Coleman Integration Versus Schneider Integration on Semistable Curves

To John Coates, on the occasion of his 60th birthday

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Abstract. The purpose of this short note is to clarify the relation between p-adic integration on curves with semistable reduction, and the filtered (Φ, N)-module attached to the curve, following the work of Coleman and Iovita.

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0.1. The filtered (Φ, N)-module attached to a semistable curve. Let K be a local field of characteristic 0 and residual characteristic p. Denote by $\mathcal{O}_K$ its ring of integers, and by κ its residue field. Denote by K_0 the fraction field of the Witt vectors of κ, and by σ its Frobenius automorphism. Thus K/K_0 is a finite, totally ramified extension.
By a *curve X over $\mathcal{O}_K$* we shall mean a proper flat scheme over $\mathcal{O}_K$ of relative dimension 1. We denote its generic fiber by X_K and its special fiber by X_κ. We assume that X has *semistable reduction*. This means that X is regular and X_κ is a reduced curve whose singularities are ordinary double points. [Some authors use a less restrictive definition, in which X need not be regular, but this will require some modifications in what we do below.] We assume also that X_κ is *split*: the irreducible components of the (geometric) special fiber, its singular points, and the two tangents at each singular point, are all defined over κ. This can be achieved if we replace K by a finite unramified extension.
Let $H = H^1_{dR}(X_K/K)$ be the first de-Rham cohomology of X_K. It can be identified with the space of *differentials of the second kind* on X_K modulo exact differentials. H is a finite dimensional vector space over K, and it carries the Hodge filtration (differentials of the first kind)

$$0 \subset F^1_{dR} = H^0(X_K, \Omega^1) \subset F^0_{dR} = H. \qquad (0.1)$$

Let $X_\kappa^\times$ be the log-scheme associated to the special fiber with its induced log-structure [Ill]. Let $D = H^1_{crys}(X_\kappa^\times/K_0)$ be its first log-crystalline cohomology [LS, H-K]. Recall that D is a finite dimensional vector space over K_0, which comes equipped with a σ-linear bijective endomorphism Φ (*Frobenius*) and a nilpotent endomorphism N (*monodromy*) satisfying the relation

$$N\Phi = p\Phi N. \tag{0.2}$$

For every prime π of K Hyodo and Kato constructed a *comparison isomorphism*

$$\rho_\pi : D \otimes_{K_0} K \simeq H \tag{0.3}$$

and the following relation holds for any two choices of a uniformizer

$$\rho_{\pi'} = \rho_\pi \circ \exp\left(\log(\pi'/\pi)N\right). \tag{0.4}$$

Note that the exponential is in fact a finite sum because N is nilpotent.
The structure $(H, F^{\cdot}_{dR}, D, \Phi, N, \rho_\pi)$ is the *filtered* (Φ, N)*-module* attached to X.

0.2. The weight decomposition. Let $\phi = \Phi^f$, where $f = [\kappa : \mathbb{F}_p]$, be the relative Frobenius, which now acts linearly on D. Write $q = p^f$ for the cardinality of κ. By [LS] (see [Mo] in higher dimensions) we have a *weight decomposition*

$$D = D^0 \oplus D^1 \oplus D^2 \tag{0.5}$$

where ϕ acts on D^i with eigenvalues which are q-Weil numbers of weight i (algebraic integers whose absolute value in any complex embedding is $q^{i/2}$). From the relation $N\phi = q\phi N$ we deduce that N must vanish on D^0 and D^1, and must map D^2 to D^0. In fact, it is known that it maps D^2 *isomorphically* onto D^0. This is a special case of the p-adic monodromy-weight conjecture.
By means of the isomorphism ρ_π we transport the weight decomposition to H,

$$H = H^0 \oplus H^1 \oplus H^2_\pi \tag{0.6}$$

where only the last summand, but not H^0 or H^1, depends on π, because N vanishes on D^0 and D^1. The weight filtration is defined by

$$F^i_W H = \sum_{j \le i} H^j. \tag{0.7}$$

Our goal is to *explain the weight decomposition of H in terms of the generic fiber only,* using two transcendental processes in rigid analysis - Schneider and Coleman integration. The main theorem is a reformulation of the work of Coleman and Iovita [Co-I1], and only the presentation, and a few trivial observations, are new. We have a vague hope that similar techniques might help to understand the weight decomposition, and in particular the monodromy-weight conjecture, for the cohomology of higher dimensional varieties as well. For p-adically uniformized varieties this was done in [dS], see also [Ito]. We also note that [Co-I2] and [GK] treat Frobenius and monodromy in more general situations, the first reference in cohomology of curves with coefficients, and the second in higher dimensions.

0.3. Schneider integration. To describe the main theorem, consider the *degeneration complex* Δ of X (also called the dual graph of the special fiber). Its *vertices* Δ_0 are the irreducible components of X_κ. Its *edges* Δ_1 are the singular points of X_κ (recall that the special fiber is assumed to be split). Each singular point, being an ordinary double point, determines two distinct analytic branches. The irreducible components on which these analytic branches lie, which may be the same, are the two end points of the edge. An *orientation* of an edge is an ordering of the two analytic branches at the singularity. We denote by $\vec{\Delta}_1$ the set of oriented edges, and by $\vec{\Delta}_1(v)$ the oriented edges originating at a vertx v. Note that if the two end points of an edge e are distinct, at most one of the oriented edges $\varepsilon, \bar{\varepsilon}$ lying over e may belong to $\vec{\Delta}_1(v)$, but if e is a loop based at v, then both of them belong there.
We introduce the space of *harmonic* 1-*cochains* on Δ which we denote by $C^1_{har}(\Delta)$. These are the maps $f : \vec{\Delta}_1 \to K$ satisfying

$$\text{(i) } f(\bar{\varepsilon}) = -f(\varepsilon), \text{ (ii) } \forall v \sum_{\varepsilon \in \vec{\Delta}_1(v)} f(\varepsilon) = 0. \tag{0.8}$$

There is a canonical isomorphism

$$\nu : C^1_{har}(\Delta) \simeq H^1(\Delta, K), \tag{0.9}$$

which sends a harmonic cochain to the singular cohomology class that it represents.
Let X^{an} denote the rigid analytic curve (over $\bar{K}$) attached to $X_{\bar{K}}$. There is a well known "*retraction*" map $r : X^{an} \to |\Delta|$. The inverse image under r of a vertex v consists of an affinoid with good reduction X_v. The *reduction* of X_v is the smooth part of Y_v, the irreducible component of X_κ labelled by v. The inverse image under r of an open edge e is an annulus X_e, isomorphic to

$$\{z |\ |\pi| < |z| < 1\}, \tag{0.10}$$

and an orientation of e determines an orientation of the annulus. All the points in X_e reduce in the special fiber to the singular point labelled by e.
If ω is a regular differential, and ε is an orientation on e, $res_\varepsilon(\omega)$ will denote the residue of ω with respect to a local parameter z on X_e compatible with ε. Clearly

$$res_{\bar{\varepsilon}}\omega = -res_\varepsilon\omega, \tag{0.11}$$

and the rigid analytic Cauchy theorem guarantees that

$$\sum_{\varepsilon \in \vec{\Delta}_1(v)} res_\varepsilon(\omega) = 0. \tag{0.12}$$

Defining

$$c_\omega(\varepsilon) = res_\varepsilon(\omega) \tag{0.13}$$

we obtain a harmonic 1-cochain c_ω. This definition extends without any difficulty to differentials of the second kind. Indeed, such a differential may be locally (Zariski) modified by an exact differential to make it regular, so on each

X_e we may assume it is regular and define its residue as before. Cauchy's theorem still holds, so c_ω is harmonic. Since all the residues along annuli of exact differentials vanish, we get a well defined map

$$H \to C^1_{har}(\Delta), \quad [\omega] \mapsto c_\omega. \tag{0.14}$$

Passing to $H^1(\Delta, K)$ we obtain the *Schneider class*

$$S_\omega = \nu(c_\omega) \in H^1(\Delta, K) \tag{0.15}$$

of ω.

0.4. Coleman integration. To define the Coleman class of a differential of the second kind ω we use Coleman's p-adic integration [Co], [Co-dS]. Let $\tilde{\Delta}$ be the tree which is the universal covering of Δ, and

$$\tilde{X}^{an} = X^{an} \times_\Delta \tilde{\Delta} \tag{0.16}$$

the rigid analytic curve which is the fiber product of X^{an} with $\tilde{\Delta}$ over Δ (the map from X^{an} to Δ being the retraction map r). We shall denote by Γ the group of deck transformations of the covering $\tilde{\Delta} \to \Delta$ (or, equivalently of $\tilde{X}^{an} \to X^{an}$). We shall continue to denote by r also the map from $\tilde{X}^{an}$ to $|\tilde{\Delta}|$. Let $\tilde{X}(\tilde{v})$, for $\tilde{v} \in \tilde{\Delta}_0$, be the inverse image under r of the *star* of a vertex $\tilde{v}$. (The star is the union of the vertex with the open edges originating at $\tilde{v}$. Note that a loop in Δ based at v lifts in $\tilde{\Delta}$ to two distinct edges starting at $\tilde{v}$. If we assume that Δ has no loops, then $\tilde{X}(\tilde{v})$ is isomorphic to $X(v)$, the inverse image of the star of v in X^{an}.) In Coleman's language $\tilde{X}(\tilde{v})$ is a wide open space, and (if v is the image of $\tilde{v}$ in Δ) $\tilde{X}_{\tilde{v}} \approx X_v$ is an underlying affinoid with good reduction in $\tilde{X}(\tilde{v})$. One can define a Coleman primitive $F_{\pi,\tilde{v}}$ of ω in $\tilde{X}(\tilde{v})$. It is a locally analytic function which satisfies $dF_{\pi,\tilde{v}} = \omega$, and is uniquely determined up to an additive constant by its behavior under a rigid analytic (overconvergent) lifting of Frobenius to $\tilde{X}_{\tilde{v}}$. As the notation suggests, $F_{\pi,\tilde{v}}$ depends (on the annuli surrounding $\tilde{X}_{\tilde{v}}$ in $\tilde{X}(\tilde{v})$) on the choice of π. For a given π, though, and neighboring vertices $\tilde{v}$, $\tilde{u}$ of $\tilde{\Delta}$, $F_{\pi,\tilde{v}} - F_{\pi,\tilde{u}}$ is constant on the annulus where it is defined. Since $\tilde{\Delta}$ is a tree, it is possible to choose the constants in such a way that the $F_{\pi,\tilde{v}}$ glue to give a primitive F_π of ω on all of $\tilde{X}^{an}$. Since ω is Γ-invariant,

$$C_{\pi,\omega}(\gamma) = \gamma(F_\pi) - F_\pi = F_\pi \circ \gamma^{-1} - F_\pi \tag{0.17}$$

is constant for every deck transformation $\gamma \in \Gamma$. The homomorphism

$$C_{\pi,\omega} \in H^1(\Gamma, K) = H^1(\Delta, K) \tag{0.18}$$

is the obstruction to descending F_π to X^{an}. It vanishes if and only if F_π lives on X^{an}, not merely on $\tilde{X}^{an}$, namely if and only if we can "Coleman integrate" ω on X.

We shall prove that

$$C_{\pi',\omega} - C_{\pi,\omega} = -\log(\pi'/\pi) S_\omega. \tag{0.19}$$

Our reformulation of the paper [Co-I] can now be stated as follows.

Theorem 0.1. *(1) One has a canonical identification $H/F^1_W H \simeq C^1_{har}(\Delta)$ via the residue homomorphism. In other words, the cochains c_ω give us all the harmonic cochains, and $c_\omega = 0$ if and only if $S_\omega = 0$, if and only if $\omega \in F^1_W H$.*
(2) One has a canonical identification $F^0_W = H^0 = H^1(\Delta, K)$ and $C_{\pi,\omega}$ is the projection of $[\omega]$ onto H^0 relative to the decomposition $H = H^0 \oplus H^1 \oplus H^2_\pi$.
(3) The map ν corresponds to the monodoromy isomorphism

$$\nu : H/F^1_W H \simeq H^0 \tag{0.20}$$

derived from N.

Corollary 0.2. *The subspace H^1 is characterized as the space of differentials of the second kind for which a global Coleman primitive exists on X, regardless of π.*

Proof. Indeed, in view of the relation between $C_{\pi,\omega}$, $C_{\pi',\omega}$ and S_ω, the following are equivalent: (i) $C_{\pi,\omega} = 0$ for all π (ii) $C_{\pi,\omega} = 0$ for two π whose ratio is not a root of unity, (iii) $C_{\pi,\omega} = 0$ for some π and $S_\omega = 0$. In view of the theorem, the last property is equivalent to $[\omega] \in H^1$. □

Another corollary is the following. Denote by $g(X_K)$ the genus of the curve and by $g(\Delta)$ the genus of Δ.

Corollary 0.3. *For generic (all but finitely many) π the image of $C_{\pi,\omega}$ is all of $H^1(\Delta, K)$, and the dimension of the space of Coleman-integrable differentials of the second kind modulo exact differentials is $2g(X_K) - g(\Delta)$.*

Proof. Since S is surjective, so is C_π for all but finitely many π. □

The Hodge filtration did not play any role so far. The position of the differentials of the first kind in H with respect to the weight decomposition is mysterious. It is known that they are transversal to H^0, and that together with $F^1_W H$ they span H, but their intersection with H^1 can be large or small. All we can say is the following.

Corollary 0.4. *For generic π, the dimension of the Coleman-integrable differentials of the first kind is $g(X_K) - g(\Delta)$. The dimension of the space of differentials of the first kind for which a Coleman primitive exists for all π is*

$$g(X_K) - 2g(\Delta) \leq \dim(H^1 \cap F^1_{dR}) \leq g(X_K) - g(\Delta). \tag{0.21}$$

Proof. Since F^1_{dR} maps *onto* $C^1_{har}(\Delta)$ under the residue map, $S|_{F^1_{dR}}$ is still surjective, so the first assertion is proved as in the previous corollary. The upper bound in the second assertion follows from it, while the lower bound is obvious by counting dimensions. □

Remark 0.1. *In [Cz], Colmez defines primitives for* every *differential of the second kind on X_K, regardless of the type of reduction. His primitives are independent of a choice of π, and in general do not coincide with Coleman's primitives, except for the case of good reduction. He embeds the curve in its Jacobian, and uses the group structure of the latter to extend his integral from* *a neighborhood of the origin to the whole Jacobian. As an example, the reader may keep in mind the case of a Tate elliptic curve, of multiplicative period q_E. Colmez' primitive of the differential of the first kind in this case would be the same as Coleman's primitive, based on a branch of the p-adic logarithm which vanishes on q_E. It is clear that for curves of higher genus no branch of the logarithm conforms to all the periods. It is precisely the consideration of Coleman's theory, as opposed to Colmez', that gives us the possibility to identify the weight decomposition in the generic fiber (granted a choice of π is fixed).*

1. THE PROOF

1.1. ESTABLISHING THE RELATION BETWEEN $C_{\pi,\omega}$ AND S_ω. Denote by $\log_\pi$ the unique logarithm on $\bar{K}^\times$ for which $\log_\pi(\pi) = 0$. We recall that Coleman's primitive $F_{\pi,\tilde{v}}$ on the wide open $\tilde{X}(\tilde{v})$ satisfies the following. If $\tilde{\varepsilon} = (\tilde{v}, \tilde{u})$ is an oriented edge of $\tilde{\Delta}$, and $\tilde{X}_{\tilde{\varepsilon}}$ the corresponding oriented annulus in $\tilde{X}^{an}$, and if z is a local parameter on $\tilde{X}_{\tilde{\varepsilon}}$, then we may expand

$$\omega|_{\tilde{X}_{\tilde{\varepsilon}}} = \sum a_n z^n dz \tag{1.1}$$

and, up to an additive constant,

$$F_{\pi,\tilde{v}}|_{\tilde{X}_{\tilde{\varepsilon}}} = \sum_{n\neq -1} a_n(n+1)^{-1} z^{n+1} + a_{-1}\log_\pi(z). \tag{1.2}$$

Since

$$\log_{\pi'}(z) - \log_\pi(z) = -\log(\pi'/\pi) ord_K(z) \tag{1.3}$$

we get that (again, up to a constant)

$$F_{\pi'} - F_\pi|_{\tilde{X}_{\tilde{\varepsilon}}} = -\log(\pi'/\pi) res_\varepsilon(\omega) ord_K(z). \tag{1.4}$$

On an affinoid $\tilde{X}_{\tilde{v}}$, Coleman's primitive is independent of π, up to a constant. Let $\gamma \in \Gamma$ and normalize $F_{\pi'}$ and F_π so that they agree on $\tilde{X}_{\tilde{v}}$. On $\tilde{X}_{\gamma^{-1}(\tilde{v})}$ we shall then have

$$F_{\pi'} - F_\pi = -\log(\pi'/\pi) \sum_{\varepsilon\in(\tilde{v},\gamma^{-1}(\tilde{v}))} res_\varepsilon \omega, \tag{1.5}$$

where the sum is over the oriented edges leading from $\tilde{v}$ to $\gamma^{-1}(\tilde{v})$. This sum is just $S_\omega(\gamma)$, because S_ω is obtained from c_ω via the connecting homomorphism

$$C^1_{har}(\Delta) = C^1_{har}(\tilde{\Delta})^\Gamma \to H^1(\Gamma, K) \tag{1.6}$$

which is associated with the short exact sequence

$$0 \to K \to \tilde{C}^0_{har}(\tilde{\Delta}) \to C^1_{har}(\tilde{\Delta}) \to 0, \tag{1.7}$$

where $\tilde{C}^0_{har}(\tilde{\Delta})$ are the 0-cochains on the tree satisfying the mean value property. It follows that

$$C_{\pi',\omega}(\gamma) - C_{\pi,\omega}(\gamma) = -\log(\pi'/\pi) S_\omega(\gamma), \tag{1.8}$$

as we had to prove.

If we assume theorem 1, then (1.8) follows also from (0.4) and the fact that $N^2 = 0$ (and vice versa). In the rest of this chapter we shall show how to derive the main theorem from the paper [Co-I].

1.2. The weight filtration on H. Determining the weight *filtration* on H in terms of the general fiber, and finding an expression for N, do not require a choice of π, or the use of Coleman integration. These will be needed only for the weight *decomposition*, to be considered in the next section.

By GAGA, H can be identified with rigid de-Rham cohomology $H^1_{dR}(X_K^{an})$. For simplicity let us assume from now on that Δ contains no loops, so we identify the wide open set $\tilde{X}(\tilde{v})$ with its image $X(v)$ in X^{an}. The covering $\mathcal{U} = \{X(v)\}$ of X^{an} is admissible and acyclic, defined over K. It follows that we may identify H with the space of rigid 1-hyper-cocycles

$$(1.9)\qquad \left\{ \begin{array}{c} (\omega_v, f_\varepsilon);\ \omega_v \in \Omega(X(v)),\ f_\varepsilon \in \mathcal{O}(X_\varepsilon),\ df_\varepsilon = \omega_v - \omega_u|_{X_\varepsilon} \\ \text{if } \varepsilon \text{ connects } u \text{ to } v, \text{and } f_{\bar{\varepsilon}} = -f_\varepsilon \end{array} \right\},$$

modulo the space of rigid 1-hyper-coboundaries: elements of the type $(df_v, f_v - f_u)$ for $f_v \in \mathcal{O}(X(v))$. Specifically, if ω is a differential of the second kind, we pick rational functions g_v so that $\omega_v = \omega - dg_v$ is holomorphic on $X(v)$, and put $f_\varepsilon = g_u - g_v$. The class $[\omega]$ is then represented by $(\omega_v, f_\varepsilon)$.

Since X_v is an affinoid with good reduction, the Frobenius morphism ϕ (of degree q) lifts to characteristic 0, to a rigid analytic mapping ϕ_v of X_v to itself. This rigid analytic Frobenius is *overconvergent:* there exists a strict neighborhood $X_v \subset X'_v \subset X(v)$ such that ϕ_v extends to a morphism of X'_v to $X(v)$. This X'_v can (and will) be chosen to consist of X_v together with an open annulus for each edge originating at v, and then the inclusion $X'_v \subset X(v)$ induces isomorphism on de Rham cohomology. We can therefore regard ϕ_v^* as an endomorphism of $H^1_{dR}(X(v))$. In fact, if we let Y_v^0 be the smooth part of Y_v, the reduction of X_v, $H^1_{dR}(X(v))$ is nothing but the Monsky-Washnitzer cohomology of Y_v^0 (tensored with K) and ϕ_v^* is its Frobenius. It is independent of the lifting.

The roots of the characteristic polynomial of ϕ_v^* on $H^1_{dR}(X(v))$ have weights 1 or 2. Moreover, there is an exact sequence

$$(1.10)\qquad 0 \to F_W^1 H^1_{dR}(X(v)) \to H^1_{dR}(X(v)) \stackrel{res}{\to} \left(\oplus_{\varepsilon \in \vec{\Delta}_v} K\right)_0 \to 0$$

where res is the residue map, and F_W^1 is the weight 1 subspace. The subscript 0 on the quotient means that we take only those elements in the direct sum whose coordinates add up to 0. On the weight 2 quotient ϕ_v^* acts by multiplication by q.

Let $F_W^0 H$ be the subspace of H represented by classes $[(0, k_\varepsilon)]$, where the k_ε are constants. It is thus isomorphic to $H^1(\Delta, K)$. Coleman and Iovita prove that under the Hyodo-Kato isomorphism this subspace is the image of the weight zero part of D (combine Lemma I.4.2 and Theorem II.5.4 of their paper).

The quotient

$$(1.11)\qquad H/F_W^0 H$$

is the image of H in $H^1_{dR}(\tilde{X}^{an})$ under pullback. It is the space of differentials of the second kind on X^{an}, modulo those which become exact on $\tilde{X}^{an}$. The residue map gives the filtration

$$(1.12)\qquad 0 \to gr^1_W H = F^1_W H/F^0_W H \to H/F^0_W H \overset{res}{\to} C^1_{har}(\Delta) \to 0,$$

which is the direct sum, over the vertices of Δ, of the short exact sequences recorded above. The surjectivity of the global residue map results from a dimension counting. Once again, Coleman and Iovita prove that under the Hyodo-Kato isomorphism, the Frobenius structure of $H/F^0_W H$ is the one described above, rigid analytically, in terms of the ϕ_v^*. (Compare how they define, in Section I.1, the Frobenius structure on

$$(1.13)\qquad Ker(H^1_{dR}(X^0) \to H^1_{dR}(X^1)^-),$$

which is our $H/F^0_W H$, and apply their Theorem II.5.4.) It follows that F^1_W is indeed the weight 1 filtration, and $C^1_{har} = gr^2_W H$. Finally, that the monodromy operator is derived from the isomorphism ν between $C^1_{har}(\Delta)$ and $H^1(\Delta, K)$ also follows from [Co-I] (combine the description of N in Section I.1.1 and the commutative diagram on p.185). This checks all the statements of our main theorem, except for the identification of the weight decomposition in terms of Coleman integration.

1.3. The weight decomposition on H. Fix a choice of π. In Section I.1 of [Co-I] the authors describe a splitting of the projection $H \to H/F^0_W H$, whose image is $H^1 \oplus H^2_\pi$. Recall that an element of $H/F^0_W H$ is a collection $\{[\omega_v]\}$ of classes $[\omega_v] \in H^1_{dR}(X(v))$, such that for any oriented edge ε, connecting u to v, $res_\varepsilon \omega_u = res_\varepsilon \omega_v$. Let $F_{\pi,v}$ be the Coleman integral of ω_v on $X(v)$, described above, which is determined up to a constant. Since the residues of ω_u and ω_v on X_ε agree, the function

$$(1.14)\qquad f_{\pi,\varepsilon} = F_{\pi,v} - F_{\pi,u} \in \mathcal{O}(X_\varepsilon)$$

is rigid analytic in the annulus. The 1-hyper-cocyle $(\omega_v, f_{\pi,\varepsilon})$ is well-defined up to a coboundary, and its class in H gives the desired splitting.

It is now easy to check that C_π vanishes on classes ω which are in the image of this splitting. Indeed, suppose the differential of the second kind ω is such that

$$(1.15)\qquad \omega = \omega_v + dg_v$$

for a meromorphic function g_v on $X(v)$, and $g_u - g_v = f_{\pi,\varepsilon} = F_{\pi,v} - F_{\pi,u}$ on X_ε. Then $F_{\pi,u} + g_u$ agree on the annuli, hence glue to give a well defined Coleman meromorphic function $F_{\pi,\omega}$ on X^{an}, which is a global primitive of ω. It follows that $C_{\pi,\omega} = 0$.

On the other hand, if we start with a 1-hypercocycle $(0, k_\varepsilon)$ where the k_ε are constants, and if ω is a differential of the second kind for which there are meromorphic functions g_v on $X(v)$ such that $\omega = dg_v$ there, and $g_v - g_u = k_\varepsilon$ for an edge connecting u to v, then $[(k_\varepsilon)] \in H^1(\Delta, K)$ is the obstruction to integrating ω globally on X^{an}, hence is equal to $C_{\pi,\omega}$.

These computations show that C_π annihilates $H^1 \oplus H^2_\pi$, and is the identity map on H^0. This completes the proof of Theorem 1.

1.4. Relation to the Neron model of the Jacobian. Even though the primitive $\tilde{F}_{\pi,\omega}$ of a differential of the second kind ω need not descend to X^{an}, we may use it to define the integral

$$\int_D \omega \tag{1.16}$$

over *certain* divisors of degree 0, namely those who specialize in X_κ to divisors which avoid the singular points and are of degree 0 on each of the irreducible components Y_v separately. This is because such a divisor D intersects each affinoid X_v in a divisor D_v of degree 0, while $F_{\pi,v}$ is well defined, up to a constant, and independently of π, on X_v. Observe that the divisors in question are precisely those whose classes in Pic^0 represent the connected component $\mathcal{J}^0$ of the Neron model of the Jacobian of X_K.

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