psi(E^1(\mathfrak{X}^f_\infty)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))$ is also of projective dimension at most one. Hence by the first part of Proposition 4.9 it is enough to show that $Fitt_R(M_1)^\# = Fitt_R(M_2)$. But this is a direct consequence of the second part of Lemma 4.3 and Proposition 2 in [13].

□

At this point we use the classical Main Conjecture in Iwasawa theory [47], i.e. the equality of fractional ideals

$$(G^*_{\psi,S}(T)) = (det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(\mathfrak{X}^f_\infty \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))$$

in $Q(\mathcal{O})[[T]]$ for any character $\psi$ of $H$ (see equality (1.2) and part 1 of Proposition 4.9). Here we note that the assumption $\mu = 0$ implies the vanishing of the $\mu$-invariants of $\mathfrak{X}^{f,\psi}_\infty$ for all (even) characters $\psi$ of $H$. Hence, using Lemma 4.13, we obtain the following equality of fractional ideals in $Q(\mathcal{O})[[T]]$:

$$\begin{aligned}(\psi(G^*_S(T)^\#)) &= (G^*_{\psi,S}(T)^\#)\\ &= (det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(E^1(\mathfrak{X}^f_\infty)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))\\ &= (\psi(det_{Q(\mathcal{O}[H])}((T+1) - m_\gamma \mid E^1(\mathfrak{X}^f_\infty)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))).\end{aligned}$$

Consequently the exact sequence

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^\# \to E^1(\mathcal{Z}^f_\infty)^\# \to E^1(\mathfrak{X}^f_\infty)^\# \to 0$$

in Theorem 4.7, and Lemma 4.10 together with the base-change property of determinants (cf. (4.1)) imply the following equality of ideals in $\mathcal{O}[[T]]$:

$$\psi((c_\infty G^*_S(T))^\#) = (\psi(det_{Q(\mathcal{O}[H])}((T+1) - m_\gamma \mid E^1(\mathcal{Z}^f_\infty)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))). \quad (4.2)$$

Before completing the proof we recall that the $\mu$-invariant $\mu(F)$ of a power series $F \in \mathcal{O}[[T]]$ is the largest exponent $\mu \geq 0$ such that $f \in (\pi^\mu)\mathcal{O}[[T]]$. For $F \in \mathbb{A}$ we define the $\mu$-invariant of $F$ to be zero if $\mu(\chi(F)) = 0$ for any $p$-adic valued character $\chi$ of $H$.

Let $F := det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid E^1(\mathcal{Z}^f_\infty)^\#)$ and $G := (c_\infty G^*_S(T))^\#$ in $\mathbb{A}$ (cf. Lemma 3.2). Then

- $\boldsymbol{\mu(F) = 0}$, since the determinantal polynomial $F \in \mathbb{Z}_p[H][[T]]$ of the projective $\mathbb{Z}_p[H]$-module $E^1(\mathcal{Z}^f_\infty)^\#$ (cf. Proposition 4.8) is monic.

- $\boldsymbol{\mu(G) = 0}$, since for any $p$-adic character $\psi$ of the group $H$, we have

$$\mu(\psi(G_S^*)) = \mu(g_{\psi,S}^* \cdot u_{\psi,S}) = 0 \quad , \quad \mu(\psi(H_S)) = 0 \quad , \quad \mu(\psi(d_\infty)) = 0.$$

  Here we note that the determinantal polynomials of the field $Q(\mathcal{O})$ and the projective $\mathbb{Z}_p[H]$-module $(\mathbb{A}/d_\infty\mathbb{A})^\#$ (cf. Proposition 4.8), which are generated by $H_S$ and $d_\infty^\#$, respectively (cf. Lemma 4.11), are monic.

- $\boldsymbol{(\psi(F)) = (\psi(G))}$, using equality (4.2).

In the terminology of [2], $R := \mathbb{Z}_p[H]$ is admissible for the abelian group $H$, i.e. $R$ is a finite product of strictly admissible rings $R_i$, which means that each $R_i$ is separated and complete in the $rad(R_i)$-adic topology and also $R_i/rad(R_i)$ is a skew field. Since the $\mu$-invariants of $F, G \in R[[T]]$ are both zero, Proposition 2.1 in [2] as an equivariant Weierstrass Preparation Theorem implies the existence of unique distinguished polynomials $f^*, g^* \in R[T]$ and units $u, v \in (R[[T]])^\times$ such that

$$F = u \cdot f^* \quad \text{and} \quad G = v \cdot g^*.$$

We apply a $p$-adic character $\psi$ of $H$ to both sides, and note that $\psi(f^*)$ and $\psi(g^*)$ are both distinguished polynomials in $\mathcal{O}[T]$, and that $\psi(u), \psi(v) \in \mathcal{O}[[T]]^\times$ are units. Hence the equality $(\psi(F)) = (\psi(G))$ together with the uniqueness of the Weierstrass decomposition yields

$$\psi(f^*) = \psi(g^*)$$

for any $p$-adic character $\psi$ of $H$. Therefore, $f^* = g^*$ and $F = uv^{-1}G$. The equality $(F) = (G)$ now implies the following:

$$\begin{aligned}(\iota(c_\infty G_S^*)) &= (det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\#)) \\ &= Fitt_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^\#) \qquad \text{by Lemma 4.10} \\ &= Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f)^\# \qquad \text{by Lemma 4.4.}\end{aligned} \tag{4.3}$$

Consequently, the equality

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f) = (c_\infty G_S^*)$$

holds, and this completes the proof of the Equivariant Main Conjecture 4.1 under the assumptions of the classical Main Conjecture and of $\mu = 0$.

Theorem 4.14. *The Equivariant Main Conjecture 4.1 follows from the classical Main Conjecture in Iwasawa theory under the assumption $\mu = 0$.*

Remark 4.15. *For any odd prime $p$, or for the prime 2 if $F$ is an absolutely abelian number field, the classical Main Conjecture holds, and hence the Equivariant Main Conjecture 4.1 is verified under the assumption $\mu = 0$.*

We recall that the assumption $\mu = 0$ holds for any absolute abelian number field $E$, i.e. for any number field $E$ whose Galois group over $\mathbb{Q}$ is abelian, by the result of Ferrero and Washington in [10]. Hence

COROLLARY 4.16. *If $E$ is an absolute abelian number field, the Equivariant Main Conjecture 4.1 holds unconditionally.*

## 5 THE COATES-SINNOTT CONJECTURE AS AN APPLICATION

Let $E/F$ be an abelian extension of number fields with Galois group $G$, let $n \geq 2$ be an integer, and let $p$ be an arbitrary prime. Let $S$ be a finite set of primes in $F$ containing the primes above $p$, the primes ramified in $E$ and the infinite primes, and let $S_f$ denote the set of all finite primes in $S$. Let

$$\Theta^S_{E/F}(s) = \sum_{\chi \in \hat{G}} L^S_{E/F}(s, \chi^{-1}) \cdot e_\chi$$

be the $G$-equivariant $S$-incomplete $L$-function associated to $E/F$. We recall that for an integer $n \geq 1$ by a result of Siegel [41]

$$\Theta^S_{E/F}(1-n) \in \mathbb{Q}[G],$$

and furthermore, by Deligne and Ribet [8] or by Cassou-Noguès [4]

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

For $n \geq 1$ the $n$-th higher Stickelberger ideal is defined as follows:

$$Stick^S_{E/F}(n) := Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

REMARK 5.1. *The classical theorem of Stickelberger states that*

$$Stick^S_{E/\mathbb{Q}}(1) \subseteq Ann_{\mathbb{Z}[G]}(Cl(\mathcal{O}_E)),$$

*where $Cl(\mathcal{O}_E)$ denotes the class group of $\mathcal{O}_E$. Brumer conjectured that the same holds for any abelian extension $E/F$ of number fields.*

The original formulation of the Coates-Sinnott Conjecture is as follows:

CONJECTURE 5.2. *(The Coates-Sinnott Conjecture, $K$-theoretic version). Let $E/F$ be an abelian Galois extension of number fields with Galois group $G$, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subseteq Ann_{\mathbb{Z}[G]}(K_{2n-2}(\mathcal{O}_E)).$$

As a consequence of the work of Voevodsky (cf. [45]) the Quillen-Lichtenbaum Conjecture holds, i.e. for all odd primes $p$ the étale Chern characters defined by Soulé [42]

$$ch^{(p)}_{i,n} : K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}_p \to H^i_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n))$$

are isomorphisms for $i = 1, 2$, and all $n \geq 2$. Here $\mathcal{O}'_F = \mathcal{O}_F[1/p]$. The surjectivity of the Chern characters was proved by Soulé for even $n$ (see [42]) and by Dwyer and Friedlander in general (see [7]).

For the prime 2 the situation is in general different. The deviation between $K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}_2$ and $H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_2(n))$ has been determined by Rognes and Weibel [37]. In [21] it was suggested to replace the $K$-groups $K_{2n-2}(\mathcal{O}_F)$ by the motivic cohomology groups $H^2_{\mathcal{M}}(E, \mathbb{Z}(n))$, because the latter groups have the advantage that their $p$-parts are isomorphic to $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))$ for all primes $p$ ([19], Theorem 2.4). This leads to the following motivic version of the Coates-Sinnott Conjecture:

Conjecture 5.3. *(The Coates-Sinnott Conjecture, motivic version). Let $E/F$ be an abelian Galois extension of number fields with Galois group $G$, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subseteq Ann_{\mathbb{Z}[G]}(H^2_{\mathcal{M}}(E, \mathbb{Z}(n))).$$

The explicit results of Rognes-Weibel show that the motivic version implies the $K$-theoretic version. Moreover, the validity of the motivic version is equivalent to the validity of the following $p$-adic version for all primes $p$:

Conjecture 5.4. *(The Coates-Sinnott Conjecture, $p$-adic version). Let $E/F$ be an abelian Galois extension of number fields with Galois group $G$, let $p$ be prime, and let $n \geq 2$. Then*

$$Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))).$$

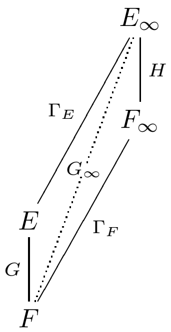
We remark that by the functional equation of $L$-functions (see for example [1]) $L_{E/F}(s, \chi)$ vanishes at negative integers $1-n$ for $n \geq 2$, unless $F$ is a totally real number field and $\chi(-1) = (-1)^n$. Therefore only the following cases are of interest:

- $E$ is a totally real number field and $n \geq 2$ is even.
- $E$ is a totally complex number field, $F$ is totally real and $n \geq 2$ is odd.

We will consider the first case, namely that $E/F$ is an abeian extension of totally real fields with Galois group $G$, and that $n \geq 2$ is even. We show that in this case the $p$-adic version of the Coates-Sinnott Conjecture follows from the Equivariant Main Conjecture for all primes $p$ assuming $\mu = 0$. For odd primes this has been done by Nguyen Quang Do [28] and independently by Greither-Popescu [15] (see also [2] for a slightly weaker result).

We first assume without loss of generality for the proof of the Coates-Sinnott Conjecture that *$E$ is the maximal real subfield of $E(\zeta_{2p})$* (this assumption clearly holds for $p = 2$, and for odd primes $p$ one can see for instance Lemma 6.14 in [15]). Here $\zeta_{2p}$ is a primitive $2p$-th root of unity. We recall the set up from Section 2: Let $E_\infty$ (resp. $F_\infty$) be the cyclotomic $\mathbb{Z}_p$-extension of $E$ (resp. $F$)

with Galois group $\Gamma_E$ (resp. $\Gamma_F$) over $E$ (resp. over $F$). We denote by $G_\infty$ the Galois group of $E_\infty/F$, by $H$ the Galois group of $E_\infty/F_\infty$, and by $\Gamma = \overline{<\gamma>}$ the image of $\Gamma_F$ under the splitting map in (2.1). The following diagram illustrates the situation:




Since $G_\infty$ is abelian, $G_\infty = H \times \Gamma$ and the completed group ring $\mathbb{A} = \mathbb{Z}_p[[G_\infty]]$ is identified with $\mathbb{Z}_p[H][[T]]$ under the identification (2.2). We let $d_\infty \in \Delta G_\infty$ be a non-zero divisor so that $\mathbb{A}/d_\infty\mathbb{A}$ is a finitely generated $\mathbb{Z}_p$-free module, e.g. $d_\infty = \gamma - 1$. By Theorem 4.7 we obtain an exact sequence of finitely generated $\mathbb{A}$-torsion modules

$$0 \to \mathbb{Z}_p \to (\mathbb{A}/d_\infty\mathbb{A})^\# \to E^1(\mathcal{Z}^f_\infty)^\# \to E^1(\mathfrak{X}^f_\infty)^\# \to 0 \tag{5.1}$$

in which the middle terms are of projective dimensions at most one. Using the equalities in (4.3) we also have

$$\begin{aligned} Fitt_{\mathbb{A}}((\mathbb{A}/d_\infty\mathbb{A})^\#(n)) &= ((\iota \circ t_n)(d_\infty)) \\ Fitt_{\mathbb{A}}(E^1(\mathcal{Z}^f_\infty)^\#(n)) &= ((\iota \circ t_n)(c_\infty G_S^*)). \end{aligned} \tag{5.2}$$

We note that the sequence (5.1) is also an exact sequence of finitely generated $\Lambda$-modules, where $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Moreover, the exact sequence (5.1) is an exact sequence of finitely generated $\mathbb{Z}_p$-free modules since we have assumed $\mu = 0$.

Lemma 5.5. *Let $G_E^{S_f}$ be the Galois group of the maximal algebraic pro-$p$-extension of $E$ unramified outside the primes above $S_f$, over $E$. Under the assumption that $E$ is the maximal real subfield of $E(\zeta_{2p})$, we have the following:*

1. *$H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq \mathbb{Z}_p(n)_{\Gamma_E}$ for $n \geq 2$*
2. *$H^2(G_E^{S_f}, \mathbb{Z}_p(n)) \simeq (E^1(\mathfrak{X}^f_\infty)^\#(n))_{\Gamma_E}$ for even $n \geq 2$ (under the hypothesis $\mu = 0$)*

*Proof.* 1. It is enough to take the $\Gamma_E$-invariants and $\Gamma_E$-coinvariants of the exact sequence $0 \to \mathbb{Z}_p(n) \to \mathbb{Q}_p(n) \to \mathbb{Q}_p(n)/\mathbb{Z}_p(n) \to 0$ to get a 6 term exact sequence in which $\mathbb{Q}_p(n)_{\Gamma_E} = \mathbb{Q}_p(n)^{\Gamma_E} = 0$ for $n \geq 2$. We note that

$H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \mathbb{Q}_p/\mathbb{Z}_p(n)^{\Gamma_E}$.

2. Let $E_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $E$, with Galois group $\Gamma_E$. As before we denote by $\Omega_E^{S_f}$ the maximal algebraic pro-$p$-extension of $E$ unramified outside the primes above $S_f$. We have the following isomorphisms:

$$\begin{aligned} E^1(\mathfrak{X}_\infty^f)^\#(n)_{\Gamma_E} &\simeq Hom(\mathfrak{X}_\infty^f, \mathbb{Z}_p)(n)_{\Gamma_E} && \text{By part 3 in Lemma 4.3} \\ &\simeq Hom(\mathfrak{X}_\infty^f(-n), \mathbb{Z}_p)_{\Gamma_E} \\ &\simeq Hom(\mathfrak{X}_\infty^f(-n)_{\Gamma_E}, \mathbb{Q}_p/\mathbb{Z}_p) && \text{By Lemma 5.18 in [14]} \\ &\simeq Hom(\mathfrak{X}_\infty^f, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E} \\ &\simeq Hom(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E}. \end{aligned}$$

If we assume that

$$Hom(\mathfrak{X}_\infty^f, \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq H^1(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n)), \tag{5.3}$$

then - using the fact that $cd_p(\Gamma_E) = 1$ - we can continue the isomorphisms above as follows:

$$\begin{aligned} H^1(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E} &\simeq H^1(Gal(\Omega_E^{S_f}/E), \mathbb{Q}_p/\mathbb{Z}_p(n)) \\ &\simeq H^2(G_E^{S_f}, \mathbb{Z}_p(n)), \end{aligned}$$

where the last isomorphism follows from the finiteness of the groups $H^1(G_E^{S_f}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and $H^2(G_E^{S_f}, \mathbb{Z}_p(n))$ for even $n$ and the totally real field $E$ (cf. Corollary 2.5 in [20] and Proposition 2.3 in [44]).

Hence, to complete the proof it is enough to show that the claim (5.3) is true. Clearly

$$Hom(\mathfrak{X}_\infty^f, \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq Hom(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))$$

since $\mathfrak{X}_\infty^f$ is the abelianization of $Gal(\Omega_E^{S_f}/E_\infty)$. Now we notice that the Galois group $Gal(\Omega_E^{S_f}/E_\infty)$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(n)$, since $n$ is even, and therefore

$$Hom(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq H^1(Gal(\Omega_E^{S_f}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))$$

□

At this point we recall the following lemma from Iwasawa theory (See Lemma 6.3 in [5], where a special case is proved, or [31], Chapter V, §3, Ex. 3):

Lemma 5.6. *Let $M$ be an $\mathcal{O}[[\Gamma]]$-torsion module, where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$, and let $F(T)$ be the characteristic polynomial of $M$. The following are equivalent:*

1. *$M^\Gamma$ is finite.*

2. *$M_\Gamma$ is finite.*

3. $F(0) \neq 0$

*If these conditions hold, then*

$$\frac{|M^{\Gamma}|}{|M_{\Gamma}|} = |F(0)|_v = p^{-f \cdot v(F(0))},$$

*where $v$ is the normalized valuation, i.e. $v(\pi) = 1$ for a uniformizer $\pi \in \mathcal{O}$, and $f$ is the residue degree of $\pi$ over $p$.*

As a consequence of Lemma 5.5 we see that $(E^1(\mathfrak{X}^f_\infty)^{\#}(n))_{\Gamma_E}$ and $\mathbb{Z}_p(n)_{\Gamma_E}$ are both finite and so by Lemma 5.6 $(E^1(\mathfrak{X}^f_\infty)^{\#}(n))^{\Gamma_E}$ and $\mathbb{Z}_p(n)^{\Gamma_E}$ are both trivial for even $n \geq 2$. We note that both $\mathbb{Z}_p(n)$ and $E^1(\mathfrak{X}^f_\infty)^{\#}(n)$ have no non-trivial finite $\Lambda$-submodules. For $E^1(\mathfrak{X}^f_\infty)^{\#}(n)$ this follows from the fact that by Lemma 4.3 $E^1(\mathfrak{X}^f_\infty)^{\#}$ is isomorphic to the adjoint of $\mathfrak{X}^f_\infty$ and as such it does not have any non-trivial finite $\Lambda$-submodules (see [16], Section 1.3). Moreover, $((\mathbb{A}/d_\infty\mathbb{A})^{\#}(n))^{\Gamma_E} = 0$, since for $n \geq 2$ we have $\kappa(\gamma)^n \neq 1$. Therefore, $((\mathbb{A}/d_\infty\mathbb{A})^{\#}(n))_{\Gamma_E}$ is again finite by Lemma 5.6. As a result the $\Gamma_E$-coinvariants of $E^1(\mathcal{Z}^f_\infty)^{\#}(n)$ are also finite and similarly $(E^1(\mathcal{Z}^f_\infty)^{\#}(n))^{\Gamma_E} = 0$, for any even $n \geq 2$. Hence by taking the $\Gamma_E$-coinvariants of the exact sequence (5.1) we obtain the following exact sequence of finite $\mathbb{Z}_p[G]$-modules for any even $n \geq 2$:

$$0 \to H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to ((\mathbb{A}/d_\infty\mathbb{A})^{\#}(n))_{\Gamma_E} \to (E^1(\mathcal{Z}^f_\infty)^{\#}(n))_{\Gamma_E} \to H^2(G_E^{S_f}, \mathbb{Z}_p(n)) \to 0, \tag{5.4}$$

where the two middle $\mathbb{Z}_p[G]$-modules are of projective dimension at most one as a consequence of the last part of Theorem 4.7 and the facts that $\mathbb{Z}_p[G_\infty]^{\Gamma_E} = 0$ and $\mathbb{Z}_p[G_\infty]_{\Gamma_E} = \mathbb{Z}_p[G]$. Furthermore, following the equalities in (5.2), we have

$$Fitt_{\mathbb{Z}_p[G]}(((\mathbb{A}/d_\infty\mathbb{A})^{\#}(n))_{\Gamma_E}) = ((\pi \circ \iota \circ t_n)(d_\infty)),$$
$$Fitt_{\mathbb{Z}_p[G]}((E^1(\mathcal{Z}^f_\infty)^{\#}(n))_{\Gamma_E}) = ((\pi \circ \iota \circ t_n)(c_\infty G_S^*),$$

where $\pi : \mathbb{A} \to \mathbb{Z}_p[G]$ is the projection mapping $\gamma - 1$ to 0.

Now we take advantage of the following Proposition due to Burns-Greither, which relates the Fitting ideals of the modules of a 4-term exact sequence under some assumptions:

PROPOSITION 5.7. *([2], Lemma 5) Let $R := \mathbb{Z}_p[G]$, for a finite abelian group $G$ and a prime number $p$. Assume that we have an exact sequence of finite $R$-modules*

$$0 \to A \to P \to P' \to A' \to 0.$$

*Further, assume that $pd_{\mathbb{Z}_p[G]}P \leq 1$ and $pd_{\mathbb{Z}_p[G]}P' \leq 1$. Then, we have*

$$Fitt_R(A^*) \cdot Fitt_R(P') = Fitt_R(A') \cdot Fitt_R(P),$$

*where the Pontryagin dual $A^* := Hom(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is endowed with the covariant G-action.*

Applying Proposition 5.7 to the exact sequence (5.4) of finite $\mathbb{Z}_p[G]$-modules yields the following equality:

$$\begin{aligned}&Fitt_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^*)Fitt_{\mathbb{Z}_p[G]}((E^1(\mathcal{Z}_\infty^f)^\#(n))_{\Gamma_E})\\ &\qquad = Fitt_{\mathbb{Z}_p[G]}((\mathbb{A}/d_\infty\mathbb{A})^\#(n)_{\Gamma_E})Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n))).\end{aligned}$$

Property 5 of Fitting ideals (section 4) shows that the Fitting ideal of $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^*$ is the same as the annihilator ideal of $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))$. This, together with (5.2), yields the following equality of fractional ideals in $\mathbb{Z}_p[G]$:

$$\begin{aligned}&Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n)))\\ &\qquad = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi \circ \iota \circ t_n)(c_\infty/d_\infty \cdot G_S^*)).\end{aligned}$$

Finally we have $d_\infty/c_\infty = (\gamma - 1)e + (1 - e)$, which can be identified by (2.2) with $Te + 1 - e = H_S(T)$, and obtain the following theorem:

THEOREM 5.8. *We have the following equality of ideals of $\mathbb{Z}_p[G]$:*

$$Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n))) = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi \circ \iota \circ t_n)(\frac{G_S^*}{H_S})).$$

Now let $p$ be an odd prime. In this case we have an equality of the Galois groups $G_E^S$ and $G_E^{S_f}$, as we noticed before. Furthermore, by Lemma 3.1 we have $G_S^* = G_S$ under the assumption $\mu = 0$. On the other hand for odd primes $p$, since $E$ is the maximal real subfield of $E(\zeta_{2p})$, we have $(\pi \circ \iota \circ t_n)(G_S/H_S) = \Theta_{E/F}^S(1 - n)$ for any even $n$ by Corollary 3.4. Therefore from Theorem 5.8 we obtain

$$Fitt_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta_{E/F}^S(n).$$

This implies the $p$-adic version of the Coates-Sinnott Conjecture for odd primes. Here we note that $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n)) \subseteq H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))$.

For $p = 2$, we use the following lemma:

LEMMA 5.9. *Let $E/F$ be an abelian extension of totally real fields with Galois group G, and let $n \geq 2$ be an integer. Let $r_1(F) = [F : \mathbb{Q}]$. Then we have the following exact sequence of $\mathbb{A}$-modules for $\mathbb{A} = \Lambda[H]$:*

$$0 \to (\mathbb{A}/2\mathbb{A})^{r_1(F)} \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0.$$

*Proof.* We have the following commutative diagram by class field theory (cf. [18]) for the finite sets $S$ and $S_f$ of primes in $F$:

$$\begin{array}{ccccccccc} D_E & \rightarrowtail & \hat{U}_E^S & \to & \prod_{v\in S}\prod_{w|v}\hat{E}_w & \to & Gal(M_E^S/H_E^S) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ D'_E & \rightarrowtail & \hat{U}_E^{S_f} & \to & \prod_{v\in S_f}\prod_{w|v}\hat{E}_w & \to & Gal(M_E^{S_f}/H_E^{S_f}) & \to & 0, \end{array}$$

where $D_E$ and $D'_E$ are the kernels of the corresponding maps and are bounded by the Leopoldt defect $\delta_E$, $\hat{U}_E^S$ (resp. $\hat{U}_E^{S_f}$) is the $p$-adification of the $S$-unit (resp. $S_f$-unit) group of the ring of integers of $E$, $\hat{E}_w$ is the $p$-adic completion of the local field $E_w$, $M_E^S$ (resp. $M_E^{S_f}$) is the maximal abelian pro-$p$-extension of $E$ unramified outside the primes in $S$ (resp. in $S_f$), and $H_E^S$ (resp. $H_E^{S_f}$) is the Hilbert $S$-class ($S_f$-class) field of $E$. Here the $S_f$-unit group means the group of the totally real $S$-units. Since $\hat{E}_w$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for a real prime $w$, and $G$ acts transitively on the set

$$\{\hat{E}_w \quad | \quad w : \text{infinite primes of } E \text{ lying above } v\}$$

for any infinite prime $v$ of $F$, we obtain the following exact sequence of $\Lambda[H]$-modules:

$$0 \to D_E \to D'_E \to \prod_{w\in S\setminus S_f} \mathbb{Z}[G]/2 \to Gal(M_E^S/M_E^{S_f}) \to 0,$$

where $\prod_{w\in S\setminus S_f}\mathbb{Z}[G]/2 \simeq (\mathbb{Z}[G]/2)^{r_1(F)}$. Since the real primes are unramified in $E_\infty/E$, we can write the exact sequence above for the unique intermediate fields $E_n$ of $E_\infty/E$ with $G_n := Gal(E_n/E) \simeq \mathbb{Z}/2^n\mathbb{Z}$ for all $n \geq 0$, as follows:

$$0 \to D_{E_n} \to D'_{E_n} \to (\mathbb{Z}/2\mathbb{Z}[G_n])^{r_1(F)} \to Gal(M_{E_n}^S/M_{E_n}^{S_f}) \to 0.$$

Now the claim is that we have the isomorphism

$$\varprojlim(\mathbb{Z}/2\mathbb{Z}[G_n])^{r_1(F)} \simeq (\mathbb{A}/2\mathbb{A})^{r_1(F)}$$

for $\mathbb{A} = \Lambda[H]$. Since $\Lambda/2\Lambda \simeq \varprojlim \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}$, it suffices to show for a fixed real prime $v$ of $F$ that the inverse limits of $\{\prod_{v_n|v}\mathbb{Z}/2\mathbb{Z}\}$ and $\{\mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}\}$ are isomorphic. For this we inductively define an isomorphism

$$f_n : \prod_{v_n|v}\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}$$

compatible with the norm maps as follow: Let $f_0$ be the identity and assume we have defined the isomorphisms $f_m$ compatible with the norm maps for all $m \leq n$. Let $v_{n+1}$ and $v'_{n+1}$ be the extensions of $v_n$ to $F_{n+1}$. We define

$$f_{n+1} : \prod_{v_{n+1}}\mathbb{Z}/2\mathbb{Z} \cdot \prod_{v'_{n+1}}\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[T]/T^{2^{n+1}}$$

as follows:

$$f_{n+1}(a_1,\cdots,a_{2^n},b_1,\cdots,b_{2_n}) = f_n(a_1+b_1,\cdots,a_{2^n}+b_{2^n})+T^{2^n}f_n(a_1,\cdots,a_{2^n}).$$

Now we have the commutative diagram

$$\begin{array}{ccc} \prod_{v_n|v}\mathbb{Z}/2\mathbb{Z} & \overset{f_n}{\rightarrow} & \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n} \\ \downarrow & & \downarrow \\ \prod_{v_{n+1}|v}\mathbb{Z}/2\mathbb{Z} & \overset{f_{n+1}}{\rightarrow} & \mathbb{Z}/2\mathbb{Z}[T]/T^{2^{n+1}} \end{array}$$

for any $n \geq 0$, and hence

$$\varprojlim_n \prod_{v_n|v} \mathbb{Z}/2\mathbb{Z} \simeq \Lambda/2\Lambda.$$

This completes the proof of the claim. Now the exact sequence

$$0 \to Gal(M_E^S/M_E^{S_f}) \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0$$

yields an exact sequence

$$(\mathbb{A}/2\mathbb{A})^{r_1(F)} \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0.$$

We note that under the assumption of the weak Leopoldt Conjecture for $E_\infty/E$ the cokernels of $D_{E_n} \to D'_{E_n}$, for all $n \geq 0$, are finite elementary 2-groups of order bounded independent of $n$ (see for example [31], Chapter X, §3). Finally, since $\Lambda/2\Lambda$ has no non-trivial finite submodules, we obtain the following exact sequence of $\Lambda[H]$-modules:

$$0 \to (\mathbb{A}/2\mathbb{A})^{r_1(F)} \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0.$$

□

We now take $\Gamma_E$-coinvariants and then the Pontryagin dual of the exact sequence of Lemma 5.9. The same calculation as in the proof of the second part of Lemma 5.5 leads to the exact sequence

$$0 \to H^2(G_E^{S_f},\mathbb{Z}_2(n)) \to H^2(G_E^S,\mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z}[G])^{r_1(F)}(n)$$

of $\mathbb{Z}_2[G]$-modules. This yields

$$2^{r_1(F)}Ann_{\mathbb{Z}_2[G]}(H^2(G_E^{S_f},\mathbb{Z}_2(n))) \subseteq Ann_{\mathbb{Z}_2[G]}(H^2(G_E^S,\mathbb{Z}_2(n))).$$

Consequently by Theorem 5.8 we obtain

$$2^{r_1(F)}Ann_{\mathbb{Z}_p[G]}(H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi\circ\iota\circ t_n)(\frac{G_S^*}{H_S})) \\ \subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S,\mathbb{Z}_2(n))).$$

Under the assumption $\mu = 0$, we have $G_S = 2^{r_1(F)} \cdot G_S^*$ for the prime 2 (cf. Lemma 3.1). Consequently $\Theta_{E/F}^S(1-n) = 2^{r_1(F)}((\pi \circ \iota \circ t_n)(\frac{G_S^*}{H_S}))$ (cf. 3.4) for any even integer $n \geq 2$ and as a result,

$$Ann_{\mathbb{Z}_2[G]}(H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))) \cdot \Theta_E^S(1-n) \subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))).$$

Finally, we note that $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_2(n)) \subseteq H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))$. Hence, the 2-adic version of the Coates-Sinnott Conjecture 5.4 holds. This finishes the proof of the following result:

THEOREM 5.10. *Let $E/F$ be an abelian extension of totally real number fields with Galois group $G$, and let $n \geq 2$ be an even integer. Then the motivic version - and therefore the original version - of the Coates-Sinnott Conjecture holds under the assumptions that $\mu = 0$ and that the 2-primary part of the classical Main Conjecture in Iwasawa theory is valid.*

We note that both assumptions are true if $E$ is abelian over $\mathbb{Q}$, and therefore we obtain the following unconditional result:

COROLLARY 5.11. *Let $E$ be a totally real absolute abelian field. For an abelian extension $E/F$ with Galois group $G$ and even $n \geq 2$, the Coates-Sinnott Conjecture 5.3 holds.*

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