Toeplitz Operators on Higher Cauchy-Riemann Spaces

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Abstract. We develop a theory of Toeplitz, and to some extent Hankel, operators on the kernels of powers of the boundary d-bar operator, suggested by Boutet de Monvel and Guillemin, and on their analogues, somewhat better from the point of view of complex analysis, defined using instead the covariant Cauchy-Riemann operators of Peetre and the second author. For the former, Dixmier class membership of these Hankel operators is also discussed. Our main tool are the generalized Toeplitz operators (with pseudodifferential symbols), in particular there appears naturally the problem of finding parametrices of matrices of such operators in situations when the principal symbol fails to be elliptic.

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1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^d$ and $L^2_{\text{hol}}(\Omega)$ the Bergman space of all holomorphic functions in $L^2(\Omega)$. For $\phi \in L^\infty(\Omega)$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $L^2_{\text{hol}}(\Omega)$ defined by

$$\mathbf{T}_\phi f = \mathbf{\Pi}(\phi f), \qquad f \in L^2_{\text{hol}}(\Omega),$$

where $\mathbf{\Pi} : L^2(\Omega) \to L^2_{\text{hol}}(\Omega)$ is the orthogonal projection (the Bergman projection). Similarly, if Ω has smooth boundary $\partial\Omega$, one has the Hardy space $H^2(\partial\Omega)$ consisting of all functions in $L^2(\partial\Omega)$ (with respect to the surface

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measure on $\partial\Omega$) whose Poisson extension into Ω is holomorphic, and for $\phi \in L^\infty(\partial\Omega)$ the Toeplitz operator T_ϕ on $H^2(\partial\Omega)$ is defined by

$$T_\phi u = \Pi(\phi u), \qquad u \in H^2(\partial\Omega),$$

where $\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is the orthogonal projection (the Szegö projection). There are also Hankel operators $\mathbf{H}_\phi : L^2_{\text{hol}}(\Omega) \to L^2(\Omega) \ominus L^2_{\text{hol}}(\Omega)$ and $H_\phi : H^2(\partial\Omega) \to L^2(\Omega) \ominus H^2(\partial\Omega)$ defined as

$$\mathbf{H}_\phi f = (I - \mathbf{\Pi})(\phi f), \quad H_\phi u = (I - \Pi)(\phi u), \qquad \text{respectively.}$$

Toeplitz and Hankel operators, and their various generalizations, have been extensively studied for the last three decades, and have turned out to play important role in many subjects ranging from operator theory and complex function theory to geometry and mathematical physics, see e.g. [14], [16], [19], [18] and the references therein for a sample.

The spaces $L^2_{\text{hol}}(\Omega)$ on which $\mathbf{T}_\phi$ and $\mathbf{H}_\phi$ act can alternatively be characterized as the kernel of the operator $\overline{\partial}$ in $L^2(\Omega)$, where, as usual, $\overline{\partial}$ denotes the operator assigning to a function f on Ω the $(0,1)$-form

$$\overline{\partial} f := \sum_{j=1}^d \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j.$$

For $d > 1$, one has a similar characterization of the Hardy space $H^2(\partial\Omega)$ as the kernel of the operator $\overline{\partial}_b$ in $L^2(\partial\Omega)$, where for a function u on $\partial\Omega$, $\overline{\partial}_b u$ is the restriction of du to the antiholomorphic complex tangent space $\mathcal{T}''(\partial\Omega)$, consisting of all vectors X on $\partial\Omega$ of the form

$$X = \sum_{j=1}^d X_j \frac{\partial}{\partial \overline{z}_j}, \qquad X_j \in \mathbf{C},$$

which are tangent to $\partial\Omega$. Fixing a positively-signed defining function ρ for Ω, so that $\rho > 0$ on Ω and $\rho = 0 < |\nabla\rho|$ on $\partial\Omega$, the last condition just means that $X\rho = 0$, and $\mathcal{T}''(\partial\Omega)$ is spanned by the (linearly dependent) vector fields

$$(1) \qquad L_{jk} := \frac{\partial\rho}{\partial\overline{z}_j}\frac{\partial}{\partial\overline{z}_k} - \frac{\partial\rho}{\partial\overline{z}_k}\frac{\partial}{\partial\overline{z}_j}, \qquad 1 \le j < k \le d;$$

so $\overline{\partial}_b u = 0$ means that $L_{jk} u = 0$ for all j, k.

A generalization of the Hardy space due to Boutet de Monvel and Guillemin [3, §15.3] are the subspaces $\mathcal{B}_m$ in $L^2(\partial\Omega)$, $m = 1, 2, \dots$, of all functions u annihilated by $\overline{\partial}_b^m$, in the sense that

$$L_{j_1k_1} L_{j_2k_2} \dots L_{j_mk_m} u = 0 \qquad \text{for all } j_1, \dots, j_m, k_1, \dots, k_m.$$

Clearly $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset L^2(\partial\Omega)$ and $\mathcal{B}_1 = H^2(\partial\Omega)$. It was shown in [3] that microlocally, the associated "higher Szegö projectors" $\Pi_m : L^2(\partial\Omega) \to \mathcal{B}_m$ are of the same type as Π, and, hence, in this sense, so are the corresponding Toeplitz operators $T_\phi^{\mathcal{B}_m} : u \mapsto \Pi_m(\phi u)$ on $\mathcal{B}_m$.
The drawback of the spaces $\mathcal{B}_m$, however, is that in spite of the microlocal equivalence of the Szegö projectors just mentioned, they fail to be invariant under biholomorphic maps. For instance, in the simplest possible situation when $\Omega = \mathbf{B}^2$, the unit ball of $\mathbf{C}^2$, one checks easily that the function $\overline{z}_2$ belongs to $\mathcal{B}_2 = \operatorname{Ker} L^2_{12}$, but if ϕ_a is the automorphism of $\mathbf{B}^2$ interchanging the origin with a point $a \in \mathbf{B}^2$, then $\overline{z}_2 \circ \phi_a \notin \mathcal{B}_2$ if $a \neq 0$. (See Section 3 below for the details.)
The aim of this paper is, firstly, to show that in spite of not being biholomorphically invariant, the spaces $\mathcal{B}_m$, $m > 1$, have Toeplitz and Hankel operators which behave very similarly as in the classical case $m = 1$; and secondly, to propose a different generalization, the so-called higher Cauchy-Riemann spaces $\mathcal{C}_m$, $m = 1, 2, \dots$, which are well-behaved under biholomorphic maps, and study the associated Toeplitz operators.
For the first part, we work out only the case of $\Omega = \mathbf{B}^2$, the unit ball in $\mathbf{C}^2$. Our main result is the following.

Theorem 1. *The Toeplitz operator $T_f^{\mathcal{B}_2}$ on $\mathcal{B}_2(\partial\mathbf{B}^2)$ is unitarily equivalent to the operator*

$$\begin{bmatrix} T_f & 0 \\ 0 & T_f \end{bmatrix} + \text{lower order term}$$

on the direct sum $H^2(\partial\mathbf{B}^2) \oplus H^2(\partial\mathbf{B}^2)$.

Here the "lower order term" means a 2×2 matrix of generalized Toeplitz operators of order at most $-\frac{1}{2}$; see again Section 3 below for more details.
As a corollary to the theorem, we also get a similar result for the product $\mathcal{H}_{\overline{f}}^* \mathcal{H}_g$ of two Hankel operators on $\mathcal{B}_2(\partial\mathbf{B}^2)$, and a formula for the Dixmier trace of $(\mathcal{H}_{\overline{f}}^* \mathcal{H}_g)^2$ that can be compared to the one for ordinary Hankel operators $H_{\overline{f}}^* H_g$ from [11].
Concerning the second part, consider, quite generally, a Kähler metric $g_{j\overline{k}}$ on Ω, and let $g^{\overline{l}j}$ be the inverse matrix to $g_{j\overline{k}}$ (so the Kähler form is given by $g^{\overline{l}k} d\overline{z}_l \wedge dz_k$). The Cauchy-Riemann operator, introduced by Peetre (cf. [8] and [15]), is the map from functions into holomorphic vector fields on Ω given by

$$\overline{D} f := g^{\overline{l}k} \overline{\partial}_l f,$$

where we have started to employ the Einstein summation convention of summing automatically over any index that occurs twice, and also to write for brevity $\overline{\partial}_l = \partial / \partial \overline{z}_l$. One can iterate this construction and set, for $m = 1, 2, \dots$,

$$\overline{D}^m f := g^{\overline{l}_m k_m} \overline{\partial}_{l_m} g^{\overline{l}_{m-1} k_{m-1}} \overline{\partial}_{l_{m-1}} \dots g^{\overline{l}_1 k_1} \overline{\partial}_{l_1} f.$$

It turns out that $(\overline{D}^m f)^{k_m \dots k_1}$ is symmetric in the indices $k_m, \dots, k_1$ [15], and in fact coincides with the contravariant derivative $f^{/k_m \dots k_1}$ with respect to the Hermitian connection [8]. The m-th Cauchy-Riemann space $\mathcal{C}_m$ is, by definition, the kernel of $\overline{D}^m$:

$$\mathcal{C}_m := \{f : \overline{D}^m f = 0 \text{ on } \Omega\}.$$

Clearly $\mathcal{C}_1$ comprises precisely of holomorphic functions, and is also independent of the metric $g_{j\overline{k}}$. For $m > 1$, $\mathcal{C}_m$ depends on $g_{j\overline{k}}$ (although this fact is not reflected by the notation). Now there are various holomorphically invariant Kähler metrics associated to a given bounded strictly pseudoconvex domain with smooth boundary, such as the Bergman metric, the Poincare (Kähler-Einstein) metric, the metric coming from the invariant Szegö kernel, and so forth. Taking any of these for $g_{j\overline{k}}$, by the very nature of their construction the spaces $\mathcal{C}_m$, unlike $\mathcal{B}_m$, will be invariant under biholomorphisms.
Choosing a (positive smooth) weight w on Ω, let $\mathcal{C}_{m,w} := \mathcal{C}_m \cap L^2(\Omega, w)$ (since differential operators are closed, this is a closed subspace of $L^2(\Omega, w)$), and let $\mathcal{T}_\phi^{(m,w)} : f \mapsto \mathbf{\Pi}^{(m,w)}(\phi f)$, where $\mathbf{\Pi}^{(m,w)} : L^2(\Omega, w) \to \mathcal{C}_{m,w}$ is the orthogonal projection, be the associated Toeplitz operator on $\mathcal{C}_{m,w}$ with symbol $\phi \in L^\infty(\Omega)$. (Thus $\mathcal{T}_\phi^{(m,w)}$ again depends also on the choice of the metric $g_{j\overline{k}}$, although this is not reflected by the notation.) We will actually assume that Ω is bounded, strictly pseudoconvex and with smooth boundary, that $\phi \in C^\infty(\overline{\Omega})$ is smooth on the closure $\overline{\Omega}$ of Ω, that the weight w is of the form

$$w = \rho^\nu, \qquad \nu \in \mathbf{R},$$

where ρ is a (fixed) positively-signed defining function for Ω and ν is large enough; and that $g_{j\overline{k}}$ is given by a Kähler potential Ψ,

$$g_{j\overline{k}} = \partial_j \overline{\partial}_k \Psi,$$

where Ψ is of the form

$$\Psi \approx \sum_{j=0}^{\infty} (\rho^M \log \rho)^j \eta_j, \qquad \eta_j \in C^\infty(\overline{\Omega}),$$

with an integer $M \geq 2$; see Section 4 for the details. Note that all the metrics $g_{j\overline{k}}$ mentioned in the penultimate paragraph are of this kind.

Theorem 2. *Let Ω, w and $g_{j\overline{k}}$ be as stated above. Assume that $\nu > 1$ and that $-\rho$ is strictly plurisubharmonic near $\partial\Omega$ and $|\partial\rho| = 1$ on $\partial\Omega$. Then the Toeplitz operator $\mathcal{T}_\phi^{(2,w)}$ is unitarily equivalent to the operator*

$$\mathcal{T}_\phi^{(2,w)} \cong \bigoplus_{j=0}^{d} T_{\phi|\partial\Omega} + \text{lower order term} \tag{2}$$

on the direct sum $\oplus_{j=0}^d H^2(\partial\Omega)$ of $(d+1)$ copies of $H^2(\partial\Omega)$.

Here the "lower order term" means a $(d+1)\times(d+1)$ matrix of sums of generalized Toeplitz operators of orders and $-\frac{1}{2}$ and -1, see Section 4 for the details.

The hypothesis $\nu > 1$ is needed to have $\mathcal{C}_{2,w}$ nontrivial: for $\nu \leq 1$, $\mathcal{C}_{2,w}$ contains just the function constant zero.

For general $m \geq 2$, (2) holds too, except that the hypothesis on ν becomes $\nu > 2m-3$ and instead of $d+1$ copies of $T_{\phi|\partial\Omega}$, one gets $\binom{d+m-1}{m-1}$ copies (and matrices of the corresponding size).

Our main tool are Toeplitz operators with pseudodifferential symbols, or generalized Toeplitz operators, on $H^2(\partial\Omega)$ whose theory was worked out by Boutet de Monvel and Guillemin in [3]. The proof of Theorem 1 follows what is now already a more or less standard Ansatz (cf. e.g. [2] and [11]) once we identify $\mathcal{B}_2$ explicitly by parameterizing it by two copies of $H^2(\partial\Omega)$. To some extent this is also true for Theorem 2, however the main difficulty there is that we are confronted with inverting a matrix of generalized Toeplitz operators whose principal symbol is not invertible (i.e. the matrix is not elliptic).

The proof of Theorem 1 is presented in Section 3, after reviewing various prerequisites in Section 2. The proof of Theorem 2 occupies Section 4, while the simplest case of the ball is worked out in detail in Section 5.

As already introduced above, we write simply ∂_j, $\overline{\partial}_j$ for $\partial/\partial z_j$ and $\partial/\partial\overline{z}_j$, respectively. Throughout the paper, abusing the notation slightly, we will also denote the restriction $\phi|_{\partial\Omega}$ of a function $\phi \in C^\infty(\overline{\Omega})$ to $\partial\Omega$ just again by ϕ.

2. Background

2.1 Pseudodifferential operators. Throughout the rest of this paper, Ω will be a bounded strictly pseudoconvex domain in $\mathbf{C}^d$, $d > 1$, with smooth ($= C^\infty$) boundary, and ρ a positively signed defining function for Ω, i.e. $\rho \in C^\infty(\overline{\Omega})$, $\rho > 0$ on Ω, and $\rho = 0 < |\nabla\rho|$ on $\partial\Omega$. Denote by η the restriction to $\partial\Omega$ of the 1-form $\operatorname{Im}(-\partial\rho) = (\overline{\partial}\rho - \partial\rho)/2i$. The strict pseudoconvexity of Ω guarantees that the half-line bundle

$$\Sigma := \{(x,\xi) \in \mathcal{T}^*(\partial\Omega) : \xi = t\eta_x, t > 0\}$$

is a symplectic submanifold of the cotangent bundle $\mathcal{T}^*(\partial\Omega)$.

By a classical (or polyhomogeneous) pseudodifferential operator (ψdo for short) P on $\partial\Omega$ of order m we will mean a ψdo whose total symbol in any local coordinate chart has an asymptotic expansion $p(x,\xi) \sim \sum_{j=0}^\infty p_{m-j}(x,\xi)$, where p_{m-j} is C^∞ in x,ξ and positively homogeneous of degree $m-j$ in ξ for $|\xi| > 1$. Here m can be any real number, and "$\sim$" means that the difference $p - \sum_{j=0}^{k-1} p_{m-j}$ should belong to the Hörmander class S^{m-k}, for each $k = 0,1,2,\dots$. The function $p_m(x,\xi)$ is called the (leading or principal) symbol of P, denoted $\sigma_m(P)$ (or just $\sigma(P)$ if the order m is clear from the context), and the set of all ψdo's of order m will be denoted by Ψ^m. Operators in $\bigcap_{m\in\mathbf{R}} \Psi^m$ are the smoothing operators, i.e. those with C^∞ Schwartz kernel; and we will write $P \sim Q$ if $P - Q$ is smoothing.

Unless explicitly stated otherwise, all ψdo's henceforth will be classical. In Section 4, we will also need ψdo's with symbols whose degrees of homogeneity go down by $\frac{1}{2}$ instead of 1; for the purposes of this paper, we will call these *demi-classical* ψdo's. In other words, a demi-classical ψdo of order m is the sum of a classical ψdo of order m and a classical ψdo of order $m-\frac{1}{2}$.

2.2 Generalized Toeplitz operators. For $Q \in \Psi^m$ the generalized Toeplitz operator (or gTo for short) T_Q is defined as

$$T_Q = \Pi Q|_{H^2(\partial\Omega)}.$$

Alternatively, one can view T_Q as the operator

$$T_Q = \Pi Q \Pi$$

on all of $L^2(\partial\Omega)$. In both cases, T_Q is a densely defined operator (its domain contains the Sobolev space $W^m(\partial\Omega)$), and extends to a continuous map from the Sobolev space $W^s(\partial\Omega)$ into $W^{s-m}_{\text{hol}}(\partial\Omega)$, for any $s \in \mathbf{R}$.
Generalized Toeplitz operators are known to enjoy the following properties.

- (P1) They form an algebra, i.e. $T_P T_Q = T_R$ for some ψdo R.
- (P2) In fact, for any T_Q there exists a ψdo P of the same order as Q such that $T_P = T_Q$ and $P\Pi = \Pi P$.
- (P3) If P, Q are of the same order and $T_P = T_Q$, then $\sigma(P)$ and $\sigma(Q)$ coincide on the half-line bundle Σ. One can thus define unambiguously the order of T_Q as $\operatorname{ord}(T_Q) := \inf\{\operatorname{ord}(P) : T_P = T_Q\}$, and the (principal) symbol $\sigma(T_Q) := \sigma(Q)|_\Sigma$ if $\operatorname{ord}(Q) = \operatorname{ord}(T_Q)$. (The symbol is undefined if $\operatorname{ord}(T_Q) = -\infty$.)
- (P4) $\operatorname{ord}(T_P T_Q) = \operatorname{ord}(T_P) + \operatorname{ord}(T_Q)$, $\sigma(T_P T_Q) = \sigma(T_P)\sigma(T_Q)$, and $\sigma([T_P, T_Q]) = \frac{1}{i}\{\sigma(T_P), \sigma(T_Q)\}_\Sigma$ where $\{\cdot,\cdot\}_\Sigma$ denotes the Poisson bracket on Σ.
- (P5) If $\operatorname{ord}(T_Q) \le 0$, then T_Q is bounded on $L^2(\partial\Omega)$; if $\operatorname{ord}(T_Q) < 0$, it is even compact.
- (P6) If $Q \in \Psi^m$ and $\sigma_m(Q)|_\Sigma = 0$, then there exists $P \in \Psi^{m-1}$ with $T_P = T_Q$. If $T_Q \sim 0$, then there exists $P \sim 0$ such that $T_P = T_Q$.
- (P7) One says that T_Q is elliptic if $\sigma(T_Q)$ does not vanish. Then T_Q has a parametrix, i.e. there exists a gTo T_P, with $\operatorname{ord}(T_P) = -\operatorname{ord}(T_Q)$ and $\sigma(T_P) = \sigma(T_Q)^{-1}$, such that $T_P T_Q \sim T_Q T_P \sim I_{H^2(\partial\Omega)}$.
- (P8) If an elliptic gTo T_P is in addition positive self-adjoint as an operator on $H^2(\partial\Omega)$, then its complex power T_P^z, $z \in \mathbf{C}$, defined using the spectral theorem, is again a gTo, of order $z \operatorname{ord}(T_P)$ and with symbol equal to $\sigma(T_P)^z$; in particular, the inverse T_P^{-1} and the positive square roots $T_P^{1/2}, T_P^{-1/2}$ are gTo's.

We refer to the book [3], especially its Appendix, and to the paper [2] for the proofs and for additional information on generalized Toeplitz operators.

In addition to classical ψdo's and gTo's, we will also need the more general class $\Psi_{\log}$ of log-classical (or log-polyhomogeneous) ψdo's and gTo's, whose total symbol in any local coordinate chart satisfies

$$p(x,\xi) - \sum_{j=0}^{k-1} p_{m-j}(x,\xi) \in S^{m-k+\epsilon} \quad \forall \epsilon > 0,\ k = 0,1,2,\dots,$$

where p_{m-j} are of the form

$$p_{m-j}(x,\xi) = \sum_{k=0}^{k_j} p_{m-j,k}(x, \tfrac{\xi}{|\xi|}) |\xi|^{m-j} (\log|\xi|)^k \tag{3}$$

for $|\xi| > 2$, for some (finite) integers k_j. Such ψdo's arise naturally as logarithms of complex powers of elliptic classical ψdo's, and similarly for the corresponding gTo's. The properties (P1)–(P8) above remain in force for log-classical gTo's, except in (P7), (P8) and the first part of (P5) one must assume that $k_0 = 0$ (i.e. that the principal symbol is log-free). The reader is referred e.g. to [10], and the references therein, for the details.
Again, in Section 4 we will also need the demi-classical analogues of log-pluri-homogeneous ψdo's and gTo's, i.e. with symbols whose degrees of homogeneity go down by $\frac{1}{2}$ instead of 1; everything above extends also to this case.

2.3 Boutet de Monvel calculus. Let $\mathbf{K}$ denote the Poisson extension operator, i.e. $\mathbf{K}$ solves the Dirichlet problem

$$\Delta \mathbf{K} u = 0 \text{ on } \Omega, \quad \mathbf{K}u|_{\partial\Omega} = u. \tag{4}$$

(Thus $\mathbf{K}$ acts from functions on $\partial\Omega$ into functions on Ω. Here Δ is the ordinary Laplacian.) By the standard elliptic regularity theory [13], $\mathbf{K}$ is continuous from $W^s(\partial\Omega)$ into $W^{s+\frac{1}{2}}(\Omega)$, for any real s; in particular, it is continuous from $L^2(\partial\Omega)$ into $L^2(\Omega)$, and thus has a continuous Hilbert space adjoint $\mathbf{K}^* : L^2(\Omega) \to L^2(\partial\Omega)$. The composition

$$\mathbf{K}^*\mathbf{K} =: \Lambda$$

is known to be a positive selfadjoint elliptic ψdo on $\partial\Omega$ of order -1. We have by definition $\Lambda^{-1}\mathbf{K}^*\mathbf{K} = I_{L^2(\partial\Omega)}$; comparing this with (4) we see that the restriction

$$\gamma := \Lambda^{-1}\mathbf{K}^*|_{\operatorname{Ran}\mathbf{K}}$$

is the operator of "taking the boundary values" of a harmonic function. The operators

$$\Lambda_w := \mathbf{K}^* w \mathbf{K},$$

with w a smooth function on $\overline{\Omega}$, are governed by a calculus developed by Boutet de Monvel in [1]. For typographical reasons, we will often write $\Lambda[w]$ instead of Λ_w. It was shown that for w of the form

$$w = \rho^s g, \qquad g \in C^\infty(\overline{\Omega}),\ s > -1,$$

$\Lambda[w]$ is a ψdo on $\partial\Omega$ of order $-s-1$, with principal symbol

$$\sigma(\Lambda_w)(x,\xi) = \frac{\Gamma(s+1)|\eta_x|^s}{2|\xi|^{s+1}} g(x). \tag{5}$$

More generally, $\Lambda[g\rho^s(\log\rho)^k]$ is a log-classical ψdo on $\partial\Omega$ whose leading symbol $p_{-s-1}(x,\xi)$ has the form (3) with $k_0 = k$; see e.g. [12].

2.4 The Levi form. We denote by $\mathcal{T}'' \equiv \mathcal{T}''(\partial\Omega) \subset \mathcal{T}(\partial\Omega) \otimes \mathbf{C}$ the antiholomorphic complex tangent space to $\partial\Omega$, i.e. elements of $\mathcal{T}''_x$, $x \in \partial\Omega$, are vectors $X = \sum_{j=1}^d X_j \frac{\partial}{\partial \overline{z}_j}$, $X_j \in \mathbf{C}$, such that $X\rho = 0$. (This notation follows [4, p. 141].) The holomorphic complex tangent space $\mathcal{T}'$ is defined similarly, and the whole complex tangent space $\mathcal{T}(\partial\Omega) \otimes \mathbf{C}$ is spanned by $\mathcal{T}'$, $\mathcal{T}''$ and the vector

$$E := \sum_{j=1}^d \frac{\partial\rho}{\partial\overline{z}_j}\frac{\partial}{\partial z_j} - \frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial\overline{z}_j}$$

(the "complex normal" direction).
The boundary d-bar operator $\overline{\partial}_b : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega \to \mathcal{T}''^*)$ is defined as the restriction

$$\overline{\partial}_b f := df|_{\mathcal{T}''},$$

or, more precisely, $\overline{\partial}_b f = d\tilde{f}|_{\mathcal{T}''}$ for any smooth extension $\tilde{f}$ of f to a neighbourhood of $\partial\Omega$ in $\mathbf{C}^d$ (the right-hand side is independent of the choice of such extension). Recall that the Levi form is the Hermitian form on $\mathcal{T}'$ defined by

$$L'(X,Y) = -\sum_{j,k=1}^d \frac{\partial^2\rho}{\partial z_j \partial\overline{z}_k} X_j \overline{Y}_k \qquad \text{if } X = \sum_{j=1}^d X_j \frac{\partial}{\partial z_j},\ Y = \sum_{k=1}^d Y_k \frac{\partial}{\partial z_k}.$$

The strong pseudoconvexity implies that L' is positive definite. Similarly one has the positive-definite Levi form L'' on $\mathcal{T}''$ defined by

$$L''(X,Y) := -\sum_{j,k=1}^d \frac{\partial^2\rho}{\partial z_k \partial\overline{z}_j} X_j \overline{Y}_k \qquad \text{if } X = \sum_j X_j \frac{\partial}{\partial\overline{z}_j},\ Y = \sum_k Y_k \frac{\partial}{\partial\overline{z}_k}.$$

In terms of the complex conjugation $X \mapsto \overline{X}$ given by $\overline{X_j \frac{\partial}{\partial z_j}} = \overline{X}_j \frac{\partial}{\partial\overline{z}_j}$, mapping $\mathcal{T}'$ onto $\mathcal{T}''$ and vice versa, the two forms are related by

$$L''(X,Y) = L'(\overline{Y},\overline{X}) \qquad \forall X,Y \in \mathcal{T}''.$$

By the usual formalism, L'' induces a positive definite Hermitian form (or, perhaps more appropriately, a positive definite Hermitian bivector) on the dual space $\mathcal{T}''^*$ of $\mathcal{T}''$; we denote it by $\mathcal{L}$. Namely, for $\alpha \in \mathcal{T}''^*$, let $Z''_\alpha \in \mathcal{T}''$ be defined by

$$L''(X, Z''_\alpha) = \alpha(X) \qquad \forall X \in \mathcal{T}''.$$

(This is possible, and Z''_α is unique, owing to the non-degeneracy of L''. Note that $\alpha \mapsto Z''_\alpha$ is conjugate-linear.) Then

$$\mathcal{L}(\alpha,\beta) = L''(Z''_\beta, Z''_\alpha) = \alpha(Z''_\beta) = \overline{\beta(Z''_\alpha)}.$$

These objects are related to the symplectic structure of Σ as follows. Note that

$$d\eta = -i\partial\overline{\partial}\rho = -i\sum_{k,l=1}^{d} \frac{\partial^2\rho}{\partial z_k \partial \overline{z}_l}\, dz_k \wedge d\overline{z}_l,$$

hence

$$d\eta(X'+X'', Y'+Y'') = iL'(X', \overline{Y''}) - iL'(Y', \overline{X''})$$

for all $X', Y' \in \mathcal{T}'$ and $X'', Y'' \in \mathcal{T}''$. It follows that $d\eta$ is a non-degenerate skew-symmetric bilinear form on $\mathcal{T}'+\mathcal{T}''$. Let us define $E_{\mathcal{T}} \in \mathcal{T}'+\mathcal{T}''$ by

$$d\eta(X, E_{\mathcal{T}}) = d\eta(X, E) \qquad \forall X \in \mathcal{T}'+\mathcal{T}''$$

(again, this is possible and unambiguous by virtue of the non-degeneracy of $d\eta$ on $\mathcal{T}'+\mathcal{T}''$), and set

$$E_\perp := \frac{E - E_{\mathcal{T}}}{\eta(E)} = \frac{E - E_{\mathcal{T}}}{i\|\eta\|^2}.$$

The vector field $E_\perp$ is usually called the Reeb vector field, and is defined by the conditions $\eta(E_\perp) = 1$, $i_{E_\perp} d\eta = 0$.

For $f, g \in C^\infty(\partial\Omega)$, if we denote by f, g also the corresponding functions on the half-line bundle Σ constant on each fiber, then one has the following formula for their Poisson bracket:

$$\frac{1}{i}\{f,g\}_\Sigma = \frac{\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b \overline{g}) - \mathcal{L}(\overline{\partial}_b g, \overline{\partial}_b \overline{f})}{t}, \qquad \xi = t\eta_x,\ t>0. \tag{6}$$

In particular, the right-hand side gives the symbol $\sigma_{-1}([T_f, T_g])$ of the commutator of two Toeplitz operators, by the property (P4). One can also show that

$$\sigma_{-1}(T_{fg} - T_f T_g) = \tfrac{1}{t}\mathcal{L}(\overline{\partial}_b g, \overline{\partial}_b \overline{f}). \tag{7}$$

More generally, identifying — once for all — the half-line bundle Σ with $\partial\Omega \times \mathbf{R}_+$ via the map $(x, t\eta_x) \mapsto (x,t)$, let F, G be the functions on Σ given by

$$F(x,t) = t^{-k} f(x), \quad G(x,t) = t^{-m} g(x).$$

Then the Poisson bracket of F and G is given by

$$\{F,G\}_\Sigma = t^{-k-m-1}\Big(i\mathcal{L}(\overline{\partial}_b f, \overline{\partial}_b \overline{g}) - i\mathcal{L}(\overline{\partial}_b g, \overline{\partial}_b \overline{f}) + mgE_\perp f - kfE_\perp g\Big). \tag{8}$$

See [11, Corollary 8] for (6) and (8), from which (7) follows in the same way as in the proof of Theorem 9 there.

2.5 Dixmier trace. Recall that if A is a compact operator acting on a Hilbert space then its sequence of singular values $\{s_j(A)\}_{j=1}^{\infty}$ is the sequence of eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in nonincreasing order. In particular if $A \gg 0$ this will also be the sequence of eigenvalues of A in nonincreasing order. For $0 < p < \infty$ we say that A is in the Schatten ideal $\mathcal{S}_p$ if $\{s_j(A)\} \in l^p(\mathbf{Z}_{>0})$. If $A \gg 0$ is in $\mathcal{S}_1$, the trace class, then A has a finite trace and, in fact, $\operatorname{tr}(A) = \sum_j s_j(A)$. If however we only know that

$$s_j(A) = O(j^{-1}) \text{ or that}$$
$$S_k(A) := \sum_{j=1}^{k} s_j(A) = O(\log(1+k))$$

then A may have infinite trace. However in this case we may still try to compute its *Dixmier trace*, $\operatorname{tr}_\omega(A)$. Informally $\operatorname{tr}_\omega(A) = \lim_k \frac{1}{\log k} S_k(A)$ and this will actually be true in the cases of interest to us. We begin with the definition. Select a continuous positive linear functional ω on $l^\infty(\mathbf{Z}_{>0})$ and denote its value on $a = (a_1, a_2, ...)$ by $\lim_\omega(a_k)$. We require of this choice that $\lim_\omega(a_k) = \lim a_k$ if the latter exists. We require further that ω be *scale invariant*; a technical requirement that is fundamental for the theory but will not be of further concern to us.

Let $\mathcal{S}^{\mathrm{Dixm}}$ be the class of all compact operators A which satisfy

$$\Big(\frac{S_k(A)}{\log(1+k)}\Big) \in l^\infty.$$

With the norm defined as the l^∞-norm of the left-hand side, $\mathcal{S}^{\mathrm{Dixm}}$ becomes a Banach space. For a positive operator $A \in \mathcal{S}^{\mathrm{Dixm}}$, we define the Dixmier trace of A, $\operatorname{tr}_\omega A$, as $\operatorname{tr}_\omega A = \lim_\omega(\frac{S_k(A)}{\log(1+k)})$; tr_ω is then extended by linearity to all of $\mathcal{S}^{\mathrm{Dixm}}$. Although this definition does depend on ω, the operators A we consider are *measurable*, that is, the value of $\operatorname{tr}_\omega A$ is independent of the particular choice of ω. We refer to [7] for details and for discussion of the role of these functionals.

It is a result of Connes [6] that if Q is a ψdo on a compact manifold M of real dimension n and $\operatorname{ord}(Q) = -n$, then $Q \in \mathcal{S}^{\mathrm{Dixm}}$ and

$$\operatorname{tr}_\omega(Q) = \frac{1}{n!(2\pi)^n} \int_{\mathcal{T}^*(M)_1} \sigma(Q).$$

(Here $\mathcal{T}^*(M)_1$ denotes the unit sphere bundle in the cotangent bundle $\mathcal{T}^*(M)$, and the integral is taken with respect to a measure induced by any Riemannian metric on M; since $\sigma(Q)$ is homogeneous of degree $-n$, the value of the integral is independent of the choice of such metric.) It was shown in [11] that for Toeplitz operators T_Q on $\partial\Omega$, the "right" order for T_Q to belong to $\mathcal{S}^{\mathrm{Dixm}}$ is not $-\dim_{\mathbf{R}} \partial\Omega = -(2d-1)$, but rather $-\dim_{\mathbf{C}} \Omega = -d$. Namely, if T is a generalized Toeplitz operator on $H^2(\partial\Omega)$ of order $-d$, then $T \in \mathcal{S}^{\mathrm{Dixm}}$, T is measurable, and

$$\operatorname{tr}_\omega T = \frac{1}{d!(2\pi)^d} \int_{\partial\Omega} \sigma_{-d}(T)(x, \eta_x)\, \eta \wedge (d\eta)^{d-1}.$$

See Theorem 3 in [11].

3. Operators on $\mathcal{B}_m$

In this section we consider Toeplitz and Hankel operators on the Boutet de Monvel-Guillemin spaces $\mathcal{B}_m$ associated to "higher Szegö projectors". We deal in detail with the case $m = 2$ on the unit ball $\mathbf{B}^2$ of $\mathbf{C}^2$, and at the end discuss what happens for $m > 2$ and general domains.

Let thus $\Omega = \mathbf{B}^2$, with the usual defining function $\rho(z) = 1 - |z|^2$. The anti-holomorphic complex tangent space $\mathcal{T}''$ is then one-dimensional, spanned by the single vector field

$$\overline{Z} := L_{12} = z_1\overline{\partial}_2 - z_2\overline{\partial}_1 \qquad \text{on } \partial\mathbf{B}^2.$$

Its adjoint with respect to the inner product in $L^2(\partial\mathbf{B}^2)$ equals (by Stokes' theorem)

$$Z := \overline{z}_2\partial_1 - \overline{z}_1\partial_2.$$

The spaces $\mathcal{B}_m$, $m = 1, 2, \dots$, are given simply by

$$\mathcal{B}_m = L^2(\partial\mathbf{B}^2) \cap \operatorname{Ker} \overline{Z}^m.$$

As already noted, $\mathcal{B}_1 = H^2(\partial\mathbf{B}^2) \equiv H^2$. Let

$$H^2_0 := \{f \in H^2 : f(0) = 0\},$$

where, abusing notation slightly, we denote by f also the holomorphic extension of $f \in H^2$ into $\mathbf{B}^2$.

Proposition 3. *Every function in $\mathcal{B}_2$ can uniquely be written in the form $f + Zg$, with $f \in H^2$ and $g \in H^2_0$.*

Proof. By a simple computation

$$\overline{Z}Z(z_1^m z_2^n) = (m+n) z_1^m z_2^n = R(z_1^m z_2^n), \tag{9}$$

where $R := z_1\partial_1 + z_2\partial_2$ is the holomorphic radial derivative. Letting $S : z_1^m z_2^n \mapsto z_1^m z_2^n/(m+n)$ stand for the inverse of R on H^2_0, we thus have $\overline{Z}ZSh = h$ for all $h \in H^2_0$. Also, by direct check, $\overline{z}_2\partial_1 S$ and $\overline{z}_1\partial_2 S$ are both bounded from H^2_0 into $L^2(\partial\mathbf{B}^2)$, hence so is ZS. Now if $u \in \mathcal{B}_2$, then $\overline{Z}u =: h$ must be a function in H^2. By Stokes' theorem,

$$h(0) = \fint_{\partial\mathbf{B}^2} h = \fint_{\partial\mathbf{B}^2} \overline{Z}u = \fint_{\partial\mathbf{B}^2} u(\overline{Z}\mathbf{1}) = 0,$$

so in fact $h \in H^2_0$. Hence $\overline{Z}(u - ZSh) = 0$, so $u - ZSh$ is holomorphic and belongs to $L^2(\partial\mathbf{B}^2)$, i.e. $u - ZSh \in H^2$. Taking $f = u - ZSh$, $g = Sh$ thus yields the desired decomposition.
Uniqueness is immediate from the fact that $\overline{Z}(f + Zg) = \overline{Z}Zg = Rg$ together with the injectivity of R on H^2_0. $\square$

The last proof actually shows that

$$A : f \oplus g \mapsto f + Zg, \qquad f \in H^2,\ g \in H^2_0,$$

is a densely defined closed operator mapping its domain $H^2 \oplus SH^2_0$ bijectively onto $\mathcal{B}_2$. By abstract operator theory, we have the polar decomposition

$$A = U(A^*A)^{1/2},$$

where $U = A(A^*A)^{-1/2}$ is a partial isometry with initial space $\overline{\operatorname{Ran} A^*} = (\operatorname{Ker} A)^\perp = H^2 \oplus H^2_0$ and final space $\overline{\operatorname{Ran} A} = \mathcal{B}_2$; that is, U is a unitary map of $H^2 \oplus H^2_0$ onto $\mathcal{B}_2$, and $UU^* = \Pi_2$ is the orthogonal projection of $L^2(\partial\mathbf{B}^2)$ onto $\mathcal{B}_2$.
For $f \in L^\infty(\partial\mathbf{B}^2)$, the Toeplitz operator $T^{\mathcal{B}_2}_f$ — which, throughout this section, we will abbreviate just to $\mathcal{T}_f$ — can thus be written as

$$\mathcal{T}_f = UU^* f UU^*,$$

and is therefore unitarily equivalent to the operator

$$U^* f U = (A^*A)^{-1/2} A^* f A (A^*A)^{-1/2}$$

on $H^2 \oplus H^2_0$. Denote by $\Pi_0 = \Pi - \langle \cdot, \mathbf{1}\rangle\mathbf{1}$ the orthogonal projection of H^2 onto H^2_0; note that $\Pi - \Pi_0$ is a smoothing operator (its Schwartz kernel equals constant 1). The complex normal vector field

$$E = \sum_{j=1}^{2} (\overline{z}_j \overline{\partial}_j - z_j \partial_j)$$

is tangential to $\partial\mathbf{B}^2$ and $E|_{H^2} = -R$, i.e. $T_E = -R$. It follows that R is an elliptic gTo of order 1 with principal symbol $|\xi|/|\eta_x|$; and, hence, also its parametrix $\Pi_0 S \Pi_0$ is a gTo, of order -1 and with principal symbol $|\eta_x|/|\xi|$. Consequently (see Proposition 16 in [9] for detailed argument), the square root

$$\check{R} := \Pi_0 S^{-1/2} \Pi_0 : z_1^m z_2^n \mapsto \begin{cases} (m+n)^{-1/2} z_1^m z_2^n, & m+n > 0, \\ 0, & m = n = 0, \end{cases}$$

is a gTo of order $-\frac{1}{2}$ with symbol $|\eta_x|^{1/2}/|\xi|^{1/2}$.
After these preliminaries, we are ready for the main result of this section.

Theorem 4. *For $f \in C^\infty(\partial\mathbf{B}^2)$,*

$$U^* f U = \begin{bmatrix} T_f & T_{fZ}\check{R} \\ \check{R}T_{\overline{Z}f} & \check{R}T_{\overline{Z}fZ}\check{R} \end{bmatrix}. \tag{10}$$

Here the off-diagonal entries are of order $-\frac{1}{2}$, while $\check{R}T_{\overline{Z}fZ}\check{R} - T_f$ is of order -1; consequently, $\mathcal{T}_f \cong U^* f U = \begin{bmatrix} T_f & 0 \\ 0 & T_f \end{bmatrix} + A$ where A is a 2×2 matrix of gTo's of orders at most $-\frac{1}{2}$.

Here and below, $T_{\overline{Z}f}$ stands for the operator $u \mapsto \Pi\overline{Z}(fu)$, i.e. f is to be understood as a multiplier; we will write $T_{(\overline{Z}f)}$ for the operator T_g with the function $g = \overline{Z}f$ (i.e. $u \mapsto \Pi(u\overline{Z}f)$).

Proof. Writing $f \oplus g \in H^2 \oplus H^2_0$ as the column vector $\begin{bmatrix} f \\ g \end{bmatrix}$, we have $A = [I, Z]$, so

$$A^*A = \begin{bmatrix} I & \Pi Z \Pi_0 \\ \Pi_0 \overline{Z}\Pi & \Pi_0 \overline{Z} Z \Pi \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Pi_0 R \Pi_0 \end{bmatrix},$$

since $\overline{Z}Z = R$ while $\overline{Z}\Pi = \Pi Z = 0$. It follows that $(A^*A)^{-1/2} = \begin{bmatrix} I & 0 \\ 0 & \check{R} \end{bmatrix}$. Next,

$$A^* f A = \begin{bmatrix} I \\ \Pi_0 \overline{Z} \end{bmatrix} f[I, Z] = \begin{bmatrix} T_f & \Pi f Z \Pi_0 \\ \Pi_0 \overline{Z} f \Pi & \Pi_0 \overline{Z} f Z \Pi_0 \end{bmatrix},$$

and so

$$U^* f U = (A^*A)^{-1/2} A^* f A (A^*A)^{-1/2} = \begin{bmatrix} T_f & T_{fZ}\check{R} \\ \check{R}T_{\overline{Z}f} & \check{R}T_{\overline{Z}fZ}\check{R} \end{bmatrix},$$

proving (10).

We have $T_{\overline{Z}f} = T_{(\overline{Z}f)} + T_{f\overline{Z}} = T_{(\overline{Z}f)}$, since $\overline{Z}\Pi = 0$; similarly

$$T_{fZ} = T_{Zf} - T_{(Zf)} = -T_{(Zf)}, \tag{11}$$

since $\Pi Z = 0$. Thus $T_{\overline{Z}f}$ and T_{fZ} are gTo's of order 0, hence $T_{fZ}\check{R}$ and $\check{R}T_{\overline{Z}f}$ are indeed of order $-\frac{1}{2}$. Finally, by the Leibniz rule and (11),

$$T_{\overline{Z}fZ} = T_{f\overline{Z}Z} + T_{(\overline{Z}f)Z} = T_f R - T_{(Z\overline{Z}f)} = -T_f T_E - T_{(Z\overline{Z}f)} \tag{12}$$

is a gTo of order 1, so that $\check{R}T_{\overline{Z}fZ}\check{R}$ is of order 0, with principal symbol $\sigma(\check{R})^2\sigma(-T_E)\sigma(T_f) = \sigma(T_f)$. Thus $\check{R}T_{\overline{Z}fZ}\check{R} - T_f$ is of order -1, and the second part of the theorem follows. $\square$

Clearly, the second part of the last theorem is precisely the statement of Theorem 1 from the Introduction.

Using the standard relation

$$\mathcal{H}^*_{\overline{f}}\mathcal{H}_g = \mathcal{T}_{fg} - \mathcal{T}_f\mathcal{T}_g, \tag{13}$$

one can also get information about the "higher Hankel" operators $\mathcal{H}_f \equiv \mathcal{H}^{(2)}_f$: $u \mapsto (I - \Pi_2)(fu)$, $u \in \mathcal{B}_2$.

Theorem 5. *For $f, g \in C^\infty(\partial\mathbf{B}^2)$,*

(14)
$$U^*\mathcal{H}_{\overline{f}}^*\mathcal{H}_g U = \begin{bmatrix} T_{fg} - T_f T_g - T_{fZ}\check{R}^2 T_{\overline{Z}g} & (T_{fgZ} - T_f T_{gZ} - T_{fZ}\check{R}^2 T_{\overline{Z}gZ})\check{R} \\ \check{R}(T_{\overline{Z}fg} - T_{\overline{Z}f}T_g - T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}g}) & \check{R}(T_{\overline{Z}fgZ} - T_{\overline{Z}f}T_{gZ} - T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}gZ})\check{R} \end{bmatrix}.$$

Here the orders of the entries are at most $\begin{bmatrix} -2 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix}$, *with*

$$\sigma_{-1}(\check{R}(T_{\overline{Z}fgZ} - T_{\overline{Z}f}T_{gZ} - T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}gZ})\check{R}) = -2(Zf)(\overline{Z}g).$$

Proof. The formula (14) follows directly from (10) and (13). In the upper left corner, $\sigma(T_{fg}) = fg = \sigma(T_f T_g)$, so $T_{fg} - T_f T_g$ is of order -1, and so is $T_{fZ}\check{R}^2 T_{\overline{Z}g}$ since T_{fZ} and $T_{\overline{Z}g}$ are of order 0 by (11) while $\check{R}$ is of order $-\frac{1}{2}$. In the bottom right corner $T_{\overline{Z}fgZ}$ has order 1 and symbol $fg/\sigma(\check{R})^2$ by (12), and so does $T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}gZ}$ (for the same reason); so their difference is of order 0, while $T_{\overline{Z}f}$ and T_{gZ} are also of order 0 by (11); so the whole entry has the same order as $\check{R}^2$, i.e. -1. Finally, in the upper right corner, $\sigma(T_{fgZ}) = -Z(fg)$ and $\sigma(T_f T_{gZ}) = -f(Zg)$ by (11), so $\sigma(T_{fgZ} - T_f T_{gZ}) = -(Zf)g$; while $\sigma(T_{fZ}\check{R}^2 T_{\overline{Z}gZ}) = -(Zf)\sigma(\check{R})^2 \frac{g}{\sigma(\check{R})^2} = -(Zf)g$ as well, all these gTo's being of order 0. Hence $T_{fgZ} - T_f T_{gZ} - T_{fZ}\check{R}^2 T_{\overline{Z}gZ}$ is of order at most -1, and the whole entry is of order at most $-1 + \operatorname{ord}(\check{R}) = -\frac{3}{2}$.

We claim that $\sigma_{-1}(T_{fg} - T_f T_g - T_{fZ}\check{R}^2 T_{\overline{Z}g}) = 0$, so the upper left corner in (14) is in fact of order -2 (at most). Note first of all that $L''(\overline{Z}, \overline{Z}) = 1$, whence by a short computation $Z''_{\overline{\partial}_b g} = -(Z\overline{g})\overline{Z}$ and $L''(\overline{\partial}_b f, \overline{\partial}_b g) = -(\overline{Z}f)(Z\overline{g})$. Therefore by (7),

$$\sigma_{-1}(T_{fg} - T_f T_g) = -\frac{1}{t}(Zf)(\overline{Z}g), \qquad \xi = t\eta_x. \tag{15}$$

Consequently,

$$\begin{aligned} &\sigma_{-1}(T_{fg} - T_f T_g - T_{fZ}\check{R}^2 T_{\overline{Z}g}) \\ &\qquad = \sigma_{-1}(T_{fg} - T_f T_g + T_{(Zf)}\check{R}^2 T_{(\overline{Z}g)}) \qquad \text{by (11)} \\ &\qquad = -\tfrac{1}{t}(Zf)(\overline{Z}g) + (Zf)\sigma_{-1}(\check{R}^2)(\overline{Z}g) \qquad \text{by (15)} \\ &\qquad = 0 \qquad \text{since } \sigma(\check{R}^2) = \frac{1}{t}, \end{aligned}$$

proving the claim.

It remains to compute the principal symbol of the bottom right corner in (14). Now by (11) and (12) again

$$\begin{aligned} &T_{\overline{Z}fgZ} - T_{\overline{Z}f}T_{gZ} - T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}gZ} \\ &\quad = T_{-fgE-(Z\overline{Z}fg)} + T_{(\overline{Z}f)}T_{(Zg)} - T_{-fE-(Z\overline{Z}f)}\check{R}^2 T_{-gE-(Z\overline{Z}g)} \\ &\qquad = -T_{fg}T_E - T_{(Z\overline{Z}f)g+(Zf)(\overline{Z}g)+(\overline{Z}f)(Zg)+f(Z\overline{Z}g)} + T_{(\overline{Z}f)}T_{(Zg)} \\ &\qquad - T_{fE}\check{R}^2 T_{gE} - T_{fE}\check{R}^2 T_{(Z\overline{Z}g)} - T(Z\overline{Z}f)\check{R}^2 T_{gE} - T_{(Z\overline{Z}f)}\check{R}^2 T_{(Z\overline{Z}g)}. \end{aligned}$$

The last summand is of order -1, while $-T_{fE}\check{R}^2 = -T_f T_E \check{R}^2 = T_f R\check{R}^2 = T_f - T_f(\Pi - \Pi_0) \sim T_f$, and similarly $\sigma_0(-\check{R}^2 T_{gE}) = \sigma_0(-T_{gE}\check{R}^2) = \sigma_0(T_g) = g$. Thus

$$\begin{aligned}\sigma_0(T_{\overline{Z}fgZ} &- T_{\overline{Z}f}T_{gZ} - T_{\overline{Z}fZ}\check{R}^2 T_{\overline{Z}gZ}) \\ &= \sigma_0(-T_{fgE} - T_{fE}\check{R}^2 T_{gE}) - (Zf)(\overline{Z}g) \\ &= \sigma_{-1}(T_{fg} + T_{fE}\check{R}^2 T_g)\sigma_1(-T_E) - (Zf)(\overline{Z}g) \\ &= \sigma_{-1}(T_{fg} - T_f T_g)\sigma_1(-T_E) - (Zf)(\overline{Z}g) \\ &= -\frac{(Zf)(\overline{Z}g)}{t}t - (Zf)(\overline{Z}g) \qquad \text{by (15)} \\ &= -2(Zf)(\overline{Z}g),\end{aligned}$$

completing the proof. □

Corollary 6. *For $f, g \in C^\infty(\partial\mathbf{B}^2)$, $(\mathcal{H}^*_{\overline{f}}\mathcal{H}_g)^2 \in \mathcal{S}^{\mathrm{Dixm}}$, is measurable, and*

$$\operatorname{tr}_\omega(\mathcal{H}^*_{\overline{f}}\mathcal{H}_g)^2 = 4\operatorname{tr}_\omega(H^*_{\overline{f}}H_g)^2 = 2\int_{\partial\mathbf{B}^2}(Zf)^2(\overline{Z}g)^2\,d\tilde{\sigma},$$

where $d\tilde{\sigma}$ stands for the normalized surface measure on $\partial\mathbf{B}^2$.

Proof. Immediate from the last theorem and Theorem 11 in [11]. □

Remark. By a similar computation as above, one can show that the principal symbol of the upper right corner in (14) is

$$\sigma_{-3/2}((T_{fgZ} - T_f T_{gZ} - T_{fZ}\check{R}^2 T_{\overline{Z}gZ})\check{R}) = (Z^2 f)(\overline{Z}g).$$

We have not tried to compute $\sigma_{-2}(T_{fg} - T_f T_g - T_{fZ}\check{R}^2 T_{\overline{Z}g})$, which is probably going to be more tricky (but is of no relevance from the point of view of e.g. Corollary 6). □

The last theorem and corollary extend also to the spaces $\mathcal{B}_m(\partial\mathbf{B}^2) = \operatorname{Ker}\overline{Z}^m \cap L^2(\partial\mathbf{B}^2)$ for $m > 2$. Indeed, generalizing (9), one has

$$\overline{Z}^m Z^m = m!R(R-1)\dots(R-m+1),$$

and as before one concludes that

$$\mathcal{B}_m = H^2 + ZH^2_0 + Z^2 H_{00} + \dots + Z^{m-1}H^2_{0^{m-1}},$$

where we denoted $H^2_{0^j} := \{f \in H^2 : \partial^\alpha f(0) = 0\ \forall|\alpha| < j\}$; and following the same argument as in the proof of Theorem 5, one gets a unitary equivalence of the Toeplitz operator $T^{\mathcal{B}_m}_f$ on $\mathcal{B}_m$ to a certain $m \times m$ matrix of gTo's on H^2. We are leaving the details to the interested reader.

As already mentioned, what we perceive as the main drawback of the spaces $\mathcal{B}_m$ — despite their similarity to the ordinary H^2 from the "microlocal" point of view — is that they fail to be invariant under biholomorphic equivalence. In fact, assume that $\phi = (\phi_1, \phi_2)$ is a biholomorphic automorphism of $\mathbf{B}^2$. Then by the chain rule

$$\overline{Z}(f \circ \phi) = \sum_{k=1}^{2} (\overline{\partial}_k f) \circ \phi \cdot \overline{Z\phi_k}.$$

Since $0 = \overline{Z}\mathbf{1} = \overline{Z}(\phi_1\overline{\phi}_1 + \phi_2\overline{\phi}_2) = \phi_1\overline{Z\phi}_1 + \phi_2\overline{Z\phi}_2$, we see that

$$\overline{Z}(f \circ \phi) = a_\phi \cdot (\overline{Z}f) \circ \phi,$$

where $a_\phi = -\frac{\overline{Z\phi}_1}{\phi_2} = \frac{\overline{Z\phi}_2}{\phi_1}$. Hence

$$\overline{Z}^2(f \circ \phi) = \overline{Z}[a_\phi \cdot (\overline{Z}f) \circ \phi] = a_\phi^2 \cdot (\overline{Z}^2 f) \circ \phi + \overline{Z}a_\phi \cdot (\overline{Z}f) \circ \phi.$$

Consequently, $f \in \mathcal{B}_2 \implies f \circ \phi \in \mathcal{B}_2$ would mean that $\overline{Z}a_\phi \cdot (\overline{Z}f) \circ \phi = 0$ $\forall f \in \mathcal{B}_2$, so — taking f so that $\overline{Z}f$ are e.g. the coordinate functions — $\overline{Z}a_\phi = 0$, whence $\overline{Z}^2\overline{\phi}_1 = \overline{Z}^2\overline{\phi}_2 = 0$. The last condition is easily seen to be fulfilled only if ϕ is an affine map, showing that $\mathcal{B}_2 \circ \phi \not\subset \mathcal{B}_2$ for other ϕ.

Yet another drawback of the spaces $\mathcal{B}_m$ is their dependence on the — in some sense arbitrary — choice of the special vector fields L_{jk}, $1 \le j < k \le d$, in (1) (reducing to just $L_{12} = \overline{Z}$ for $d = 2$). Namely, the "natural" definition would rather be

$$\begin{gathered}\tilde{\mathcal{B}}_m := \{u \in L^2(\partial\Omega) : \overline{X}_1 \dots \overline{X}_m u = 0 \\ \text{for any } C^\infty \text{ sections } X_1, \dots, X_m \text{ of } \mathcal{T}''(\partial\Omega)\},\end{gathered}$$

i.e. $\overline{X}_1 \dots \overline{X}_m u = 0$ for any m-tuple of anti-holomorphic complex tangential vector fields $\overline{X}_1, \dots, \overline{X}_m$ on $\partial\Omega$. However, unfortunately, one has $\tilde{\mathcal{B}}_m = H^2(\partial\Omega)$ for all $m \ge 1$. In fact, e.g. for $\Omega = \mathbf{B}^2$ and $m = 2$ again, the condition $f \in \tilde{\mathcal{B}}_2$ means that

$$\overline{Z}a\overline{Z}f = 0 \qquad \forall a \in C^\infty(\partial\Omega),$$

or $(\overline{Z}a)(\overline{Z}f) + a\overline{Z}^2 f = 0$ for all a. Taking $a = \mathbf{1}$ gives $\overline{Z}^2 f = 0$, so $(\overline{Z}a)(\overline{Z}f) = 0$ for all a; and taking $a = Zz_j$, so that $\overline{Z}a = Rz_j = z_j$, then yields $z_j\overline{Z}f = 0$ $\forall j$. Thus $\overline{Z}f = 0$ and $f \in H^2(\partial\mathbf{B}^2)$, as claimed.

We therefore proceed to describe a different variant of the spaces $\mathcal{B}_m$, which does not suffer from the above deficiencies.

4. Toeplitz operators on $\mathcal{C}_m$

Throughout this section, we assume that Ω is a bounded strictly pseudoconvex domain in $\mathbf{C}^d$, $d>1$, with smooth boundary $\partial\Omega$. We fix once for all a positively-signed defining function ρ for Ω, i.e. $\rho \in C^\infty(\overline{\Omega})$ satisfies $\rho>0$ on Ω and $\rho = 0 < |\nabla\rho|$ on $\partial\Omega$. Assume that Ω is equipped with a Kähler metric

$$g_{j\overline{k}} = \partial_j \overline{\partial}_k \Psi,$$

where Ψ is a real-valued strictly-plurisubharmonic function (Kähler potential) on Ω, which we assume to be of the form

$$\Psi \approx \log\frac{1}{\rho} + \sum_{j=0}^{\infty} (\rho^M \log\rho)^j \eta_j \tag{16}$$

with an integer $M \geq 2$ and $\eta_j \in C^\infty(\overline{\Omega})$. Here "$\approx$" is to be understood in the sense of "resolution of singularities", i.e. it means that the difference $\Psi - \log\frac{1}{\rho} - \sum_{j=1}^{N-1}$ should belong to $C^{MN-1}(\overline{\Omega})$ and vanish on $\partial\Omega$ to order $MN-1$. It is known that e.g. the Bergman metric on Ω is of this form (with $M=d+1$), as is the "Szegö metric" corresponding to $\Psi(z) = \frac{1}{d}\log K_{\mathrm{Sz}}(z,z)$ where K_{Sz} is the invariantly defined Szegö kernel on Ω (then $M=d$), and likewise the Poincare metric (i.e. the Kähler-Einstein metric) on Ω corresponding to $\Psi = \log\frac{1}{u}$ with u the solution of the Monge-Ampere equation (then $M = d+1$ again); see for instance the survey [5] and the references therein. Finally, we equip Ω with the weight

$$w = \rho^\nu, \qquad \nu \in \mathbf{R},$$

where ν will be sufficiently large as precised further below.

As described in the introduction, we then have the kernels of powers of the Cauchy-Riemann operator

$$\mathcal{C}_m := \{ f \text{ on } \Omega : \ \overline{D}^m f = 0 \}$$

and the associated "higher Cauchy-Riemann spaces"

$$\mathcal{C}_{m,w} := \mathcal{C}_m \cap L^2(\Omega, w),$$

with their Toeplitz and Hankel operators

$$\mathcal{T}_\phi^{(m,w)} : u \mapsto \mathbf{\Pi}^{(m,w)}(\phi u), \qquad \mathcal{H}_\phi^{(m,w)} : u \mapsto (I - \mathbf{\Pi}^{(m,w)})(\phi u), \qquad u \in \mathcal{C}_{m,w},$$

where $\phi \in L^\infty(\Omega)$ and $\mathbf{\Pi}^{(m,w)} : L^2(\Omega,w) \to \mathcal{C}_{m,w}$ is the orthogonal projection. We will usually write just $\mathcal{T}_\phi, \mathcal{H}_\phi$ instead of $\mathcal{T}_\phi^{(m,w)}, \mathcal{H}_\phi^{(m,w)}$ if there is no danger of confusion.

Clearly $\mathcal{T}_\phi$ and $\mathcal{H}_\phi$ have the usual properties of Toeplitz and Hankel operators, namely they depend linearly on ϕ, $\mathcal{T}_{\mathbf{1}} = I$, $\mathcal{T}_\phi^* = \mathcal{T}_{\overline{\phi}}$, and $\|\mathcal{T}_\phi\| \leq \|\phi\|_\infty$ and similarly for $\mathcal{H}_\phi$ (by Cauchy-Schwarz).

In the context of bounded symmetric domains, the next assertion is proved as Proposition 2.4 in Shimura [17].

Proposition 7. *One has $f \in \mathcal{C}_2$ if and only if f can be written in the form*

$$f = h^k \partial_k \Psi + h \tag{17}$$

where h^k, $k = 1, \dots, d$, and h are holomorphic functions; the representation (17) *is unique.*
More generally, $f \in \mathcal{C}_m$ if and only if f can be written in the form

$$f = \sum_{j=0}^{m-1} h^{k_1 \dots k_j} \Psi_{k_1} \dots \Psi_{k_j},$$

with $h^{k_1 \dots k_j}$, $1 \le k_1 \le \dots \le k_j \le d$, holomorphic on Ω; and this representation is unique.

Here we have introduced the shorthand

$$\Psi_k := \partial_k \Psi,$$

and also started using the (Einstein) summation convention of automatically summing over any index that appears twice in the formula.

Proof. For $m = 2$, we have by the definition of $\overline{D}$

$$\overline{D}(h^k \Psi_k) = g^{\bar{l}m} \overline{\partial}_l (h^k \Psi_k) = g^{\bar{l}m} h^k (\overline{\partial}_l \Psi_k) = g^{\bar{l}m} h^k g_{k\bar{l}} = \delta_k^m h^k = h^m$$

since h^k are holomorphic. Thus indeed $\overline{D}^2(h^k \Psi_k) = 0$, and since $\overline{D} h = 0$, any f of the form (17) belongs to $\mathcal{C}_2$. Conversely, given $f \in \mathcal{C}_2$, we must have $\overline{D} f = h^k$ for some $h^k \in \operatorname{Ker} \overline{D}$ i.e. for some holomorphic functions h^k, $k = 1, \dots, d$, on Ω. Then by the above computation, $\overline{D}(f - h^k \Psi_k) = 0$, i.e. $h := f - h^k \Psi_k$ is a holomorphic function, so f is of the form (17). Finally, if f is of the form (17) and $f = 0$, then $h^k = \overline{D} f = 0$, hence $h = f = 0$, proving uniqueness.
The proof for general m is the same. $\square$

Note as a corollary that if $f \in \mathcal{C}_m$ and g is holomorphic, then $gf \in \mathcal{C}_m$. It follows, in particular, that for g holomorphic, the Toeplitz operator $\mathcal{T}_g$ is just the operator of "multiplication by g", and in fact

$$\mathcal{T}_\phi \mathcal{T}_g = \mathcal{T}_{\phi g}, \quad \mathcal{T}_{\overline{g}} \mathcal{T}_\phi = \mathcal{T}_{\overline{g}\phi}, \quad \mathcal{H}_g = 0, \qquad \text{for } g \text{ holomorphic and any } \phi. \tag{18}$$

As in Section 3, our strategy now will be to transfer the Toeplitz operators $\mathcal{T}_\phi$ to (the direct sum of copies of) the Hardy space. To avoid too many indices, we will again deal only with the case $m = 2$. Let $\kappa : (\bigoplus_{j=1}^d H^2(\partial\Omega)) \oplus H^2(\partial\Omega) \to L^2(\Omega, w)$ be the operator defined by

$$\kappa \begin{bmatrix} u_j \\ u \end{bmatrix} = \sum_{j=1}^d \Psi_j \mathbf{K} u_j + \mathbf{K} u \tag{19}$$

where $\mathbf{K}$ is the Poisson extension operator from §2.3. Take again the polar decomposition of κ,

$$\kappa = U(\kappa^*\kappa)^{1/2},$$

where U is a partial isometry with initial space $\overline{\operatorname{Ran}\kappa^*} = (\operatorname{Ker}\kappa)^\perp = 0^\perp = \oplus^{d+1}H^2(\partial\Omega)$ and final space $\overline{\operatorname{Ran}\kappa}$, i.e. U is a unitary operator from $\oplus^{d+1}H^2(\partial\Omega)$ onto $\mathcal{C}_{2,w}$ by the last proposition. Also, $U^*U = I$ on $\oplus^{d+1}H^2(\partial\Omega)$ while $UU^* = \mathbf{\Pi}^{(2,w)}$, the projection onto $\mathcal{C}_{2,w}$. The Toeplitz operator $\mathcal{T}_\phi = UU^*\phi|_{\operatorname{Ran}UU^*}$ is therefore unitarily equivalent to the operator

$$U^*\phi U = (\kappa^*\kappa)^{-1/2}\kappa^*\phi\kappa(\kappa^*\kappa)^{-1/2} \tag{20}$$

on the direct sum $\oplus^{d+1}H^2(\partial\Omega)$ of $d+1$ copies of $H^2(\partial\Omega)$.

Lemma 8. *Let $\nu > 1$. Writing as in* (19) *the elements of $\oplus^{d+1}H^2(\partial\Omega)$ as column vectors $\begin{bmatrix} u_j \\ u \end{bmatrix} = [u_1, u_2, \dots, u_d, u]^t$, we have*

$$\kappa^*\phi\kappa = \begin{bmatrix} T_{\Lambda[\overline{\Psi}_k\phi w\Psi_j]} & T_{\Lambda[\overline{\Psi}_k\phi w]} \\ T_{\Lambda[\phi w\Psi_j]} & T_{\Lambda[\phi w]} \end{bmatrix}. \tag{21}$$

(So here the right-hand side is a $(d+1)\times(d+1)$ matrix of gTo's on $H^2(\partial\Omega)$, with $j = 1, \dots, d$ the column index and $k = 1, \dots, d$ the row index.)

Proof. For any $u, v \in C^\infty_{\text{hol}}(\overline{\Omega})$, we have

$$\begin{aligned} \langle \phi\mathbf{K}u, \mathbf{K}v\rangle_{L^2(\Omega,w)} &= \langle w\phi\mathbf{K}u, \mathbf{K}v\rangle_{L^2(\Omega)} \\ &= \langle \mathbf{K}^*w\phi\mathbf{K}u, v\rangle_{L^2(\partial\Omega)} \\ &= \langle \Lambda[\phi w]u, v\rangle_{L^2(\partial\Omega)} \\ &= \langle T_{\Lambda[\phi w]}u, v\rangle_{H^2(\partial\Omega)}, \end{aligned}$$

and similarly for $\overline{\Psi}_k\phi w\Psi_j$, $\overline{\Psi}_k\phi w$ and $\phi w\Psi_j$ in the place of ϕw. By (19), the claim follows. $\square$

For brevity, let us denote the collection of all functions in $C^\infty(\Omega)$ of the form $\sum_{j=0}^\infty \eta_j(\rho^m\log\rho)^j$ as in (16) by $\mathcal{A}_M$ (thus $\Psi - \log\frac{1}{\rho} \in \mathcal{A}_M$), and also denote

$$\rho_j := \partial_j\rho.$$

By the Leibniz rule, we have $\rho\Psi_j \in \rho_j + \rho\mathcal{A}_{M-1} \subset \mathcal{A}_M$. From the facts reviewed in §2.3, we thus conclude that for $\phi \in C^\infty(\overline{\Omega})$, $\Lambda[\phi w\Psi_j] = \Lambda[\phi\rho^{\nu-1}(\rho\Psi_j)]$ is an operator in $\Psi^{-\nu}_{\log}$, with log terms appearing only at orders $-\nu - M$ and lower (in particular, there is no log term in the leading symbol), and with principal symbol $\sigma_{-\nu}(\Lambda[\phi w\Psi_j]) = \frac{\Gamma(\nu)|\eta|^{\nu-1}\phi\rho_j}{2|\xi|^\nu}$. Similarly for the other entries in (21),

$$\sigma_{-\nu}(\Lambda[\overline{\Psi}_k\phi w]) = \frac{\Gamma(\nu)|\eta|^{\nu-1}\overline{\rho}_k\phi}{2|\xi|^\nu}, \quad \sigma_{1-\nu}(\Lambda[\overline{\Psi}_k\phi w\Psi_j]) = \frac{\Gamma(\nu-1)|\eta|^{\nu-2}\overline{\rho}_k\phi\rho_j}{2|\xi|^{\nu-1}},$$

$$\sigma_{-\nu-1}(\Lambda[\phi w]) = \frac{\Gamma(\nu+1)|\eta|^\nu\phi}{2|\xi|^{\nu+1}}.$$

In particular, for $\phi = \mathbf{1}$, the operator $\kappa^*\kappa$ has for its entries gTo's of orders $\begin{bmatrix} 1-\nu & -\nu \\ -\nu & -\nu-1 \end{bmatrix}$, with leading symbol $\sigma_{1-\nu}(\kappa^*\kappa) = \frac{\Gamma(\nu-1)|\eta|^{\nu-2}}{2|\xi|^{\nu-1}} \begin{bmatrix} \overline{\rho}_k\rho_j & 0 \\ 0 & 0 \end{bmatrix}$. This is obviously not elliptic, so it is not clear at first whether $(\kappa^*\kappa)^{-1}$, not to say $(\kappa^*\kappa)^{-1/2}$, are given by generalized Toeplitz operators. Our next task is to show that in fact they are; the main role in this result is played by the "sub-principal" order terms of $\kappa^*\kappa$.

Denote by $Q = [Q_{kj}]_{j,k=1}^d$ the $d \times d$ matrix of gTo's

$$Q_{kj} := T_{\Lambda[\overline{\Psi}_k w \Psi_j]} - T_{\Lambda[\overline{\Psi}_k w]} T^{-1}_{\Lambda[w]} T_{\Lambda[w\Psi_j]} \tag{22}$$

(where as before j is the column index and k the row index). Since $T_{\Lambda[w]}$ is elliptic of order $-\nu-1$, it follows from the formulas for symbols above that Q_{kj} are gTo's of order $1-\nu$ (for $\nu > 1$), with principal symbols

$$\begin{aligned} \sigma_{1-\nu}(Q_{kj}) &= \sigma_{1-\nu}(\Lambda[\overline{\Psi}_k w \Psi_j]) - \frac{\sigma_{-\nu}(\Lambda[\overline{\Psi}_k w])\sigma_{-\nu}(\Lambda[w\Psi_j])}{\sigma_{-\nu-1}(\Lambda[w])}\Big|_\Sigma \\ &= \frac{\Gamma(\nu-1)|\eta|^{\nu-2}}{2\nu|\xi|^{\nu-1}}\overline{\rho}_k\rho_j. \end{aligned} \tag{23}$$

Denote

$$Z_{kj} := Q_{kj} - \frac{1}{\nu} T_{\overline{\rho}_k} T_{\Lambda[\rho^{\nu-2}]} T_{\rho_j}. \tag{24}$$

In view of (23) and (5), we have $\sigma_{1-\nu}(Z_{kj}) = 0$, so in fact Z_{kj} is a gTo of order at most $-\nu$.

In addition to our positively signed defining function ρ, we will also use the negatively signed defining function $r := -\rho$, and denote again by

$$r_j := \partial_j r = -\rho_j, \quad r_{\overline{k}} := \overline{\partial}_k r = -\overline{\rho}_k, \qquad r_{j\overline{k}} := \partial_j \overline{\partial}_k r = -\partial_j\overline{\partial}_k \rho$$

its partial derivatives as indicated.

Proposition 9. *Assume that $\nu > 1$. Then there exists a function $c \in C^\infty(\partial\Omega)$ such that, using again the identification $(x, t\eta_x) \in \Sigma \cong \partial\Omega \times \mathbf{R}_+ \ni (x,t)$,*

$$\sigma_{-\nu}(Z_{kj}) = \tag{25}$$
$$\frac{\sigma(T_{\Lambda[\rho^{\nu-1}]})}{\nu-1}\Big(r_{j\overline{k}} + \frac{|\eta|}{\nu} r_{\overline{k}} \mathcal{L}(\overline{\partial}_b r_j, \overline{\partial}_b \frac{1}{|\eta|}) + \frac{|\eta|}{\nu} r_j \mathcal{L}(\overline{\partial}_b \frac{1}{|\eta|}, \overline{\partial}_b r_{\overline{k}}) + c r_{\overline{k}} r_j\Big).$$

Proof. Denote, quite generally, for $\phi, \psi \in C^\infty(\partial\Omega)$,

$$\sigma_{-\nu}(T_{\Lambda[\overline{\phi}\rho^{\nu-2}\psi]} - T_{\Lambda[\overline{\phi}\rho^{\nu-1}]} T^{-1}_{\Lambda[\rho^\nu]} T_{\Lambda[\rho^{\nu-1}\psi]} - \frac{1}{\nu} T_{\overline{\phi}} T_{\Lambda[\rho^{\nu-2}]} T_\psi) =: Q(\overline{\phi}, \psi). \tag{26}$$

Again, first of all, by the same computation as in (23), we see that the operator on the left-hand side of (26) is a gTo of order $1-\nu$ with $\sigma_{1-\nu}$ vanishing, hence, indeed, is in fact a gTo of order $-\nu$, so the $\sigma_{-\nu}$ in (26) makes sense.
Next, from the relation $\langle T_{\Lambda[w]}u, v\rangle = \int_\Omega wu\overline{v}\,dz$, $u, v \in H^2(\partial\Omega)$, and the fact that T_f is just the operator of "multiplication by f" when f is holomorphic, we have

$$T_{\Lambda[wf]} = T_{\Lambda[w]}T_f, \quad T_{\Lambda[\overline{g}w]} = T_{\overline{g}}T_{\Lambda[w]} \tag{27}$$

for holomorphic f, g and arbitrary w. For our bilinear form Q from (26), this means that

$$Q(\overline{g}\overline{\phi}, \psi f) = \overline{g}Q(\overline{\phi}, \psi)f \qquad \text{for } f, g \text{ holomorphic.}$$

Since, by general theory, $Q(\overline{\phi}, \psi)$ depends only on finitely many derivatives of ϕ and ψ (see the bottom of p. 616 in [11] for the detailed argument), it is therefore enough to evaluate $Q(\overline{\phi}, \psi)$ for $\overline{\phi} = f$ and $\psi = \overline{g}$ with f, g holomorphic. In that case we have, using (27) again,

$$\begin{aligned} Q(f, \overline{g}) &= \sigma_{-\nu}(T_{\Lambda[f\rho^{\nu-2}\overline{g}]} - T_{\Lambda[f\rho^{\nu-1}]}T^{-1}_{\Lambda[\rho^\nu]}T_{\Lambda[\rho^{\nu-1}\overline{g}]} - \frac{1}{\nu}T_f T_{\Lambda[\rho^{\nu-2}]}T_{\overline{g}}) \\ &= \sigma_{-\nu}(T_{\overline{g}}T_{[\nu-2]}T_f - T_{[\nu-1]}T_f T^{-1}_{[\nu]}T_{\overline{g}}T_{[\nu-1]} - \frac{1}{\nu}T_f T_{[\nu-2]}T_{\overline{g}}), \end{aligned}$$

where, for typographical reasons, we have started writing just $[\nu]$ for $\Lambda[\rho^\nu]$. Abusing notation slightly, we will also write $\{\cdot,\cdot\}$ for $\frac{1}{i}\{\cdot,\cdot\}_\Sigma$, so that (P4) reads simply $\sigma([T_P, T_Q]) = \{\sigma(T_P), \sigma(T_Q)\}$. Then

$$\sigma_{-\nu}(T_{\overline{g}}T_{[\nu-2]}T_f - T_f T_{[\nu-2]}T_{\overline{g}}) = \{g, [\nu-2]\}f + [\nu-2]\{\overline{g}, f\} + \{[\nu-2], f\}\overline{g},$$

where we started denoting by $[\nu]$ also the symbol of $T_{[\nu]}$. Similarly,

$$\begin{aligned} &\sigma_{-\nu-1}(T_{[\nu-1]}T_f - T_f T_{[\nu-1]}) = \{[\nu-1], f\}, \\ &\sigma_{-\nu-1}(T_{\overline{g}}T_{[\nu-1]} - T_{[\nu-1]}T_{\overline{g}}) = \{\overline{g}, [\nu-1]\}, \end{aligned}$$

and, consequently,

$$\begin{aligned} Q(f, \overline{g}) = \{g, [\nu-2]\}f &+ [\nu-2]\{\overline{g}, f\} + \{[\nu-2], f\}\overline{g} \\ &- \{[\nu-1], f\}\frac{[\nu-1]}{[\nu]}\overline{g} - \{\overline{g}, [\nu-1]\}\frac{[\nu-1]}{[\nu]}f \\ &+ \sigma_\nu(T_f T_{[\nu-2]}T_{\overline{g}} - T_f T_{[\nu-1]}T^{-1}_{[\nu]}T_{[\nu-1]}T_{\overline{g}} - \frac{1}{\nu}T_f T_{[\nu-2]}T_{\overline{g}}). \end{aligned}$$

The last summand equals just $fQ(\mathbf{1}, \mathbf{1})\overline{g}$. Since $[\nu] = \frac{\Gamma(\nu+1)}{2|\eta|t^{\nu+1}}$ by (5), we have $\frac{[\nu-1]}{[\nu]} = \frac{t}{\nu}$. By the Leibniz rule for the Poisson bracket

$$\{\overline{g}, ab\} = a\{\overline{g}, b\} + \{\overline{g}, a\}b,$$

we have $\{\overline{g}, t^\nu\} = \nu t^{\nu-1}\{\overline{g}, t\}$, and so, using again (5),

$$\begin{aligned}
\{\overline{g}, [\nu-2]\} &= \frac{\Gamma(\nu-1)}{2}\{\overline{g}, \frac{t^{1-\nu}}{|\eta|}\} \\
&= \frac{\Gamma(\nu-1)}{2|\eta|}(1-\nu)t^{-\nu}\{\overline{g}, t\} + \frac{\Gamma(\nu-1)}{2}t^{1-\nu}\{\overline{g}, \frac{1}{|\eta|}\}, \\
\frac{t}{\nu}\{\overline{g}, [\nu-1]\} &= \frac{\Gamma(\nu)}{2\nu|\eta|}(-\nu)t^{-\nu}\{\overline{g}, t\} + \frac{\Gamma(\nu)}{2\nu}t^{1-\nu}\{\overline{g}, \frac{1}{|\eta|}\},
\end{aligned}$$

whence

$$\{\overline{g}, [\nu-2]\} - \frac{[\nu-1]}{[\nu]}\{\overline{g}, [\nu-1]\} = \frac{[\nu-2]|\eta|}{\nu}\{\overline{g}, \frac{1}{|\eta|}\}.$$

Similarly for the corresponding brackets with f. Finally by (6), when either f or g is holomorphic, we have

$$\{\overline{g}, f\} = \frac{1}{t}\mathcal{L}(\overline{\partial}_b\overline{g}, \overline{\partial}_b\overline{f}).$$

Putting everything together, we thus arrive at

$$Q(f, \overline{g}) = \tag{28}$$

$$[\nu-2]\frac{f\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b\overline{g}, \overline{\partial}_b\frac{1}{|\eta|}) + \overline{g}\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b\frac{1}{|\eta|}, \overline{\partial}_b\overline{f}) + \mathcal{L}(\overline{\partial}_b\overline{g}, \overline{\partial}_b\overline{f})}{t} + f\overline{g}Q(\mathbf{1}, \mathbf{1}),$$

and so

$$Q(\overline{\phi}, \psi) = \tag{29}$$

$$[\nu-2]\frac{\overline{\phi}\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b\psi, \overline{\partial}_b\frac{1}{|\eta|}) + \psi\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b\frac{1}{|\eta|}, \overline{\partial}_b\phi) + \mathcal{L}(\overline{\partial}_b\psi, \overline{\partial}_b\phi)}{t} + \overline{\phi}\psi Q(\mathbf{1}, \mathbf{1}),$$

Now $\sigma_{-\nu}(Z_{kj}) = Q(\overline{\rho}_k, \rho_j)$. We claim that

$$\mathcal{L}(\overline{\partial}_b\rho_j, \overline{\partial}_b\rho_k) = r_{j\overline{k}} + qr_jr_{\overline{k}} \tag{30}$$

with some $q \in C^\infty(\partial\Omega)$. To see this, assume we are at a point of $\partial\Omega$ where e.g. $r_1 \neq 0$, so that

$$R_j := \partial_j - \frac{r_j}{r_1}\partial_1, \quad \overline{R}_k := \overline{\partial}_k - \frac{r_{\overline{k}}}{r_{\overline{1}}}\overline{\partial}_1, \qquad j = 2, \dots, d,$$

form a basis for $\mathcal{T}'$ and $\mathcal{T}''$, respectively. By a simple computation,

$$L'(R_k, R_n) = R_kr_{\overline{n}} - \frac{r_{\overline{n}}}{r_{\overline{1}}}R_kr_{\overline{1}} =: \ell_{k\overline{n}}.$$

Let $\ell^{\overline{m}k}$ be the inverse matrix of $\ell_{k\overline{n}}$ (which is positive definite by the strict pseudoconvexity). Thus (employing the summation convention)

$$\ell^{\overline{k}m}\overline{R}_k r_j = \delta_j^m + \frac{r_j}{r_1}\ell^{\overline{k}m}\overline{R}_k r_1. \tag{31}$$

Remembering the definition of the dual Levi form, we have $\mathcal{L}(\overline{\partial}_b\rho_j, \overline{\partial}_b\rho_k) = \overline{\partial}_b\rho_j(\overline{X}) = \overline{X}\rho_j$ where $\overline{X} \in \mathcal{T}''$ is characterized by $L''(\overline{R}_n, \overline{X}) = \overline{\partial}_b\rho_k(\overline{R}_n) = \overline{R}_n\rho_k$ for all $n = 2, \ldots, d$. Writing temporarily $X = \sum_j c_j\overline{R}_j$ this yields, since $L''(\overline{R}_n, \overline{R}_j) = \overline{L'(R_j, R_n)} = \ell_{j\overline{n}}$,

$$\overline{c}_j\ell_{j\overline{n}} = \overline{R}_n\rho_k,$$

or, using (31),

$$\overline{c}_m = \ell^{\overline{n}m}\overline{R}_n\rho_k = -\ell^{\overline{n}m}\overline{R}_n r_k = -\delta_k^m - \frac{r_k}{r_1}\ell^{\overline{n}m}\overline{R}_n r_1.$$

Hence, using (31) one more time,

$$\begin{aligned}
\mathcal{L}(\overline{\partial}_b\rho_j, \overline{\partial}_b\rho_k) &= c_m\overline{R}_m\rho_j = (\delta_k^m + \frac{r_{\overline{k}}}{r_{\overline{1}}}\ell^{\overline{m}n}R_n r_{\overline{1}})\overline{R}_m r_j \\
&= \overline{R}_k r_j + \frac{r_{\overline{k}}}{r_{\overline{1}}}(R_n r_{\overline{1}})(\delta_j^n + \frac{r_j}{r_1}\ell^{\overline{m}n}\overline{R}_m r_1) \\
&= (\overline{R}_k r_j + \frac{r_{\overline{k}}}{r_{\overline{1}}}R_j r_{\overline{1}}) + \frac{r_{\overline{k}}r_j}{r_1 r_{\overline{1}}}\ell^{\overline{m}n}(R_n r_{\overline{1}})(\overline{R}_m r_1) \\
&= r_{j\overline{k}} + \frac{r_{\overline{k}}r_j}{r_1 r_{\overline{1}}}[\ell^{\overline{m}n}(R_n r_{\overline{1}})(\overline{R}_m r_1) - r_{1\overline{1}}] \\
&= r_{j\overline{k}} + r_j r_{\overline{k}} q,
\end{aligned}$$

with

$$q := \frac{1}{r_1 r_{\overline{1}}}[\ell^{\overline{m}n}(R_n r_{\overline{1}})(\overline{R}_m r_1) - r_{1\overline{1}}] \tag{32}$$

(on the piece of $\partial\Omega$ where $r_1 \neq 0$; note that q is real-valued). Thus (30) holds. Inserting (30) into (29), noting that $\frac{[\nu-2]}{t} = \frac{[\nu-1]}{\nu-1}$, we finally get

(33)
$$\begin{aligned}
\sigma_{-\nu}(Z_{kj}) &= Q(\overline{\rho}_k, \rho_j) \\
&= \frac{[\nu-1]}{\nu-1}\Big(r_{\overline{k}}\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b r_j, \overline{\partial}_b\frac{1}{|\eta|}) + r_j\frac{|\eta|}{\nu}\mathcal{L}(\overline{\partial}_b\frac{1}{|\eta|}, \overline{\partial}_b r_k) + r_{j\overline{k}} + q r_j r_{\overline{k}}\Big) \\
&\qquad + r_j r_{\overline{k}} Q(\mathbf{1}, \mathbf{1}),
\end{aligned}$$

completing the proof of the proposition (with $c = q + \frac{\nu-1}{[\nu-1]}Q(\mathbf{1}, \mathbf{1})$). $\square$

Throughout the rest of this section, we will assume that on $\partial\Omega$

$$[r_{j\overline{k}}]_{j,k=1}^d \text{ is positive definite, and } |\eta| = 1. \tag{34}$$

Such defining functions exist in abundance: indeed, it is standard that there exists a (negatively signed) defining function r' for Ω such that r' is strictly plurisubharmonic, i.e. $[r'_{j\overline{k}}]_{j,k=1}^d$ is positive definite, on $\overline{\Omega}$. If $g \in C^\infty(\overline{\Omega})$ and $r = r'g$, then on $\partial\Omega$

$$|\eta|^2 = r_k r_{\overline{k}} = |g|^2 r'_k r'_{\overline{k}},$$

and

$$r_{j\overline{k}} = g r'_{j\overline{k}} + g_j r'_{\overline{k}} + r'_j g_{\overline{k}} = g r'_{j\overline{k}} + \Big(\frac{\partial g}{\partial n} \otimes \frac{\partial r'}{\partial n}\Big)_{j\overline{k}}$$

(where $\partial/\partial n$ stands for the normal derivative; note that the tangential derivatives of r' vanish). Thus taking any positive g with

$$g = (r'_k r'_{\overline{k}})^{-1/2} \quad \text{and} \quad \frac{\partial g}{\partial n} = 0$$

on $\partial\Omega$ produces an r satisfying (34).

Corollary 10. *Assume that* (34) *holds. Then*

$$\sigma_{-\nu}(Z_{kj}) = \frac{[\nu-1]}{\nu-1}\,(r_{j\overline{k}} + c r_j r_{\overline{k}}), \qquad c \in C^\infty(\partial\Omega).$$

Proof. If $|\eta| = 1$, then $\overline{\partial}_b \frac{1}{|\eta|} = 0$. □

Proposition 11. *Assume that $\nu > 1$ and r satisfies* (34)*. Then*

(i) *the inverse of the $d \times d$ matrix of gTo's $Q = [Q_{kj}]_{j,k=1}^d$ is given by $S = [S_{jm}]_{j,m=1}^d$, where S_{jm} are gTo's of order ν;*

(ii) *the inverse of the $(d+1)\times(d+1)$ matrix of gTo's $\kappa^*\kappa$ is a matrix all of whose entries are gTo's of order ν, except the bottom right entry which is a positive selfadjoint elliptic gTo of order $\nu+1$.*

Proof. Denote, for brevity, $\mathcal{R}_j := T_{\rho_j}$ and $\check{\mathcal{R}} = \frac{1}{\nu}T_{[\nu-2]} + \frac{1}{\nu-1}T_{c[\nu-1]}$, with the c from (25). By the last proposition and its corollary, $\tilde{Y}_{kj} := Q_{kj} - \mathcal{R}_k^*\check{\mathcal{R}}\mathcal{R}_j$ is a gTo of order $-\nu$ with $\sigma_{-\nu}(\tilde{Y}_{kj}) = \frac{[\nu-1]}{\nu-1} r_{j\overline{k}}$. Since the matrix $[r_{j\overline{k}}]_{j,k=1}^d$ is positive definite by hypothesis, there exists a matrix S_1 of (finite rank) smoothing operators such that $[Y_{kj}] := [\tilde{Y}_{kj}] + S_1$ is positive selfadjoint on $\oplus^d H^2(\partial\Omega)$. (Sketch of proof: looking at the symbols shows that $(\nu-1)[\tilde{Y}_{kj}] = T_{[\nu-1]}^{1/2}[T_{r_{j\overline{k}}}]^{1/2}(I+K)[T_{r_{j\overline{k}}}]^{1/2}T_{[\nu-1]}^{1/2}$ with K of order -1, i.e. K is a compact selfadjoint operator. Thus K has at most finitely many eigenvalues, each of finite multiplicity, in the interval $(-\infty, -\frac{1}{2})$; the corresponding spectral projection $P_{(-\infty,-\frac{1}{2})}$ is thus a finite rank smoothing operator.

Take $S_0 := P_{(-\infty,-\frac{1}{2})}KP_{(-\infty,-\frac{1}{2})}$ (a finite rank smoothing operator too, with $I+K+S_0 \gg \frac{1}{2}I$) and $S_1 := \frac{1}{\nu-1}T^{1/2}_{[\nu-1]}[T_{r_{j\overline{k}}}]^{1/2}S_0[T_{r_{j\overline{k}}}]^{1/2}T^{1/2}_{[\nu-1]}$ (a finite rank smoothing operator, with $[\tilde{Y}_{kj}]+S_1 \gg \frac{1}{2}\frac{1}{\nu-1}T^{1/2}_{[\nu-1]}[T_{r_{j\overline{k}}}]T^{1/2}_{[\nu-1]} \gg 0$).) Consequently, by the (matrix variant of the) property (P8) in §2.2, the powers Y^z of $Y=[Y_{kj}]^d_{j,k=1}$, with any $z\in\mathbf{C}$, are gTo's, with entries of order $-z\nu$; in particular, the inverse $Y^{-1}=W$, $W=[W_{jm}]^\infty_{j,m=1}$, is matrix of gTo's, and so are the square roots $Y^{1/2}$ and $W^{1/2}=Y^{-1/2}$.
For the matrix $Q=[Q_{kj}]$ above, we can then write, in the obvious block matrix notation ($\mathcal{R}$ is to be viewed as a row vector)

$$Q = Y+\mathcal{R}^*\check{\mathcal{R}}\mathcal{R}-S_1 = Y^{1/2}(I+z^*\check{\mathcal{R}}z-S_2)Y^{1/2}, \qquad z:=\mathcal{R}W^{1/2}, \tag{35}$$

with $S_2=W^{1/2}S_1W^{1/2}$ smoothing. Recall from (21) that

$$\kappa^*\kappa = \begin{bmatrix} T_{\Lambda[\overline{\Psi}_k w\Psi_j]} & T_{\Lambda[\overline{\Psi}_k w]} \\ T_{\Lambda[w\Psi_j]} & T_{\Lambda[w]} \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{36}$$

Note that $D=T_{[\nu]}$, as well as all of $\kappa^*\kappa$, are positive selfadjoint, so have an (unbounded) inverse. Using the decomposition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}, \tag{37}$$

we thus see that $A-BD^{-1}C=Q$ must also be positive selfadjoint. Hence, by (35), so must be $(I+z^*\check{\mathcal{R}}z-S_2)$.
The operator

$$H := zz^* = \mathcal{R}W\mathcal{R}^*$$

is an elliptic gTo on $H^2(\partial\Omega)$ of order ν, with $\sigma_\nu(H)=|[r_{j\overline{k}}]^{-1/2}[r_{\overline{k}}]|^2>0$. Being of the form zz^*, it is also automatically nonnegative; by a similar argument as above, it follows that the projection H_0 onto $\operatorname{Ker}H$ is a finite rank smoothing operator, and $H+H_0$ is a positive selfadjoint elliptic gTo of order ν; hence $H^{1/2}=(H+H_0)^{1/2}-H_0=:h$ is a gTo (of order $\frac{\nu}{2}$), and so is $h_- := (H+H_0)^{-1/2}-H_0$ (of order $-\frac{\nu}{2}$), with $hh_-=h_-h=I-H_0$. Taking polar decomposition of z^*, we obtain

$$z^* = \mathcal{V}h$$

with some $\mathcal{V}$ a column matrix of gTo's of order 0 and a partial isometry from $H^2(\partial\Omega)$ into $\oplus^d H^2(\partial\Omega)$, with $\mathcal{V}^*\mathcal{V}=I-H_0$, and $z^*\check{\mathcal{R}}z=\mathcal{V}^*h\check{\mathcal{R}}h\mathcal{V}$. Again, the operator $I+h\check{\mathcal{R}}h$ is a selfadjoint gTo of order 1 with symbol $\sigma_1(h\check{\mathcal{R}}h)=\frac{[\nu-2]}{\nu}\sigma_\nu(H)>0$, so there exists a finite rank smoothing operator S_3 such that $I+h\check{\mathcal{R}}h+S_3\gg 0$; and by (P8), $(I+h\check{\mathcal{R}}h+S_3)^{-1/2}$ is an elliptic gTo of order $-\frac{1}{2}$. Set

$$b := h_-^2 - h_-(I+h\check{\mathcal{R}}h+S_3)^{-1/2}h_-,$$

this is an elliptic demi-classical gTo of order $-\nu$ (the first summand is elliptic of order $-\nu$, the second summand is of order $-\nu-\frac12$); and

$$\begin{aligned} X'' &:= I - \mathcal{V}hbh\mathcal{V}^* \\ &= I - \mathcal{V}(I-H_0)[I-(I+h\check{\mathcal{R}}h+S_3)^{-1/2}](I-H_0)\mathcal{V}^*, \end{aligned}$$

which differs by a finite-rank smoothing operator from

$$I - \mathcal{V}[I-(I+h\check{\mathcal{R}}h+S_3)^{-1/2}]\mathcal{V}^* = (I-\mathcal{V}\mathcal{V}^*) + \mathcal{V}(I+h\check{\mathcal{R}}h+S_3)^{-1/2}\mathcal{V}^*.$$

The last operator is selfadjoint, equals I on $\operatorname{Ker}\mathcal{V}^*$, while on $(\operatorname{Ker}\mathcal{V}^*)^\perp = \operatorname{Ran}\mathcal{V}$ it is unitarily equivalent to $\mathcal{V}^*\mathcal{V}(I+h\check{\mathcal{R}}h+S_3)^{-1/2}\mathcal{V}^*\mathcal{V}$, which differs by a finite-rank smoothing operator from $(I+h\check{\mathcal{R}}h+S_3)^{-1/2}$, an elliptic gTo of order $-\frac12$. It follows one more time that there exists a finite-rank smoothing operator S_4 such that

$$X' := X'' + S_4 = I - \mathcal{V}hbh\mathcal{V}^* + S_4$$

is a matrix of demi-classical gTo's of order 0 which is also positive selfadjoint as an operator on $\oplus^d H^2(\partial\Omega)$. Now

$$\begin{aligned} X'^2 &\sim (I-\mathcal{V}hbh\mathcal{V}^*)^2 \\ &= I + \mathcal{V}h(-2b+bh\mathcal{V}^*\mathcal{V}hb)h\mathcal{V}^* \\ &\sim I + \mathcal{V}h(-2b+bh^2b)h\mathcal{V}^* \\ &= I + \mathcal{V}[(I-hbh)^2-I]\mathcal{V}^* \\ &\sim I + \mathcal{V}[\underbrace{(I+h\check{\mathcal{R}}h+S_3)^{-1}-I}_{=:G}]\mathcal{V}^*, \end{aligned}$$

and so

$$\begin{aligned} X'^2(I+z^*\check{\mathcal{R}}z-S_2) &= X'^2(I+\mathcal{V}h\check{\mathcal{R}}h\mathcal{V}^*-S_2) \\ &\sim (I+\mathcal{V}G\mathcal{V}^*)(I+\mathcal{V}h\check{\mathcal{R}}h\mathcal{V}^*) \\ &= I + \mathcal{V}(G+h\check{\mathcal{R}}h+G\mathcal{V}^*\mathcal{V}h\check{\mathcal{R}}h)\mathcal{V}^* \\ &\sim I + \mathcal{V}(G+h\check{\mathcal{R}}h+Gh\check{\mathcal{R}}h)\mathcal{V}^* \\ &= I + \mathcal{V}[(I+G)(I+h\check{\mathcal{R}}h)-I]\mathcal{V}^* \\ &\sim I + \mathcal{V}[(I+G)(I+h\check{\mathcal{R}}h+S_3)-I]\mathcal{V}^* = I. \end{aligned}$$

Consequently,

$$X'^2(I+z^*\check{\mathcal{R}}z-S_2) = I + S_5$$

with some smoothing operator S_5. Since we have managed that both terms on the left-hand side are positive, hence invertible, so must be the right-hand side, thus $(I+S_5)^{-1}$ is a gTo, and

$$(I+S_5)^{-1}X'^2(I+z^*\check{\mathcal{R}}z-S_2) = I,$$

i.e. $(I+S_5)^{-1}X'^2$ is a left inverse of $(I+z^*\check{\mathcal{R}}z-S_2)$. Taking adjoints, we see that $X'^2(I+S_5)^{-1}$ is a right inverse, hence

$$S' := (I+S_5)^{-1}X'^2 = X'^2(I+S_5)^{-1} \tag{38}$$

is the two-sided inverse for $(I+z^*\check{\mathcal{R}}z-S_2)$; thus finally by (35),

$$S := W^{1/2}S'W^{1/2}$$

is the inverse of Q, proving part (i).
Note that although X' is only demi-classical, the operator G above, and thus also X'^2, is classical, hence so are also S' and S.
For (ii), recall from (37) that

$$\begin{aligned}(\kappa^*\kappa)^{-1} &= \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}\begin{bmatrix} Q^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} Q^{-1} & -Q^{-1}BD^{-1} \\ -D^{-1}CQ^{-1} & D^{-1}+D^{-1}CQ^{-1}BD^{-1} \end{bmatrix}.\end{aligned}$$

Thus the matrix of gTo's

$$\begin{bmatrix} S & -SBD^{-1} \\ -D^{-1}CS & D^{-1}+D^{-1}CSBD^{-1} \end{bmatrix} \qquad (\text{note that } C=B^*)$$

is the inverse for $\kappa^*\kappa$.
Now $D^{-1}CS$ is a gTo of order $(\nu+1)+(-\nu)+\nu=\nu+1$, with symbol

$$\begin{aligned}\sigma_{\nu+1}(D^{-1}CS) &= \sigma_{\nu+1}(D^{-1}CW^{1/2}X'^2W^{1/2}) \\ &= \sigma_{\nu+1}(D^{-1})\sigma_{-\nu/2}(CW^{1/2}X')\sigma_{\nu/2}(X'W^{1/2}),\end{aligned}$$

with

$$\begin{aligned}\sigma_{-\nu/2}(CW^{1/2}X') &= \sigma_{-\nu}(C)\sigma_{\nu/2}(W^{1/2})\sigma_0(X') \\ &= [\nu-1]\mathcal{R}\cdot\sigma_{\nu/2}(W^{1/2})\sigma_0(I-\mathcal{V}hbh\mathcal{V}^*) \\ &= [\nu-1]\sigma_{\nu/2}(z)\sigma_0(I-\mathcal{V}hbh\mathcal{V}^*) \\ &= [\nu-1]\sigma_{\nu/2}(h\mathcal{V}^*)\sigma_0(I-\mathcal{V}hbh\mathcal{V}^*) \\ &= [\nu-1]\sigma_{\nu/2}(h)\sigma_0(\mathcal{V}^*-\mathcal{V}^*\mathcal{V}hbh\mathcal{V}^*) \\ &= [\nu-1]\sigma_{\nu/2}(h)\sigma_0((I-hbh)\mathcal{V}^*) \qquad (\text{since } \mathcal{V}^*\mathcal{V}\sim I) \\ &= [\nu-1]\sigma_{\nu/2}(h)\sigma_0(I-hbh)\sigma_0(\mathcal{V}^*) \\ &= 0,\end{aligned} \tag{39}$$

since $I-bhb\sim(I+h\check{\mathcal{R}}h+S_3)^{-1/2}$ is of order $-\frac{1}{2}$. Thus $D^{-1}CS$ is in fact of order ν, and similarly for SBD^{-1}, while $D^{-1}CSBD^{-1}=(D^{-1}CS)Q(SBD^{-1})$

has order $\nu+(1-\nu)+\nu=\nu+1$, same as D^{-1}. Consequently, $(\kappa^*\kappa)^{-1}$ is a $(d+1)\times(d+1)$ matrix of gTo's with orders $\begin{bmatrix}\nu & \nu\\ \nu & \nu+1\end{bmatrix}$, as asserted.

Finally, as $C=B^*$,

$$\begin{aligned}\sigma_{\nu+1}&(D^{-1}+D^{-1}CSBD^{-1})\\ &=\sigma_{\nu+1}(D^{-1}+D^{-1}CW^{1/2}X'^2W^{1/2}C^*D^{-1})\\ &=\sigma_{\nu+1}(D^{-1})+|\sigma_{\nu+1}(D^{-1})\sigma_{-\nu/2-1/2}(CW^{1/2}X')|^2>0,\end{aligned}$$

showing the ellipticity of the bottom right corner and thus completing the proof of part (ii). $\square$

Proposition 12. *Under the same hypothesis as in Proposition 11, there exists an isometry V on $\oplus^{d+1}H^2(\partial\Omega)$ such that $V(\kappa^*\kappa)^{-1/2}$ is a $(d+1)\times(d+1)$ matrix of demi-classical gTo's.*

Proof. Keeping the notations from the previous proof, we have from the commutativity of the two factors on the right-hand side of (38) that $I+S_5$ is positive and

$$S':=(I+S_5)^{-1/2}X'^2(I+S_5)^{-1/2}.$$

Thus the operator

$$X:=X'(I+S_5)^{-1/2}W^{1/2}$$

satisfies

$$X^*X=W^{1/2}S'W^{1/2}=S=Q^{-1}.$$

Finally, with the notation (37), set

$$\mathcal{Z}:=\begin{bmatrix}X & -XBD^{-1}\\ 0 & D^{-1/2}\end{bmatrix}. \tag{40}$$

Then

$$\begin{aligned}\mathcal{Z}^*\mathcal{Z}&=\begin{bmatrix}X^* & 0\\ -D^{-1}CX^* & D^{-1/2}\end{bmatrix}\begin{bmatrix}X & -XBD^{-1}\\ 0 & D^{-1/2}\end{bmatrix}\\ &=\begin{bmatrix}S & -SBD^{-1}\\ -D^{-1}CS & D^{-1}+D^{-1}CSBD^{-1}\end{bmatrix}\end{aligned}$$

is precisely the matrix of $(\kappa^*\kappa)^{-1}$ from the preceding proposition. Taking polar decomposition, it follows that $\mathcal{Z}=V(\kappa^*\kappa)^{-1/2}$ with V a partial isometry with initial space $\overline{\operatorname{Ran}(\kappa^*\kappa)^{-1/2}}=(\operatorname{Ker}(\kappa^*\kappa)^{-1/2})^\perp=0^\perp$, i.e. an isometry on $\oplus^{d+1}H^2(\partial\Omega)$. $\square$

We pause to note what are the orders of the entries in $\mathcal{Z}$: while those of X have orders $\frac{\nu}{2}$, the bottom right entry is a positive selfadjoint elliptic gTo of order $\frac{\nu+1}{2}$ with symbol $[\nu]^{-1/2}$. For $XBD^{-1}=(D^{-1}CX^*)^*$, a brute count gives order $\frac{\nu}{2}+1$; however, $D^{-1}CX^* \sim D^{-1}CW^{1/2}X'$ and we have seen in (39) that $\sigma_{-\nu/2}(CW^{1/2}X')=0$ so that $CW^{1/2}X'$ is actually of order $-\frac{\nu}{2}-\frac{1}{2}$; thus XBD^{-1} is actually of order $\frac{\nu}{2}+\frac{1}{2}$, i.e. the same as $D^{-1/2}$.

Proof of Theorem 2. We have seen in (20) that the Toeplitz operator $\mathcal{T}_\phi$ on $\mathcal{C}_{2,w}$ is unitarily equivalent to $(\kappa^*\kappa)^{-1/2}\kappa^*\phi\kappa(\kappa^*\kappa)^{-1/2}$ on $\oplus^{d+1}H^2(\partial\Omega)$. By the last proposition, the latter is in turn equivalent to (modulo smoothing operators)

$$V^*\mathcal{Z}\kappa^*\phi\kappa\mathcal{Z}^*V \cong \mathcal{Z}\kappa^*\phi\kappa\mathcal{Z}^*$$

with $\mathcal{Z}$ from (40). Writing now (21) as $\kappa^*\phi\kappa = \begin{bmatrix} A_\phi & B_\phi \\ C_\phi & D_\phi \end{bmatrix}$ (similarly to (36)) we have by (40) (noting again that $B^*=C$)

$$\mathcal{Z}\kappa^*\phi\kappa\mathcal{Z}^* = \begin{bmatrix} X & -XBD^{-1} \\ 0 & D^{-1/2} \end{bmatrix} \begin{bmatrix} A_\phi & B_\phi \\ C_\phi & D_\phi \end{bmatrix} \begin{bmatrix} X^* & 0 \\ -D^{-1}CX^* & D^{-1/2} \end{bmatrix}$$

which equals

$$\begin{bmatrix} \begin{matrix} X(A_\phi - BD^{-1}C_\phi - B_\phi D^{-1}C + \\ +BD^{-1}D_\phi D^{-1}C)X^* \end{matrix} & X(B_\phi - BD^{-1}D_\phi)D^{-1/2} \\ D^{-1/2}(C_\phi - D_\phi D^{-1}C) & D^{-1/2}D_\phi D^{-1/2} \end{bmatrix}.$$

However, from (21) we see that $\sigma(A_\phi)=\phi\sigma(A)$, so $A_\phi = T_\phi A + A'_\phi$ with A'_ϕ of order 1 less than A i.e. $-\nu$; and similarly $B_\phi = BT_\phi + B'_\phi$, $C_\phi = T_\phi C + C'_\phi$ and $D_\phi = DT_\phi + D'_\phi$. It follows that

$$\begin{aligned} D^{-1/2}D_\phi D^{-1/2} &= D^{1/2}T_\phi D^{-1/2} + D^{-1/2}D'_\phi D^{-1/2} \\ &= T_\phi + ([D^{1/2}, T_\phi]D^{-1/2} + D^{-1/2}D'_\phi D^{-1/2}), \end{aligned}$$

where the term in the parentheses is of order -1. Next,

$$\begin{aligned} R_\phi :&= X(B_\phi - BD^{-1}D_\phi)D^{-1/2} \\ &= X[(BT_\phi + B'_\phi) - BD^{-1}(DT_\phi + D'_\phi)]D^{-1/2} \\ &= X[B'_\phi - BD^{-1}D'_\phi]D^{-1/2}, \end{aligned} \tag{41}$$

which is of order $-\frac{1}{2}$; similarly for $D^{-1/2}(C_\phi - D_\phi D^{-1}C)X^* = R^*_{\overline{\phi}}$. Finally,

$$\begin{aligned} &X(A_\phi - BD^{-1}C_\phi - B_\phi D^{-1}C + BD^{-1}D_\phi D^{-1}C)X^* \\ &\quad = X[(T_\phi A + A'_\phi) - BD^{-1}(T_\phi C + C'_\phi) - (BT_\phi + B'_\phi)D^{-1}C \\ &\qquad\qquad + BD^{-1}(DT_\phi + D'_\phi)D^{-1}C]X^* \\ &\quad = X[(T_\phi A + A'_\phi) - BD^{-1}(T_\phi C + C'_\phi) - B'_\phi D^{-1}C + BD^{-1}D'_\phi D^{-1}C]X^* \\ &\quad = X[T_\phi A + A'_\phi - [BD^{-1}, T_\phi]C - T_\phi BD^{-1}C - BD^{-1}C'_\phi \\ &\qquad\qquad - B'_\phi D^{-1}C + BD^{-1}D'_\phi D^{-1}C]X^* \end{aligned}$$

$$\begin{aligned}
&= X[T_\phi(A-BD^{-1}C)+\\
&\quad \underbrace{A'_\phi-[BD^{-1},T_\phi]C-BD^{-1}C'_\phi-B'_\phi D^{-1}C+BD^{-1}D'_\phi D^{-1}C}_{=:W'_\phi}]X^*\\
&= X[T_\phi Q+W'_\phi]X^*\\
&= [X,T_\phi]QX^*+T_\phi[X,QX^*]+T_\phi QX^*X+XW'_\phi X^*\\
&= T_\phi+[X,T_\phi]QX^*+T_\phi[X,QX^*]+XW'_\phi X^*, \qquad (42)
\end{aligned}$$

since $QX^*X=I$ by the construction of X. The last line is a $d\times d$ matrix of gTo's of order 0, and their symbols — $M(\phi)=[M_{kj}(\phi)]_{j,k=1}^d$, say — depend linearly on derivatives of ϕ of at most first order, i.e. are of the form

$$M_{kj}(\phi)=a_{kj}\phi+L_{kj}\phi$$

with some $a_{kj}\in C^\infty(\partial\Omega)$ and smooth vector-fields L_{kj} (not necessarily tangential) on $\partial\Omega$. We thus conclude that

$$\mathcal{T}_\phi\cong\mathcal{Z}\kappa^*\phi\kappa\mathcal{Z}^*=\begin{bmatrix}T_{M(\phi)} & 0\\ 0 & T_\phi\end{bmatrix}+\text{lower order term}, \qquad (43)$$

where the “lower order term” is a matrix of gTo's of order $-\frac12$. Now, first of all, $\mathcal{T}_{\mathbf{1}}=I$, which implies that $a_{kj}=\delta_{kj}\mathbf{1}$. Secondly, remembering that, by (18), $\mathcal{T}_{\phi f}=\mathcal{T}_\phi\mathcal{T}_f$ for f holomorphic, we see that for all such f,

$$(\phi I+L\phi)(fI+Lf)=\phi fI+L(\phi f),$$

or, using the Leibniz rule, $(L\phi)(Lf)=0$. Taking $\phi=\overline{f}$ shows that $Lf=0$ for all holomorphic f. Since $\mathcal{T}_\phi^*=\mathcal{T}_{\overline{\phi}}$, we similarly get $L\overline{g}$=0 for all holomorphic g. Thus L contains neither holomorphic nor anti-holomorphic derivatives, i.e. $L=0$. This completes the proof. □

Remark. An alternative proof of $L=0$ can be given by brute force computation from (42) using (39). □

Remark. We have been somewhat silent about the log terms in the various ψdo's and gTo's, so here we spell them out: clearly $\breve{\mathcal{R}}$, c and $\mathcal{R}_j$ contain no log terms, while $Q_{kj}\in\Psi^{1-\nu}_{\log}$ have log terms starting at order $1-\nu-M$ (i.e. at distance M from the leading order). The operators $Y_{kj}\in\Psi^{-\nu}_{\log}$ have log terms starting at distance $M-1$ from the leading order already (since they were basically fabricated from Q_{kj} by cancelling the leading order term); and similarly for W, z, H, S, b, X and $\mathcal{Z}$, as well as for the right-hand side of (43). □

Remark. It should be noted that, in general, even if a Toeplitz operator T_ϕ is positive selfadjoint on $H^2(\partial\Omega)$ — hence, having an inverse there with the same properties — then this inverse need not be a gTo in general (i.e. if T_ϕ is not in addition elliptic); hence, in particular, the effort needed for proving Proposition 11. An example is $\phi = |z_1|^2$ on the unit ball in $\mathbf{C}^d$, $d > 1$: its inverse would have to be of order 0 with symbol $1/|z_1|^2$, but $T_{1/|z_1|^2}$ is not a well defined operator. Note that, in fact, for the ball the entries in the matrix (36) for $\kappa^*\kappa$ are precisely of the above type: for instance, the (1,1)-entry in (36) is then $T_{\overline{z}_1} T_{[\nu-2]} T_{z_1}$, hence with symbol $[\nu-2]|z_1|^2$ of the kind as above. It is noteworthy that although each entry is non-manageable in this way, the entire matrix $\kappa^*\kappa$ is nonetheless invertible as a gTo by Proposition 11. Similar remarks apply to the square root $T_\phi^{1/2}$ and Proposition 12 (see also the remark at the very end of the next section). $\square$

Remark. Note that the only place where (34) was used was in the proof of Proposition 11, to ensure the existence of powers of $[Y_{kj}]$. One can get this even if the second part of (34), i.e. $|\eta| = 1$, is dropped, as soon as ν is sufficiently large: namely, thanks to the factors of $\frac{|\eta|}{\nu}$ in (33), the matrix

$$\Big[r_{j\overline{k}} + \frac{|\eta|}{\nu} \Big(r_{\overline{k}} \mathcal{L}(\overline{\partial}_b r_j, \overline{\partial}_b \frac{1}{|\eta|}) + r_j \mathcal{L}(\overline{\partial}_b \frac{1}{|\eta|}, \overline{\partial}_b r_k) \Big) \Big]$$

will be positive whenever $[r_{j\overline{k}}]$ is and ν gets large enough, because the second summand becomes negligible compared to $[r_{j\overline{k}}]$. We are not sure what happens when (34) is dropped completely (i.e. if r is not strictly-PSH near $\partial\Omega$), though we expect that at least Theorem 2 will still remain in force. $\square$

Remark. We have left aside the case of dimension $d = 1$. In that case, many things simplify considerably: namely, the operator $Q = A - BD^{-1}C$ in (37) is then an elliptic positive selfadjoint gTo, so it follows immediately that so is also its inverse Q^{-1} and inverse square root $Q^{-1/2} = X$; the constructions in the proofs of Propositions 11 and 12 are thus not needed, while the statements of these two propositions remain in force, and so does that of Theorem 2. $\square$

Remark. We pause to remark that the function q from (32) occurs on at least one more interesting occasion. Namely, denote by $\mathcal{E}$ the holomorphic vector field on $\partial\Omega$ given by

$$\mathcal{E} f = \frac{1}{i} E_\perp f \qquad \text{for holomorphic } f,$$

where $E_\perp$ is the Reeb vector field from §2.4; in other words, by (8),

$$\mathcal{E} f = \{t, f\} \qquad \text{for holomorphic } f.$$

(This operator makes appearance e.g. when one computes the symbol of operators like the B'_ϕ, D'_ϕ in the proof of Theorem 2.) Then one can show that

$$\mathcal{E} r_{\overline{k}} = -q r_{\overline{k}}, \qquad k = 1, \dots, d.$$

The function q seems to be an interesting object from the point of view of complex geometry of strictly pseudoconvex domains, and even more so its normalized version $|\eta|q$ which does not depend on the choice of the defining function r. $\square$

Finally, it should be clear how to proceed for analogues of Theorem 2 for the spaces $\mathcal{C}_{m,w}$ with general $m > 2$.

5. The case of the ball

In this final section, we work out the situation from Theorem 2, i.e. the Toeplitz operators on $\mathcal{C}_{2,w}$, more explicitly for the unit ball $\Omega = \mathbf{B}^d$ of $\mathbf{C}^d$, with the standard weights $w = \rho^\nu$, $\rho(z) = 1 - |z|^2$. (Note that this defining function satisfies the hypothesis (34).) In particular, we show that in this case the inverse square root $(\kappa^*\kappa)^{-1/2}$ is itself a gTo (so the isometry V in Proposition 12 is not needed). We are unable to prove whether this is also the case for general strictly pseudoconvex domains Ω and their defining functions from Section 4.

For multiindices α and β, the familiar formula for integration over the unit sphere

$$\langle z^\alpha, z^\beta\rangle_{H^2(\partial\mathbf{B}^d)} = \int_{\partial\mathbf{B}^d} z^\alpha \overline{z}^\beta \, d\sigma(z) = \frac{2\pi^d \delta_{\alpha\beta}\alpha!}{\Gamma(d+|\alpha|)} \tag{44}$$

(where $d\sigma$ stands for the surface measure on $\partial\mathbf{B}^d$) implies

$$\int_{\mathbf{B}^d} \rho(z)^\nu z^\alpha \overline{z}^\beta \, dz = \frac{\alpha!\delta_{\alpha\beta}\pi^d}{(\nu+1)_{d+|\alpha|}}. \tag{45}$$

Here $(\nu)_k := \nu(\nu+1)\dots(\nu+k-1)$ is the Pochhammer symbol. It follows that

$$\begin{aligned} T_{\Lambda[\rho^\nu]} z^\alpha &= \sum_\beta \langle T_{\Lambda[\rho^\nu]} z^\alpha, z^\beta\rangle \frac{z^\beta}{\langle z^\beta, z^\beta\rangle} \\ &= \sum_\beta \Big(\int_{\mathbf{B}^d} \rho^\nu z^\alpha \overline{z}^\beta \, dz \Big) \frac{\Gamma(d+|\beta|)}{2\pi^d \beta!} z^\beta \qquad \text{by (44)} \\ &= \frac{\Gamma(d+|\alpha|)}{2(\nu+1)_{d+|\alpha|}} \qquad \text{by (45).} \end{aligned} \tag{46}$$

Similarly, we compute how the operators in the matrix (36) act on the basis $\{z^\alpha\}$:

$$\begin{aligned} T_{\Lambda[z_k \rho^{\nu-2}\overline{z}_j]} z^\alpha &= \sum_\beta \frac{(\alpha+e_k)!\delta_{\alpha+e_k,\beta+e_j}\pi^d}{(\nu-1)_{d+|\alpha|+1}} \frac{\Gamma(d+|\beta|)}{2\pi^d\beta!} z^\beta \\ &= \frac{(\alpha_j+\delta_{kj})\Gamma(d+|\alpha|)}{2(\nu-1)_{d+|\alpha|+1}} z^{\alpha+e_k-e_j}, \\ T_{\Lambda[\rho^{\nu-2}\overline{z}_j]} z^\alpha &= \frac{\alpha_j \Gamma(d+|\alpha|-1)}{2(\nu)_{d+|\alpha|}} z^{\alpha-e_j}, \\ T_{\Lambda[z_k\rho^{\nu-2}]} z^\alpha &= \frac{\alpha_j \Gamma(d+|\alpha|+1)}{2(\nu)_{d+|\alpha|+1}} z^{\alpha+e_k}, \end{aligned}$$

where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at k-th position. It follows that

$$T_{\Lambda[z_k\rho^{\nu-2}]}T^{-1}_{\Lambda[\rho^\nu]}T_{\Lambda[\rho^{\nu-2}\overline{z}_j]}z^\alpha = \frac{\alpha_j}{\nu}\frac{\Gamma(d+|\alpha|)}{2(\nu)_{d+|\alpha|}}\,z^{\alpha+e_k-e_j}$$

and so $Q_{kj} = T_{\Lambda[z_k\rho^{\nu-2}\overline{z}_j]} - T_{\Lambda[z_k\rho^{\nu-2}]}T^{-1}_{\Lambda[\rho^\nu]}T_{\Lambda[\rho^{\nu-2}\overline{z}_j]}$ satisfies

$$Q_{kj}z^\alpha = \frac{\Gamma(d+|\alpha|)}{2(\nu)_{d+|\alpha|}}\Big[\frac{\alpha_j+\delta_{kj}}{\nu-1} - \frac{\alpha_j}{\nu}\Big]\,z^{\alpha+e_k-e_j}.$$

On the other hand, $T_{\overline{z}_j}z^\alpha = \frac{\alpha_j}{d+|\alpha|-1}z^{\alpha-e_j}$, whence

$$T_{z_k}T_{\Lambda[\rho^{\nu-2}]}T_{\overline{z}_j}z^\alpha = \frac{\alpha_j}{d+|\alpha|-1}\frac{\Gamma(d+|\alpha|-1)}{2(\nu-1)_{d+|\alpha|-1}}z^{\alpha+e_k-e_j}.$$

Thus for the operator $Z_{kj} = Q_{kj} - \frac{1}{\nu}T_{z_k}T_{[\nu-2]}T_{\overline{z}_j}$ from (24) we get

$$Z_{kj}z^\alpha = \frac{\Gamma(d+|\alpha|)}{2(\nu)_{d+|\alpha|}}\Big[\frac{\delta_{kj}}{\nu-1} - \frac{2\nu-1}{\nu(\nu-1)}\frac{\alpha_j}{d+|\alpha|-1} - \frac{\alpha_j}{(d+|\alpha|-1)^2}\Big]\,z^{\alpha+e_k-e_j}.$$

Consequently,

$$Z_{kj} = \frac{\delta_{kj}}{\nu-1}T_{[\nu-1]} - T_{z_k}DT_{\overline{z}_j}$$

where the operator

$$D : z^\alpha \mapsto \Big[\frac{2\nu-1}{\nu(\nu-1)} + \frac{1}{d+|\alpha|}\Big]\frac{\Gamma(d+|\alpha|+1)}{2(\nu)_{d+|\alpha|+1}}z^\alpha$$

satisfies

$$D = \frac{2\nu-1}{\nu(\nu-1)}(T_{[\nu-1]} - T_{[\nu]}) + \frac{1}{\nu}T_{[\nu]} = \frac{2\nu-1}{\nu(\nu-1)}T_{[\nu-1]} - \frac{1}{\nu-1}T_{[\nu]}.$$

Thus, in full accordance with Proposition 9,

$$\sigma_{-\nu}(Z_{kj}) = \frac{[\nu-1]}{\nu-1}\delta_{kj} - \frac{2\nu-1}{\nu(\nu-1)}[\nu-1]\,\overline{z}_jz_k.$$

Indeed, $r_{j\overline{k}} = \delta_{kj}$, while, by a routine computation, $q = -\mathbf{1}$ and

$$\begin{aligned}
&\big(\tfrac{\nu-1}{\nu}T_{[\nu-2]} - T_{[\nu-1]}T^{-1}_{[\nu]}T_{[\nu-1]}\big)z^\alpha \\
&\qquad = \frac{\Gamma(d+|\alpha|)}{2}\Big[\frac{\nu-1}{\nu}\frac{1}{(\nu-1)_{d+|\alpha|}} - \frac{(\nu+1)_{d+|\alpha|}}{(\nu)^2_{d+|\alpha|}}\Big]z^\alpha \\
&\qquad = -\frac{1}{\nu}\frac{\Gamma(d+|\alpha|)}{2(\nu)_{d+|\alpha|}}z^\alpha = -\frac{1}{\nu}T_{[\nu-1]}z^\alpha,
\end{aligned}$$

so $Q(\mathbf{1},\mathbf{1}) = \sigma_{-\nu}(\frac{\nu-1}{\nu}T_{[\nu-2]} - T_{[\nu-1]}T_{[\nu]}^{-1}T_{[\nu-1]}) = -\frac{[\nu-1]}{\nu}$ and

$$c = \frac{\nu-1}{[\nu-1]}Q(\mathbf{1},\mathbf{1}) + q = -\frac{2\nu-1}{\nu}.$$

We also pause to note that the Reeb vector field is given simply by $E_\perp = \frac{1}{i}E = \frac{1}{i}(\overline{z}_j\overline{\partial}_j - z_j\partial_j)$, and $\mathcal{E} = z_j\partial_j$ is just the holomorphic radial derivative. Furthermore, $\Pi E = E\Pi$ and $T_E = -\mathcal{E}$.
GTo's that are invariant under the action of the unitary group $U(d)$ of rotations of $\mathbf{B}^d$ are precisely the diagonal operators

$$T_{\boldsymbol{s}} : z^\alpha \mapsto s_{|\alpha|}z^\alpha \tag{47}$$

on $H^2(\partial\mathbf{B}^d)$, with eigenvalue sequence $\boldsymbol{s} = (s_k)_{k=0}^\infty$ possessing an asymptotic expansion

$$s_k \sim \sum_{j=0}^\infty c_j(k+1)^{m-j} \qquad \text{as } k \to +\infty,$$

with some $c_j \in \mathbf{C}$, $j = 0, 1, 2, \ldots$, where m is the order of $T_{\boldsymbol{s}}$ and $\sigma_m(T_{\boldsymbol{s}}) = c_0 t^m$; alternatively, $T_{\boldsymbol{s}} \sim \sum_{j=0}^\infty c_j(I+\mathcal{E})^{m-j}$. One has

$$T_{\boldsymbol{s}}T_{z_k} = T_{z_k}T_{\boldsymbol{s}'}, \tag{48}$$

where $T_{\boldsymbol{s}'}$ is again a rotation-invariant gTo with eigenvalue sequence $s'_k = s_{k+1}$, i.e. $\boldsymbol{s}' = S^*\boldsymbol{s}$ where S^* is the "backward shift" operator on sequences; alternatively, $T_{\boldsymbol{s}'} = \sum_{k=1}^d T_{\overline{z}_k}T_{\boldsymbol{s}}T_{z_k}$.
In particular, this applies to $T_{[\nu]} = T_{\boldsymbol{s}_{[\nu]}}$ where, by (46),

$$\boldsymbol{s}_{[\nu]} = \Big(\frac{\Gamma(d+k)}{2(\nu+1)_{d+k}}\Big)_{k=0}^\infty, \tag{49}$$

and so

$$\begin{aligned} Q_{kj} = Z_{kj} + \frac{1}{\nu}T_{z_k}T_{[\nu-2]}T_{\overline{z}_j} &= \frac{\delta_{kj}}{\nu-1}T_{[\nu-1]} + T_{z_k}(\tfrac{1}{\nu}T_{[\nu-2]} - D)T_{\overline{z}_j} \\ &= \frac{1}{\nu-1}T_{[\nu-1]}^{1/2}(\delta_{kj}I + T_{z_k}T_{\boldsymbol{s}}T_{\overline{z}_j})T_{[\nu-1]}^{1/2}, \end{aligned}$$

with

$$\boldsymbol{s} = (\nu-1)(S^*\boldsymbol{s}_{[\nu-1]})^{-1}(\tfrac{1}{\nu}\boldsymbol{s}_{[\nu-2]} - \tfrac{2\nu-1}{\nu(\nu-1)}\boldsymbol{s}_{[\nu-1]} + \tfrac{1}{\nu-1}\boldsymbol{s}_{[\nu]}) \tag{50}$$

(the multiplication and inverse of sequences being understood pointwise; note that any two operators of the form (47) commute). From the simple formula

$$(\delta_{kj}I + T_{z_k}AT_{\overline{z}_j})(\delta_{jm}I + T_{z_j}BT_{\overline{z}_m}) = \delta_{km}I + T_{z_k}(A+B+AB)T_{\overline{z}_m} \tag{51}$$

(because $\sum_{j=1}^d T_{\overline{z}_j} T_{z_j} = T_{\sum_j |z_j|^2} = T_{\mathbf{1}} = I$) we thus see that the matrix $Q = [Q_{kj}]_{j,k=1}^d$ has the inverse $S = [S_{jm}]_{j,m=1}^d$ given by the gTo's

$$S_{jm} = (\nu-1) T_{[\nu-1]}^{-1/2} (\delta_{jm} I - T_{z_j} T_{\boldsymbol{x}} T_{\overline{z}_m}) T_{[\nu-1]}^{-1/2} \tag{52}$$

where $\boldsymbol{x} = \boldsymbol{s}/(1+\boldsymbol{s})$ (i.e. $x_k = s_k/(1+s_k)$ for all k); furthermore, $S = X^* X$ for $X = [X_{jm}]_{j,m=1}^d$ where

$$X = (\nu-1)^{1/2} (\delta_{jm} I - T_{z_j} T_{\boldsymbol{y}} T_{\overline{z}_m}) T_{[\nu-1]}^{1/2}$$

with $y_k = 1 - (1-x_k)^{1/2} = 1 - (1+s_k)^{-1/2}$ (note that, by a direct check from (50) and (49), $s_k > 0$ for all k). This offers explicit expressions for the various operators constructed in Propositions 11 and 12.

One can even do a little better and show that the positive square root $S^{1/2}$ of the matrix S is a matrix of gTo's. Namely, using again (48) we can rewrite (41) as

$$S_{jm} = (\nu-1)(\delta_{jm} T_{[\nu-1]}^{-1} - T_{z_j} T_{\boldsymbol{w}} T_{\overline{z}_m})$$

where $\boldsymbol{w} = (S^* \boldsymbol{s}_{[\nu-1]})^{-1} \boldsymbol{x}$. By (48) and the same computation as in (51),

$$(\delta_{kj} T_{\boldsymbol{a}} - T_{z_k} T_{\boldsymbol{b}} T_{\overline{z}_j})(\delta_{jm} T_{\boldsymbol{a}} - T_{z_j} T_{\boldsymbol{b}} T_{\overline{z}_m}) = \delta_{km} T_{\boldsymbol{a}}^2 - T_{z_k} T_{2(S^*\boldsymbol{a})\boldsymbol{b} - \boldsymbol{b}^2} T_{\overline{z}_m}.$$

Thus taking $\boldsymbol{a} = \boldsymbol{s}_{[\nu-1]}^{-1/2}$ and $\boldsymbol{b} = S^*\boldsymbol{a} - \sqrt{(S^*\boldsymbol{a})^2 - \boldsymbol{w}}$ we see that the matrix of gTo's $\mathcal{T} = (\nu-1)^{1/2} [\delta_{jm} T_{[\nu-1]}^{-1/2} - T_{z_j} T_{\boldsymbol{b}} T_{\overline{z}_m}]_{j,m=1}^d$ satisfies $\mathcal{T}^2 = S$. (Note that $(S^*\boldsymbol{a})^2 - \boldsymbol{w} = (S^* \boldsymbol{s}_{[\nu-1]})^{-1}/(1+\boldsymbol{s})$ is a sequence with positive elements; similarly one checks from $\mathcal{T} = (\nu-1)^{1/2} T_{[\nu-1]}^{-1/4} [\delta_{jm} - T_{z_j} T_{\boldsymbol{c}} T_{\overline{z}_m}] T_{[\nu-1]}^{-1/4}$ with $\boldsymbol{c} = (S^* \boldsymbol{s}_{[\nu-1]})^{1/2} \boldsymbol{b} = 1 - (1+\boldsymbol{s})^{-1/2} \in (0,1)$ that $\mathcal{T}$ is positive selfadjoint as an operator on $\oplus^d H^2(\partial \mathbf{B}^d)$.)

Remark. In general, it is not true that if T is a positive selfadjoint gTo (not necessarily elliptic), then its positive square root $T^{1/2}$ is also a gTo. A counterexample is furnished by $T = T_{\overline{z}_1} T_{z_1} = T_{|z_1|^2}$ on $H^2(\partial \mathbf{B}^d)$, $d > 1$. Namely, if $T^{1/2} =: T_P$, then T_P has to be of order 0 with $\sigma_0(P)^2 = |z_1|^2$, which has no solutions in $C^\infty(\partial\Omega)$. $\square$

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