$\mathcal{L}$-Invariant for Siegel–Hilbert Forms

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Received: January 23, 2015

Communicated by Otmar Venjakob

Abstract. We prove a formula for the Greenberg–Benois $\mathcal{L}$-invariant of the spin, standard and adjoint Galois representations associated with Siegel–Hilbert modular forms. In order to simplify the calculation, we give a new definition of the $\mathcal{L}$-invariant for a Galois representation V of a number field $F \neq \mathbb{Q}$; we also check that it is compatible with Benois' definition for $\mathrm{Ind}_F^{\mathbb{Q}}(V)$.

2000 Mathematics Subject Classification: 11R23, 11F80, 11F46, 11S25

Keywords and Phrases: Iwasawa Theory, L-invariants, p-adic L-functions, p-adic families of automorphic forms

1 Introduction

Since the historical results of Kummer and Kubota–Leopold on congruences for Bernoulli numbers, people have been interested in studying the p-adic variation of special values of L-functions.

More precisely, fix a motive M over $\mathbb{Q}$. We suppose that M is Deligne critical at $s = 0$ and that there exists a Deligne's period $\Omega(M)$ such that $\frac{L(M,0)}{\Omega(M)}$ is algebraic. Fix a prime p and two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Let V be the p-adic realization of M and suppose that V is semistable (à la Fontaine). Thanks to work of Coates and Perrin-Riou, we have precise conjectures on how the special values should behave p-adically; we fix a regular sub-module of V. This corresponds to the choice of a sub-(φ, N)-module of $\mathcal{D}_{\mathrm{st}}(V)$ which is a section of the exponential map

$$\mathcal{D}_{\mathrm{st}}(V) \to t(V) \cong \frac{\mathcal{D}_{\mathrm{st}}(V)}{\mathrm{Fil}^0 \mathcal{D}_{\mathrm{st}}(V)}.$$

Let h be the valuation of the determinant of φ on D. We can state the following conjecture;

Conjecture 1.1. *There exists a formal series $L_p^D(V,T) \in \mathbb{C}_p[[T]]$ which grows as $\log_p^h$ such that for all non-trivial, finite-order characters $\varepsilon : 1 + p\mathbb{Z}_p \to \mu_{p^\infty}$ we have*

$$L_p^D(V, \varepsilon(1+p) - 1) = C_\varepsilon(D)\frac{L(M \otimes \varepsilon, 0)}{\Omega(M)}.$$

Moreover, for $\varepsilon = \mathbf{1}$ we have

$$L_p^D(V, 0) = E(D)\frac{L(M,0)}{\Omega(M)},$$

where $E(D)$ is an explicit product of Euler-type factors depending on D and $(\mathcal{D}_{\mathrm{st}}(V)/D)^{N=0}$.

It may happen that one of the factors of $E(D)$ vanishes and then we say that trivial zeros appear. Since the seminal work of [MTT86], people have been interested in describing the p-adic derivative of $L_p^D(V,(1+p)^s-1)$ when trivial zeros appear.
We suppose for simplicity that $L(M,0)$ is not vanishing. We have the following conjecture by Greenberg and Benois;

Conjecture 1.2. *Let t be the number of vanishing factors of $E(D)$. Then*

- $\mathrm{ord}_{s=0} L_p^D(V,(1+p)^s - 1) = t,$
- $L_p^D(V,0)^* = \mathcal{L}(V^*(1), D^*)E^*(D)\frac{L(M,0)}{\Omega(M)}.$

Here $E^*(D)$ is the product of non-vanishing factors of $E(D)$ and $\mathcal{L}(V^*(1), D^*)$ is a number, defined in purely Galois theoretical terms (see Section 3.1), for the dual Galois representation $V^*(1)$.

The error factor $\mathcal{L}(V,D)$ is quite mysterious. It has been calculated in only few cases for the symmetric square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms by Hida and Harron–Jorza. Unless V is an elliptic curve over $\mathbb{Q}$ with multiplicative reduction at p we can not prove the non-vanishing of $\mathcal{L}(V,D)$.
The aim of this paper is to calculate it in some new cases; let F be a totally real field (we make no assumptions on the ramification at p) and π be an automorphic representation of $\mathrm{GSp}_{2g/F}$ of weight $\underline{k} = (k_\tau)_\tau$, where τ runs through the real embeddings of F and $(k_\tau) = (k_{1,\tau}, \ldots, k_{g,\tau}; k_0)$ (note that k_0 does not depend on τ). We say that π is parallel of weight k, $k \in \mathbb{Z}_{\geq 0}$ if $k_{i,\tau} = k$ for all τ and $i = 1, \ldots, g$ and $k_0 = gk$.
We suppose that it has Iwahoric level at all $\mathfrak{p} \mid p$. We suppose moreover that $\pi_{\mathfrak{p}}$ is either Steinberg (see Definition 4.8) or spherical. We partition consequently the prime ideals of F above p in $S^{\mathrm{Stb}} \cup S^{\mathrm{Sph}}$.
We have conjecturally two Galois representations associated with π, namely

the spinorial one V_{spin} and the standard one V_{sta}. Let V be one of these two representations. We choose for each prime $\mathfrak{p}$ of F dividing p a regular sub module $D_{\mathfrak{p}}$ of $\mathcal{D}_{\mathrm{st}}(V_{|G_{F_{\mathfrak{p}}}})$.

Consider a family of Siegel–Hilbert modular forms as in [Urb11] passing through π. Let us denote by $\beta_{\mathfrak{p}}(\kappa)$ the eigenvalue of the normalized Hecke operators $U_{1,\mathfrak{p}}$ (see Definition 4.9) on this family. Let $S^{\mathrm{Sph},1} = S^{\mathrm{Sph},1}(V,D)$ be the subset of S^{Sph} for which $(\mathcal{D}_{\mathrm{st}}(V_{\mathfrak{p}})/D_{\mathfrak{p}})^{N=0}$ does contain the eigenvalue 1. Conjecturally, it is empty for the spin representation. The eigenvalues 1 always appears in $\mathcal{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ for V the standard representation but it may appear in $D_{\mathfrak{p}}$ (this is already the case for the symmetric square of a modular form).

Let t_{Stb} be the cardinality of S^{Stb} and t_{Sph} be the cardinality of $S^{\mathrm{Sph},1}$. We define $f_{\mathfrak{p}} = [F_{\mathfrak{p}}^{\mathrm{ur}} : \mathbb{Q}_p]$.

THEOREM 1.3. *Let π be as above, of parallel weight k. Let $V = V_{\mathrm{spin}}$ and suppose hypothesis* LGP *of Section 4.2, then the expected number of trivial zeros for $L_p^D(V(k-1),T)$ is t_{Stb} and*

$$\mathcal{L}(V(k-1),D) = \prod_{\mathfrak{p}\in S^{\mathrm{Stb}}} -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p}}(k)}{\mathrm{d}k}_{|k=\underline{k}}.$$

Let $V = V_{\mathrm{std}}$, then the conjectural number of trivial zero for $L_p^D(V,T)$ is $t_{\mathrm{Stb}} + t_{\mathrm{Sph}}$ and

$$\mathcal{L}(V,D) = \mathcal{L}(V,D)^{\mathrm{Sph}} \prod_{\mathfrak{p}\in S^{\mathrm{Stb}}} -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p}}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

where $\mathcal{L}(V,D)^{\mathrm{Sph}}$ is a priori global factor. It is 1 if $t_{\mathrm{Sph}} = 0$.

In Section 4.2 we shall provide also a formula for the $\mathcal{L}$-invariant of $V_{\mathrm{std}}(s)$ $(\min(k-g-1,g-1) \geq s \geq 1)$.

The proof of the theorem is not different from the one of [Ben10, Theorem 2] which in turn is similar to the original one of [GS93].

Let now $g = 2$. Let t be the number of primes above p in F. We consider the $2t$-dimensional eigenvariety for $\mathrm{GSp}_{4/F}$ with variables $k = \{k_{\mathfrak{p},1}, k_{\mathfrak{p},2}\}_{\mathfrak{p}}$(see Section 5) and let us denote by $F_{\mathfrak{p},i}(k)$ $(i = 1,2)$ the first two graded pieces of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{spin}})$. The 10-dimensional Galois representation $\mathrm{Ad}(V_{\mathrm{spin}})$ has a natural regular sub-(φ,N)-module induced by the p-refinement of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{spin}})$ and which we shall denote by D_{Ad}. With this choice of regular sub module, $\mathrm{Ad}(V_{\mathrm{spin}})$ presents $2t$ conjectural trivial zeros. In Section 5 we prove the following theorem;

THEOREM 1.4. *Let π be an automorphic form of weight $\underline{k}$. Suppose that hypothesis* LGP *of Section 4.2 is verified for V_{spin} and the point corresponding* *to π in the eigenvariety $\mathcal{X}'$ (as defined in Section 5) is étale over the weight space. We have then*

$$\mathcal{L}(\mathrm{Ad}(V_{\mathrm{spin}}(\pi)), D_{\mathrm{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \begin{pmatrix} \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,1}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,1}} \\ \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,2}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,2}} \end{pmatrix}_{1 \leq i,j \leq t |_{k=\underline{k}}}.$$

We remark that this theorem is the first to really go beyond GL_2 and its representations Sym^n.

The motivation for Theorem 1.3 lies in a generalization of [Ros15] to Siegel forms. In *loc. cit.* we use Greenberg–Stevens method to prove a formula for the derivative of the symmetric square p-adic L-function and calculate the analytic $\mathcal{L}$-invariant and the same method of proof could possibly be generalized to finite slope Siegel forms thanks to the overconvergent Maß-Shimura operators and overconvergent projectors of Z. Liu's thesis.
With some work, it could also be generalized to totally real field where p is inert, as already done for the symmetric square [Ros13].

We hope to calculate the $\mathcal{L}$-invariant for V_{std} and $\mathrm{Ad}(V_{\mathrm{spin}})$ for more general forms in a future work.

In Section 2 we recall the theory of (φ, Γ)-module over a finite extension of $\mathbb{Q}_p$. It will be used in Section 3 to generalize the definition of the $\mathcal{L}$-invariant à la Greenberg–Benois to Galois representations V over general number field F (note that we do not suppose p split or unramified). This definition does not require one to pass through $\mathrm{Ind}_F^{\mathbb{Q}}(V)$ to calculate the $\mathcal{L}$-invariant which in turn simplifies explicit calculation. We shall check that this definition coincides with Benois' definition for $\mathrm{Ind}_F^{\mathbb{Q}}(V)$.
We prove the above-mentioned theorems in Section 4 and 5, inspired mainly by the methods of [Hid07].

Acknowledgement This paper is part of the author's PhD thesis and we would like to thank J. Tilouine for his constant guidance and attention. We would like to thank A. Jorza for telling us that the study of the $\mathcal{L}$-invariant in the Steinberg case was within reach. We would also like to thank D. Hansen, É. Urban and S. Shah for useful conversations and the anonymous referee for his/her remarks and corrections.
The paper has been written while the author was a PhD Fellow of the Fund for Scientific Research - Flanders, at KU Leuven. Part of it has been written during a visit at Columbia University which the author would like to thank for the excellent working condition. During this work, the author has been supported by a FWO travel grant (V4.260.14N) and an ANR grant (ANR-10-BLANC 0114 ArShiFo).

2 Some results on rank one (φ, Γ)-module

Let L be a finite extension of $\mathbb{Q}_p$. The aim of this section is to recall certain results concerning (φ, Γ)-modules over the Robba ring $\mathcal{R}_L$. Let L_0 be the maximal unramified extension contained in L. Let L_0' be the maximal unramified extension contained in $L_\infty := L(\mu_{p^\infty})$ and $L' = L \cdot L_0'$. Let $e_L := [L(\mu_{p^\infty}) : L_0(\mu_{p^\infty})] = [\Gamma_{\mathbb{Q}_p} : \Gamma_L]$, where $\Gamma_L := \mathrm{Gal}(L_\infty/L)$. We define

$$\begin{aligned} \mathbf{B}^{\dagger,r}_{L,\mathrm{rig}} = & \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi_L^n | a_n \in L_0', \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \right. \\ & \left. \text{is holomorphic on } p^{-\frac{1}{e_L r}} \leq |X|_p < 1 \right\}, \\ \mathbf{B}^{\dagger}_{L,\mathrm{rig}} := & \bigcup_r \mathbf{B}^{\dagger,r}_{L,\mathrm{rig}}, \end{aligned}$$

where π_L is a certain uniformizer coming from the theory of field of norms. Note that $\mathbf{B}^{\dagger}_{L,\mathrm{rig}}$ is classically called the Robba ring of L_0'. For sake of notation, we shall denote write $\mathcal{R}_L := \mathbf{B}^{\dagger}_{L,\mathrm{rig}}$. We hope that this will cause no confusion in what follows.

We have an action of φ on $\mathcal{R}_L$. If $L = L_0$, there is no ambiguity and we have:

$$\varphi(\pi_L) = (1 + \pi_L)^p - 1, \quad \varphi(a_n) = \varphi_{L_0'}(a_n).$$

Otherwise the action on π_L is more complicated.

Similarly, we have a Γ_L-action. If $L = L_0$ we have

$$\gamma(\pi_L) = (1 + \pi_L)^{\chi_{\mathrm{cycl}}(\gamma)} - 1,$$

where χ_{cycl} is the cyclotomic character. If L is ramified we also have an action of Γ_L on the coefficients given by

$$\gamma(a_n) = \sigma_\gamma(a_n)$$

where σ_γ is the image of γ via

$$\Gamma_L \to \Gamma_L/\Gamma_{L'} \stackrel{\cong}{\to} \mathrm{Gal}(L_0'/L_0).$$

If a_n is fixed by φ and Γ_L, then is it in $\mathbb{Q}_p$. We have $\mathrm{rk}_{\mathcal{R}_{\mathbb{Q}_p}} \mathcal{R}_L = [L_\infty : \mathbb{Q}_{p,\infty}]$.

Let $\delta : L^\times \to E^\times$ be a continuous character. Let $\mathcal{R}_L(\delta)$ be the rank one (φ, Γ_L)-module defined as follows; fix a uniformizer ϖ_L of L and write $\delta = \delta_0 \delta_1$ with $\delta_0|_{\mathcal{O}_L^\times} := \delta|_{\mathcal{O}_L^\times}$, $\delta_0(\varpi_L) := 1$ and δ_1 is trivial on $\mathcal{O}_L^\times$ and $\delta_1(\varpi_L) := \delta(\varpi_L)$. As δ_0 is a unitary character, it defines by class field theory a unique one dimensional Galois representation $\tilde{\delta}_0$. Fontaine's theorem on the equivalence of category between (φ, Γ_L)-modules and Galois representations [Fon90] gives

us a one dimensional (φ, Γ_L)-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(\tilde{\delta}_0)$.
We define $\mathcal{R}_L(\delta_1) := \mathcal{R}_L \otimes_{\mathbb{Q}_p} Ee_{\delta_1}$ so that $\varphi^{f_L}(e_{\delta_1}) = \delta_1(\varpi_L)e_{\delta_1}$ (here f_L is the degree of L_0 over $\mathbb{Q}_p$), $\gamma(e_{\delta_1}) = e_{\delta}$ and φ does not act on the E-coefficient. Finally, we define $\mathcal{R}_L(\delta) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(\tilde{\delta}_0) \otimes_{\mathcal{R}_L} \mathcal{R}_L(\delta_1)$.

We now classify the cohomology of such a (φ, Γ_L)-modules. It will be useful to calculate it explicitly in terms of $C_{\varphi,\gamma}$-complexes [Ben11, §1.1.5]. We fix then a generator γ_L of Γ_L; if clear from the context, we shall drop the subscript $_L$ and write simply γ.

PROPOSITION 2.1. *We have $H^0(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ for all τ; in this case we have $H^0(\mathcal{R}_L(\delta)) \cong E$. We shall denote its basis by $t^{-\underline{m}} \otimes e_\delta$, where*

$$t^{-\underline{m}} = (t^{-m_\tau}) \in \prod_\tau B^+_{\mathrm{dR}} \otimes_{L,\tau} E.$$

If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

If $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

Otherwise

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p].$$

We have $H^2(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$; in this case we have $H^2(\mathcal{R}_L(\delta)) \cong E$.

Note that when we choose $t^{-\underline{m}}$ as a basis we are implicitly using the fact that we can embed certain sub-rings of $\mathcal{R}_L$ into B^+_{dR} (see [Ben11, §1.2.1]).

Proof. The same results is stated in [Nak09, Proposition 2.14, 2.15, Lemma 2.16] for $E-B$-pairs, but the proof for (φ, Γ)-modules is the same.
Recall that have a canonical duality [Liu08] given by cup product

$$H^i(D) \times H^{2-i}(D^*(\chi_{\mathrm{cycl}})) \to H^2(\chi_{\mathrm{cycl}}).$$

The last fact is then a direct consequence. □

This allows us to define a canonical basis of $H^2(\mathcal{R}_L(|\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}))$.
We define $H^1_{\mathrm{f}}(D)$ as the H^1 of the complex

$$\mathcal{D}_{\mathrm{cris}}(D) \to t_D \oplus \mathcal{D}_{\mathrm{cris}}(D)$$

and we have immediately [Nak09, Proposition 2.7]

$$\dim_E H^1_{\mathrm{f}}(D) = \dim_E(H^0(D)) + \dim_E t_D. \tag{2.2}$$

Hence

Lemma 2.3. *If* $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ *with* $m_\tau \leq 0$*, then*

$$\dim_E H^1_{\mathrm{f}}(\mathcal{R}_L(\delta)) = 1.$$

If $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ *with* $k_\tau \geq 1$*, then*

$$\dim_E H^1_{\mathrm{f}}(\mathcal{R}_L(\delta)) = d.$$

Proposition 2.4. *Let* D *be a semi-stable* (φ, Γ)*-module over* $\mathcal{R}_L$ *with non-negative Hodge–Tate weight. Suppose that* $\mathcal{D}_{\mathrm{st}}(D) = \mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$*. Then* D *is crystalline,*

$$D \cong \oplus \mathcal{R}_L(\delta_i)$$

with $\delta_i(z) = \prod_\tau \tau(z)^{m_{i,\tau}}$*,* $m_{i,\tau} \leq 0$ *and* $\mathcal{D}_{\mathrm{st}}(D) = \mathcal{D}_{\mathrm{cris}}(D) = H^0(D)$*.*

Proof. We follow closely the proof [Ben11, Proposition 1.5.8]. As $N\varphi = p\varphi N$ we obtain immediately that $N = 0$, hence D is crystalline.
Let r be the rank of D over $\mathcal{R}_L$. We write the Hodge–Tate weight as $(\underline{m_i})_{i=1}^r$ where $\underline{m_i} = (m_{i,\tau})_\tau$.
We prove the proposition by induction; the case $r = 1$ is easy.
If D is not split, for $r = 2$, we can suppose, as D is de Rham, that for each τ we have $-m_{1,\tau} \leq -m_{2,\tau}$, hence $\underline{m_1} = 0$ by twisting. Let δ be defined by $\prod_\tau \tau(z)^{m_\tau}$. So we have an extension of $\mathcal{R}_L(\delta)$ by $\mathcal{R}_L$. Let d_2 be a lift to D of a basis of $\mathcal{R}_L$. As $\varphi = 1$ we have $\varphi d_2 = d_2$. As the extension is crystalline we know that γ acts trivially too, hence the extension splits.
Suppose now $r > 2$. Take v in the $\mathrm{Fil}^{-\underline{m_0}}\mathcal{D}_{\mathrm{st}}(D)$, the smallest filtered piece of $\mathcal{D}_{\mathrm{st}}(D)$. We can associate to it $\mathcal{R}_L(\delta)$, where $\delta(z) = \prod_\tau \tau(z)^{m_{0,\tau}}$. We have

$$0 \to \mathcal{R}_L(\delta) \to D \to D' \to 0.$$

By inductive hypothesis $D' \cong \oplus_{i=1}^{d-1} \mathcal{R}_L(\delta_i)$. We can write

$$\mathrm{Ext}(D', \mathcal{R}_L(\delta)) = \oplus_{i=1}^{d-1} \mathrm{Ext}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta))$$

and we are reduced to the case $r = 2$ which has already been dealt. □

We now want to calculate $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$ for $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$. We recall the following lemma [Ben11, Lemma 1.4.3]

Lemma 2.5. *The extension* $\mathrm{cl}(a, b)$ *in* $H^1(\mathcal{R}_L(\delta))$ *corresponding to the couple* (a, b) *is crystalline if and only if the equation* $(1 - \gamma)x = b$ *has a solution in* $\mathcal{R}_L(\delta)\left[\frac{1}{t}\right]$

The following proposition in an immediate consequence of the above lemma [Ben11, Theorem 1.5.7 (i)] (see also the construction of [Nak09] at page 900)

Proposition 2.6. *Let* e_δ *be a basis for* $\mathcal{R}_L(\delta)$*. Then* $x_{\underline{m}} = \mathrm{cl}(t^{-\underline{m}}, 0)e_\delta$ *is a basis of* $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$*.*

Remark 2.7. *If δ is the trivial character then x_0 corresponds (via class field theory) to the unramified $\mathbb{Z}_p$-extension of* $\mathrm{Hom}(G_L, E^\times) \cong H^1(G_L, E)$.

We now have to cut out another "canonical" one-dimensional subspace in $H^1(\mathcal{R}_L(\delta))$ which trivially intersects $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$ (and reduces to the cyclotomic $\mathbb{Z}_p$-extension in the sense of the previous remark).
We recall that for $L = \mathbb{Q}_p$ Benois has defined in [Ben11, Proposition 1.5.9] a canonical complement $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$ of $H^1_{\mathrm{f}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$ inside $H^1(\mathcal{R}_{\mathbb{Q}_p}(z^m))$. He has also defined a canonical basis y_m of $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$.
We hence define the extension

$$y_{\underline{m}} := \frac{1}{e_L} \log_p(\chi_{\mathrm{cycl}}(\gamma_L))\mathrm{cl}(0, t^{-\underline{m}})e_\delta.$$

When $L = \mathbb{Q}_p$, this is the same element y_m as defined by Benois.
We can calculate cohomology of induced $(\varphi, \Gamma_{\mathbb{Q}_p})$-module. Indeed, we now consider two p-adic fields K and L, L a finite extension of K. The main reference for this part is [Liu08, §2.2]. Let D be a (φ, Γ_L)-module, we define

$$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D) = \{f : \Gamma_K \to D | f(hg) = hf(g) \ \forall h \in \Gamma_L\}.$$

It has rank $[L : K]\mathrm{rk}_{\mathcal{R}_L}(D)$ over $\mathcal{R}_K$; indeed $\mathcal{R}_L$ is a $\mathcal{R}_K$-module of rank $[L : K]/|\Gamma_K/\Gamma_L| = [L'_0 : K'_0]$. (The unramified part of L/K plus the ramified part which is disjoint by K_∞. See after [Liu08, Theorem 2.2].) If D comes from a G_L-representation V we have

$$\mathbf{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{G_L}^{G_K}(V)) = \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)).$$

We have then the equivalent of Shapiro's lemma

$$H^i(D) \cong H^i(\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)).$$

Moreover, the aforementioned duality for (φ, Γ)-modules is compatible with induction [Liu08, Theorem 2.2].
If $D \cong \mathcal{R}_L(\delta)$ is free of rank one, then we have an explicit description of $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$. Let $e_\infty = |\Gamma_K/\Gamma_L|$, we write $\left\{\omega^i\right\}_{i=0}^{e_\infty - 1}$ for $(\Gamma_K/\Gamma_L)^\wedge$. The $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$ is the $\mathcal{R}_L$-span of f_i, where $f_i(g) = \omega^i(g)\delta(\chi_{\mathrm{cycl}}(g))e_\delta$.

We go back to the previous setting where $K = \mathbb{Q}_p$ (hence $e_\infty = e_L$). Suppose $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ and let $D = \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\mathbb{Q}_p}}(\mathcal{R}_L(\delta))$. Note that in this case $\mathcal{D}_{\mathrm{st}}(D) \cong E^{f_L}$ is a filtered φ-module where φ acts as a permutation of length f_L. To $\mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$ corresponds (by Proposition 2.4 over $\mathbb{Q}_p$) a rank-one (φ, Γ)-module $\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})$, for m_0 the minimum of the m_τ's (hence $-m_0$ is the greatest Hodge–Tate weight of D).
The identifications

$$H^0(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})) = \mathcal{D}_{\mathrm{st}}(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))^{\varphi=1} = \mathcal{D}_{\mathrm{st}}(D)^{\varphi=1} = H^0(D) = H^0(\mathcal{R}_L(\delta))$$

induces (via the maps cl(0,) and cl(,0)) an injection

$$H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})) \hookrightarrow H^1(\mathrm{Ind}_L^{\mathbb{Q}_p}(\mathcal{R}_L(\delta))). \tag{2.8}$$

which sends x_{m_0} to $x_{\underline{m}}$ and y_{m_0} to $y_{\underline{m}}$.

We consider a (φ, Γ)-module M which sits in the non-split exact sequence

$$0 \to M_0 := \oplus_{i=1}^r \mathcal{R}_L(\delta_i) \to M \to M_1 := \oplus_{i=1}^r \mathcal{R}_L(\delta_i') \to 0, \tag{2.9}$$

where $\delta_i(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{m_{i,\tau}}$ with $m_{i,\tau} \geq 1$ for all τ and $\delta_i'(z) = \prod_\tau \tau(z)^{k_{i,\tau}}$ with $k_{i,\tau} \leq 0$ for all τ. We say that M is of type $U_{m,k}$ if the image of M in $H^1(M_1)$ is crystalline.

Proposition 2.10. *Suppose that M as above is not of type $U_{m,k}$. Then we have $\dim_E(H^1(M)) = 2[L:\mathbb{Q}_p]r$ and $H^2(M) = H^0(M) = 0$. Moreover, if we write*

$$0 \to H^0(M_1) \stackrel{\Delta_0}{\to} H^1(M_0) \stackrel{f_1}{\to} H^1(M) \stackrel{g_1}{\to} H^1(M_1) \stackrel{\Delta_1}{\to} H^2(M_0) \to 0$$

we have $H^1(M_0) = \mathrm{Im}(\Delta_1) \oplus H^1_{\mathrm{f}}(M_0)$, $\mathrm{Im}(f_1) = H^1_{\mathrm{f}}(M)$ and $H^1(M_1) = \mathrm{Im}(g_1) \oplus H^1_{\mathrm{f}}(M_1)$.

Proof. We have $H^0(M) = 0$ by definition of M. Note that $M^*(\chi_{\mathrm{cycl}})$ is a module of the same type, hence $H^2(M) = H^0(M^*(\chi_{\mathrm{cycl}})) = 0$. We can write

$$0 \to H^0(M_1) \to H^1(M_0) \stackrel{f_1}{\to} H^1(M) \stackrel{g_1}{\to} H^1(M_1) \to H^2(M_0) \to 0$$

and conclude by Proposition 2.1.
Note that $\dim_E H^1_{\mathrm{f}}(M) = rd$ by (2.2).
By hypothesis, we have that $\mathrm{Im}(\Delta_1) \cap H^1_{\mathrm{f}}(M_0) = 0$ and the first statement follows from dimension counting.
The third statement follows from duality.
For the second statement $H^1_{\mathrm{f}}(M_0)$ injects into $H^1_{\mathrm{f}}(M)$. As both have the same dimension, we conclude. ☐

We give the following key lemma for the definition of the $\mathcal{L}$-invariant

Lemma 2.11. *The intersection of $T := \mathrm{Im}(H^1(M))$ and $\mathrm{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})))$ in $\mathrm{Im}(H^1(M_1))$ is one dimensional.*

Proof. The intersection is non-empty as the sum of their dimension is $d+2$ and $\mathrm{Im}(H^1(M_1))$ has dimension $d+1$. We have that $H^1_{\mathrm{f}}(M_1)$ is contained in the image of $H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$ via (2.8) and by the previous proposition the former is not in the image of g_1 and we are done. ☐

In particular, we deduce that T surjects into the image of $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$.

3 $\mathcal{L}$-invariant over number fields

Let F be a number field. We consider a global Galois representation

$$V : G_F \to \mathrm{GL}_n(E)$$

where E is p-adic field. We suppose that it is unramified outside a finite number of places S containing all the p-adic places. We suppose moreover that it is semistable at all places above p (*i.e.* $\mathcal{D}_{\mathrm{st}}(V_{|F_{\mathfrak{p}}})$ is of rank n over $F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$, being $F_{\mathfrak{p}}^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Q}_p$ contained in $F_{\mathfrak{p}}^{\mathrm{ur}}$).
In this section we generalize Greenberg–Benois definition of the $\mathcal{L}$-invariant for V whenever it presents trivial zeros. Note that we do not require p split or unramified in F.
Let t be the number of trivial zeros. The classical definition by Greenberg [Gre94] describes the $\mathcal{L}$-invariant as the "slope" of a certain t-dimension subspace of $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p^t)$ which is a $2t$-dimensional space with a canonical basis given by ord_p and $\log_p$.
In our setting, the main obstacle is that the cohomology of the (φ,Γ)-module $\mathcal{R}_{F_{\mathfrak{p}}}$ is no longer two-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida's work for symmetric powers of Hilbert forms [Hid07], we consider the image of $H^1(\mathcal{R}_{\mathbb{Q}_p})$ inside $H^1(\mathcal{R}_{F_{\mathfrak{p}}})$.
If t denotes the number of expected trivial zeros, we show that we can define, similarly to [Ben11], a t-dimensional subspace of $H^1(G_{F,S}, V)$ whose image in $H^1(\mathcal{R}_{\mathbb{Q}_p})$ has trivial intersection with the crystalline cocycle. This is enough to define the $\mathcal{L}$-invariant; we further check that our definition is compatible with Benois'.

3.1 Definition of the $\mathcal{L}$-invariant

We define local cohomological conditions L_v in order to define a Selmer group; we denote by G_v a fixed decomposition group at v in $G_{F,S}$ and by I_v the inertia. For $v \nmid p$ we define

$$L_v := \mathrm{Ker}\left(H^1(G_v, V) \to H^1(I_v, V)\right).$$

If $v \mid p$ we define

$$L_v := H^1_{\mathrm{f}}(F_v, V) = \mathrm{Ker}(H^1(G_v, V) \to H^1(G_v, V \otimes_E \mathbf{B}_{\mathrm{cris}})).$$

If $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ denotes the (φ,Γ)-module associated with V we also have $L_p = H^1_{\mathrm{f}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. We define then the Bloch-Kato Selmer group

$$H^1_{\mathrm{f}}(V) := \mathrm{Ker}\left(H^1(G_{F,S}, V) \to \prod_{v \in S} \frac{H^1(G_v, V)}{L_v}\right).$$

We make the following additional hypotheses:

C1) $H^1_{\mathrm{f}}(V) = H^1_{\mathrm{f}}(V^*(1)) = 0$,

C2) $H^0(G_{F,S}, V) = H^0(G_{F,S}, V^*(1)) = 0$,

C3) φ on $\mathbf{D}_{\mathrm{st}}(V_{|F_{\mathfrak{p}}})$ is semisimple at $1 \in F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$ and $p^{-1} \in F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$ for all $\mathfrak{p} \mid p$,

C4) $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{|F_{\mathfrak{p}}})$ has no saturated sub-quotient of type $U_{m,k}$ for all $\mathfrak{p} \mid p$.

Note that if V satisfies the previous four conditions, so does $V^*(1)$.
The first two conditions tell us that the Poitou–Tate sequence reduces to

$$H^1(G_{F,S}, V) \cong \bigoplus_{v \in S} \frac{H^1(G_v, V)}{H^1_{\mathrm{f}}(F_v, V)}. \tag{3.1}$$

For each $\mathfrak{p} \mid p$ we denote by $V_{\mathfrak{p}}$ the restriction to $G_{F_{\mathfrak{p}}}$ of V. We choose a regular sub-module $D_{\mathfrak{p}} \subset \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ and define a filtration $(D_{\mathfrak{p},i})$ of $\mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$.

$$D_{\mathfrak{p},i} = \begin{cases} 0 & i = -2, \\ (1 - p^{-1}\varphi) D_{\mathfrak{p}} + N(D_{\mathfrak{p}}^{\varphi=1}) & i = -1, \\ D_{\mathfrak{p}} & i = 0, \\ D_{\mathfrak{p}} + \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})^{\varphi=1} \cap N^{-1}(D_{\mathfrak{p}}^{\varphi=p^{-1}}) & i = 1, \\ \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}}) & i = 2. \end{cases} \tag{3.2}$$

We have that $D_{\mathfrak{p},1}/D_{\mathfrak{p},-1}$ coincides with the eigenvectors of φ on $\mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ of eigenvalue 1 (resp. p^{-1}) and which are in the kernel (resp. in the image) of N. This filtration induces a filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})$. Namely, we pose

$$F_i \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}) \cap (D_{\mathfrak{p},i} \otimes \mathcal{R}_{F_{\mathfrak{p}},\log}[t^{-1}]).$$

We define

$$W_{\mathfrak{p}} := F_1 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}) / F_{-1} \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}).$$

The same proof as [Ben11, Proposition 2.1.7] tells us that we can find a unique decomposition

$$W_{\mathfrak{p}} = W_{\mathfrak{p},0} \bigoplus W_{\mathfrak{p},1} \bigoplus M_{\mathfrak{p}}$$

such that $t_{\mathfrak{p},0} = \dim_E H^0(W_{\mathfrak{p}}^*(1)) = \mathrm{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_{\mathfrak{p},0}$, $t_{\mathfrak{p},1} = \dim_E H^0(W_{\mathfrak{p}}) = \mathrm{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_{\mathfrak{p},1}$ and $M_{\mathfrak{p}}$ sits in a sequence

$$0 \to M_{\mathfrak{p},0} \xrightarrow{f} M_{\mathfrak{p}} \xrightarrow{g} M_{\mathfrak{p},1} \to 0$$

such that $\mathrm{gr}^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})) = W_{\mathfrak{p},0} \oplus M_{\mathfrak{p},0}$ and $\mathrm{gr}^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})) = W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}$. Moreover $M_{\mathfrak{p}}$ is non-split; by construction we have $H^0(M_{\mathfrak{p}}) = H^2(M_{\mathfrak{p}}) = 0$ and if the exact sequence were split we would have $H^0(M_{\mathfrak{p}}) \neq 0$ and $H^2(M_{\mathfrak{p}}) \neq 0$.

We can prove exactly in the same way as [Ben11, Proposition 2.1.7 (i)] that C4 implies $\mathrm{rank}_{\mathcal{R}_{F_\mathfrak{p}}} M_{\mathfrak{p},1} = \mathrm{rank}_{\mathcal{R}_{F_\mathfrak{p}}} M_{\mathfrak{p},0}$.
In order to define the $\mathcal{L}$-invariant we shall follow verbatim Benois' construction. For sake of notation, we write $\mathbf{D}_\mathfrak{p}^\dagger$ for $\mathbf{D}_{\mathrm{rig}}^\dagger(V_\mathfrak{p})$. We obtain from [Ben11, Proposition 1.4.4 (i)]

$$H^1_{\mathrm{f}}(\mathrm{gr}^2(\mathbf{D}_\mathfrak{p}^\dagger)) = H^0(\mathrm{gr}^2(\mathbf{D}_\mathfrak{p}^\dagger)) = 0.$$

We deduce the following isomorphism

$$H^1_{\mathrm{f}}(F_1\mathbf{D}_\mathfrak{p}^\dagger) = H^1_{\mathrm{f}}(\mathbf{D}_\mathfrak{p}^\dagger) = H^1_{\mathrm{f}}(F_\mathfrak{p}, V). \tag{3.3}$$

As the Hodge–Tate weights of $F_{-1}\mathbf{D}_\mathfrak{p}^\dagger$ are < 0, we obtain from [Ben11, Proposition 1.5.3 (i)] and Poiteau–Tate duality $H^2(F_{-1}\mathbf{D}_\mathfrak{p}^\dagger) = 0$. Using the long exact sequence associated with

$$0 \to F_{-1}\mathbf{D}_\mathfrak{p}^\dagger \to F_1\mathbf{D}_\mathfrak{p}^\dagger \to W_\mathfrak{p} \to 0$$

we see that

$$\frac{H^1(W_\mathfrak{p})}{H^1_{\mathrm{f}}(W_\mathfrak{p})} = \frac{H^1(F_{-1}\mathbf{D}_\mathfrak{p}^\dagger)}{H^1_{\mathrm{f}}(F_\mathfrak{p}, V)}.$$

As Greenberg and Benois do, we make the extra assumption that

C5) $W_{\mathfrak{p},0} = 0$ for all $\mathfrak{p} \mid p$.

Using Proposition 2.4 we can write $\mathrm{gr}^1(\mathbf{D}_\mathfrak{p}^\dagger) = \oplus_{i=1}^{t_{\mathfrak{p},1}+r_\mathfrak{p}} \mathcal{R}_{F_\mathfrak{p}}(\prod_{\tau_\mathfrak{p}} \tau_\mathfrak{p}(z)^{m_{i,\tau_\mathfrak{p}}})$.
We define the $2(t_{\mathfrak{p},1} + r_\mathfrak{p})$-dimensional subspace obtained as the image of

$$\mathrm{Ind}_\mathfrak{p} := \left\{ \sum_{i=0}^{t_{\mathfrak{p},1}+r_\mathfrak{p}} Ex_{\underline{m_i}} + Ey_{\underline{m_i}} \right\} \subset H^1(\mathrm{gr}^1(\mathbf{D}_\mathfrak{p}^\dagger)). \tag{3.4}$$

We define

$$T_\mathfrak{p} = (H^1(F_1\mathbf{D}_\mathfrak{p}^\dagger) \cap \mathrm{Ind}_\mathfrak{p}) / H^1_{\mathrm{f}}(F_\mathfrak{p}, V).$$

It has dimension $t_{\mathfrak{p},1} + r_\mathfrak{p}$.
Write $t = \sum_\mathfrak{p} t_{\mathfrak{p},1} + r_\mathfrak{p}$. We have a unique t-dimensional subspace $H^1(D, V)$ of $H^1(G_{F,S}, V)$ projecting via (3.1) to $\oplus_\mathfrak{p} T_\mathfrak{p}$. We have an isomorphism (cfr. [Ben11, Proposition 1.5.9])

$$\mathrm{Ind}_\mathfrak{p} \cong \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}) \oplus \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}) \cong E^{t_{\mathfrak{p},1}+r_\mathfrak{p}} \oplus E^{t_{\mathfrak{p},1}+r_\mathfrak{p}},$$

where the first (resp. the second) factor is identified with $E^{t_{\mathfrak{p},1}+r_\mathfrak{p}}$ via the basis $\left\{ x_{\underline{m_i}} \right\}$ (resp. $\left\{ y_{\underline{m_i}} \right\}$). We shall denote the two projections by $\iota_{\mathrm{f},\mathfrak{p}}$ and $\iota_{\mathrm{c},\mathfrak{p}}$.
We denote by ι_{f} (resp. ι_{c}) the projection of $H^1(D, V)$ to E^t via $\oplus \iota_{\mathrm{f},\mathfrak{p}}$ (resp. $\oplus \iota_{\mathrm{c},\mathfrak{p}}$). By the remark after Lemma 2.11 and the definition of $T_\mathfrak{p}$, we have that ι_{c} is surjective.
Summing up, we can give the following definition;

Definition 3.5. *The $\mathcal{L}$-invariant of the pair (V, D) is*

$$\mathcal{L}(V, D) := \det(\iota_{\mathrm{f}} \circ \iota_{\mathrm{c}}^{-1}),$$

where the determinant is calculated w.r.t. the basis $(x_{\underline{m_i}}, y_{\underline{m_j}})_{1 \leq i,j \leq t}$.

Remark 3.6. *There is no a priori reason for which $\mathcal{L}(V, D)$ should be non-zero.*

In the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ we see from the description of $H^1(F_1\mathbf{D}^{\dagger}_{\mathfrak{p}})$ that the space $T_{\mathfrak{p}}$ depends only on $V_{|F_{\mathfrak{p}}}$ exactly as in the classical case.

3.2 Comparison with Benois' definition

Fix a global field F and let $\{\mathfrak{p}\}$ be the set of primes above p.
Let G_p denote a fixed decomposition group at p in $G_{\mathbb{Q}}$ and let $\mathfrak{p}_0$ be the corresponding place of F. Let $G_{\mathfrak{p}_0,F}$ be the decomposition group at $\mathfrak{p}_0$ in G_F. For each other place $\mathfrak{p}$ above p in F, we have $G_{\mathfrak{p}} = \tau_{\mathfrak{p}} G_p \tau_{\mathfrak{p}}^{-1}$. We shall denote by $G_{\mathfrak{p},F}$ the corresponding decomposition group in G_F. Consider a p-adic Galois representation

$$V : G_F \to \mathrm{GL}_n(E).$$

We shall suppose E big enough to contain the Galois closure of $F_{\mathfrak{p}}$, for all $\mathfrak{p}$. As before, we suppose V semistable at all primes above p. We have then

$$\mathrm{Ind}_F^{\mathbb{Q}}(V)_{|G_p} \cong \bigoplus_{\mathfrak{p}} \tau_{\mathfrak{p}}^{-1} \mathrm{Ind}_{G_{\mathfrak{p},F}}^{G_{\mathfrak{p}}} V_{|G_{\mathfrak{p},F}}$$

where $\tau_{\mathfrak{p}} \in G_p \setminus \mathrm{Hom}(F, \overline{\mathbb{Q}})$.
Consider the (φ, Γ)-module

$$\mathbf{D}^{\dagger} := \mathbf{D}^{\dagger}_{\mathrm{rig}}\left(\mathrm{Ind}_F^{\mathbb{Q}} V\right).$$

We let D be the regular (φ, N)-module of $\mathcal{D}_{\mathrm{st}}(\mathbf{D}^{\dagger})$ induced by $\{D_{\mathfrak{p}}\}_{\mathfrak{p}}$. As before we have a filtration $(F_i\mathbf{D}^{\dagger})$ on $\mathbf{D}^{\dagger}$ induced by the filtration on D. We denote by W the quotient $F_1\mathbf{D}^{\dagger}/F_{-1}\mathbf{D}^{\dagger}$. Note that it is semistable. We write $W = W_0 \oplus M \oplus W_1$. We suppose that V satisfies the hypotheses C1-C5 of the previous section.

Lemma 3.7. *Let M be as in (2.9). We have*

$$0 \to \mathrm{Ind}(M_0) \to \mathrm{Ind}(M) \to \mathrm{Ind}(M_1) \to 0.$$

We can now compare our definition of $\mathcal{L}$-invariant with Benois'.

Proposition 3.8. *We have a commutative diagram*

$$\begin{array}{ccccc} H^1(G_{\mathbb{Q},S},\mathrm{Ind}(V)) & \leftarrow & H^1(\mathrm{Ind}(V),\mathrm{Ind}(D)) & \xrightarrow{\mathrm{Res}_p} & \frac{H^1(F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)))}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}(V))} = \frac{H^1(F_{-1}\mathbf{D}^\dagger)}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}(V))}. \\ \downarrow & & \downarrow & & \downarrow \iota_p \\ H^1(G_{F,S},V) & \longleftarrow & H^1(V,D) & \xrightarrow{\oplus_{\mathfrak{p}}\mathrm{Res}_{\mathfrak{p}}} & \prod_{\mathfrak{p}} T_{\mathfrak{p}} \end{array}$$

whose vertical arrows are isomorphism.

Proof. We follow [Hid06, §3.4.4]. Recall that we wrote $\mathbf{D}^\dagger_{\mathfrak{p}}$ for $\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}})$. Shapiro's lemma tells us that

$$\frac{H^1(G_p,\mathrm{Ind}_F^{\mathbb{Q}}V)}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}_F^{\mathbb{Q}}V)} \stackrel{\iota_p}{\cong} \bigoplus_{\mathfrak{p}} \frac{H^1(\mathbf{D}^\dagger_{\mathfrak{p}})}{H^1_{\mathrm{f}}(\mathbf{D}^\dagger_{\mathfrak{p}})}.$$

We are left to show that $H^1(F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)))$ is sent by ι_p into $(H^1(F_1\mathbf{D}^\dagger_{\mathfrak{p}})\cap\mathrm{Inv}_{\mathfrak{p}})$ and we shall conclude by dimension counting.
We have then an injection

$$F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)) \hookrightarrow \oplus_{\mathfrak{p}}\mathrm{Ind}(F_1(\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}}))).$$

Then clearly the image of ι_p lands in $H^1(F_1\mathbf{D}^\dagger_{\mathfrak{p}})$. But we have also the injection

$$\mathrm{gr}^1(\mathbf{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}V)) \hookrightarrow \oplus_{\mathfrak{p}}\mathrm{Ind}(\mathrm{gr}^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}})))$$

which by (2.8) tells us that the image of ι_p lands in $\mathrm{Inv}_{\mathfrak{p}}$ and we are done. □

Corollary 3.9. *We have* $\mathcal{L}(V,D) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(V),\mathrm{Ind}_F^{\mathbb{Q}}(D))$.

4 Siegel–Hilbert modular forms, the local case

The calculation of the $\mathcal{L}$-invariant requires to produce explicit cocycles in $H^1(D,V)$; when V appears in $\mathrm{Ad}(V')$ for a certain representation V' we can sometimes use the method of Mazur and Tilouine [MT90] to produce these cocycles. This has been done in many case for the symmetric square [Hid04, Mok12] and generalized to symmetric powers of the Galois representation associated with Hilbert modular forms in [Hid07, HJ13]. The main limit of this approach is that for most representations V it is computationally heavy to obtain V as the quotient of an adjoint representation.
In the case $\mathbf{D}^\dagger_{\mathrm{rig}}(V) = W = M$ the situation is way simpler; if $t = 1$ it has been proved in [Ben10] that to produce the cocycle in $H^1(V,D)$ it is enough to find deformations of $V|_{\mathbb{Q}_p}$.
We shall generalized the method of Benois to our situation in the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. This will allow us to give a complete formula for the $\mathcal{L}$-invariant of the Galois representations associated with a Siegel–Hilbert modular form which is Steinberg at all primes above p.

4.1 The case $t_{\mathfrak{p}} = r_{\mathfrak{p}} = 1$

We now suppose that $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. For sake of notation, in this section we shall drop the index $\mathfrak{p}$. In particular, in this subsection $F = F_{\mathfrak{p}}$. All that we have to do is to check that the calculation of [Ben11, Theorem 2] works in our setting.
We write as before

$$0 \to M_0 \to M \to M_1 \to 0$$

and, only in this subsection, we shall write δ for the character defining M_0 and ψ for the character defining M_1. We suppose $\delta = \delta' \circ \mathrm{N}_{F/\mathbb{Q}_p}$ for $\delta'(z) = |z|_p z^k$ with $k \geq 1$ and $\psi = \psi' \circ \mathrm{N}_{F/\mathbb{Q}_p}$ with $\psi'(z) = z^m$ with $m \leq 0$. We consider an infinitesimal deformation

$$0 \to M_{0,A} \to M_A \to M_{1,A} \to 0,$$

over $A = E[T]/(T^2)$. We suppose that $M_{0,A}$ (resp. $M_{1,A}$) is an infinitesimal deformation of M_0 (resp. M_1) which still factors through $\mathrm{N}_{F/\mathbb{Q}_p}$.
We shall write δ_A, δ'_A, ψ_A and ψ'_A for the corresponding one-dimensional character.

Theorem 4.1. *Suppose that* $\mathrm{d}\log_p(\delta'_A {\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})) \neq 0$*; then*

$$\mathcal{L}(M, M_0) = -\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))\frac{f^{-1}\mathrm{d}\log_p(\delta_A\psi_A^{-1})(\varpi)}{\mathrm{d}\log_p(\delta'_A{\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))}.$$

Proof. Recall the definition of Ind in (3.4). We have a vector $v = ax_m + by_m$ in $H^1(F_1\mathbf{D}^{\dagger}) \cap \mathrm{Ind}$. By definition $\mathcal{L}(M) = ab^{-1}$. The extension $M_{j,A}$ provides us with connecting morphisms $B_j^i : H^i(M_j) \to H^{i+1}(M_j)$. We have by definition

$$\begin{aligned} B_1^0(t^{-m}e_m) =& \mathrm{cl}(\mathrm{dlog}(\psi'_A)(p)t^{-m}e_m, \mathrm{dlog}(\psi'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))t^{-m}e_m) \\ =& \mathrm{dlog}(\delta'_A)(p)x_m + \mathrm{dlog}(\delta'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))y_m. \end{aligned} \tag{4.2}$$

As in [Ben10, §3.2] we consider the dual extension

$$0 \to M_1^*(\chi_{\mathrm{cycl}}) \to M^*(\chi_{\mathrm{cycl}}) \to M_0^*(\chi_{\mathrm{cycl}}) \to 0,$$

and we shall denote with a * the corresponding map in the long exact sequence of cohomology.
We have hence $\ker(\Delta_1) \perp \mathrm{Im}(\Delta_0^*)$ under duality, and a map

$$H^1(M_1^*) \to H^1(\mathcal{R}_{\mathbb{Q}_p}(|z|z^{1-m})).$$

By duality again, we deduce that the image of Δ_0^* inside the target of the above arrow is

$$a\alpha_{1-m} + b\beta_{1-m},$$

where α_{1-m} (resp. β_{1-m}) is the dual of x_m (resp. y_m) as in [Ben10, Proposition 1.1.5].
We now consider the map

$$B_1^{1*} : H^1(M_1^*(\chi_{\mathrm{cycl}})) \to H^2(M_1^*(\chi_{\mathrm{cycl}})) = H^2(\mathcal{R}_{\mathbb{Q}_p}(|z|z^m)) \cong E.$$

We can use [Ben10, Proposition 2.4] to see that after the above identification of H^2 with E we have

$$B_1^{1*}(\alpha_{1-m}) = c\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))^{-1}\mathrm{d}\log_p({\psi'_A}^{-1}(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})), \tag{4.3}$$

$$B_1^{1*}(\beta_{1-m}) = c\mathrm{d}\log_p({\psi'_A}^{-1}(p)), \tag{4.4}$$

where $c \in E^\times$. We consider the following anti-commutative diagram

$$\begin{array}{ccc} H^0(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_0^*} & H^1(M_1^*(\chi_{\mathrm{cycl}})) \\ \downarrow{\scriptstyle B_0^{1*}} & & \downarrow{\scriptstyle B_1^{1*}} \\ H^1(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_1^*} & H^2(M_1^*(\chi_{\mathrm{cycl}})) \end{array}$$

which means

$$B_1^{1*}\Delta_0^* = -\Delta_1^* B_0^{1*}.$$

We calculate this identity on $t^{1-k}e_{1-k}$. Applying (4.3) and (4.4) to ${\psi'_A}^{-1}\chi_{\mathrm{cycl}}$, (4.2) to ${\delta'_A}^{-1}\chi_{\mathrm{cycl}}$ and using [Ben10, (3.6)] which says

$$\Delta_1^* B_0^{1*}(t^{1-k}) = c\left(b\mathrm{d}\log_p(\delta'_A)(p) + a\mathrm{d}\log_p(\delta'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))\right)$$

we get

$$b^{-1}a = -\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))\frac{\mathrm{d}\log_p(\delta'_A{\psi'_A}^{-1})(p)}{\mathrm{d}\log_p(\delta'_A{\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))}.$$

We conclude as $\delta'_A(p)^f = \delta_A(\varpi)$. □

Remark 4.5. *In particular, this theorem proves that this definition of $\mathcal{L}$-invariant is compatible with the Fontaine-Mazur one [Pot14, Zha14].*

4.2 Calculation of the $\mathcal{L}$-invariant for Steinberg forms

We fix a totally real field F. Let I be the set of real embeddings. Fix two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

as before. We partition $I = \sqcup_{\mathfrak{p}} I_{\mathfrak{p}}$ according to the p-adic place which each embedding induces. We shall denote by $q_{\mathfrak{p}} = p^{f_{\mathfrak{p}}}$ the residual cardinality for each prime ideal $\mathfrak{p}$. We consider an irreducible representation π of $\mathrm{GSp}_{2g/F}$, algebraic of weight $k = (k_\tau)_\tau$, where $(k_\tau) = (k_{\tau,1}, \ldots, k_{\tau,g}; k_0)$ (k_0 is a parallel weight for $\mathrm{Res}_F^{\mathbb{Q}}(\mathbb{G}_m)$) with $k_{\tau,1} \leq k_{\tau,2} \ldots \leq k_{\tau,g}$. If $k_{\tau,1} \geq g+1$ for all τ, then the weight is cohomological. The cohomological weight of π is then

$$(\mu_\tau)_\tau = (k_\tau)_\tau - (g+1, \ldots, g+1; 0)_\tau.$$

For parallel weights k, we shall choose $k_0 = gk$.
We now describe the conjectural Galois representation associated with π. We have a spin Galois representation V_{spin} (whose image is contained in GL_{2^g}) and a standard Galois representation V_{sta} (whose image is contained in GL_{2g+1}) given respectively by the spinorial and the standard representation of $\mathrm{GSpin}_{2g+1} = {}^L\mathrm{GSp}_{2g}$.
Thanks to the work of Scholze [Sch15] we now dispose of the standard Galois representation (see for example [HJ13, Theorem 18]). We also know the existence of the spin representation in many cases [KS14].
We now recall some expected properties of these Galois representations. Our main reference is [HJ13, §3.3]. We will make the following assumption on π at p;

for each $\mathfrak{p} \mid p$ either $\pi_{\mathfrak{p}}$ is spherical or Steinberg.

We explain what we mean by Steinberg. Consider the Satake parameters at $\mathfrak{p}$, normalized as in [BS00, Corollary 3.2], $(\alpha_{\mathfrak{p},1}, \ldots, \alpha_{\mathfrak{p},g})$. We have the following theorem on Iwahori-spherical representation of $\mathrm{GSp}_{2g}(F_{\mathfrak{p}})$ [Tad94, Theorem 7.9].

Theorem 4.6. *Let $\alpha_1, \ldots, \alpha_g, \alpha$ be $g+1$ character of $F_{\mathfrak{p}}^\times$. Let $B_{\mathrm{GSp}_{2g}}$ be the Borel subgroup of $\mathrm{Sp}_{2g}(F_{\mathfrak{p}})$. Then $\mathrm{Ind}_{B_{\mathrm{GSp}_{2g}}}^{\mathrm{GSp}_{2g}(F_{\mathfrak{p}})}(\alpha_1 \times \cdots \times \alpha_g \rtimes \alpha)$ is not irreducible if and only if one of the following conditions is satisfied:*

i) There exist at least three indexes i such that α_i has exact order two and the α_i's are mutually distinct;

ii) There exists i such that $\alpha_i = {|\mathrm{N}(\ )|_{\mathfrak{p}}}^{\pm 1}$;

iii) There exist i and j such that $\alpha_i = {|\mathrm{N}(\ )|_{\mathfrak{p}}}^{\pm 1}{\alpha_j}^{\pm 1}$.

Remark 4.7. *As shown in [HJ13, Lemma 19], such a points are contained in a proper subset of the Hecke eigenvariety for GSp_{2g}.*

Definition 4.8. *We say that $\pi_{\mathfrak{p}}$ is Steinberg if $\alpha_i = |\mathrm{N}(\ )|_{\mathfrak{p}}^{i-1}\alpha_1$.*

If $\pi_{\mathfrak{p}}$ is Steinberg at p, then $\alpha_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}) = q_{\mathfrak{p}}^i \alpha_{\mathfrak{p},1}(\varpi_{\mathfrak{p}})$.
Trivial zeros appear also for automorphic forms which are only partially Steinberg at $\mathfrak{p}$ and can be dealt exactly at the same way as the parallel one but for the sake of notation we prefer not to deal with them.

To each $g+1$ non-zero elements $(t_1, \dots, t_g; t_0) \in (A^\times)^{g+1}$ we associate the diagonal matrix

$$u(t_1, \dots, t_g; t_0) := (t_1, \dots, t_g, t_0 t_g^{-1}, \dots, t_0 t_1^{-1})$$

of $\mathrm{GSp}_{2g}(A)$.
For $1 \leq i \leq g-1$ we denote by $u_{\mathfrak{p},i}$ the diagonal matrix associated with $(1, \dots, 1, \varpi_{\mathfrak{p}}^{-1}, \dots, \varpi_{\mathfrak{p}}^{-1}; \varpi_{\mathfrak{p}}^{-2})$, where $\varpi_{\mathfrak{p}}$ appears i times; we also denote by $u_{\mathfrak{p},0}$ the diagonal matrix corresponding to $(1, \dots, 1; \varpi_{\mathfrak{p}}^{-1})$.

DEFINITION 4.9. *The Hecke operators* $U_{\mathfrak{p},i}$*, for* $1 \leq i \leq g$ *are defined as the double coset operator* $[\mathrm{Iw} u_{\mathfrak{p},g-i} \mathrm{Iw}]$.

We have that $U_{\mathfrak{p},g}$ is the "classical" U_p operator [BS00, §0]. We shall say then that π is of finite slope for $U_{\mathfrak{p},g}$ if $U_{\mathfrak{p},g}$ has eigenvalue $\alpha_{\mathfrak{p},0} \neq 0$ on $\pi_{\mathfrak{p}}$.

We are interested to study the possible p-stabilization of π (*i.e.* Iwahori fixed vectors). If $\pi_{\mathfrak{p}}$ is unramified at $\mathfrak{p}$, we have then $2^g g!$ choices (see [HJ13, Lemma 16] or [BS00, Proposition 9.1]). If $\pi_{\mathfrak{p}}$ is Steinberg, we have instead only one possible choice, as the monodromy N has maximal rank.

Suppose that we can lift π to an automorphic representation $\pi^{(2^g)}$ of GL_{2^g}. We suppose also that we can lift π to an automorphic representation $\pi^{(2g+1)}$ of GL_{2g+1}.
Let $V = V_{\mathrm{spin}}$ (resp. V_{sta}) be the Galois representation associated with $\pi^{(2^g)}$ (resp. $\pi^{(2g+1)}$). We make the following assumption

LGP) V is semistable at all $\mathfrak{p} \mid p$ and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for f as above. Arthur's transfer from GSp_{2g} to GL_{2g+1} has been proven in [Xu] (note that it is now unconditional [MW]) and for $V = V_{\mathrm{sta}}$ this hypothesis is then verified thanks to [Car14, Theorem 1.1]. These hypotheses are also satisfied in many cases for $V = V_{\mathrm{spin}}$ in genus 2 (see [AS06, PSS14]).
Roughly speaking, we require that

$$\mathrm{WD}(V_{|F_{\mathfrak{p}}})^{\mathrm{ss}} \cong \iota_n^{-1} \pi_{\mathfrak{p}}^{(n)},$$

where $\mathrm{WD}(V_{|F_{\mathfrak{p}}})$ is the Weil-Deligne representation associated with $V_{|F_{\mathfrak{p}}}$ à la Berger, $\pi_{\mathfrak{p}}^{(n)}$ is the component at $\mathfrak{p}$ of $\pi^{(n)}$, and ι_n is the local Langlands correspondence for $\mathrm{GL}_n(F_{\mathfrak{p}})$ geometrically normalized ($n = 2g+1$ when V is the standard representation and $n = 2^g$ when V is the spinorial representation).
When $\pi_{\mathfrak{p}}$ is an irreducible quotient of $\mathrm{Ind}_B^{\mathrm{GSp}_{2g}}(\alpha_{\mathfrak{p},1} \otimes \cdots \otimes \alpha_{\mathfrak{p},g})$ we have that

the Frobenius eigenvalues on $\mathrm{WD}(V_{\mathrm{spin}_{|F_{\mathfrak{p}}}})^{\mathrm{ss}}$ are the 2^g numbers

$$\left(\alpha_{\mathfrak{p},0} \prod_{\substack{0\leq r\leq g\\ 1\leq i_1<\ldots<i_r\leq g}} \alpha_{\mathfrak{p},i_1}(\varpi_{\mathfrak{p}})\cdots\alpha_{\mathfrak{p},i_r}(\varpi_{\mathfrak{p}})\right).$$

The ones on $\mathrm{WD}(V_{\mathrm{sta}|F_{\mathfrak{p}}})^{\mathrm{ss}}$ are

$$\left(\alpha_{\mathfrak{p},g}^{-1}(\varpi_{\mathfrak{p}}),\ldots,\alpha_{\mathfrak{p},1}^{-1}(\varpi_{\mathfrak{p}}),1,\alpha_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}),\ldots,\alpha_{\mathfrak{p},g}(\varpi_{\mathfrak{p}})\right).$$

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel) if we are Steinberg or be trivial otherwise. (This is also a consequence of the weight-monodromy conjecture for V.)
Let $\mathfrak{p}$ be a p-adic place of F and let τ be a complex place in $I_{\mathfrak{p}}$. The Hodge–Tate weights of $V_{\mathrm{spin}_{|F_{\mathfrak{p}}}}$ at τ are then

$$\left(\frac{k_0}{2}+\frac{1}{2}\sum_{i=1}^{g}\varepsilon(i)(k_{i,\tau}-i)\right)_{\varepsilon},$$

where ε ranges among the 2^g maps from $\{1,\ldots,g\}$ to $\{\pm 1\}$.
The one of $V_{\mathrm{sta}_{|F_{\mathfrak{p}}}}$ are $(1-k_{\tau,g},\ldots,g-k_{\tau,1},0,k_{\tau,1}-g,\ldots,k_{\tau,g}-1)$.
Thanks to work of Tilouine-Urban [TU99], Urban [Urb11], Andreatta-Iovita-Pilloni [AIP15] we have families of Siegel modular forms;

Theorem 4.10. *Let* $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times \times ((\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times)^g, \mathbb{C}_p^\times\right)$ *be the weight space. There exist an affinoid neighborhood* $\mathcal{U}$ *of* $\kappa_0 = \left((z,(z_i)_{i=1}^g)\mapsto z^{k_0}\prod_{\tau\in I}\prod_i \tau(z_i)^{k_{i,\tau}}\right)$ *in* $\mathcal{W}$, *an equidimensional rigid variety* $\mathcal{X}=\mathcal{X}_\pi$ *of dimension* $dg+1$, *a finite surjective map* $w:\mathcal{X}\to U$, *a character* $\Theta:\mathcal{H}^{Np}\to\mathcal{O}(\mathcal{X})$, *and a point* x *in* $\mathcal{X}$ *above* $\underline{k}$ *such that* $x\circ\Theta$ *corresponds to the Hecke eigensystem of* π.
Moreover, there exists a dense set of points x *of* $\mathcal{X}$ *coming from classical cuspidal Siegel–Hilbert automorphic forms of weight* $(k_{i,\tau};k_0)$ *which are regular and spherical at* p.

Remark 4.11. *Assuming Leopoldt's conjecture, the multiplicative group appearing in the definition of* $\mathcal{W}$ *is, up to a finite subgroup,* $((\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times)^{g+1}/\overline{\mathcal{O}_F^\times}$ *(i.e. the* $\mathbb{Z}_p$*-points of the torus of* $\mathrm{Res}_F^{\mathbb{Q}}(\mathrm{GSp}_{2g})$ *modulo the* $\mathbb{Z}_p$*-points of the center).*

This allows us to define two pseudo-representations $R_? : G_{\mathbb{Q}}\to\mathcal{O}(\mathcal{X})$, for $?=$ spin, sta, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Suppose now that $V_?$ is absolutely irreducible (this is conjectured to hold when π is Steinberg at least at one prime); we have then, shrinking $\mathcal{U}$ around $\underline{k}$ if necessary, a *big* Galois representation $\rho_?$ with value in $\mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ such that $\mathrm{Tr}(\rho_?) = R_?$ [BC09, page 214].

For $1 \leq j < g$ we define $\lambda_{\mathfrak{p}}(u_{\mathfrak{p},g-j}) = \varpi_{\mathfrak{p}}^{\sum_{\tau \in I_{\mathfrak{p}}} k_{\tau,1}+\cdots+k_{\tau,j}-k_0}$ and $\lambda_{\mathfrak{p}}(u_{\mathfrak{p},0}) = \varpi_{\mathfrak{p}}^{\sum_{\tau \in I_{\mathfrak{p}}}(k_{\tau,1}+\cdots+k_{\tau,g}-k_0)/2}$. We have analytic functions $\beta_{\mathfrak{p},j} := \Theta(U_{\mathfrak{p},j}|\lambda_{\mathfrak{p}}(u_{\mathfrak{p},g-j})|_p) \in \mathcal{O}(\mathcal{X})$. We proceed now as in [HJ13]. We recall the following theorem [Liu13, Theorem 0.3.4];

THEOREM 4.12. *Let $\rho : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ be a continuous representation. Suppose that there exist $\kappa_1(x), \ldots, \kappa_n(x)$ in $F_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} \mathcal{O}(\mathcal{X})$, $F_1(x), \ldots, F_d(x)$ in $\mathcal{O}(\mathcal{X})$, and a Zariski dense set of points $Z \subset \mathcal{X}$ such that*

- *for any x in $\mathcal{X}$, the Hodge–Tate weights of ρ_x are $\kappa_1(x), \ldots, \kappa_n(x)$;*
- *for any z in Z, ρ_z is crystalline;*
- *for any z in Z, $\kappa_{\tau,1}(z) < \ldots < \kappa_{\tau,n}(z)$, for all $\tau \in I_{\mathfrak{p}}$;*
- *for any z in Z, the eigenvalues of $\varphi^{f_{\mathfrak{p}}}$ on $\mathcal{D}_{\mathrm{cris}}(V_z)$ are $\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{\kappa_{\tau,1}(z)} F_1(z), \ldots, \prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{\kappa_{\tau,n}(z)} F_n(z)$;*
- *for any C in $\mathbb{R}$, defines $Z_C \subset Z$ as the set of points z such that for all $I, J \subset \{1, \ldots, n\}$ such that $|\sum_{i \in I} \kappa_{i,\tau}(z) - \sum_{j \in J} \kappa_{\tau,j}(z)| > C$ for all $\tau \in I_{\mathfrak{p}}$. We require that for all $z \in Z$ and $C \in \mathbb{R}$, Z_C accumulates at z.*
- *for $1 \leq i \leq n$ there exist character $\chi_i : \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \to \mathcal{O}(\mathcal{X})^{\times}$ such that its derivative at 1 is κ_i and at each $z \in Z$ we have $\chi_i(u) = \prod_{\tau} \tau(u)^{\kappa_{\tau,i}(z)}$.*

Then, for all x in $\mathcal{X}$ non-critical and regular ($\kappa_1(x) < \ldots < \kappa_n(x)$ and the eigenvalues of φ on $\bigwedge^i \mathcal{D}_{\mathrm{cris}}(V_x)$ are distinct for all i) there exists a Zariski neighbourhood U of x such that ρ_U is trianguline and its graded pieces are $\mathcal{R}_U(\chi_i)$.

Here the rank one (φ, Γ)-module $\mathcal{R}_U(\chi_i)$ over U is defined similarly as in Section 2 following [Liu13, §0.2].
We can apply this theorem and show that the (φ, Γ)-module associated with $\rho_{?|G_{\mathbb{Q}_p}}$ is trianguline. We now explicit the triangulation, given in [HJ13, §3.3].

As seen before, a p-stabilization of $\pi_{\mathfrak{p}}$ corresponds to a permutation ν and a map ε.
The eigenvalues of $\varphi^{f_{\mathfrak{p}}}$ are given by

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_1+\mu_{1,\tau}} \beta_{\mathfrak{p},1},$$

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_i+\mu_{i,\tau}} \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}},$$

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_g+\mu_{g,\tau}} \frac{\beta_{\mathfrak{p},g-1}}{\beta_{\mathfrak{p},g}^2}$$

where c_i's are a positive integer independent of the weight.
We define the following characters of $F_{\mathfrak{p}}$ with value in $\mathcal{O}(\mathcal{X})$:

$$\chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p},1},$$
$$\chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}) = \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}},$$
$$\chi_{\mathfrak{p},g}(\varpi_{\mathfrak{p}}) = \frac{\beta_{\mathfrak{p},g-1}}{\beta^2_{\mathfrak{p},g}},$$

and $\chi_{\mathfrak{p},i}(u) = \prod_{\tau \in I_{\mathfrak{p}}} \tau(u)^{c_i + \mu_{i,\tau}}$.

From [HJ13, Lemma 19] we have that the graded pieces of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{sta}|_{\mathfrak{p}}})$ are then given by the characters $\chi_{\mathfrak{p},g}, \ldots, 1, \ldots, \chi^{-1}_{\mathfrak{p},g}$.

Concerning V_{spin}, we number the subsets of $\{1, \ldots, g\}$ as $I_1, I_2, \ldots, I_{2^g}$. Each I_j corresponds to a map $\varepsilon_j : \{1, \ldots, g\} \to \pm 1$.
We have then the graded pieces $\delta_{\mathfrak{p},j}$ are given by the characters

$$\delta_{\mathfrak{p},\varepsilon_j}(u) = \prod_{\tau \in I_{\mathfrak{p}}} \tau(u)^{d_j + \frac{k_0 + \sum_i \varepsilon_j(i) k_{i,\tau}}{2}},$$
$$\delta_{\mathfrak{p},\varepsilon_j}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p},g} \prod_{i \in I_j} \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}).$$

Let V be either V_{Sta} or V_{spin}. If $\pi_{\mathfrak{p}}$ is Steinberg, there is only one choice of a regular (φ, N)-sub-module $D_{\mathfrak{p}}$ of $\mathbf{D}_{\mathrm{st}}(V_{G_{F_{\mathfrak{p}}}})$, where V is one of the two representations associated with π described above. If the form is not Steinberg at $\mathfrak{p}$ many different regular sub-module can be chosen.
In any case, we expect (and we shall assume in the follow) that there is at most one trivial zero for each $\mathfrak{p}$. Consider now the representation π of parallel weight $\underline{k}$ (*i.e.* associated with $\mathrm{N}_{F/\mathbb{Q}}(\det^{\underline{k}})$, $\underline{k} \in \mathbb{Z}$) as in the introduction.
We give a preliminary proposition on the factorization of the $\mathcal{L}$-invariant. Recall the set $S^{\mathrm{Sph},1}$ and S^{Stb} defined in the introduction, we have the following;

Proposition 4.13. *We have the following factorization*

$$\mathcal{L}(V, D) = \mathcal{L}(V, D)^{\mathrm{Sph}} \prod_{\mathfrak{p} \in S^{\mathrm{Stb}}} \mathcal{L}(V, D)_{\mathfrak{p}},$$

where $\mathcal{L}(V, D)^{\mathrm{Sph}}$ comes from the prime in S^{sph} and the factors $\mathcal{L}(V, D)_{\mathfrak{p}}$ are local.

Proof. We follow [Hid07, §1.3]. In the notation of Section 3, we write $W_1 = \oplus_{\mathfrak{p} \in S^{\mathrm{Stb}}} W_{\mathfrak{p},1}$ and $M_1 = \oplus_{\mathfrak{p} \in S^{\mathrm{Sph},1}} M_{\mathfrak{p},1}$. We are left to show that the endomorphism $\iota_{\mathrm{f}} \circ \iota_{\mathrm{c}}^{-1}$ of $\mathcal{D}_{\mathrm{cris}}(W_1 \oplus M_1) \cong E^t$ keeps stable $\mathcal{D}_{\mathrm{cris}}(M_1)$ and on the quotient it respects the direct sum decomposition $\oplus_{\mathfrak{p} \in S^{\mathrm{Stb}}} \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1})$.

Consider a prime $\mathfrak{p}_0 \in S^{\mathrm{Stb}}$ and a cocycle $c \in H^1(D, V)$ such that $\mathrm{res}_{\mathfrak{p}}(c) = 0$ for all $\mathfrak{p} \neq \mathfrak{p}_0$. This means that $\mathrm{res}_{\mathfrak{p}}(c) \in H^1_{\mathrm{f}}(F_{\mathfrak{p}}, V) = H^1_{\mathrm{f}}(F_{\mathfrak{p}}, M_{\mathfrak{p}})$ (by (3.3)). We have hence $\iota_{\mathrm{c},\mathfrak{p}}(c) = 0$ for all primes $\mathfrak{p} \neq \mathfrak{p}_0$ as H^1_{c} is the direct sum complement of H^1_{f} (see [Ben11, Proposition 1.5.9]).
If $\mathfrak{p}$ in S^{Stb} by Proposition 2.10 we also have $\iota_{\mathrm{f},\mathfrak{p}}(c) = 0$.
The proposition then follows from standard linear algebra as in [Hid07, Corollary 1.9]. □

REMARK 4.14. *A key ingredient in the proof of the factorization at Steinberg places is that each prime ideal brings a single trivial zero.*

We now consider the case $V = V_{\mathrm{sta}}$. We have a contribution to trivial zeros from the $\pi_{\mathfrak{p}}$'s which are Steinberg and possibly from the $\pi_{\mathfrak{p}}$ which are spherical. In particular, if we choose the regular sub-module coming from an ordinary filtration, we always have a trivial zero coming from each place.
For all $1 \leq s \leq \min(k - g - 1, g - 1)$ we have also e_{Stb} trivial zeros for $V(s)$.

THEOREM 4.15. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V, D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\mathrm{d} \log_p \beta_{\mathfrak{p},1}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

where k is the parallel weight variable.
For $1 \leq s \leq \min(k - g - 1, g - 2)$ we also have

$$\mathcal{L}(V(s), D(s))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\mathrm{d} \log_p (\beta_{\mathfrak{p},s-1} \beta^{-1}_{\mathfrak{p},s}(k))}{\mathrm{d}k}_{|k=\underline{k}}$$

and if $g - 1 \leq k - g - 1$ we have

$$\mathcal{L}(V(g-1), D(g-1))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\mathrm{d} \log_p (\beta_{\mathfrak{p},g-1} \beta^{-2}_{\mathfrak{p},g}(k))}{\mathrm{d}k}_{|k=\underline{k}}.$$

Proof. We note that we can specialize to a parallel family, so that no contribution from the denominator appears. We can apply Theorem 4.1 for $\delta_A \psi_A^{-1}(\varpi_{\mathfrak{p}}) = \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}})$. The factor $\log_p(u)$ disappears because of the change of variable $T \mapsto u^k - 1$ (u any topological generator of $\mathbb{Z}_p^\times$). □

REMARK 4.16. *The presence of $f_{\mathfrak{p}}$ in the denominator can be explained in terms of the p-adic L-function for the induced representation, its missing Euler factors at p and Conjecture 1.2. See [Hid09, pag. 1348].*

From now on, $V = V_{\mathrm{spin}}(k-1)$ ($s = k - 1$ is the only critical integer); if $\pi_{\mathfrak{p}}$ is spherical it should not give any trivial zeros (as the corresponding p-adic representation is conjectured to be crystalline and consequently the β_i's are Weil numbers of non-zero weight).
So we are left to see what happen at the primes Steinberg at $\mathfrak{p}$. Twisting by $\beta_{\mathfrak{p},g}$ the triangulated (φ, Γ)-module of ρ_{spin} we are in the hypothesis of Theorem 4.1 and we have

Theorem 4.17. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V,D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p},1}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

where k is a parallel weight variable.

5 The case of the adjoint representation

We prove Theorem 1.4 of the introduction. We consider only the case $g = 2$. Fix an automorphic representation π of weight $\underline{k} = (\underline{k}_{\tau,1}, \ldots, \underline{k}_{\tau,g}; \underline{k}_0)_\tau$ and let $V = V_{\mathrm{spin}}$ be the spin representation associated with π. Let $\rho = \rho_{\mathrm{spin}}$ be the corresponding big Galois representation.

We specializes the eigenvariety $\mathcal{X}$ of Theorem 4.10 to the subspace of the weight space given by the equations $k_{i,\tau} = k_{i,\tau'}$ if τ and τ' induce the same p-adic place $\mathfrak{p}$ and $k_0 = \underline{k}_0$. We shall denote the new variable by $k_{\mathfrak{p},i}$ and this eigenvariety by $\mathcal{X}'$. For simplicity, we rewrite the graded pieces of V as

$$\begin{aligned}
\delta_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},1}^{-1}(k),\ \delta_{\mathfrak{p},1}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},1}+k_{\mathfrak{p},2}-3}{2}},\\
\delta_{\mathfrak{p},2}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},2}^{-1}(k),\ \delta_{\mathfrak{p},2}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},2}-k_{\mathfrak{p},1}+1}{2}},\\
\delta_{\mathfrak{p},3}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},2}(k),\ \delta_{\mathfrak{p},3}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},2}+k_{\mathfrak{p},1}-1}{2}},\\
\delta_{\mathfrak{p},4}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},1}(k),\ \delta_{\mathfrak{p},4}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},1}-k_{\mathfrak{p},2}+3}{2}}
\end{aligned}$$

where $k = (k_{\mathfrak{p},1}, k_{\mathfrak{p}_2}; k_0)_{\mathfrak{p}}$.
The representation space of $\mathrm{Ad}(V)$ is given by the matrices

$$\mathfrak{Sp}_4 = \left\{X \in \mathfrak{SL}_4 | X^t J + JX = 0\right\}.$$

The p-stabilization on V induces a natural p-stabilization and consequently a regular sub-module D_{Ad} on $\mathrm{Ad}(V_{\mathrm{spin}})$. We have

$$\begin{aligned}
D_{\mathrm{Ad}-1} &= \{\text{nilpotent } X\},\\
D_{\mathrm{Ad}0} &= \{\text{unipotent } X\}.
\end{aligned}$$

The basis for the space $D_{\mathrm{Ad}0}/D_{\mathrm{Ad}-1}$ is given by the two diagonal matrices $d_1 = [-1,0,0,1]$ and $d_2 = [0,-1,1,0]$. We shall denote by $d_{\mathfrak{p},i}$ these matrices when seen as a vector for $\mathrm{Ad}(V_{\mathfrak{p}})$.

Proposition 5.1. *Suppose that* C1-C4 *holds for V. Suppose that the classical E-point x in the eigenvariety $\mathcal{X}'$ corresponding to π is étale above the weight space. Then, the space $\mathcal{L}(D_{\mathrm{Ad}}, V)$ is generated by the image of $\left(\frac{\mathrm{d}\log_p \delta_{\mathfrak{p},i}}{\mathrm{d}k_{\mathfrak{p}',j}} d_{\mathfrak{p},i}\right)_{\mathfrak{p}',j=1,2}$.*

Proof. The proof is standard and goes back to [MT90], so we shall only sketch it. Let $A = E[T]/(T^2)$. Consider an infinitesimal deformation of ρ given by

$$\rho_A = V \oplus \rho';$$

note that ρ' can be written as the first order truncation of $\frac{\partial \rho}{\partial v}$, where v is any direction in the weight space.
From ρ_A we can construct a cocyle $c_{x,A}$ defined by

$$G_F \ni \sigma \mapsto \rho'(\sigma)V^{-1}(\sigma).$$

It is easy to check that this defines a cocycle with values in $V \otimes V^*$. Moreover its image lands in $\mathrm{Ad}(V) \subset V \otimes V^*$ as the determinant is fixed (by our choice of the Hodge–Tate weight on $\mathcal{X}'$). Writing explicitly the matrix for the (φ, Γ)-module associated with ρ_A we obtain

$$\begin{pmatrix} \frac{\partial \delta_{\mathfrak{p},1}}{\partial v} & * & * & * \\ & \frac{\partial \delta_{\mathfrak{p},2}}{\partial v} & * & * \\ & & \frac{\partial \delta_{\mathfrak{p},3}}{\partial v} & * \\ & & & \frac{\partial \delta_{\mathfrak{p},4}}{\partial v} \end{pmatrix}_{|k=\underline{k}} \begin{pmatrix} \delta_{\mathfrak{p},1}^{-1} & * & * & * \\ & \delta_{\mathfrak{p},2}^{-2} & * & * \\ & & \delta_{\mathfrak{p},3}^{-1} & * \\ & & & \delta_{\mathfrak{p},4}^{-1} \end{pmatrix}_{|k=\underline{k}}$$

In particular, they are upper triangular and their projection via ι_{f} onto the vector $d_{\mathfrak{p},1}$ is $\frac{\mathrm{d} \log_p F_{\mathfrak{p},1}(k)}{\mathrm{d}v}_{|k=\underline{k}}$. Similarly for $d_{\mathfrak{p},2}$.
We also have that the projection via ι_{c} onto $d_{\mathfrak{p},1}$ is $-\frac{\partial (k_{\mathfrak{p},1}+k_{\mathfrak{p},2})/2}{\partial v}_{|k=\underline{k}}$.
By hypothesis, the projection to the weight space is étale at x and hence $\left\{ \frac{\partial}{\partial k_{\mathfrak{p},i}} \right\}_{\mathfrak{p},i=1,2}$ is a base of the tangent space at x in $\mathcal{X}'$ and we are done. □

We can now prove Theorem 1.4 which we recall;

THEOREM 5.2. *Let π be an automorphic form of weight k. Suppose that hypothesis* LGP *is verified for V_{spin} and the point corresponding to π in the eigenvariety $\mathcal{X}'$ is étale over the weight space. We have then*

$$\mathcal{L}(\mathrm{Ad}(V_{\mathrm{spin}}), D_{\mathrm{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \begin{pmatrix} \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,1}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,1}} \\ \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,2}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,2}} \end{pmatrix}_{1 \leq i,j \leq t_{|k=\underline{k}}}.$$

Proof. By hypothesis we can use Proposition 5.1, so we just have to follow the proof of [Hid06, Theorem 3.73]. The matrix of ι_{f} is exactly what appears in the Theorem, while the matrix of ι_{c} can be directly calculated using the formula $\frac{\mathrm{d} \log_p (u^{\pm k_{\mathfrak{p},i}})}{\mathrm{d} k_{\mathfrak{p}',j}} = \pm \delta_{\mathfrak{p},\mathfrak{p}'} \delta_{i,j}$ (where $\delta_{a,b}$ here is Kronecker delta) and gives a contribution of 2^{-1} for each prime ideal $\mathfrak{p}$. □

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