Combinatorics of the Basic Stratum of Some Simple Shimura Varieties

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Abstract. We express the cohomology of the basic Newton stratum of some unitary Shimura varieties associated to division algebras in terms of automorphic representations of the group in the Shimura datum.

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Introduction

Let S be a Shimura variety of PEL type, defined over its reflex field E [8, 9]. The variety S parametrizes Abelian varieties with additional PEL structures (Polarization, Endomorphisms, Level structures) [17, §5]. In this article we compute for certain simple S the ℓ-adic cohomology of the basic stratum, also called supersingular locus, at a prime of good reduction. Let us make this more precise.

Let $\mathfrak{p}$ be a prime of E where S has good reduction; this means in particular that the PEL type moduli problem extends over the ring of integers $\mathcal{O}_{E_\mathfrak{p}}$ of $E_\mathfrak{p}$, and that this extended problem is representable by a smooth and quasi-projective scheme $\mathcal{S}$ over $\mathcal{O}_{E_\mathfrak{p}}$ [17, §5]. Let $\mathbb{F}_q$ be the residue field of E at $\mathfrak{p}$. Write $S_\mathfrak{p} := \mathcal{S} \otimes \mathbb{F}_q$, and write for each $x \in S_\mathfrak{p}(\overline{\mathbb{F}}_q)$, $\mathcal{A}_x/\overline{\mathbb{F}}_q$ for the corresponding Abelian variety with additional PEL structures. The Newton strata of $S_\mathfrak{p}$ are defined as follows: Two points $x, y \in S_\mathfrak{p}(\overline{\mathbb{F}}_q)$ lie in the same Newton stratum if and only if the p-divisible group $\mathcal{A}_x[p^\infty]$ with its additional PE-structures is isogenous to $\mathcal{A}_y[p^\infty]$. Thus the Newton strata of $S_\mathfrak{p}$ are the loci in $S_\mathfrak{p}$ where the isogeny class of $\mathcal{A}_x[p^\infty]$ is constant. The Newton strata of $S_\mathfrak{p}$ are varieties defined over $\mathbb{F}_q$, and stable under the Hecke correspondences. Thus their cohomology carries an action of the Hecke algebra and the Frobenius operator. For various arithmetic applications, one is interested in computing this cohomology. There is one Newton stratum which is particularly important. This stratum is called the *basic stratum*, or *supersingular* locus. The cohomology of the other strata is expected to be induced from Levi subgroups of lower rank. Thus, in this article we focus on the basic stratum, although one could extend the methods of this article to also derive results for the non-basic strata [21].

Before we continue, we give the definition of the basic Newton stratum. For each point $x \in S_{\mathfrak{p}}(\overline{\mathbb{F}}_q)$, the isogeny class of the corresponding p-divisible group with PE structure $\mathcal{A}_x[p^\infty]$ is determined by its rational Dieudonné module $\mathbb{D}(\mathcal{A}_x[p^\infty])_{\mathbb{Q}}$. This Dieudonné module is an "isocrystal with additional structure" and, as such, it is determined by its slope morphism ν [14]. If the centralizer of the slope morphism ν is G, then the isocrystal is called *basic*. Among (the isomorphism classes of) the isocrystals $\mathbb{D}(\mathcal{A}_x[p^\infty])_{\mathbb{Q}}$ for $x \in S(\overline{\mathbb{F}}_q)$ there is one *unique* basic isocrystal, and the locus in $S_{\mathfrak{p}}$ corresponding to this isocrystal is the *basic stratum* $B \subset S_{\mathfrak{p}}$. For example consider the modular curve $Y_1(N)$ (we take $N \geq 4$), let p be prime number not dividing N and consider the supersingular locus B of $Y_1(N)_{\mathbb{F}_p}$. Then $B(\overline{\mathbb{F}}_p)$ is the set of pairs $(E, \eta) \in Y_1(N)(\overline{\mathbb{F}}_p)$ of elliptic curves E equipped with a point η of order N where E is *supersingular*. A famous result of Deligne-Rapoport [10] is a geometric description of the basic stratum B, as follows. Consider the modular curve Y associated to the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(p) \subset \mathrm{GL}_2(\mathbb{Z})$ (for notation, see [11]). Deligne and Rapoport construct a model for the curve Y over $\mathbb{Z}_p$ (they introduce Y as a moduli space), and in particular they may reduce Y modulo p. When reduced modulo p, the curve Y is isomorphic to the union of two copies of $Y_1(N)_{\mathbb{F}_p}$, fibered over the supersingular locus. Since the curve Y has semistable reduction, one may compare its cohomology over the generic fibre with the cohomology of the special fibre. This yields a description of the Hecke/Galois module $\mathrm{H}^0_{\mathrm{et}}(B_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ in terms of weight 2 modular forms. This description of $\mathrm{H}^0_{\mathrm{et}}(B_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ can also be deduced from the Langlands/Kottwitz method and application of the trace formula, as we explain in an introductory chapter to our thesis [19, Chap. 1].

For higher dimensional Shimura varieties it is difficult to describe the basic stratum geometrically. Several complications occur because usually the basic stratum B is a non-finite, non-smooth variety[1]. There are geometric descriptions on the structure of the basic stratum, but only under very restricted conditions. For example Vollaard and Wedhorn [32] obtained a geomteric description of the basic stratum, if $G_{\mathbb{R}}$ is the group $GU(n-1,1)$, and the prime p of (good) reduction is non-split. These restricted classes of Shimura varieties are still very interesting as they are certain arithmetic quotients of the n-dimensional complex unit ball. However, one would like to know what the basic stratum looks like in general. Since general geometric results seem currently out of reach, it is therefore interesting to see that it *is* possible to describe

[1]On a positive side, the basic stratum B should always be a projective variety. However, the author must admit that he is unaware of a proof nor a reference of this fact for general non-projective Shimura varieties.

the cohomology of the basic stratum for general classes of Shimura varieties. In this article we answer the question for the simple, unitary Shimura varieties of Kottwitz at split primes of good reduction.

This article is a sequel to the article [20]. In [20] we have already studied the above problem, but under strong simplifying conditions on the signatures. The varieties are associated to certain division algebras over $\mathbb{Q}$ with involution of the second kind; we call such varieties Kottwitz varieties. In [20] we assumed that the signatures of the unitary group $U \subset G$ are coprime with n. This hypothesis greatly simplifies the geometry and cohomology of B; completely new phenomena occur when the hypothesis is dropped. In this article we solve all the resulting problems when one removes the hypothesis from the theorem in case the prime p of reduction is split in the center of the division algebra defining the Kottwitz variety. A consequence of our final result is an automorphic expression for the ℓ-adic cohomology of the basic stratum of Kottwitz's varieties at split primes of good reduction. The expressions are in terms of: (1) Automorphic forms on the group G of the Shimura datum, (2) The factor at $\mathfrak{p}$ of their associated Galois representations, and (3) Polynomials in q^α of combinatorial nature, associated to certain non-crossing lattice paths in the plane $\mathbb{Q}^2$. In particular we describe the zeta function of the basic stratum in terms of the objects (1), (2) and (3) above.

The first part of the proof is the same as the one we carried out in [20], and we will be only very brief on that part of the argument. This first part roughly consists of 3 steps:

- Truncate the formula of Kottwitz for Shimura varieties of PEL-type [17] to count the number of points in the basic stratum.
- Pseudo-stabilize the expression using the arguments from [16] (cf. [15]).
- Compare the resulting stable expression with the geometric side of the trace formula.

Carrying out this program, we obtained in [20] the formula

$$\sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}\ (f^{\infty p} \times \Phi_{\mathfrak{p}}^{\alpha}, \mathrm{H}^i_{\mathrm{et}}(B_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)) = |\mathrm{Ker}^1(\mathbb{Q}, G)| \cdot \mathrm{Tr}\ (f, \mathcal{A}(G)). \tag{0.1}$$

In this formula we have

- $\mathcal{A}(G) = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is the space of automorphic forms on G with respect to inverse ω of the central character of ξ;
- $\Phi_{\mathfrak{p}}$ is the geometric Frobenius of $\overline{\mathbb{F}}_q$ over $\mathbb{F}_q$;
- (one fixes the choice of an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ to compare the left hand side with the right hand side);
- The function f is a tensor product $f_\infty \otimes f^{\infty p} \otimes f_p$, of a function f_∞ on $G(\mathbb{R})$, a function $f^{\infty p}$ on $G(\mathbb{A}_{\mathrm{f}}^p)$, and a function f_p on $G(\mathbb{Q}_p)$, where
  - $f^{\infty p}$ is an arbitrary K^p-bi-invariant, locally constant compactly supported complex valued function on $G(\mathbb{A}_{\mathrm{f}}^p)$.

  - The function f_∞ on $G(\mathbb{R})$ is up to scalar the Euler-Poincaré function corresponding to the representation ξ (for the value of the scalar, see [16, Lem. 3.2]).
  - The component f_p is a the pointwise product $\chi_c^G f_\alpha$, where
    * the function χ_c^G is the characteristic function on $G(\mathbb{Q}_p)$ of the set of compact elements in $G(\mathbb{Q}_p)$;
    * the function f_α is the *function of Kottwitz*. It arises by base change from $G(E_{\mathfrak{p},\alpha})$ to $G(\mathbb{Q}_p)$ of a spherical function ϕ_α on $G(E_{\mathfrak{p},\alpha})$; the function ϕ_α is the characteristic function of the subset $G(\mathcal{O}_{E_{\mathfrak{p},\alpha}})\mu(p^{-1})G(\mathcal{O}_{E_{\mathfrak{p},\alpha}}) \subset G(E_{\mathfrak{p},\alpha})$, with respect to the choice (any choice) of reductive model of G over $\mathbb{Z}_p$.
- $\mathrm{Ker}^1(\mathbb{Q}, G)$ is the set of classes $\sigma \in \mathrm{H}^1(\mathbb{Q}, G)$ such that for all rational places v of $\mathbb{Q}$ the restricted class $\sigma_v \in \mathrm{H}^1(\mathbb{Q}_v, G)$ is trivial.

Once Equation (0.1) is established, the second step is to compute $\mathrm{Tr}\ (\chi_c^G f, \pi)$ for each automorphic representation π occurring in $\mathcal{A}(G)$, and it is this second step that we carried out only partially in [20]. The problem is local at p: One needs a satisfactory expression for the traces $\mathrm{Tr}\ (\chi_c^G f_\alpha, \pi_p)$ against the local representations π_p of $G(\mathbb{Q}_p)$. We will not solve this local problem completely in this article: We only solve the problem in sufficient generality for global applications. More precisely, we only compute the compact trace $\mathrm{Tr}\ (\chi_c^G f_\alpha, \pi_p)$ for the class of rigid representations of $G(\mathbb{Q}_p)$. All representations π_p that occur as a factor at p of an automorphic representation π of G are rigid. The rigid representations form a very restricted subclass of all smooth representations of $G(\mathbb{Q}_p)$. Using base change, Jacquet-Langlands and the Moeglin-Waldspurger description of the discrete spectrum of GL_n, we describe the rigid representations precisely. Then, we compute the compact traces of f_α against all rigid representations.

Let us explain this last computation in more detail. Since our group $G(\mathbb{Q}_p)$ is isomorphic to a product of general linear groups, the computation quickly reduces to a computation with general linear groups. Furthermore, it is quite easy to see that $\mathrm{Tr}\ (\chi_c^G f_\alpha, \pi_p)$ vanishes unless π_p has an invariant vector for the action of the Iwahori subgroup (the representation π_p is *semistable*). Thus cuspidal representations π_p do not appear in the description. A semistable representation π of $\mathrm{GL}_n(\mathbb{Q}_p)$ is called *standard* if it is isomorphic to a *product* (see §2) of essentially square integrable representations. The computation of the compact trace $\mathrm{Tr}\ (\chi_c^G f_\alpha, \pi)$ on a square-integrable representation is easy, and using van Dijk's formula adapted for compact traces [20, Prop. 3], we easily deduce formulas for compact traces on the standard representations. Inside the Grothendieck group of smooth $G(\mathbb{Q}_p)$-representations of finite length, any semistable irreducible representation π may be[2] (uniquely) written as a sum $\pi = \sum_I c_I \cdot I$ where I ranges over the standard representations, and the coefficients $c_I \in \mathbb{C}$ are 0 for all but finitely many I. Consequently the

[2] Zelevinsky proved in [33] that the standard representations form a basis of the Grothendieck group of the category of smooth $\mathrm{GL}_n(\mathbb{Q}_p)$-representations of finite length.

compact trace Tr $(\chi_c^G f, \pi)$ equals $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$. There are two steps to compute Tr $(\chi_c^G f, \pi)$: (Prob1) Know the coefficients c_I and (Prob2) Make the sum $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$. The first problem (Prob1) is related to the Kazhdan-Lusztig conjecture[3]. The "Kazhdan-Lusztig theorem" of Beilinson-Bernstein [3] (and [13]) interprets the multiplicity of any given irreducible representation π in the representation I, as the dimension of certain intersection cohomology spaces, and also as the value at $q = 1$ of certain Kazhdan-Lusztig polynomials. Combining all these theorems gives expressions for the sum $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$; however they are far from satisfactory. The coefficients c_I, even though they are known, depend on dimensions of complicated cohomology spaces. Furthermore, the coefficients c_I can be negative, and the sum $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$ has in general a lot of redundancy. Thus Problem 2 seems (to the author) very non-trivial.
To compute the sum $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$ we restrict our attention to the rigid representations only. Any rigid representation is a product of unramified twists of Speh representations, and therefore we reduce to the class of Speh representations. A deep theorem of Tadic completely resolves the first problem (Prob1) for the Speh representations. The coefficients c_I turn out to be $-1, 0$ or 1 for these representations (precise statement in Theorem 2.3). Still the sum $\sum_I c_I \mathrm{Tr}\ (\chi_c^G f, I)$ remains very redundant; in fact most of the terms cancel out against each other. To evaluate the sum, we give an interpretation of the traces Tr $(\chi_c^G f, I)$ as certain lattice paths in $\mathbb{Q}^2$. Using a variation on a classical lemma in combinatorics [30, Thm. 7.2.1], we see that only the non-intersecting paths remain in the sum, all other terms cancel out.

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1. Notation

In Sections 1–13 we will work (almost) exclusively with the group GL_n over a non-Archimedean local field. Let us set up the principal notation.

- p is a prime number.
- F is a non-Archimedean local field with residue characteristic equal to p.
- $\overline{F}$ is a fixed algebraic closure of F.

[3]This conjecture is a theorem, see [5, Thm. 8.6.23].

- $\mathcal{O}_F$ is the ring of integers of F.
- $\varpi_F \in \mathcal{O}_F$ is a prime element of F.
- $q = \#(\mathcal{O}_F/\varpi_F)$ is the cardinality of the residue field of F.
- n is a positive integer.
- $G_n = \mathrm{GL}_n(F)$ (often we write simply $G = G_n$ when n is understood).
- $K = \mathrm{GL}_n(\mathcal{O}_F)$.
- $\mathcal{H}(G)$ for the Hecke algebra of complex valued, locally compact constantly supported functions on G. The product $f * h$ of two functions $f, h \in \mathcal{H}(G)$ is defined by the convolution integral $(f * h)(g) = \int_G f(x)h(x^{-1}g)\mathrm{d}x$ with measure $\mathrm{d}x$ normalized so that the group $K \subset G$ has volume 1. The algebra $\mathcal{H}(G)$ is associative, but it does not have a unit element.
- $\mathcal{H}_0(G)$ is the K-spherical Hecke algebra of G, *i.e.* the algebra of $f \in \mathcal{H}(G)$ such that f is invariant under left and right K-translations.
- We write 1_K for the characteristic function of the subset $K \subset G$. Then 1_K is the unit element of the algebra $\mathcal{H}_0(G)$.
- The group $P_0 \subset G$ is the standard upper triangular Borel subgroup of G with $P = TN_0$, where T is the diagonal torus of G, and N_0 is the group of upper triangular unipotent matrices in G.
- a parabolic subgroup P of G *standard* if it is upper triangular, and we write $P = MN$ for its standard Levi decomposition.

A *partition* of n is a finite, *non-ordered* list of non-negative numbers whose sum is equal to n. A *composition* of n is a finite, *ordered* list of positive numbers whose sum is equal to n. To each a composition (n_a) of n we attach the standard parabolic subgroup

$$P_{(n_a)} \overset{\text{def}}{=} \left\{ \begin{pmatrix} g_1 & & * \\ & \ddots & \\ 0 & & g_k \end{pmatrix} \in G_n \,\middle|\, g_a \in G_{n_a} \right\} \subset G_n.$$

We write $M_{(n_a)} = G_{n_1} \times G_{n_2} \times \cdots \times G_{n_k}$ for the standard Levi-subgroup of $P_{(n_a)}$.

Finally, to each parabolic subgroup $P \subset G$ we attach the sign $\varepsilon_P := (-1)^{\dim A_P/A_G}$, where A_G (resp. A_P) is the center of G (resp. P). For $P = P_{(n_a)}$ we have $\varepsilon_P = (-1)^{n-k}$.

2. Tadic's determinantal formula

We recall an important character formula of Tadic for the Speh representations. This formula is a crucial ingredient for our computations.

Let $m, m' \in \mathbb{Z}_{\geq 1}$. If π (resp. π') is a smooth admissible representation of G_m (resp. $G_{m'}$), then we write $\pi \times \pi'$ for the $G_{m+m'}$-representation parabolically induced (unitary induction) from the representation $\pi \otimes \pi'$ of the standard Levi subgroup consisting of two blocks, one of size m, and the other one of size m'. The tensor product $\pi \otimes \pi'$ in the above formula is taken along the blocks of this Levi subgroup. We write $\mathcal{R}$ for the direct sum $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{Groth}(G_n)$ with the convention that G_0 is the trivial group. The group G_0 has one unique irreducible representation σ_0 (the space $\mathbb{C}$, with trivial action). The operation "direct sum of representations" together with the product "$\times$" turns the vector space $\mathcal{R}$ into a commutative $\mathbb{C}$-algebra with σ_0 as unit element. We call it the *ring of Zelevinsky*.

We write ν for the absolute value morphism from $\mathrm{GL}_1(F) = F^\times$ to $\mathbb{C}^\times$. By a *segment* $S = \langle x, y\rangle$ we mean a set of numbers $\{x, x+1, \dots, y\}$ where $x, y \in \mathbb{Q}$, $y - x \in \mathbb{Z}$, and where we need to explain the conventions in case $y \le x$. In case y is strictly smaller than $x-1$, then $\langle x, y\rangle = \emptyset$; in case x is equal to y, the segment $\langle x, y\rangle = \{x\}$ has one element. We have one unusual convention: For $y = x - 1$ we define the segment $\langle x, y\rangle$ to be the set $\{\star\}$ of one element containing a distinguishing symbol "$\star$". The *length* $\ell(S)$ of a segment $S = \langle x, y\rangle$ is defined to be $y - x + 1$. Thus the segment $\{\star\}$ has length 0, the segment $\{x\}$ has length 1, the segment $\{x, x+1\}$ has length 2, etc.

For any segment $\langle x, y\rangle$ with $y \ge x$ we write $\Delta\langle x, y\rangle$ for the unique irreducible quotient of the induced representation $\nu^x \times \nu^{x+1} \times \cdots \times \nu^y$. We define $\Delta\{\star\}$ to be σ_0 (the one-dimensional representation of the trivial group $\mathrm{GL}_0(F)$), and we define $\Delta\langle x, y\rangle$ to be 0 in case $y < x - 1$. For any segment S of non-negative length the object ΔS is a representation of the group $\mathrm{GL}_n(F)$, where n is the length of S.

For the standard properties of segments we refer to Zelevinsky's work [33] (cf. [28]), but note that our conventions are slightly different, because we allow rational numbers in the segments and we have the segment $\{\star\}$.

For any finite ordered list of segments $S_1, S_2, \dots, S_t$ we have the product representation $\pi := (\Delta S_1) \times (\Delta S_2) \times \cdots \times (\Delta S_t)$. Observe that, due to our conventions, in case $S_a = \{\star\}$ for some a, then ΔS_a is the unit in $\mathcal{R}$, and $\pi = \prod_{b=1, b\neq a}^t \Delta S_b \in \mathcal{R}$. In case $S_a = \emptyset$ for some index a, then we have $\pi = 0$ in $\mathcal{R}$.

Definition 2.1. We write St_G for the *Steinberg representation* of the group G. This representation is by definition the (unique) irreducible quotient of the space of locally constant compactly supported complex valued functions on $P_0\backslash G$, on which G acts by right translations.

Definition 2.2. Let t, h be positive integers such that $n = th$. We define $\mathrm{Speh}(h, t)$ to be the (unique) irreducible quotient of the representation $\mathrm{St}_{G_h}\nu^{\frac{t-1}{2}} \times \cdots \times \mathrm{St}_{G_h}\nu^{\frac{1-t}{2}}$. This representation has t segments, $S_a = \langle x_a, y_a\rangle$, $a = 1, \dots, t$, where $x_a = \frac{t-h}{2} - (a-1)$, $y_a = \frac{t+h}{2} - a$. Observe that, for each index a, we have $\ell S_a = h$. Furthermore, for each index $a < t$, we have $x_{a+1} = x_a - 1$ and $y_{a+1} = y_a - 1$.

Let $S_1 = \langle x_1, y_1\rangle, S_2 = \langle x_2, y_2\rangle, \dots, S_t = \langle x_t, y_t\rangle$ be an ordered list of segments defining a representation of the group $G = \mathrm{GL}_n(F)$. Let $\mathfrak{S}_t$ be the symmetric group on $\{1, 2, \dots, t\}$. For any $w \in \mathfrak{S}_t$ we define the number n_a^w to be $y_a - x_{w(a)} + 1$. One checks easily that $\sum_{a=1}^k n_a^w = n$. The numbers n_a^w need not be positive. We define $\mathfrak{S}'_t \subset \mathfrak{S}_t$ to be subset consisting of those permutations $w \in \mathfrak{S}_t$ such that the numbers n_a^w are positive or 0. If the permutation w lies in the subset $\mathfrak{S}'_t \subset \mathfrak{S}_t$, then (n^w_a) is a composition of n. Assuming that $w \in \mathfrak{S}'_t$ we will write $P_w = M_w N_w$ for the parabolic subgroup of G corresponding to the composition (n^w_a).

Let $w \in \mathfrak{S}'_t$. We define the segments $S^w_1 := \langle x_{w(1)}, y_1\rangle, S^w_2 := \langle x_{w(2)}, y_2\rangle, \ldots, S^w_t := \langle x_{w(t)}, y_t\rangle$. We have $\ell(S^w_a) = n^w_a$. We let Δ_w be the representation of M_w defined by $(\Delta S^w_1) \otimes \cdots \otimes (\Delta S^w_t)$, where the tensor product is taken along the blocks of M_w. The representation I_w is defined to be the product $\Delta S^w_1 \times \Delta S^w_2 \times \cdots \times \Delta S^w_t$, *i.e.* it is the (unitary) parabolic induction $\mathrm{Ind}^G_{P_w} \Delta_w$ of Δ_w to G. In case $w \in \mathfrak{S}_t \backslash \mathfrak{S}'_t$ we define both Δ_w and I_w to be 0.

Remark. It is possible that $S^w_a = \{\star\}$ for some permutation w. In that case the representation ΔS^w_a is the unit element σ_0 of $\mathcal{R}$, and thus can be left out of the product that defined I_w.

In this notation we have the following theorem:

Theorem 2.3 (Tadic). *Let π be a Speh representation of G and let $S_1 = \langle x_1, y_1\rangle, S_2 = \langle x_2, y_2\rangle, \ldots S_t = \langle x_t, y_t\rangle$ be its segments. The representation π satisfies Tadic's determinantal formula*

$$\pi = \sum_{w \in \mathfrak{S}_t} \mathrm{sign}(w) I_w \in \mathcal{R}. \tag{2.1}$$

Proof. This theorem was first proved by Tadic in [31] for Speh representations with a difficult argument. Chenevier and Renard give a simpler proof of Theorem 2.3 in [4]. Also Badulescu gives a simpler proof in the note [2] using the Moeglin-Waldspurger algorithm [25]. Recently Lapid and Minguez [23, Thm. 1] extended the formula to the larger class of ladder representations. □

Remark. By the definition of the subset $\mathfrak{S}'_t \subset \mathfrak{S}_t$ we have for all $w \in \mathfrak{S}_t$ that $I_w \neq 0$ if and only if $w \in \mathfrak{S}'_t$, and thus we may as well index over the elements $w \in \mathfrak{S}'_t$ in the sum in the above theorem. In the cases where the inclusion $\mathfrak{S}'_t \subset \mathfrak{S}_t$ is strict, the subset $\mathfrak{S}'_t$ is practically never a subgroup of $\mathfrak{S}_t$, it will neither be closed under composition nor contain inverses of elements.

The reason we call the formula of Theorem 2.3 the *determinantal formula of Tadic*, is the observation of Chenevier and Renard [4], who recognized that Equation (2.1) can be written as a determinant of a $t \times t$-matrix over the ring $\mathcal{R}$:

$$\pi = \det \begin{pmatrix} \Delta\langle x_1, y_1\rangle & \Delta\langle x_1, y_2\rangle & \cdots & \Delta\langle x_1, y_t\rangle \\ \Delta\langle x_2, y_1\rangle & \Delta\langle x_2, y_2\rangle & \cdots & \Delta\langle x_2, y_t\rangle \\ \vdots & \vdots & & \vdots \\ \Delta\langle x_t, y_1\rangle & \Delta\langle x_t, y_2\rangle & \cdots & \Delta\langle x_t, y_t\rangle \end{pmatrix} \in \mathcal{R}.$$

3. The Satake transform

Let $P = MN \subset G$ be a standard parabolic subgroup of G, and let $f \in \mathcal{H}(G)$ be a locally constant function on G. Then we define the constant term $f^{(P)} \in$

$\mathcal{H}(M)$ by the formula

$$f^{(P)}(m) = \delta_P^{1/2}(m) \cdot \int_N f(mn)\mathrm{d}n,$$

for all $m \in M$, and where

- δ_P is the *modulus character* of P, *i.e.* $\delta_P(m) = |\det(m, Lie(N))|$;
- the integral is taken with respect to the left Haar measure of N giving the group $K \cap N$ measure 1.

In case f is K-spherical, then $f^{(P)}$ is $K_M = M(\mathcal{O}_F)$-spherical, and the constant term $f \mapsto f^{(P)}$ defines a morphism of algebras $\mathcal{H}_0(G) \to \mathcal{H}_0(M)$. In case $T = M$ we have the obvious isomorphism $\mathcal{H}_0(T) \overset{\sim}{\to} \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, under which the characteristic function of the subset $q^{-e_1}\mathcal{O}_F^\times \times q^{-e_2}\mathcal{O}_F^\times \times \cdots q^{-e_n}\mathcal{O}_F^\times \subset (F^\times)^n = T$, corresponds to the monomial $X^{e_1}X^{e_2}\cdots X^{e_n}$ in $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ for all $(e_i) \in \mathbb{Z}^n$. The composition of the constant term $\mathcal{H}_0(G) \to \mathcal{H}_0(T)$ with the isomorphism $\mathcal{H}_0(T) \cong \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is called the *Satake morphism*, and is denoted $\mathcal{S}_G$ or simply $\mathcal{S}$ if G is understood. Satake proved [29] that the map is injective, and has image equal to the algebra $A = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ is the Weyl group of T in G; it is the symmetric group on the set $\{1, \ldots, n\}$ acting on $X_1, \ldots, X_n$ through its natural permutation action.
If $P = MN \subset G$ is a standard parabolic subgroup corresponding to the composition (n_a), then we have a commutative diagram

$$\begin{array}{ccccc} f & & \mathcal{H}_0(G) & \xrightarrow{\mathcal{S}_G} & \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_n} \\ \downarrow & & \downarrow & & \downarrow \\ f^{(P)} & & \mathcal{H}_0(M) & \xrightarrow{\mathcal{S}_M} & \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}} \end{array}$$

The conceptual way to think about the algebra $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is as the symmetric algebra $\mathbb{C}[X_*(T)]$ on the lattice of cocharacters of T. Similarly A is the algebra of Weyl group invariant elements in $\mathbb{C}[X_*(T)]$. Write $\widehat{G}, \widehat{T}, \widehat{M}$ for the complex dual groups of G, T, M. Then $\mathbb{C}[X_*(T)]$ is the algebra of regular algebraic functions on $\widehat{T}$, A is the algebra of algebraic functions on $\widehat{T}/\mathfrak{S}_n$, and the Hecke algebra of M is the space of regular algebraic functions on $\widehat{T}/\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}$.
If π is an unramified representation of G, then the space of K-invariants π^K is one-dimensional, and induces a surjection $\mathcal{H}_0(G) \to \mathrm{End}(\pi^K) = \mathbb{C}$. We get an element $\varphi_{G,\pi} \in \mathrm{SpecMax}(\mathcal{H}_0(G)) = \widehat{T}/\mathfrak{S}_n$ which may be identified with a semisimple element $\varphi_{G,\pi} \in \widehat{G}$, well-defined up to conjugacy. This element is called the *Hecke matrix* of π. For example, if π is the trivial representation of G, then an element $f \in \mathcal{H}_0(G)$ acts on π by multiplication with the degree $\deg(f) = \int_G f(g)\mathrm{d}g$. By the Iwasawa decomposition $G = KP_0$ we have

$$\int_G f(g)\mathrm{d}g = \int_{P_0}\int_K f(kp)\mathrm{d}p\mathrm{d}k = \int_{P_0} f(p)\mathrm{d}p = \int_{T_0} \delta_{P_0}^{-1/2}(t) \cdot f^{(P_0)}(t)\mathrm{d}t$$

$$= \langle \mathcal{S}(f), \rho_G \rangle,$$

where $\rho_G = \left(\frac{n-1}{2}, \frac{n-1}{2}, \dots, \frac{1-n}{2}\right) \in X^*(T) \otimes \mathbb{Z}[1/2]$ is the half sum of the positive roots of G. Equivalently, $\langle \mathcal{S}(f), \rho_G \rangle$ is the evaluation of $\mathcal{S}(f) \in \mathbb{C}[\widehat{T}/\mathfrak{S}_n]$ at the point

$$\varphi_{G,\mathrm{Triv}} = \left(q^{\frac{n-1}{2}}, \dots, q^{\frac{1-n}{2}} \right) \in \widehat{T}, \tag{3.1}$$

(the Hecke matrix of the trivial representation).

4. Weyl chambers and truncation of Hecke functions

Let $P \subset G$ be a parabolic subgroup corresponding to the composition (n_a) of n. Let k be the length of n_a. Write $\mathfrak{a}_P$ for the space $X_*(A_P) \otimes \mathbb{R}$, where $A_P \subset P$ is the center of P. We write $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$. Let $\mathfrak{a}_P \subset \mathfrak{a}_0$ be the set of $(x_i) \in \mathbb{R}^n$ so that $x_1 = \ldots = x_{n_1}$, …, $x_{n_1+\ldots+n_{k-1}+1} = \ldots = x_n$. The *acute Weyl chamber* $\mathfrak{a}_P^+$ in $\mathfrak{a}_P$ is the set of $(x_a) \in \prod_{a=1}^k \mathbb{R}^{n_a} = \mathbb{R}^n$ such that for all indices $a = 1, \dots, k-1$ we have $x_a \in \mathbb{R}^{n_a}$, and $\frac{1}{n_a}\sum_{i=1}^{n_a} x_{a,i} > \frac{1}{n_{a+1}} \sum_{i=1}^{n_{a+1}} x_{a+1,i}$. The *obtuse Weyl chamber* ${}^+\mathfrak{a}_P$ in $\mathfrak{a}_P$ is the set of $(x_i) \in \mathbb{R}^n$ so that for all a we have

$$x_1 + \ldots + x_{n_a} > \frac{n_1 + n_2 + \ldots + n_a}{n}(x_1 + x_2 + \ldots + x_n).$$

Let $X = X_1^{e_1} \cdots X_n^{e_n} \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = \mathbb{C}[X_*(T)] = A$ be a monomial. We may view $X_*(T)$ as a lattice inside the space $\mathfrak{a}_0$. Let $\pi_P \colon \mathfrak{a}_0 \to \mathfrak{a}_P$ be the canonical projection. Let $f \mapsto \chi_N f$ be the endomorphism of A sending $X \in X_*(T)$ to X if $\pi_P(X) \in \mathfrak{a}_P^+$ and to 0 if $\pi_P(X) \notin \mathfrak{a}_P^+$. Similarly, let $f \mapsto \widehat{\chi}_N f$ be the endomorphism of A sending $X \in X_*(T)$ to X if $\pi_P(X) \in {}^+\mathfrak{a}_P$, and to 0 if $\pi_P(X) \notin {}^+\mathfrak{a}_P$). Via the Satake isomorphism $\mathcal{H}_0(T) \cong A$ we transport the endomorphisms $f \mapsto \chi_N f$, $f \mapsto \widehat{\chi}_N f$ to endomorphisms of $\mathcal{H}_0(T)$.

5. Functions of Kottwitz

We introduce the functions of Kottwitz $f_{n\alpha s}$ that will be used later in the paper:

Definition 5.1. Let n and α be positive integers, and let s be a non-negative integer with $s \leq n$. We call the number s the *signature*, and we call the number α the *degree*. The function $f_{n\alpha s} \in \mathcal{H}_0(G)$ is the spherical function whose Satake transform is

$$q^{\alpha s(n-s)/2} \sum_{\nu \in \mathfrak{S}_n \cdot \mu_s} [\nu]^\alpha = q^{\alpha s(n-s)/2} \sum_{I \subset \{1,\dots,n\}, \#I = s} \prod_{i \in I} X_i^\alpha \in A. \tag{5.1}$$

We put $f_{n\alpha s} = 0$ when $n, \alpha, s \in \mathbb{Z}_{\geq 0}$ are such that $n < s$. We will call $f_{n\alpha s}$ a *simple Kottwitz function*.

6. Compact traces

We introduce the notion of compact trace for the group G (cf. [6], [20, §2.2], [20, p. 509]). An element $g \in G$ is *compact* if the roots $\alpha_i \in \overline{F}$ of its characteristic polynomial $\mathrm{charpol}(g) = \det(1 - Xg) \in F[X]$ all have the same absolute value. We write χ_c^G for the characteristic function of the set of compact elements in G. If g lies inside a standard Levi subgroup M of G, then there are two inequivalent notions of compactness: (1) $g \in M$ can be M-compact, (2) $g \in G$ can be G-compact. Write $M = \prod_i G_{n_a}$, then the element $g \in M$ is M-compact if $g_a \in G_{n_a}$ is compact for all a. We write χ_c^M for the characteristic function on M of the set of M-compact elements.

There is a notion of compactness and compact traces for general unramified reductive groups G': A semisimple element $\gamma \in G'$ is *compact* if for some (any) maximal torus T' in G' containing γ the absolute value $|\alpha(\gamma)|$ equals 1 for all roots α of T' in $Lie(G')$. A non-semisimple element of G' is *compact* if its semisimple part is compact. Let π be a smooth G-representation of finite length and f a locally constant, compactly supported function on G. Harish-Chandra proved that there is a locally integrable function θ_π on G such that for all $f \in \mathcal{H}(G)$ the trace $\mathrm{Tr}\ (f, \pi)$ is given by the integral $\int_{G_{\mathrm{rs}}} f(g)\theta_\pi(g) \mathrm{d}g$ where $G_{\mathrm{rs}} \subset G$ is the subset of regular semisimple elements. In this paper we are interested in the *compact trace*, $\mathrm{Tr}_{\mathrm{c}}(f, \pi)$, which is defined by taking the integral over the set G_{c} compact, regular semisimple elements, $\int_{G_{\mathrm{c}}} f(g)\theta_\pi(g)\mathrm{d}g$ (cf. [6]). Equivalently, write $\chi_c^G f$ for the pointwise product of f with χ_c^G; then $\mathrm{Tr}_{\mathrm{c}}(f, \pi) = \mathrm{Tr}\ (\chi_c^G f, \pi)$.

7. Lattice paths and the Steinberg representation

Fix an integer s with $0 \leq s \leq n$, and $\alpha \in \mathbb{Z}_{\geq 1}$. We express the compact trace of the functions $f_{n\alpha s}$ on the Steinberg representation in terms of certain lattice paths in $\mathbb{Q}^2$.

Let A^+ be the polynomial ring $\mathbb{C}[q^a | a \in \mathbb{Q}]$ of rational, formal powers of the variable q. Equivalently, A^+ is the complex group ring $\mathbb{C}[\mathbb{Q}^+]$ of the additive group $\mathbb{Q}^+$ underlying $\mathbb{Q}$.

Definition 7.1. A *path* L in $\mathbb{Q}^2$ is a sequence of points $\vec{v}_0, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ such that $\vec{v}_{i+1} - \vec{v}_i = (1, 0)$ (*east*), or $\vec{v}_{i+1} - \vec{v}_i = (1, 1)$ (*north-east*). The starting point of L is $\vec{v}_0$ and the end point is $\vec{v}_r$; the number r is the *length*. An eastward step $(1, 0)$ has weight 1 and a north-eastward step $(a, b) \to (a + 1, b + 1)$ has weight $q^{-\alpha \cdot a} \in A^+$. The *weight of the path* L is defined to be the product in A^+ of the weights of its steps.

Remark. We allow paths of length zero; such a path consists of one point $\vec{v}_0$ and no steps. The weight of a path of length 0 is equal to 1. The paths of length 0 correspond to compact traces on the special segments $\{\star\}$ introduced earlier.

Definition 7.2. Let L be a path in $\mathbb{Q}^2$. Connect the starting point $\vec{v}_0$ of L with its end point $\vec{v}_r$ via a straight line ℓ. Then L is called a *Dyck path* if all of its points $\vec{v}_a$ lie on or below the line ℓ in the plane $\mathbb{Q}^2$. The Dyck path is called *strict* if none of its points $\vec{v}_a$ other than the initial and end point, lies on the line ℓ. Let $\vec{x}, \vec{y}$ be two points in $\mathbb{Q}^2$. Then we write $\mathrm{Dyck_s}(\vec{x}, \vec{y}) \in A^+$ for the sum of the weights of all the strict Dyck paths that go from the point $\vec{x}$ to the point $\vec{y}$. We call the polynomial $\mathrm{Dyck_s}(\vec{x}, \vec{y})$ the *strict Dyck polynomial*.

There are also non-strict Dyck polynomials $\mathrm{Dyck}(\overline{x}, \overline{y})$ but we are not concerned with those in this section; they will become important later, when we compute the compact traces on the trivial representation.
Using obtuse and acute Weyl chambers we will define truncations of the Satake transforms of elements in $\mathcal{H}_0(G)$. Those truncations will be best understood graphically. We first extend the notion of a path slightly to the concept of a graph.

DEFINITION 7.3. A *graph* in $\mathbb{Q}^2$ is a sequence of points $\vec{v}_0, \vec{v}_1, \dots, \vec{v}_r$ with $\vec{v}_{i+1} - \vec{v}_i = (1, e_i)$, where e_i is an integer.

Let $\ell_{s,n} \subset \mathbb{Q}^2$ be the line of slope $\frac{s}{n}$ going through the origin. We will often abuse notation, and write simply $\ell = \ell_{s,n}$. If $x \in \mathbb{Q}$, then we write $\ell(x)$ for the point $(x, \frac{s}{n}x)$ on the line ℓ.

DEFINITION 7.4. To a monomial $X = X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n} \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}]$, with $e_i \in \mathbb{Z}$ and $\sum_{i=1}^n e_i = s$ we associate the graph $\mathcal{G}_X$ with points $\vec{v}_0 := \ell(\frac{1-n}{2})$, and for $i = 1, \dots, n$, the point $\vec{v}_i$ is defined by

$$\vec{v}_0 + \big(i, e_n + e_{n-1} + \ldots + e_{n-(i+1)}\big) \in \mathbb{Q}^2.$$

DEFINITION 7.5. We define the *weight of a step* $(a, b) \to (a+1, b+e)$ to be $q^{-\alpha \cdot e \cdot a} \in A^+$, and the *weight of a graph* is the product of the weights of its steps.

LEMMA 7.6. *Let* $X = X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n} \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}]$ *be a monomial of degree* $\sum_{i=1}^n e_i = s$. *Let* $\mathcal{G} = \mathcal{G}_X$ *be the graph of* X. *Then*

- *the end point of the graph is* $\ell(\frac{n-1}{2} + 1) \in \mathbb{Q}^2$;
- *the evaluation of* X *at the point* $\varphi_{G,\mathrm{Triv}} \in \widehat{T}$ *is equal the weight of the graph* $\mathcal{G}_X$.

Furthermore, the following 3 statements are equivalent:

(*i*) $\widehat{\chi}_{N_0} X = X$;
(*ii*) $e_1 + e_2 + \ldots + e_i > \frac{s}{n} i$, *for all indices* $i < n$;
(*iii*) *the graph* $\mathcal{G}_X$ *lies strictly below the line* ℓ.

Proof. The fact stated in the first bullet point is trivial. The second point follows from Equation (3.1). The equivalence of (*i*)-(*iii*) is also formal (details can be found in the proof of [20, Prop. 5]). □

LEMMA 7.7. *Consider the representation* $\pi = 1_T(\delta_{P_0}^{1/2})$ *of the group* T. *Let* f *be a function in the spherical Hecke algebra of* T. *Then the trace of* f *against* π *is equal to the evaluation of* $\mathcal{S}(f) \in A$ *at the point* $(q^{(1-n)/2}, q^{(3-n)/2}, \dots, q^{(n-1)/2}) \in \widehat{T}$.

Proof. The character $\delta_{P_0}^{1/2}$ on T maps an element $(t_1, t_2, \ldots, t_n)$ to the complex number $|t_1|^{(n-1)/2}|t_2|^{(n-3)/2} \cdots |t_n|^{(n-1)/2}$. To any (rational) cocharacter $\nu \in X_*(T)$ we may associate the composition $(\delta_{P_0}^{1/2} \circ \nu)\colon F^\times \to T \to \mathbb{C}^\times$. We evaluate this composition at the prime element $\varpi_F \in F^\times$. Thus we have an element of the set $\mathrm{Hom}(X_*(T), \mathbb{C}^\times) = \widehat{T}(\mathbb{C})$ where the last isomorphism is given by $X_*(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \ni \nu \otimes z \mapsto \nu(z) \in \widehat{T}(\mathbb{C})$. We have $T = (F^\times)^n$ and thus we have the standard basis e_i on $X_*(T)$. This corresponds to the standard basis e_i on $X_*(\widehat{T})$. If we take $\nu = e_i$ in $(\delta_{P_0}^{1/2} \circ \nu)(\varpi_F)$ then we get $(\delta_{P_0}^{1/2} \circ e_i)(\varpi_F) = |\varpi_F|^{(n-1)/2-i+1} = q^{(1-n)/2+i-1}$. This completes the verification. □

Lemma 7.8. *The compact trace* $\mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, \mathrm{St}_G)$ *on the Steinberg representation is equal to the polynomial* $(-1)^{n-1} q^{s(n-s)\alpha/2} \cdot \mathrm{Dyck}_{\mathrm{s}}(\ell((1-n)/2), \ell((n-1)/2+1)) \in A^+$.

Proof. This lemma is a translation of [20, Prop. 7] using Lemma 7.6. □

Compact traces are compatible with twists:

Lemma 7.9. *Let* χ *be an unramified character of* $F^\times$, π *a smooth irreducible* G *representation. Then* $\mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, \pi \otimes \chi) = \chi(\varpi_F^{\alpha s}) \cdot \mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, \pi)$.

Proof. [20, Lem. 2]. □

Lemma 7.10. *Assume that* π *is an essentially square integrable representation of the form* ΔS, *where* $S = \langle x, y\rangle$ *is a segment of length* n. *The compact trace* $\mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, \Delta\langle x, y\rangle)$ *equals* $(-1)^{n-1} \cdot q^{s(n-s)\alpha/2} \cdot \mathrm{Dyck}_{\mathrm{s}}(\ell(x), \ell(y+1))$.

Proof. The representation $(\Delta S) \otimes \nu^{-x+(1-n)/2}$ is the Steinberg representation, and so Lemma 7.8 applies to it. The result then follows from Lemma 7.9. □

8. Lattice t-paths and standard representations

We describe the compact traces on the standard representations of G using 't-paths'.

Definition 8.1. Let t be a positive integer. Let $\vec{x} = (\vec{x}_a)$ and $\vec{y} = (\vec{y}_a)$ be two ordered lists of points in $\mathbb{Q}^2$, both of length t. A *t-path from $\vec{x}$ to $\vec{y}$* is the datum consisting of, for each index $a \in \{1, 2, \ldots, t\}$, a path L_a from the point $\vec{x}_a$ to the point $\vec{y}_a$.

A t-path (L_a) is called a *Dyck t-path* if all the paths L_a are Dyck paths. The Dyck path (L_a) is called *strict* if, for each index a, no point $\vec{v}_i$ of L_a other than $\vec{v}_0$ and $\vec{v}_r$ lies on the line ℓ. The *weight* $\mathrm{weight}(L_a)$ of a t-path (L_a) is the product of the weights of the paths L_a, where a ranges over the set $\{1, 2, \ldots, t\}$. We extend the definition of the strict Dyck polynomial $\mathrm{Dyck}_{\mathrm{s}}(\vec{x}, \vec{y}) \in A^+$ also to t-paths: The polynomial $\mathrm{Dyck}_{\mathrm{s}}(\vec{x}, \vec{y}) \in A^+$ is by definition the sum of the weights of the strict Dyck t-paths from the points $(\vec{x}_a)$ to the points $(\vec{y}_a)$. The Dyck polynomial $\mathrm{Dyck}_{\mathrm{s}}(\vec{x}, \vec{y})$ decomposes into the product $\prod_{a=1}^t \mathrm{Dyck}_{\mathrm{s}}(\vec{x}_a, \vec{y}_a) \in A^+$.

Lemma 8.2. *Let $S_1 = \langle x_1, y_1\rangle, S_2 = \langle x_2, y_2\rangle, \dots, S_t = \langle x_t, y_t\rangle$ be a list of segments and let I be the representation $(\Delta S_1) \times (\Delta S_2) \times \cdots \times (\Delta S_t)$. The compact trace* $\mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, I)$ *is equal to* $(-1)^{n-t} q^{s(n-s)\alpha/2} \mathrm{Dyck_s}(\vec{x}, \vec{y})$, *where for the indices $a = 1, \dots, t$ we have $\vec{x}_a := \ell(x_a)$ and $\vec{y}_a := \ell(y_a + 1)$.*

Proof. Let P be the parabolic subgroup of G corresponding to the composition $n = \sum_{a=1}^t \ell(S_a)$ of n. Let χ_M^G be the characteristic function on M of the subset of elements $m = (m_a) \in \prod_{a=1}^t G_{\ell(S_a)} = M$ such that $\prod_{a=1}^b |\det(m_a)| = |\det(m)|^{\frac{1}{n}\sum_{a=1}^b n_a}$ for all $b \leq t$. Then $\chi_c^G = \chi_M^G \chi_c^M$ holds on M. By the integration formula of van Dijk for compact traces [20, Prop. 3] we have

$$\begin{aligned} \mathrm{Tr}\ (\chi_c^G f, I) &= \mathrm{Tr}\ \left(\chi_c^G f_{n\alpha s}^{(P)}, (\Delta S_1) \times (\Delta S_2) \times \cdots \times (\Delta S_t)\right) \\ &= \mathrm{Tr}\ \left(\chi_c^M \chi_M^G f_{n\alpha s}^{(P)}, (\Delta S_1) \times (\Delta S_2) \times \cdots \times (\Delta S_t)\right). \end{aligned} \tag{8.1}$$

By [20, Prop. 4] the constant term $\chi_c^G f_{n\alpha s}^{(P)}$ is equal to $q^{\alpha C(\vec{n},\vec{s})} f_{n\alpha s_1} \otimes f_{n\alpha s_2} \otimes \cdots \otimes f_{n\alpha s_t}$ where $\vec{n} := (n_a)_{a=1}^t$, $\vec{s} := (s_a)_{a=1}^t$, $s_a := n_a s/n$, and $C(\vec{n}, \vec{s}) := \frac{1}{2}s(n-s) - \frac{1}{2}\sum_{a=1}^t s_a(n_a - s_a)$. The constant term $\chi_c^G f_{n\alpha s}^{(P)}$ vanishes in case one of the numbers s_a is non-integral. We have $(\chi_c^G f_{n\alpha s}^{(P)})^{(P_0 \cap M)} = \chi_M^G \chi_c^M f_{n\alpha s}^{(P_0)}$. Consequently, one may rewrite the trace in Equation (8.1) to the product $q^{\alpha C(\vec{n},\vec{s})} \prod_{a=1}^t \mathrm{Tr}\ (\chi_c^{G_{n_a}} f_{n_a \alpha s_a}, \Delta S_a)$. By Lemma 7.10 we obtain

$$q^{\alpha C(\vec{n},\vec{s})} \prod_{a=1}^t (-1)^{n_a - 1} q^{s_a(n_a - s_a)\alpha/2} \mathrm{Dyck_s}(\ell(x_a), \ell(y_a + 1)).$$

Note that the vertical distance between the point $\vec{y}_a$ and the point $\vec{x}_a$ has to be integral before paths can exist. Therefore the expression in this last Equation simplifies to the one stated in the lemma and the proof is complete. □

9. Non-crossing paths

We express the compact traces on Speh representations in terms of non-crossing lattice paths.

We call a t-path (L_a) *crossing* if there exists a couple of indices a, b with $a \neq b$ such that the path L_a has a point $\vec{v} \in \mathbb{Q}^2$ in common with the path L_b. There is an important condition:

- The point $\vec{v}$ of crossing must appear in the list of points $\vec{v}_{a,i}$ that define L_a and it must also occur in the list of points $\vec{v}_{b,i}$ that define L_b

(Because we work with rational coordinates, the point of intersection could a priori be a point lying halfway a step of a path (for example). We are ruling out such possibilities.)

We write $\mathrm{Dyck_s^+}(\vec{x}, \vec{y})$ for the sum of the weights of the *non-crossing* strict Dyck t-paths. Let π be the Speh representation of G associated to the Zelevinsky segments $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \dots, \langle x_t, y_t\rangle$ with $x_1 > x_2 > \dots > x_t$ and $y_1 > y_2 > \dots > y_t$. We define the points $\vec{x}_a := \ell(x_a) \in \mathbb{Q}^2$ and $\vec{y}_a := \ell(y_a + 1) \in \mathbb{Q}^2$, for $a = 1, 2, \dots, t$. The group $\mathfrak{S}_t$ acts on the free $\mathbb{Q}^2$-module $\mathbb{Q}^{2t} = (\mathbb{Q}^2)^t$ by

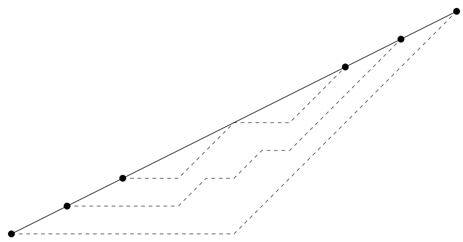

FIGURE 1. An example of a 3-path corresponding to the representation π of $\mathrm{GL}_{54}(F)$ defined by the segments $\langle 3, 20\rangle$, $\langle 2, 19\rangle$ and $\langle 1, 18\rangle$. We take $s = 27$ and we take the permutation $w = (13) \in \mathfrak{S}'_3$. The 3 dots on the lower left hand corner are the points $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ in $\mathbb{Q}^2$ respectively; the points $\vec{y}_1, \vec{y}_2$ and $\vec{y}_3$ are in the upper right corner. Observe that this 3-path is non-strict.

sending the a-th standard basis vector $e_a \in (\mathbb{Q}^2)^t$ to the basis vector $e_{w(a)} \in (\mathbb{Q}^2)^t$. Thus if we have the vector $\vec{x} \in \mathbb{Q}^{2t}$, then we get the new vector $\vec{x}^w$ whose a-th coordinate $\vec{x}^w_a \in \mathbb{Q}^{2t}$ is equal to $w(a)$-th coordinate of the vector $\vec{x}$. Note that the difference $\frac{s}{n} \cdot (y_a + 1) - \frac{s}{n} \cdot x_a$ need not be integral, in this case there do not exist paths from the point $\vec{x}^w_a \in \mathbb{Q}^2$ to $\vec{y}_a \in \mathbb{Q}^2$.
Let π be a Speh representation of type (h, t). The points $\vec{x}_a \in \mathbb{Q}^2$ and $\vec{y}_a \in \mathbb{Q}^2$ lie on the line $\ell \subset \mathbb{Q}^2$, and the point $\vec{x}_a$ lies on the left of the point $\vec{y}_a$ with horizontal distance $y_a + 1 - x_a = \ell(S_a) = h$. The two lists of points may overlap: There could exist couples of indices (a, b) such that $\vec{x}_a = \vec{y}_b$. All points $\vec{x}_a$ and $\vec{y}_b$ are *distinct* if we have $h \geq t$ (cf. Figure 1).
Assume $h \geq t$. Then, because all the points $\vec{x}_a$, $\vec{y}_b$ are distinct, there is no permutation $w \in \mathfrak{S}_t$ such that one of the segments $S^w_a = \langle x_{w(a)}, y_a\rangle$ is empty or equal to $\{\star\}$ for some index a. In particular we have $\mathfrak{S}'_t = \mathfrak{S}_t$.

DEFINITION 9.1. To any point $\vec{v} \in \mathbb{Q}^2$ we associate the *invariant* $\rho(\vec{v}) := p_2(\vec{v}) \in \mathbb{Q}/\mathbb{Z}$ where $p_2 \colon \mathbb{Q}^2 \to \mathbb{Q}$ is projection on the second coordinate.

Remark. The horizontal distance between the point $\vec{x}_b$ and the point $\vec{y}_a$ is integral for all indices. Therefore the invariant of the first coordinate is not of interest. However, the vertical distance is the number $s^w_a = \frac{s}{n} n^w_a \in \mathbb{Q}$, which certainly need not be integral.

Using this invariant we define a particular permutation $w_0 \in \mathfrak{S}_t$:

DEFINITION 9.2. Assume $h \geq t$ and *assume* that for each invariant $\rho \in \mathbb{Q}/\mathbb{Z}$ the number of indices a such that the point $\vec{x}_a$ has invariant ρ is equal to the number of indices a such that the point $\vec{y}_a$ has invariant ρ. The element

$w_0 \in \mathfrak{S}_t$ is the unique permutation such that for all indices a, b we have

$$\left(a < b \quad \text{and} \quad \rho(\vec{x}_a) = \rho(\vec{x}_b)\right)$$
$$\Longrightarrow \left(w_0^{-1}(a) > w_0^{-1}(b) \quad \text{and} \quad \rho(\vec{y}_a) = \rho(\vec{y}_b) = \rho(\vec{x}_a)\right).$$

Remark. Observe that the permutation w_0 depends on the integer s because the heights of the points $\vec{x}_a$, $\vec{y}_a$, and therefore also their invariants depend on s.

Remark. If our assumption on the invariants $\rho(\vec{x}_a)$ and $\rho(\vec{y}_a)$ in Definition 9.2 is *not* satisfied, then the permutation w_0 cannot exist because it has to induce bijections between sets of different cardinality.

One could also define the permutation $w_0 \in \mathfrak{S}_t$ inductively: First the index $w_0^{-1}(t) \in \{1, 2, 3, \ldots, t\}$ is the minimal index b such that the points $\vec{x}_t$ and $\vec{y}_b$ have the same invariant. Next, the index $w_0^{-1}(t-1) \in \{1, 2, 3, \ldots, t\}$ is the minimal index b, different from $w_0^{-1}(t)$, such that $\vec{x}_a$ and $\vec{y}_b$ have the same invariant. And so on: $w_0^{-1}(t-i) \in \{1, 2, 3, \ldots, t\}$ is the minimal index b different from the previously chosen indices $w_0^{-1}(t)$, $w_0^{-1}(t-1)$, $\ldots$, $w_0^{-1}(t-i+1)$, such that the points $\vec{y}_b$ and $\vec{x}_{t-i}$ have the same invariant.

Lemma 9.3. *Let π be a Speh representation with parameters h, t with $h \geq t$. Let d be the greatest common divisor of n and s and write m for the quotient n/d. Define the points $\vec{x}_a := \ell(x_a)$ and $\vec{y}_a := \ell(y_a + 1)$. The following two statements are equivalent:*

- (i) *for each invariant $\rho \in \mathbb{Q}/\mathbb{Z}$ the number of indices a such that the point $\vec{x}_a$ has invariant ρ is equal to the number of indices a such that the point $\vec{y}_a$ has invariant ρ*
- (ii) *m divides t or m divides h*

Proof. We first prove that "$m|t \Rightarrow (i)$". We have

$$\rho(\vec{x}_{a+1}) = \rho(\vec{x}_a) - \tfrac{s}{n} \in \mathbb{Q}/\mathbb{Z} \tag{9.1}$$

and the same relation for the points $\vec{y}_a$. Therefore, if m divides t, then the possible classes of the points $\vec{x}_a$ are equally distributed over the subset $\frac{s}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$, and every invariant occurs precisely t/m times. The same statement also holds for the points $\vec{y}_a$, and in particular (i) is true. This proves the claim.
We prove that "$m|h \Rightarrow (i)$". Assume $m|h$. Then the invariants of $\vec{x}_a$ and $\vec{y}_a$ are the same for all indices a. Thus (i) is true.
We prove that "$(m \nmid t$ and $m \nmid h) \Rightarrow ((i)$ is false)". Assume $m \nmid t$ and $m \nmid h$. We first reduce to the case where $t < m$. Assume $t \geq m$. Consider the lists of elements $\rho(\vec{x}_1)$, $\rho(\vec{x}_2)$, $\ldots$, $\rho(\vec{x}_m)$, and $\rho(\vec{y}_1)$, $\rho(\vec{y}_2)$, $\ldots$, $\rho(\vec{y}_m)$ of $\mathbb{Q}/\mathbb{Z}$. By Equation (9.1) every possible class in $\frac{s}{n}\mathbb{Z}/\mathbb{Z}$ occurs precisely once in both lists. Thus, the truth value of (i) is not affected if we remove the elements $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ and $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m$ from the respective lists. Renumber the indices and repeat this argument until $t < m$. Because we assumed that t did

not divide m there remains a positive number of elements in the list $\vec{x}_a$ and $\vec{y}_a$. We renumber so that the indices range from 1 to t. Then we have reduced to the case where $1 \leq t < m$. Now look at the two lists $\rho(\vec{x}_1), \rho(\vec{x}_2), \dots, \rho(\vec{x}_t)$ and $\rho(\vec{y}_1), \rho(\vec{y}_2), \dots, \rho(\vec{y}_t)$. In both lists every class in $\mathbb{Q}/\mathbb{Z}$ occurs *at most* once. We assumed that m does not divide h, and therefore $\rho(\vec{x}_1) \neq \rho(\vec{y}_1)$. If there does not exist an index b such that $\rho(\vec{x}_1) = \rho(\vec{y}_b)$, then (i) is false and we are done. Thus assume $\rho(\vec{x}_1) = \rho(\vec{y}_b)$ for some $1 < b \leq t$. By Equation (9.1) we then have $\rho(\vec{y}_{b-1}) = \rho(\vec{y}_b) + \frac{s}{n} = \rho(\vec{x}_1) + \frac{s}{n}$. The invariant $\xi := \rho(\vec{x}_1) + \frac{s}{n} \in \mathbb{Q}/\mathbb{Z}$ does not occur in the list $\rho(\vec{x}_1), \rho(\vec{x}_2), \dots, \rho(\vec{x}_t)$. Thus, we have found an invariant, namely ξ, occurring once in the list of invariants of the elements $\vec{y}_a$ and does not occur in the list of invariants of the elements $\vec{x}_a$. This contradicts (i) and completes the proof. □

Theorem 9.4. *Let π be a Speh representation with parameters h, t with $h \geq t$. Let d be the greatest common divisor of n and s and write m for the quotient n/d. Define the points $\vec{x}_a := \ell(x_a)$ and $\vec{y}_a := \ell(y_a + 1)$ (for x_a, y_a, see Def. 2.2). The compact trace* $\mathrm{Tr}\ (\chi_c^G f_{n\alpha s}, \pi)$ *on π is non-zero if and only if m divides t or m divides h, and if the compact trace is non-zero, then it is equal to* $(-1)^{n-t}\mathrm{sign}(w_0) q^{s(n-s)\alpha/2} \mathrm{Dyck}_{\mathrm{s}}^{+}(\vec{x}^{w_0}, \vec{y})$, *where the permutation $w_0 \in \mathfrak{S}_t$ depends on s and is defined in Definition 9.2.*

Proof. For a technical reason we assume that $0 < s < n$ in this proof. In case $s = 0$ we have $f_{n\alpha s} = 1_{\mathrm{GL}_n(\mathcal{O}_F)}$. All the elements in $\mathrm{GL}_n(\mathcal{O}_F)$ are compact and therefore $\chi_c^G f_{n\alpha 0} = f_{n\alpha 0}$. The compact trace becomes the usual trace and the theorem is easy. A similar argument applies in case $s = n$. Thus we may indeed assume $0 < s < n$.

To ease notation, we write $f = f_{n\alpha s}$. By Theorem 2.3 the compact trace $\mathrm{Tr}\ (\chi_c^G f, \pi)$ is equal to the combinatorial sum $\sum_{w \in \mathfrak{S}_t} \mathrm{sign}(w) \mathrm{Tr}\ (\chi_c^G f, I_w)$ for any Hecke operator $f \in \mathcal{H}(G)$. We apply it to the Kottwitz functions $f = f_{n\alpha s}$. We have $\mathfrak{S}_t' = \mathfrak{S}_t$ because $h \geq t$. Let $w \in \mathfrak{S}_t$. Thus we have the formula

$$\mathrm{Tr}\ (\chi_c^G f, \pi) = \sum_{w \in \mathfrak{S}_t} \mathrm{sign}(w) \cdot \mathrm{Tr}\ (\chi_c^G f^{(P_w)}, I_w) \tag{9.2}$$

By Lemma 8.2 we get for $f = f_{n\alpha s}$,

$$\mathrm{Tr}\ (\chi_c^G f, \pi) = q^{s(n-s)\alpha/2} \sum_{w \in \mathfrak{S}_t} \mathrm{sign}(w) \cdot \varepsilon_{P_0 \cap M_w} \cdot \mathrm{Dyck}_{\mathrm{s}}(\vec{x}^w, \vec{y}). \tag{9.3}$$

We apply a standard combinatorial argument[4] to simplify the right hand side. Put the *lexicographical order* "$<$" on $\mathbb{Q}^2$, *i.e.* for all $\vec{u}, \vec{v} \in \mathbb{Q}^2$ we have $\vec{u} < \vec{v}$ if and only if ($\vec{u}_1 < \vec{v}_1$ or ($\vec{u}_1 = \vec{v}_1$ and $\vec{u}_2 < \vec{v}_2$)). Let (L_a) be a strict Dyck t-path from the points $\vec{x}^w$ to the points $\vec{y}$, and assume that (L_a) has at least one

[4]The *Lindström-Gessel-Viennot lemma.* The argument appears in many (almost) equivalent forms in the literature. We learned and essentially copied it from Stanley's book [30, Thm 2.7.1]. Note however that, strictly speaking, the Theorem 2.7.1 there does not apply as stated at this point in our argument. In the paragraph that follows we show that Stanley's argument may be adapted so that it does apply to our situation.

point of crossing. In particular (L_a) is the datum consisting of, for each index a, a path from $\vec{x}_{w(a)}$ to $\vec{y}_a$. Let $\vec{v} \in \mathbb{Q}^2$ be the point chosen among the points of crossing minimal for the lexicographical order on $\mathbb{Q}^2$. Let (a, b) a couple of different indices, minimal for the lexicographical order on the set of all such couples, such that $\vec{v}$ lies on the path L_a and also on the path L_b. We define a new path L'_a, defined by following the steps of L_b until the point $\vec{v}$ and then following the steps of the path L_a. We define L'_b by following L_a until the point $\vec{v}$ and then continuing the path L_b. For the indices c with $c \neq a, b$ we define $L'_c := L_c$. Observe that (L'_a) is a t-path from the points $\vec{x}^{(ab)w}$ to the points $\vec{y}$. Furthermore, it is a Dyck path (with respect to this *new* configuration of points), and we have $\text{weight}(L_a) = \text{weight}(L'_a)$ because the weight is the product of the weights of the steps, and only the order of the steps has changed in the construction $(L_a) \mapsto (L'_a)$. The construction is self-inverse: If we apply the construction to the path (L'_a) then we re-obtain (L_a). Both paths (L_a) and (L'_a) occur in the sum of Equation (9.3). The sign $\varepsilon_{P_0 \cap M_w}$ is equal to $(-1)^{n-1}(-1)^{t-1}(-1)^{\#\{c \in \{1,2,\ldots,t\} \mid \vec{x}_{w(c)} = \vec{y}_c\}}$. By the assumption that $h \geq t$, the points in the list $\vec{x}$ are all different to the points in the list $\vec{y}$, and therefore the sign $\varepsilon_{P_0 \cap M_w}$ equals $(-1)^{n-t}$ (and does not depend on the permutation w). The sign of the permutation w is opposite to the sign of $(ab)w$. Consequently, the contributions of the paths (L_a) and (L'_a) to Equation (9.3) cancel, and only the non-crossing paths remain in the sum. We find

$$\text{Tr}\,(\chi_c^G f, \pi) = (-1)^{n-t} q^{s(n-s)\alpha/2} \sum_{w \in \mathfrak{S}_t} \text{sign}(w) \cdot \text{Dyck}_s^+(\vec{x}^w, \vec{y}). \tag{9.4}$$

We need a second notion of crossing paths, called *topological intersection*. Here we mean that, when the t-path L is drawn in the plane $\mathbb{Q}^2$ there is a point $\vec{x} \in \mathbb{Q}^2$ lying on two paths L_a, L_b occurring in L. Because we allow rational coordinates, topological intersection is not the same as crossing: It is easy to give an example of a 2-path has one topological intersection point $\vec{x} \in \mathbb{Q}^2$ but the point $\vec{x}$ does not occur in the lists of points $\vec{v}_{1,0}, \vec{v}_{1,1}, \ldots, \vec{v}_{1,r_1}, \vec{v}_{2,0}, \vec{v}_{2,1}, \ldots, \vec{v}_{2,r_2}$ defining the 2-path. Such paths are considered non-crossing under our definition, even though they may have topological intersection points[5].

We claim that there is at most one permutation $w \in \mathfrak{S}_t$ such that the polynomial $\text{Dyck}^+(\vec{x}^w, \vec{y})$ is non-zero, and that this permutation is the one we defined in Definition 9.2. Let $\mathfrak{S}''_t$ be the set of all permutations such that $\text{Dyck}^+(\vec{x}^w, \vec{y}) \neq 0$, and assume that $\mathfrak{S}''_t$ contains an element $w \in \mathfrak{S}''_t$. We first make the following observation:

(Obs) To any point $\vec{v} \in \mathbb{Q}^2$ we associated the invariant $\rho(\vec{v}) = p_2(\vec{v}) \in \mathbb{Q}/\mathbb{Z}$. The horizontal distance between the point $\vec{x}_{w(a)}$ and the point $\vec{y}_a$ is the number n_a^w. The vertical distance is the number $s_a^w = \frac{s}{n} n_a^w \in \mathbb{Q}$. Because $w \in \mathfrak{S}''_t$ there exists a path from the point $\vec{x}_{w(a)}$ to the point

[5]If one uses the wrong, topological notion of intersection, then the proof breaks at 9 lines below Equation (9.3): The constructed 'path' (L'_a) is not a path.

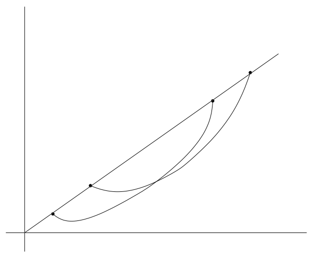

Figure 2. The leftmost point $\vec{x}_t$ is connected to the third point $\vec{y}_{w^{-1}(t)}$, and the second point $\vec{x}_a$ is connected to the last point $\vec{y}_b$. Any 2-path staying below the line ℓ must self-intersect topologically.

$\vec{y}_a$. Consequently s_a^w is integral. This implies that $\rho(\vec{x}_{w(a)}) = \rho(\vec{y}_a)$ for all indices a and in particular the invariant of the point $\vec{x}_{w(a)}$ is independent of $w \in \mathfrak{S}''_t$.

By assumption $\mathrm{Dyck}^+(\vec{x}^w, \vec{y}) \neq 0$ and thus there exists a non-crossing Dyck path (L_a) connecting $\vec{x}^w$ with $\vec{y}$. We show inductively that w is uniquely determined. We start with showing that the index $w^{-1}(t) \in \{1, 2, \ldots, t\}$ is determined. We claim that $w^{-1}(t) \in \{1, 2, \ldots, t\}$ is the minimal index such that the point $\vec{y}_{w^{-1}(t)}$ has the same invariant as $\vec{x}_t$. To see that this claim is true, suppose for a contradiction that it is false, *i.e.* assume the index $w^{-1}(t)$ is *not* minimal. There is an index b strictly smaller than $w^{-1}(t)$ such that $\vec{y}_b$ has the same invariant as $\vec{x}_t$. By the observation (Obs) there exists an index $a \neq t$ such that $\vec{x}_a$ has the same invariant as $\vec{x}_t$ and such that $\vec{x}_a$ is connected to $\vec{y}_b$. Draw a picture (see Figure 2) to see that the paths L_a and L_t must intersect topologically. But, by construction, the invariants of $\vec{x}_a$ and $\vec{x}_t$ are the same. Therefore, any topological intersection point of the paths L_a and L_t is a point of crossing. The paths L_a and L_b are crossing and thus (L_a) is crossing. This is a contradiction, and therefore the claim is true. Thus the value $w^{-1}(t)$ is determined.

We now look at the index $t-1$. The point $\vec{x}_{t-1}$ is connected to the point $\vec{y}_{w^{-1}(t-1)}$. We claim that $w^{-1}(t-1) \in \{1, 2, \ldots, t\}$ is the minimal index, different from $w^{-1}(t)$, such that $\vec{y}_{w^{-1}(t-1)}$ has the same invariant as $\vec{x}_{t-1}$. The proof of this claim is the same as the one we explained for the index t. We may repeat the same argument for the remaining indices $t-2$, $t-3$, etc. Consequently w is uniquely determined by its properties, and equal to the permutation w_0 defined in Definition 9.2.

We proved that if the set $\mathfrak{S}''_t$ is non-empty, then it contains precisely one element, and this element is equal to w_0. Therefore, if the compact trace does not vanish, then m must divide t or m divides h by Lemma 9.3. Inversely, assume that m divides t or m divides h. The permutation $w_0 \in \mathfrak{S}_t$ exists by Lemma 9.3. We claim that $\mathrm{Dyck}_{\mathrm{s}}^+(\vec{x}^{w_0}, \vec{y}) \neq 0$, so that $w_0 \in \mathfrak{S}''_t$. To prove

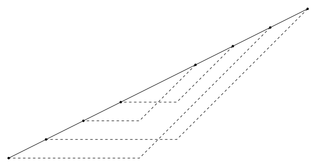

Figure 3. An example of a non-crossing 4-path (L_a) in case $\frac{s}{n} = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$ and $t = 4$. For each a, the path L_a first takes $n_a^{w_0} - s_a^{w_0}$ horizontal steps and then $s_a^{w_0}$ vertical steps. Note that paths with the same invariant do not intersect.

this, it suffices to construct one non-crossing t-path from the points $\vec{x}^{w_0}$ to the points $\vec{y}$. This is easy (see Figure 3): Let a be an index, and write $n_a^{w_0}$ for the horizontal distance between $\vec{x}_a^{w_0}$ and $\vec{y}_a$ and $s_a^{w_0}$ for the vertical distance. The path L_a from $\vec{x}_a^{w_0}$ to $\vec{y}_a$ is defined to be the path taking $n_a^{w_0} - s_a^{w_0}$ eastward steps, and then $s_a^{w_0}$ northeastward steps. Then (L_a) is a strict non-crossing t-path and therefore $\mathrm{Dyck}_{\mathrm{s}}^+(\vec{x}^{w_0}, \vec{y})$ is non-zero. This completes the proof. $\square$

10. A dual formula

We also want to compute the compact trace on the representation $\mathrm{Speh}(h, t)$ in case $h \leq t$. The argument for Theorem 9.4 extends to the case where $h \leq t$. This computation is more complicated, because the permutation $w \in \mathfrak{S}_t$ that contributes to Equation (9.4) is no longer unique and the sign $\varepsilon_{P_0 \cap M_w}$ in Equation (9.3) depends on w. We don't reproduce the computation here, because there is a more elegant approach using the duality of Zelevinsky. The ring of Zelevinsky $\mathcal{R}$ has an involution ι, called the *Zelevinsky involution* which was first defined by Zelevinsky in [33]. Aubert [1] gave a refined definition of this involution via a character formula involving Jacquet modules (which, moreover, makes sense for all reductive groups). Aubert defines the involution by $X^\iota := \sum_{P=MN} \varepsilon_P \mathrm{Ind}_P^G(X_N(\delta_P^{-1/2}))$, for all $X \in \mathcal{R}$, where

- X_N is the non-normalized Jacquet module, *i.e.* to a G-representation X it attaches the space of N-coinvariants X_N (which is an M-representation). The functor $X \mapsto X_N$ is additive, and thus extends to all objects $X \in \mathbb{R}$.
- Ind_P^G is *unitary* induction (sending unitary representations of M to unitary representation of G)
- $\varepsilon_P = (-1)^{\dim(A_P/A_G)}$, where A_P is the split center of P and A_G is the split center of G

(With 'involution' we mean that ι is an automorphism of the complex algebra $\mathcal{R}$ and it is of order two: $\iota^2 = \mathrm{Id}_{\mathcal{R}}$.)

The Zelevinsky dual of a Speh representation with parameters (h, t) is a Speh representation with the role of the parameters inversed, thus of type (t, h). Furthermore, taking the Zelevinsky dual of the formula of Tadic yields a new character formula, now in terms of duals of standard representations. Of course, the Zelevinsky dual of a standard representation is not standard, rather it is an unramified twist of products in $\mathcal{R}$ of one dimensional representations. Therefore, we compute first the compact trace on the one dimensional representations, then use van Dijk's theorem [20, Prop. 3] to obtain formulas for products in $\mathcal{R}$ of one dimensional representations, and finally use the dual of Tadic's formula to compute the compact traces on Speh representations with $h \leq t$ (opposite inequality to Theorem 9.4). We will then have computed the formula for all Speh representations.

11. The trivial representation

We compute the compact traces of spherical Hecke operators acting on the trivial representation of G. In Section 12 we will use this result, and the dual of Tadic's formula, to compute the compact traces of the Kottwitz functions against the representations that were exlcuded in Theorem 9.4.

We work with general unramified reductive groups G. In this paper we need the result only for $G = \mathrm{GL}_n(F)$, but the computation is valid in general. We fix notation, conventions and definitions on roots and convexes for unramified groups (cf. [24, §1] and [22, Chap. 1]):

- Fix $P_0 \subset G$ a Borel subgroup with Levi decomposition $P_0 = TN_0$. Standardize the parabolic subgroups of G and their Levi-decomposition with respect to $P_0 = TN_0$.
- Let P be a standard parabolic subgroup of G, then we write
  - A_P for the split center of P;
  - $\varepsilon_P = (-1)^{\dim(A_P/A_G)}$;
  - $\mathfrak{a}_P := X_*(A_P) \otimes \mathbb{R}$.
- If $P \subset P'$ then we have $A_{P'} \subset A_P$ and thus an induced map $\mathfrak{a}_{P'} \to \mathfrak{a}_P$.
- $\mathfrak{a}_P^{P'}$ is the quotient of $\mathfrak{a}_P$ by $\mathfrak{a}_{P'}$.
- $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^G = \mathfrak{a}_{P_0}^G$.
- Δ is the set of simple roots of A_{P_0} occurring in the Lie algebra of N_0.
- For each root α in Δ we have a coroot $\alpha^\vee$ in $\mathfrak{a}_0^G$.
- We write $\Delta_P \subset \Delta$ for the subset of $\alpha \in \Delta$ acting non-trivially on A_P.
- For $\alpha \in \Delta_P \subset \Delta$ we send the coroot $\alpha^\vee \in \mathfrak{a}_0^G$ to the space $\mathfrak{a}_P^G$ via the canonical surjection $\mathfrak{a}_0^G \to \mathfrak{a}_P^G$. The set of these restricted coroots $\alpha^\vee|_{\mathfrak{a}_P^G}$ with α ranging over Δ_P form a basis of the vector space $\mathfrak{a}_P^G$. By definition the set of *fundamental weights* $\{\varpi_\alpha^G \in \mathfrak{a}_P^{G*} \mid \alpha \in \Delta_P\}$ is the basis of $\mathfrak{a}_P^{G*} = \mathrm{Hom}(\mathfrak{a}_P^G, \mathbb{R})$ dual to the basis $\{\alpha^\vee_{\mathfrak{a}_P^G}\}$ of coroots.
- We have the *acute* and *obtuse* Weyl chambers of G.
  - The *acute chamber* $\mathfrak{a}_P^{G+}$ is the set of $x \in \mathfrak{a}_P^G$ such that $\langle \alpha, x \rangle > 0$ for all roots $\alpha \in \Delta_P$.

  – The *obtuse chamber* ${}^{+}\mathfrak{a}_P^G$ is the set of $x \in \mathfrak{a}_P^G$ such that we have the inequality $\langle \varpi_\alpha^G, x\rangle > 0$ for all fundamental weights ϖ_α^G, associated to $\alpha \in \Delta_P$.
  – We need another chamber ${}^{\leq}\mathfrak{a}_P^G$ defined to be the set of $x \in \mathfrak{a}_P^G$ such that for all $\alpha \in \Delta_P$ the pairing $\langle \varpi_\alpha^G, x\rangle$ is non-positive. We call this chamber the *closed opposite obtuse Weyl chamber*.
- Let ${}^{\leq}\widehat{\tau}_P^G$ be the characteristic function on $\mathfrak{a}_P$ of the closed opposite obtuse chamber.
- Let $H_M \colon M \to \mathfrak{a}_P$ be the *Harish-Chandra* mapping, normalized such that $|\chi(m)|_p = q^{-\langle \chi, H_M(m)\rangle}$ for all rational characters χ of M.
- We define the following characteristic functions on T:
  – $\xi_c^G = {}^{\leq}\widehat{\tau}_{P_0}^G \circ (\mathfrak{a}_{P_0} \to \mathfrak{a}_{P_0}^G) \circ H_{M_0}$;
  – $\chi_N = \tau_P^G \circ (\mathfrak{a}_P \to \mathfrak{a}_P^G) \circ H_M$;
  – $\widehat{\chi}_N = \widehat{\tau}_P^G \circ (\mathfrak{a}_P \to \mathfrak{a}_P^G) \circ H_M$.
- If $f \in \mathcal{H}_0(G)$ is a function whose Satake transform is the function $h \in A$, then we often abuse notation, and write $\xi_c^G h$ for the Satake transform of the function $\xi_c^G f^{(P_0)}$, and similarly for the functions $\chi_N f$ and $\widehat{\chi}_N f$ if $f \in \mathcal{H}_0(M)$ (cf. Section 4).

PROPOSITION 11.1. *Let f be a function in the Hecke algebra $\mathcal{H}_0(G)$. The compact trace* $\mathrm{Tr}\ (\chi_c^G f, 1_G)$ *is equal to* $\mathrm{Tr}\ \left(\xi_c^G f^{(P_0)}, 1_T(\delta_{P_0}^{-1/2})\right)$.

Proof. For comfort we prove the proposition under the additional assumption that G is its own derived group. We have

$$\mathrm{Tr}\ (\chi_c^G f, 1) = \sum_{P=MN} \varepsilon_P \mathrm{Tr}\ (\widehat{\chi}_N f^{(P)}, 1(\delta_P^{-1/2})).$$

Write $\widehat{A}_P$ for the complex dual torus of A_P. We write $\varphi_{M,\rho} \in \widehat{A}_P$ for the Hecke matrix of a representation ρ of M. The Hecke matrix $\varphi_{M,\delta_P^{-1/2}}$ is conjugate in $\widehat{M}$ to the Hecke matrix $\varphi_{T,\delta_P^{-1/2}\delta_{P_0\cap M}^{-1/2}} = \delta_{T,\delta_{P_0}^{-1/2}} \in \widehat{A}_{P_0} \subset \widehat{M}$. The trace on the trivial representation $\mathrm{Tr}\ (\widehat{\chi}_N f^{(P)}, 1(\delta_P^{-1/2}))$ is equal to $\mathcal{S}(\widehat{\chi}_N f^{(P_0)})(\varphi_{T,\delta_{P_0}^{-1/2}})$. Using linearity of the Satake transform we obtain

$$\mathrm{Tr}\ (\chi_c^G f, 1) = \mathcal{S}\left(\sum_{P=MN} \varepsilon_P \widehat{\chi}_N f^{(P_0)}\right)(\varphi_{T,\delta_{P_0}^{-1/2}}).$$

Thus we have to compute the function $\sum_{P=MN} \varepsilon_P \widehat{\chi}_N$ on the group T. By definition, the function $\widehat{\chi}_N$ equals $\widehat{\tau}_P^G \circ H_M$. Let W_M be the rational Weyl group of T in M. Let $t \in T$. Then $H_M(t) = \frac{1}{\# W_M} \sum_{w \in W_M} wH_T(t)$. Thus $\widehat{\chi}_N(t) = 1$ if and only if, for all roots $\alpha \in \Delta_P$ we have $\sum_{w\in W_M} \langle \varpi_\alpha^G, wH_T(t)\rangle > 0$. We have for all $\alpha \in \Delta_P$ the inequality $\langle \varpi_\alpha^G, H_T(t)\rangle > 0$ if and only if we have $\langle \varpi_\alpha^G, wH_T(t)\rangle > 0$ for all $w \in W_M$. Therefore, we have on the group T that $\widehat{\chi}_N = \widehat{\tau}_P^G \circ H_T$. Thus the alternating sum $\sum_{P=MN} \varepsilon_P \widehat{\chi}_N$ is equal to $\left(\sum_{P=MN} \varepsilon_P \widehat{\tau}_P^G\right) \circ H_T$. By inclusion-exclusion we have $\sum_{P=MN} \varepsilon_P \widehat{\tau}_P^G = {}^{\leq}\widehat{\tau}_P^G$. This proves the proposition in case $G = G_{\mathrm{der}}$. (For a group G with $G \neq G_{\mathrm{der}}$, one may repeat the above argument, replacing eaching occurance of A_P with A_P/A_G) □

12. The dual formula

Using the computations from Section 11 we give the dual version of Theorem 9.4.

Lemma 12.1. *Let $T_1 = \langle u_1, v_1\rangle$, $T_2 = \langle u_2, v_2\rangle$, …, $T_h = \langle u_h, v_h\rangle$ be a list of segments and consider the representation $J := (\Delta T_1)^\iota \times (\Delta T_2)^\iota \times \cdots \times (\Delta T_h)^\iota$. Then $\mathrm{Tr}\,(\chi_c^G f_{n\alpha s}, \pi)$ is equal to $q^{s(n-s)\alpha/2}\mathrm{Dyck}(\vec{u}, \vec{v})$, where $\vec{u}_a = \ell(u_a)$ and $\vec{v}_a = \ell(v_a + 1)$ for $a = 1, 2, \ldots, t$.*

Proof. The proof is the same as the proof for Lemma 7.8 (but now use Proposition 11.1). We repeat the argument for verification purposes. Assume first that $h = 1$ and that π is the trivial representation of G. In this case the trace $\mathrm{Tr}\left(\chi_c^G f_{n\alpha s}, \pi\right)$ is equal to $\mathrm{Tr}\left(\xi_c^G f_{n\alpha s}^{(P_0)}, 1_T(\delta_{P_0}^{-1/2})\right)$. To a monomial $X = X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n} \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots X_n^{\pm 1}]$ with $e_i \in \mathbb{Z}$ and $\sum_{i=1}^n e_i = s$ we associate the graph $\mathcal{G}_X$ with points $\vec{v}_0 := \ell((1-n)/2)$, and for $i = 1, 2, \ldots, n$, $\vec{v}_i$ equals $\vec{v}_0 + (i, e_1 + e_2 + \ldots + e_i) \in \mathbb{Q}^2$. We have $\xi_c^G X = X$ if and only if

$$e_1 + e_2 + \cdots + e_i \leq \tfrac{s}{n} i, \tag{12.1}$$

for all indices $i < n$, and $\xi_c^G X = 0$ otherwise. The evaluation of X at the point

$$\left(q^{(n-1)/2}, q^{(n-3)/2}, \ldots, q^{(1-n)/2}\right) \tag{12.2}$$

equals the weight of the graph $\mathcal{G}_X$. The trace of $f_{n\alpha s}$ against the representation $1_T(\delta_{P_0}^{-1/2})$ is equal to the evaluation of $f_{n\alpha s}$ at the point in Equation (12.2) (use Lemma 7.7 but notice that the signs are different). The monomials X occurring $\mathcal{S}(f_{n\alpha s})$ yield paths from the point $\ell((1-n)/2) \in \mathbb{Q}^2$ to the point $\ell((n-1)/2+1)$. The condition in Equation (12.1) is true if and only if the graph $\mathcal{G}_X$ lies (non-strictly) below the line ℓ. Therefore the trace $\mathrm{Tr}\,(\chi_c^G f_{n\alpha s}, 1_G)$ is equal to $q^{s(n-s)/2}\mathrm{Dyck}(\ell((1-n)/2), \ell((n-1)/2+1))$. By twisting with the character $\nu^{-x+(1-n)/2}$ as we did in Lemma 7.10 the trace $\mathrm{Tr}\,\left(\chi_c^G f, (\Delta\langle u, v\rangle)^\iota\right)$ equals $q^{s(n-s)/2}\mathrm{Dyck}(\ell(x), \ell(y+1))$ for all segments $\langle u, v\rangle$. Finally use van Dijk's induction formula for compact traces to find the compact traces on duals of standard representations as stated in the lemma (this argument is the same as the argument in Lemma 8.2). □

Theorem 12.2. *Let π be a Speh representation with parameters h, t with $h \leq t$. Let d be the greatest common divisor of n and s and write m for the quotient n/d. Let $T_a = \langle u_a, v_a\rangle$ be the segments of π^ι. Define the points $\vec{u}_a := \ell(u_a)$ and $\vec{v}_a := \ell(v_a + 1)$. The compact trace $\mathrm{Tr}\,(\chi_c^G f_{n\alpha s}, \pi)$ is non-zero if and only if m divides h or m divides t. Assume that the compact trace is non-zero, then it is equal to $(-1)^{t-1}(-1)^{n-h}\mathrm{sign}(w_0)q^{s(n-s)\alpha/2}\mathrm{Dyck}^+(\vec{u}^{w_0}, \vec{v})$, where the permutation $w_0 \in \mathfrak{S}_h$ is defined in Definition 9.2.*

Proof. Let π^ι be the dual of the representation π. After dualizing the formula of Tadic for π^ι we obtain $\pi = \sum_{w \in \mathfrak{S}_h} \mathrm{sign}(w) I_w^\iota \in \mathcal{R}$. The involution ι on $\mathcal{R}$ commutes with products. Therefore, if $T_1, \dots, T_h$ are the Zelevinsky segments of the dual representation π^ι, then I_w^ι is equal to $(\Delta T_1)^\iota \times (\Delta T_2)^\iota \times \cdots (\Delta T_h)^\iota$. By Lemma 12.1 the trace Tr $(\chi_c^G f_{n\alpha s}, I_w^\iota)$ equals $q^{s(n-s)\alpha/2}\mathrm{Dyck}(\vec{u}^w, \vec{v})$. One may now repeat the proof for Theorem 9.4, but using the dual Tadic formula instead; one only has to interchange t with h and every occurrence of the word "strict Dyck t-path" with "Dyck h-path". □

13. Return to Shimura varieties

At this point most of the original arguments of this paper are completed. In Sections 13 and 14 we shows how the computations from Sections 1–12 combine with arguments from our earlier article [20] to give results on the basic stratum in new cases.

In our article [20] we proved a formula for the basic stratum of certain Shimura varieties associated to unitary groups, subject to a technical condition on the Newton polygon of the basic stratum (that it has no non-trivial integral points). In the previous sections we have completely resolved the combinatorial issues that arise if you remove this condition in case p is totally split in the center of the division algebra. We may now essentially repeat the argument from [20] to obtain the description of the cohomology if there is no condition on the Newton polygon of the basic stratum. A large part of the argument remains the same, that part will only be sketched and we refer to [20] for the details. We will work with a *Kottwitz variety* (cf. [16]). More precisely, we fix the following list of notation and assumptions:

THE SHIMURA DATUM.

(1) Let D be a division algebra over $\mathbb{Q}$; write $n \in \mathbb{Z}_{\geq 0}$ for the positive integer such that $\dim_F(D) = n^2$.
(2) F is the center of D; we assume that F is a CM field of the form $F = \mathcal{K}F^+ \subset \overline{\mathbb{Q}}$, where F^+ is totally real, and $\mathcal{K}/\mathbb{Q}$ is quadratic imaginary.
(3) $*$ is an anti-involution on D inducing complex conjugation on F.
(4) G is the $\mathbb{Q}$-group with $G(R) = \left\{x \in D_R^\times | g^* g \in R^\times\right\}$ for every commutative $\mathbb{Q}$-algebra R.
(5) h is an $\mathbb{R}$-algebra morphism $h \colon \mathbb{C} \to D_\mathbb{R}$ such that $h(z)^* = h(\overline{z})$ for all $z \in \mathbb{C}$.
(6) We assume that the involution $x \mapsto h(i)^{-1} x^* h(i)$ on $D_\mathbb{R}$ is positive.
(7) We write X for the $G(\mathbb{R})$ conjugacy class of the restriction of h to $\mathbb{C}^\times \subset \mathbb{C}$.
(8) $\mu \in X_*(G)$ is the restriction of $h \otimes \mathbb{C} \colon \mathbb{C}^\times \times \mathbb{C}^\times \to G(\mathbb{C})$ to the factor $\mathbb{C}^\times$ of $\mathbb{C}^\times \times \mathbb{C}^\times$ indexed by the identity isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}$.
(9) $E \subset \overline{\mathbb{Q}}$ is the reflex field of the Shimura datum (G, X, h^{-1}).

THE SPECIAL FIBRE AND THE BASIC STRATUM.

(10) p is a prime number where the PEL datum fixed in (1)-(9) is unramified in the sense of [17, §5]; we assume additionally that p is *split* in the extension $\mathcal{K}/\mathbb{Q}$. In fact, for most of Sections 13 and 14 we make the following stronger assumption:

(10b) From Hypothesis 14.4 onwards, we assume that p is split in the extension $F/\mathbb{Q}$.

(11) $K \subset G(\mathbb{A}_{\mathrm{f}})$ is a compact open subgroup, small enough that the moduli problem is representable by a smooth and quasi-projective scheme over $\mathcal{O}_E \otimes \mathbb{Z}_p$, and such that K decomposes as $K^p K_p$ where K^p is a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}}^p)$ and K_p is a hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$.

(12) $S = S_K/\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ is the Shimura variety of level K corresponding to the PEL moduli problem.

(13) $\nu_p\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ is a fixed embedding, $\nu_\infty\colon \overline{\mathbb{Q}} \to \mathbb{C}$ is another fixed embedding, the fields $F, F^+, E, \mathcal{K}$ are all embedded into $\mathbb{C}$.

(14) $\mathfrak{p}$ is the E-prime induced by ν_p.

(15) $\mathbb{F}_q$ is the residue field of E at the prime $\mathfrak{p}$ and $\overline{\mathbb{F}}_q$ is the residue field of $\overline{\mathbb{Q}}$ at ν_p; for every positive integer α, $E_{\mathfrak{p},\alpha} \subset \overline{\mathbb{Q}}_p$ is the unramified extension of $E_{\mathfrak{p}}$ of degree α; $\mathbb{F}_{q^\alpha}$ is the residue field of $E_{\mathfrak{p},\alpha}$.

(16) $\iota\colon B \hookrightarrow S_{K,\overline{\mathbb{F}}_q}$ is the basic stratum [26] (cf. [12, 17, 18, 27]).

Local system.

(17) ξ is an (any) irreducible algebraic representation over $\overline{\mathbb{Q}}$ of $G_{\overline{\mathbb{Q}}}$.

(18) ℓ is a prime number and $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$ together with an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$.

(19) $\mathcal{L}$ is the ℓ-adic local system on $S_K/\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ associated to the representation $\xi \otimes \overline{\mathbb{Q}}_\ell$ of $G_{\overline{\mathbb{Q}}_\ell}$ [17, p. 393].

Test functions and harmonic analysis.

(20) We consider functions of the form $f = f_\infty \otimes f^{\infty p} \otimes f_p$, with f_∞ a function on $G(\mathbb{R})$, $f^{\infty p}$ a function on $G(\mathbb{A}_{\mathrm{f}}^p)$, f_p a function on $G(\mathbb{Q}_p)$, where:

- The function f_∞ at infinity has its stable orbital integrals prescribed by the identities of Kottwitz in [15]. The function f_∞ can be taken to be a scalar multiple of an Euler-poincaré function [16] (cf. [7]). In [16, p. 657, Lem. 3.2]) is explained that the function has the following property: Let π_∞ be an $(\mathfrak{g}, K_\infty)$-module occurring as the component at infinity of an automorphic representation π of G. The trace of f_∞ against π_∞ is equal to the Euler-Poincaré characteristic $\sum_{i=0}^{\infty} N_\infty^{-1}(-1)^i \dim \mathrm{H}^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi)$, where N_∞ is the following constant. Let Π_∞^0 be the L-packet of isomorphism classes of irreducible admissible representations of $G(\mathbb{R})$ having the same central character and infinitesimal characters as the contragredient of ξ. Then N_∞ is equal to the product of the number of representations in Π_0^∞ and the number of connected components (for the real topology) of the group $G(\mathbb{R})/Z(\mathbb{R})$, where Z is the center of G.

- The function $f_p \in \mathcal{H}(G(\mathbb{Q}_p))$ at p is a pointwise product of the form $\chi_c^G f_\alpha$, where
  - χ_c^G is the characteristic function on $G(\mathbb{Q}_p)$ of the subset of compact elements (cf. [6]).
  - The function f_α is the function of Kottwitz [15] associated to μ (cf. [20, Prop. 9]). Since the group G is unramified over $\mathbb{Q}_p$ we fix a smooth reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Q}_p$. Define $\phi_\alpha \in \mathcal{H}_0(G(E_{\mathfrak{p},\alpha}))$ to be the characteristic function of the subset $G_{\mathbb{Z}_p}(\mathcal{O}_{E_{\mathfrak{p},\alpha}})\mu(p^{-1})G_{\mathbb{Z}_p}(\mathcal{O}_{E_{\mathfrak{p},\alpha}})$. By definition, the function $f_\alpha \in \mathcal{H}_0(G(\mathbb{Q}_p))$ is the function obtained from ϕ_α by the base change morphism from $\mathcal{H}_0(G(E_{\mathfrak{p},\alpha}))$ to $\mathcal{H}_0(G(\mathbb{Q}_p))$.
- $f^{\infty p}$ is an arbitrary function in the Hecke algebra $\mathcal{H}(G(\mathbb{A}_{\mathrm{f}}^p))$ which is K^p-biinvariant.

The signatures of the unitary group.

(21) $U \subset G$ is the subgroup of elements with trivial factor of similitudes.

(22) For each infinite F^+-place v, we have signatures p_v, q_v such that $U(\mathbb{R}) \cong \prod_v U(p_v, q_v)$. We define $s_v := \min(p_v, q_v)$.

(23) The embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ induces an action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the set of infinite F^+-places. For each $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbit $\wp$ we define the number $s_\wp := \sum_{v\in\wp} s_v$, and we write $\sigma_\wp$ for the partition $(s_v)_{v\in\wp}$ of the number $s_\wp$.

The space of automorphic forms.

(24) $\mathcal{A}(G) := L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, where ω is the inverse of the central character of ξ.

14. Combining the results

We compute the factors Tr $(\chi_c^G f_\alpha, \pi_p)$ occurring in Theorem 14.3 below. We need to introduce two classes of representations:

Definition 14.1. Consider the general linear group G_n over a non-Archimedean local field. Then a representation π of G_n is called a (semistable) *rigid representation* if it is equal to a product of the form $\prod_{a=1}^k \mathrm{Speh}(x_a, y)(\varepsilon_a) \in \mathcal{R}$, where y is a divisor of n and (x_a) is a composition of n/y, and ε_a are unramified unitary characters.

Definition 14.2. A representation π of the group $G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \prod_{\wp|p} \mathrm{GL}_n(F_\wp^+)$ is called a *rigid* representation if for each F^+-place $\wp$ above p the component $\pi_\wp$ is a (semistable) rigid representation of $\mathrm{GL}_n(F_\wp^+)$ in the previous sense: $\pi_\wp = \prod_{a=1}^k \mathrm{Speh}(x_{\wp,a}, y_\wp)(\varepsilon_{\wp,a}) \in \mathcal{R}$, where two additional conditions hold: (1) $y_\wp = y_{\wp'}$ for all $\wp, \wp'|p$, and (2) the factor of similitudes

$\mathbb{Q}_p^\times$ of $G(\mathbb{Q}_p)$ acts through an unramified character on the space of π. We write $y := y_\wp$ and call the set of data $(x_{\wp,a}, \varepsilon_{\wp,a}, y)$ the *parameters* of π.

We recall a theorem from our previous article [20]:

Theorem 14.3. *Let α be a positive integer. Let $f^{\infty p} \in \mathcal{H}(G(\mathbb{A}_{\mathrm{f}}^p))$ be a K^p-biinvariant function. Assume the conditions (1)-(24) from §5.1. Then*

(14.1)

$$\sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}\ (f^{\infty p} \times \Phi_{\mathfrak{p}}^\alpha, \mathrm{H}^i_{\mathrm{et}}(B_{\overline{\mathbb{F}}_q}, \iota^*\mathcal{L})) = |\mathrm{Ker}^1(\mathbb{Q}, G)| \cdot \sum_{\substack{\pi \subset \mathcal{A}(G) \\ \pi_p \text{ rigid}}} \mathrm{Tr}\ (f, \pi),$$

the sum ranges over those irreducible subspaces $\pi \subset \mathcal{A}(G)$ such that π_p is a rigid representation. In Equation (14.1)*, the group* $\mathrm{Ker}^1(\mathbb{Q}, G)$ *the Hasse invariant of G:*

$$\mathrm{Ker}^1(\mathbb{Q}, G) := \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, G) \to \prod_v \mathrm{H}^1(\mathbb{Q}_v, G)\right),$$

where the product ranges over all places of $\mathbb{Q}$.

Erratum to [20]. I should mention that, unfortunately, I made a few typos in [20] regarding the constant $|\mathrm{Ker}^1(\mathbb{Q}, G)|$. The mistakes are easily corrected and occur only in body of the text (the statement of the main result [20, Thm. 1] is correct). Let me explain how to correct [20]. One should know that to the PEL datum that we fixed in Section 13, one can associate a priori *two* Shimura varieties: (1) The Shimura variety S_1 that represents the PEL moduli problem, and (2) The Shimura variety S_2 which is the canonical model, as in [9]. Kottwitz proves in [17, Sect. 5] that $S_1 \cong \coprod_{\mathrm{Ker}^1(\mathbb{Q},G)} S_2$. In some statements I forgot to multiply the expressions with $|\mathrm{Ker}^1(\mathbb{Q}, G)|$, thus I was counting points in the basic locus of $S_{2,\mathfrak{p}}$, instead of $S_{1,\mathfrak{p}}$. This mistake occurs at the following places of [20]: p. 490 lines 2 and 14, Prop. 10, Eqn. (19) and the equation above it, Eqn. (20), Eqn. (21), proof of Thm. 3, and Cor. 2.

Remark. In Theorem 14.3 we fixed the choice of an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, to compare the left and right hand side of Equation (14.1).

Proof. Write $T(f^{\infty p}, \alpha)$ for the left hand side of Equation (14.1). We had already found in [20, Prop. 10] that $T(f^{\infty p}, \alpha) = |\mathrm{Ker}^1(\mathbb{Q}, G)| \mathrm{Tr}\ (f, \mathcal{A}(G))$, for all sufficiently large integers α. Let $\pi \subset \mathcal{A}(G)$ be an automorphic representation of G contributing to the trace $\mathrm{Tr}\ (f, \mathcal{A}(G))$. In [20, p. 20] we explained that π may be base changed to an automorphic representation $BC(\pi)$ of the algebraic group $\mathcal{K}^\times \times D^\times$, and that, in turn, $BC(\pi)$ may be sent to an automorphic representation $\Pi := JL(BC(\pi))$ of the $\mathbb{Q}$-group $G^+ = \mathcal{K}^\times \times \mathrm{GL}_n(F)$. The representation Π is a *discrete* automorphic representation of the group $G^+(\mathbb{A})$, and Π is semistable at p. Using the Moeglin-Waldspurger classification of the discrete spectrum, a computation now shows that π_p is a rigid representation [20, Thm. 2]. □

The point of this article is to make the traces $\mathrm{Tr}\,(f_p, \pi_p) = \mathrm{Tr}\,(\chi_c^G f_\alpha, \pi_p)$ occurring in Equation (14.1) explicit. We carried out the important arguments in the previous sections, here we now only have to collect the results.

Hypothesis 14.4. From this point onwards, we work under the condition that p is split in the center F of the algebra D. (cf. Condition (10b) in the list of conditions from Section 13.)

Hypothesis 14.4 implies that $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{v \in \mathrm{Hom}(F^+, \overline{\mathbb{Q}})} \mathrm{GL}_n(\mathbb{Q}_p)$. We have by [20, Prop. 9] that

$$f_\alpha = 1_{q^{-\alpha}\mathbb{Z}_p^\times} \otimes \bigotimes_{v \in \mathrm{Hom}(F^+, \mathbb{R})} f_{n\alpha s_v}^{\mathrm{GL}_n(\mathbb{Q}_p)} \in \mathcal{H}_0(G(\mathbb{Q}_p)), \tag{14.2}$$

where the numbers s_v are the signatures of the unitary group (cf. Section 13). Let π_p be a rigid representation of $G(\mathbb{Q}_p)$. Write $\pi_p = \bigotimes_v \pi_{p,v}$, where $\pi_{p,v} = \bigotimes_{a=1}^k \mathrm{Speh}(x_{v,a}, y)(\varepsilon_{v,a})$ and where the factor of similitudes of $G(\mathbb{Q}_p)$ acts through the unramified character ε_s on the space of π_p. We compute

$$\begin{aligned}
&\mathrm{Tr}\,(\chi_c^G f_\alpha, \pi_p) = \\
&= \varepsilon_s(q^{-\alpha}) \prod_{v \in \mathrm{Hom}(F^+, \mathbb{R})} \mathrm{Tr}\,\left(\chi_c^{\mathrm{GL}_n(\mathbb{Q}_p)} f_{n\alpha s_v}, \mathrm{Ind}_{P(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \bigotimes_{a=1}^r \mathrm{Speh}(x_v, y)(\varepsilon_{v,a})\right) \\
&= \varepsilon_s(q^{-\alpha}) \left(\prod_{v \in \mathrm{Hom}(F^+, \mathbb{R})} \prod_{a=1}^r \varepsilon_{v,a}(q^{-s_v \frac{y \cdot x_a}{n} \alpha}) \right) \cdot \\
&\quad \cdot \prod_{v \in \mathrm{Hom}(F^+, \mathbb{R})} \mathrm{Tr}\,\left(\chi_c^{\mathrm{GL}_n(\mathbb{Q}_p)} f_{n\alpha s_v}, \mathrm{Ind}_{P(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \bigotimes_{a=1}^r \mathrm{Speh}(x_{v,a}, y)\right).
\end{aligned}$$

Write $\zeta_\pi^\alpha \in \mathbb{C}$ for the product $\prod_v \prod_a \varepsilon_a(q^{-s_v \frac{y \cdot x_a}{n} \alpha})$. The polynomial

$$\prod_{v \in \mathrm{Hom}(F^+, \mathbb{R})} \mathrm{Tr}\,\left(\chi_c^{\mathrm{GL}_n(\mathbb{Q}_p)} f_{n\alpha s_v}, \mathrm{Ind}_{P(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \bigotimes_{a=1}^r \mathrm{Speh}(x_{v,a}, y)\right) \in \mathbb{C}[q^\alpha], \tag{14.3}$$

is computed in Theorems 9.4 and 12.2 to be a polynomial defined by the weights of certain non-intersecting lattice paths. In particular the polynomial in Equation (14.3) vanishes unless, for all indices a, for all $v \in \mathrm{Hom}(F^+, \mathbb{R})$, the number

$$m_{v,a} := \frac{y \cdot x_{v,a}}{\gcd\left(y \cdot x_{v,a}, \frac{y \cdot x_{v,a}}{n} s_v\right)} = \frac{y \cdot x_{v,a}}{\gcd(n, s_v)}, \tag{14.4}$$

is an integer, and divides either $x_{v,a}$ or y. *Assume* that the compact trace $\mathrm{Tr}\,(\chi_c^G f_\alpha, \pi_p)$ is non-zero so that the divisibility relations at Equation (14.4) are satisfied. The number $\zeta_\pi^\alpha \in \mathbb{C}$ is defined by:

$$\varepsilon_s(q^{-\alpha}) \cdot \prod_{v \in \mathrm{Hom}(F^+, \mathbb{R})} \prod_{a=1}^r \varepsilon_{v,a}\left(q^{-s_v \frac{y \cdot x_a}{n} \alpha}\right) = \zeta_\pi^\alpha, \tag{14.5}$$

(cf. [20, Lem. 9, Eq. (24)]).

Remark. Let us comment on the assumption that p splits completely in F (Hypothesis 14.4). If p splits only in $\mathcal{K}$, then the group $G_{\mathbb{Q}_p}$ is not split, and further combinatorial complications arise. The main issue is that the function of Kottwitz f_α in Equation (14.2) is no longer a tensor product of the simple Kottwitz functions $f_{n\alpha s}$: The function f_α is a tensor product of 'composite' Kottwitz functions. (The *composite Kottwitz functions* $f_{n\alpha\sigma}$ are obtained from partitions σ of s as follows. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_r)$ be a partition of s. Then $f_{n\alpha\sigma} \in \mathcal{H}_0(G)$ is the convolution product $f_{n\alpha\sigma_1} * f_{n\alpha\sigma_2} * \cdots * f_{n\alpha\sigma_r} \in \mathcal{H}_0(G)$.)

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