Small Ramsey Numbers

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ABSTRACT: We present data which, to the best of our knowledge, includes all known nontrivial values and bounds for specific graph, hypergraph and multicolor Ramsey numbers, where the avoided graphs are complete or complete without one edge. Many results pertaining to other more studied cases are also presented. We give references to all cited bounds and values, as well as to previous similar compilations. We do not attempt complete coverage of asymptotic behavior of Ramsey numbers, but concentrate on their specific values.

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Table of Contents **2**

1. Scope and Notation

There is vast literature on Ramsey type problems starting in 1930 with the original paper of Ramsey [Ram]. Graham, Rothschild and Spencer in their book [GRS] present an exciting development of Ramsey Theory. The subject has grown amazingly, in particular with regard to asymptotic bounds for various types of Ramsey numbers (see the survey papers [GrRö, Neš, ChGra2]), but the progress on evaluating the basic numbers themselves has been unsatisfactory for a long time. In the last three decades, however, considerable progress has been obtained in this area, mostly by employing computer algorithms. The few known exact values and several bounds for different numbers are scattered among many technical papers. This compilation is a fast source of references for the best results known for specific numbers. It is not supposed to serve as a source of definitions or theorems, but these can be easily accessed via the references gathered here.

Ramsey Theory studies conditions when a combinatorial object contains necessarily some smaller given objects. The role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey Theory.

Let $G_1, G_2, \ldots, G_m$ be graphs or s-uniform hypergraphs (s is the number of vertices in each edge). $R(G_1, G_2, \ldots, G_m; s)$ denotes the m-color **Ramsey number** for s-uniform graphs/hypergraphs, avoiding G_i in color i for $1 \le i \le m$. It is defined as the least integer n such that, in any coloring with m colors of the s-subsets of a set of n elements, for some i the s-subsets of color i contain a sub-(hyper)graph isomorphic to G_i (not necessarily induced). The value of $R(G_1, G_2, \ldots, G_m; s)$ is fixed under permutations of the first m arguments. If $s = 2$ (standard graphs) then s can be omitted. If G_i is a complete graph K_k, then we may write k instead of G_i, and if $G_i = G$ for all i we may use the abbreviation $R_m(G; s)$ or $R_m(G)$. For $s = 2$, $K_k - e$ denotes a K_k without one edge, and for $s = 3$, $K_k - t$ denotes a K_k without one triangle (hyperedge).

The graph nG is formed by n disjoint copies of G, $G \cup H$ stands for vertex disjoint union of graphs, and the **join** $G + H$ is obtained by adding all the edges between vertices of G and H to $G \cup H$. P_i is a **path** on i vertices, C_i is a **cycle** of length i, and W_i is a **wheel** with $i-1$ spokes, i.e. a graph formed by some vertex x, connected to all vertices of the cycle C_{i-1} (thus $W_i = K_1 + C_{i-1}$). $K_{n,m}$ is a complete n by m bipartite graph, in particular $K_{1,n}$ is a **star** graph. The **book** graph $B_i = K_2 + \bar{K}_i = K_1 + K_{1,i}$ has $i+2$ vertices, and can be seen as i triangular pages attached to a single edge. The **fan** graph F_n is defined by $F_n = K_1 + nK_2$. For a graph G, $n(G)$ and $e(G)$ denote the number of vertices and edges, respectively, and $\delta(G)$ and $\Delta(G)$ minimum and maximum degree in G. Finally, let $\chi(G)$ be the chromatic number of G. In general we follow the notation used by West [West].

Section 2 contains the data for the classical two color Ramsey numbers $R(k, l)$ for complete graphs, section 3 for much studied two color cases of $K_n - e$, K_3, $K_{m,n}$, and section 4 for numbers involving cycles. Section 5 lists other often studied two color cases for general graphs. The multicolor and hypergraph cases are gathered in sections 6 and 7, respectively. Finally, section 8 gives pointers to cumulative data and to other surveys.

2. Classical Two-Color Ramsey Numbers

2.1. Values and bounds for $R(k,l)$, $k \le 10$, $l \le 15$

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	73 115	92 149	98 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	126 316	144 442	171 633	191 848	213 1139	239 1461	265 1878
6				102 165	113 298	132 495	169 780	179 1171	253 1804	263 2566	317 3705	 5033	401 6911
7					205 540	217 1031	241 1713	289 2826	405 4553	417 6954	511 10581	 15263	 22116
8						282 1870	317 3583	 6090	 10630	 16944	817 27490	 41525	861 63620
9							565 6588	581 12677	 22325	 39025	 64871	 89203	
10								798 23556		 81200			1265

Table I. Known nontrivial values and bounds for two color Ramsey numbers $R(k,l) = R(k,l;2)$.

k \ l	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	Kéry	Ka2 GrY	GR MZ	Ka2 GR	Ex5 RK2	Ka2 RK2	Ex12 Les	Piw1 RK2	Ex8 RK2	WW Les
4	GG	Ka1 MR4	Ex9 MR5	Ex3 Mac	Ex15 Mac	Ex17 Mac	HaKr Mac	Ex18 Spe3	SLL Spe3	2.3.e Spe3	XXR Spe3	XXR Spe3
5		Ex4 MR5	Ex9 HZ1	CET Spe3	HaKr Spe3	Ex18 Mac	Ex18 Mac	Gerb HW+	Gerb HW+	Gerb HW+	Gerb HW+	Ex17 HW+
6			Ka1 Mac	Ex17 Mac	XSR2 Mac	XXER Mac	Ex17 Mac	XXR HW+	XSR2 HW+	XXER HW+	 HW+	2.3.h HW+
7				She1 Mac	XSR2 Mac	XSR2 HZ1	2.3.h Mac	XXER HW+	XSR2 HW+	XXR HW+	 HW+	 HW+
8					BR Mac	XXER Ea1	 HZ1	 HW+	 HW+	XXER HW+	 HW+	2.3.h HW+
9						She1 ShZ1	XSR2 Ea1	 HW+	 HW+	 HW+	 HW+	
10							She1 Shi2		 Yang			2.3.h

References for Table I. HW+ abbreviates HWSYZH.

We split the data into the table of values and a table with corresponding references. In Table I, known exact values appear as centered entries, lower bounds as top entries, and upper bounds as bottom entries. For some of the exact values two references are given when the lower and upper bound credits are different.

(a) The task of proving $R(3,3) \leq 6$ was the second problem in Part I of the William Lowell Putnam Mathematical Competition held in March 1953 [Bush].

(b) Greenwood and Gleason [GG] in 1955 established the initial values $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$.

(c) Kéry [Kéry] in 1964 found $R(3,6) = 18$, but only recently an elementary and self-contained proof of this result appeared in English [Car].

(d) All the critical graphs for the numbers $R(k,l)$ (graphs on $R(k,l)-1$ vertices without K_k and without K_l in the complement) are known for $k=3$ and $l=3$, 4, 5 [Kéry], 6 [Ka2], 7 [RK3, MZ], and there are 1, 3, 1, 7 and 191 of them, respectively. All $(3,k)$-graphs, for $k \leq 6$, were enumerated in [RK3], and all (4,4)-graphs in [MR2]. There exists a unique critical graph for $R(4,4)$ [Ka2]. There are 430215 such graphs known for $R(3,8)$ [McK], 1 for $R(3,9)$ [Ka2] and 350904 for $R(4,5)$ [MR4], but there might be more of them. The graphs constructed by Exoo in [Ex9, Ex12, Ex13, Ex14, Ex15, Ex16, Ex17], and some others, are available electronically from http://ginger.indstate.edu/ge/RAMSEY.

(e) In [MR5], strong evidence is given for the conjecture that $R(5,5) = 43$ and that there exist exactly 656 critical graphs on 42 vertices.

(f) *Cyclic* (or *circular*) graphs are often used for Ramsey graph constructions. Several cyclic graphs establishing lower bounds were given in the Ph.D. dissertation by J.G. Kalbfleisch in 1966, and many others were published in the next few decades (see [RK1]). Harborth and Krause [HaKr] presented all best lower bounds up to 102 from cyclic graphs avoiding complete graphs. In particular, no lower bound in Table I can be improved with a cyclic graph on less than 102 vertices. See also item 2.3.k and section 5.16 [HaKr].

(g) The claim that $R(5,5) = 50$ posted on the web [Stone] is in error, and despite being shown to be incorrect more than once, this value is still being cited by some authors. The bound $R(3,13) \geq 60$ [XieZ] cited in the 1995 version of this survey was shown to be incorrect in [Piw1]. Another incorrect construction for $R(3,10) \geq 41$ was described in [DuHu].

(h) There are really only two general upper bound inequalities useful for small parameters, namely 2.3.a and 2.3.b. Stronger upper bounds for specific parameters were difficult to obtain, and they often involved massive computations, like those for the cases of (3,8) [MZ], (4,5) [MR4], (4,6) and (5,5) [MR5]. The bound $R(6,6) \leq 166$, only 1 more than the best known [Mac], is an easy consequence of a theorem in [Walk] (2.3.b) and $R(4,6) \leq 41$.

(i) T. Spencer [Spe3], Mackey [Mac], and Huang and Zhang [HZ1], using the bounds for minimum and maximum number of edges in (4,5) Ramsey graphs listed in [MR3, MR5], were able to establish new upper bounds for several higher Ramsey numbers, improving on all of the previous longstanding best results by Giraud [Gi3, Gi5, Gi6].

(j) Only some of the higher bounds implied by 2.3.* are shown, and more similar bounds could be derived. In general, we show bounds beyond the contiguous small values if they improve on results previously reported in this survey or published elsewhere. Some easy upper bounds implied by 2.3.a are marked as [Ea1].

(k) We have recomputed the upper bounds in Table I marked [HZ1] using the method from the paper [HZ1], because the bounds there relied on an overly optimistic personal communication from T. Spencer. Further refinements of this method are studied in [HZ2, ShZ1, Shi2]. The paper [Shi2] subsumes the main results of the manuscripts [ShZ1, Shi2]. The upper bound $R(10,12) \le 81200$ in Table I [Yang] was obtained by Yang using the method of [HWSYZH] (abbreviated in the table as HW+).

2.2. Bounds for $R(k,l)$, higher parameters

l / k	15	16	17	18	19	20	21	22	23
3	73 WW	79 WW	92 WWY1	99 Ex17	106 WWY1	111 Ex17	122 WWY1	131 WSLX2	139 XWCS
4	153 XXR	164 Gerb	200 LWXS	205 2.3.e	213 2.3.g	234 Ex17	242 SLZL	314 LSLW	
5	265 Ex17	289 2.3.h	388 XSR2	396 2.3.g	411 XSR2	424 XSR2	441 2.3.h	485 2.3.h	521 2.3.h
6	401 2.3.h	434 SLLL	548 SLLL	614 SLLL	710 SLLL	878 SLLL		1070 SLLL	
7		609 2.3.h	711 2.3.g	797 2.3.h	908 SLLL		1214 SLLL		
8	861 2.3.h		961 XSR2	1045 2.3.g	1236 2.3.g		1617 2.3.h		

l / k	24	25	26	27	28	29	30	31
3	143 WSLX1	154 WSLX2	159 WSLX1	167 WSLX1	173 WSLX2	184 WSLX2	190 WSLX2	199 WSLX2

l / k	32	33	34	35	36	37	38	39	40
3	214 WSLX2	218 ChW+	226 ChW+	231 ChW+	239 ChW+	244 ChW+	256 ChW+		

Table II. Known nontrivial lower bounds for higher two color Ramsey numbers $R(k,l)$, with references.

(a) The construction by Mathon [Mat] and Shearer [She1] (see also items 2.3.i, 6.2.k and 6.2.l), using the data obtained by Shearer [She3] for primes up to 7000, implies in particular the following diagonal lower bounds: $R(11,11) \geq 1597$, $R(13,13) \geq 2557$, $R(14,14) \geq 2989$, $R(15,15) \geq 5485$, and $R(16,16) \geq 5605$. Similarly, $R(17,17) \geq 8917$, $R(18,18) \geq 11005$ and $R(19,19) \geq 17885$ were obtained in [LSL], though the first two of these bounds follow also from the data in [She3]. The same approach does not improve on the bound $R(12,12) \geq 1639$ [XSR2].

(b) The upper bounds of 88, 99, 110, 121 133, 145, 158 on $R(3,k)$ for $15 \leq k \leq 21$, respectively, were obtained in [Les]. The lower bounds marked [XXR], [XXER], [XSR2], 2.3.e and 2.3.h need not be cyclic. Several of the Cayley colorings from [Ex17] are also non-cyclic. All other lower bounds listed in Table II were obtained by construction of cyclic graphs.

(c) The graphs establishing lower bounds marked 2.3.g can be constructed by using appropriately chosen graphs G and H with a common m-vertex induced subgraph, similarly as it was done in several cases in [XXR].

(d) Yu [Yu2] constructed a special class of triangle-free cyclic graphs establishing several lower bounds for $R(3,k)$, for $k \geq 61$. All of these bounds can be improved by the inequalities in 2.3.c and data from Tables I and II.

(e) Unpublished bound $R(4,22) \geq 314$ [LSLW] improves over 282 given in [SL]. [LSLW] includes also $R(4,25) \geq 458$. Not yet published bounds $R(3,23) \geq 139$ [XWCS] and $R(4,17) \geq 200$ [LWXS] improve over 137 and 182 obtained in [WSLX2] and [LSS1], respectively.

(f) Two special cases which improve on bounds listed in earlier revisions: $R(9,17) \geq 1411$ is given in [XXR] and $R(10,15) \geq 1265$ can be obtained by using 2.3.h.

(g) One can expect that the lower bounds in Table II are weaker than those in Table I, in the sense that some of them should not be that hard to improve, in contrast to the bounds in Table I, especially smaller ones.

2.3. General results on $R(k,l)$

(a) $R(k,l) \leq R(k-1,l) + R(k,l-1)$, with strict inequality when both terms on the right hand side are even [GG]. There are obvious generalizations of this inequality for avoiding graphs other than complete.

(b) $R(k,k) \leq 4R(k,k-2)+2$ [Walk].

(c) Explicit construction for $R(3,3k+1) \geq 4R(3,k+1)-3$, for all $k \geq 2$ [CleDa], explicit construction for $R(3,4k+1) \geq 6R(3,k+1)-5$, for all $k \geq 1$ [ChCD].

(d) Explicit triangle-free graphs with independence k on $\Omega(k^{3/2})$ vertices [Alon2, CPR]. For other constructive results in relation to $R(3,k)$ see [BBH1, BBH2, Fra1, Fra2, FrLo, Gri, KlaM1, Loc, RK3, RK4, Stat, Yu1]. See also (p) and (q) below.

(e) The study of bounds for the difference between consecutive Ramsey numbers was initiated in [BEFS], where the bound $R(k,l) \geq R(k,l-1) + 2k-3$, for $k,l \geq 3$, was established by a construction. Let $\Delta_{k,l} = R(k,l) - R(k,l-1)$. Only easy bounds on $\Delta_{k,l}$ are known, in particular $3 \leq \Delta_{3,l} \leq l$ for $k = 3$. Contrary to some claims about $\Delta_{k,l}$, it is not even known whether $\Delta_{k,k+1}/k \to \infty$ as $k \to \infty$, see [XSR2].

(f) By taking a disjoint union of two critical graphs one can easily see that $R(k,p) \geq s$ and $R(k,q) \geq t$ imply $R(k,p+q-1) \geq s+t-1$. Xu and Xie [XX1] improved this construction to yield better general lower bounds, in particular $R(k,p+q-1) \geq s+t+k-3$.

(g) For $2 \leq p \leq q$ and $3 \leq k$, if (k,p)-graph G and (k,q)-graph H have a common induced subgraph on m vertices without K_{k-1}, then $R(k,p+q-1) > n(G) + n(H) + m$. In particular, this implies the bounds $R(k,p+q-1) \geq R(k,p) + R(k,q) + k-3$ and $R(k,p+q-1) \geq R(k,p) + R(k,q) + p-2$ [XX1, XXR], with further small improvements in some cases, like the term $k-2$ instead of $k-3$ in the previous bound [XSR2].

(h) $R(2k-1,l) \geq 4R(k,l-1) - 3$ for $l \geq 5$ and $k \geq 2$, and in particular for $k=3$ we have $R(5,l) \geq 4R(3,l-1) - 3$ [XXER].

(i) If the quadratic residues Paley graph Q_p of prime order $p = 4t+1$ contains no K_k, then $R(k,k) \geq p+1$ and $R(k+1,k+1) \geq 2p+3$ [She1, Mat]. Data for larger p was obtained in [LSL]. See also 3.1.c, and items 6.2.k and 6.2.l for similar multicolor results.

(j) Study of Ramsey numbers for large disjoint unions of graphs [Bu1, Bu9], in particular $R(nK_k, nK_l) = n(k+l-1) + R(K_{k-1}, K_{l-1}) - 2$, for n large enough [Bu8].

(k) $R(k,l) \geq L(k,l) + 1$, where $L(k,l)$ is the maximal order of any cyclic (k,l)–graph. A compilation of many best cyclic bounds was presented in [HaKr].

(l) The graphs critical for $R(k,l)$ are $(k-1)$–vertex connected and $(2k-4)$–edge connected, for $k,l \geq 3$ [BePi]. This was improved to vertex connectivity k for $k \geq 5$ and $l \geq 3$ in [XSR2].

(m) All Ramsey-critical (k,l)–graphs are Hamiltonian for $k \geq l-1 \geq 1$ and $k \geq 3$, except $(k,l) = (3,2)$ [XSR2].

(n) Two color lower bounds can be obtained by using items 6.2.m, 6.2.n and 6.2.o with $r = 2$. Some generalizations of these were obtained in [ZLLS].

In the last seven items of this section we only briefly mention some pointers to the literature dealing with asymptotics of Ramsey numbers. This survey was designed mostly for small, finite, and combinatorial results, but still we wish to give the reader some useful and representative references to more traditional papers looking first of all at the infinite.

(o) In 1947, Erdős gave a simple probabilistic proof that $R(k,k) \geq c \cdot k 2^{k/2}$ [Erd1]. Spencer [Spe1] improved the constant c to $\sqrt{2}/e$. More probabilistic asymptotic lower bounds for other Ramsey numbers were obtained in [Spe1, Spe2, AlPu].

(p) The limit of $R(k,k)^{1/k}$, if it exists, is between $\sqrt{2}$ and 4 [GRS, GrRö, ChGra2].

(q) In a 1995 breakthrough Kim proved that $R(3,k) = \Theta(k^2/\log k)$ [Kim].

(r) Other asymptotic and general results on triangle-free graphs in relation to $R(3,k)$ can be found in [Boh, AlBK, AKS, Alon2, CleDa, ChCD, CPR, Gri, FrLo, Loc, She2].

(s) Explicit constructions yielding lower bounds $R(4,k) \geq \Omega(k^{8/5})$, $R(5,k) \geq \Omega(k^{5/3})$ and $R(6,k) \geq \Omega(k^2)$ [KosPR]. For the same cases classical probabilistic arguments give $\Omega(k/\log k)^{5/2})$, $\Omega(k/\log k)^3)$ and $\Omega(k/\log k)^{7/2})$, respectively [Spe2]. These were further improved in [Boh, BohK].

(t) Explicit construction of a graph with clique and independence k on $2^{c \log^2 k / \log\log k}$ vertices by Frankl and Wilson [FraWi]. Further constructions by Chung [Chu3] and Grolmusz [Grol1, Grol2]. Explicit constructions like these are usually weaker than known probabilistic results.

(u) In 2010, Conlon [Con1] obtained the best to date upper bound for the diagonal case:

$$R(k+1,k+1) \leq \begin{bmatrix} 2k \\ k \end{bmatrix} k^{-c \log k / \log\log k}$$

Other asymptotic bounds can be found, for example, in [Chu3, McS, Boh, BohK] (lower bound) and [Tho] (upper bound), and for many other bounds in the general case of $R(k,l)$ consult [Spe2, GRS, GrRö, Chu4, ChGra2, LiRZ1, AlPu, Kriv].

3. Two Colors: $K_n - e$, K_3, $K_{m,n}$

3.1. Dropping one edge from complete graph

This section contains known values and nontrivial bounds for the two color case when the avoided graphs are complete or have the form $K_k - e$, but not both are complete.

G \ H	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$	$K_{11}-e$
K_3-e	3	5	7	9	11	13	15	17	19
K_3	5	7	11	17	21	25	31	37 38	42 47
K_4-e	5	10	13	17	28	29 38	34	41	
K_4	7	11	19	27 34	37 52	77	105	143	187
K_5-e	7	13	22	31 39	40 66				
K_5	9	16	30 34	43 67	112	186	277	418	586
K_6-e	9	17	31 39	45 70	59 135				
K_6	11	21	37 53	114	205	385	621	1035	1551
K_7-e	11	28	40 66	59 135	251				
K_7	13	28 31	51 84	197	394	768	1339	2355	3766
K_8	15	42	123	306	659	1382	2562	4844	8223

Table III. Two types of Ramsey numbers $R(G,H)$,
includes all known nontrivial values.

(a) The exact values in Table III involving $K_3 - e$ are obvious, since one can easily see that $R(K_3 - e, K_k) = R(K_3 - e, K_{k+1} - e) = 2k - 1$, for all $k \geq 2$.

(b) The bound $R(K_3, K_{12} - e) \geq 46$ is given in [MPR]. Wang, Wang and Yan [WWY2] constructed cyclic graphs showing $R(K_3, K_{13} - e) \geq 54$, $R(K_3, K_{14} - e) \geq 59$ and $R(K_3, K_{15} - e) \geq 69$. It is known that $R(K_4, K_{12} - e) \geq 128$ [Shao] using one color of the (4,4,4;127)-coloring defined in [HiIr].

(c) If the quadratic residues Paley graph Q_p of prime order $p = 4t + 1$ contains no $K_k - e$, then $R(K_{k+1} - e, K_{k+1} - e) \geq 2p + 1$. In particular, $R(K_{14} - e, K_{14} - e) \geq 2987$ [LiShen]. See also item 2.3.i.

G \ H	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$	$K_{11}-e$
K_3	CH2	Clan	FRS1	GH	Ra1	Ra1	MPR MPR	WWY2 MPR
K_4-e	CH1	FRS2	McR	McR	Ea1 HZ2	Ex14	Ex14	
K_4	CH2	EHM1	Ex11 B1	Ex14 HZ2	B1	B1	B1	B1
K_5-e	FRS2	CEHMS	Ex14 Ea1	Ex14 HZ2				
K_5	BH	Ex6 Ex8	Ea1 HZ2	HZ2	B1	B1	B1	B1
K_6-e	McR	Ex14 Ea1	Ex14 HZ2	Ex14 HZ2				
K_6	McN	Ex14 B1	B1	ShZ2	B1	B1	B1	B1
K_7-e	McR	Ex14 HZ2	Ex14 HZ2	ShZ1				
K_7	Ea1 B1	Ex14 B1	B1	B1	B1	B1	B1	B1
K_8	B1	B1	B1	B1	B1	B1	B1	B1

References for Table III. B1 abbreviates Boza1.

(d) More bounds (beyond those shown in Table III) can be obtained by using Table I, an obvious generalization of the inequality $R(k,l) \le R(k-1,l) + R(k,l-1)$, and by monotonicity of Ramsey numbers, in this case $R(K_{k-1},G) \le R(K_k-e,G) \le R(K_k,G)$.

(e) All (K_3,K_k-e)-graphs for $k \le 6$ were enumerated in [Ra1], and for $k = 7$ in [Fid2].

(f) The critical graphs are unique for: $R(K_3,K_l-e)$ for $l=3$ [Tr], 6 and 7 [Ra1], $R(K_4-e,K_4-e)$ [FRS2], $R(K_5-e,K_5-e)$ [Ra3] and $R(K_4-e,K_7-e)$ [McR].

(g) The number of $R(K_3,K_l-e)$-critical graphs for $l = 4,5$ and 8 is 4, 2 and 9, respectively [MPR], and there are at least 6 such graphs for $R(K_3,K_9-e)$ [Ra1].

(h) All the critical graphs for the cases $R(K_4-e,K_4)$ [EHM1], $R(K_4-e,K_5)$ and $R(K_5-e,K_4)$ [DzFi1] are known, and there are 5, 13 and 6 of them, respectively.

(i) Full sets of (K_3,K_k-e)-graphs are available [Fid2] for the following parameters: (K_3,K_k-e) for $k \le 7$, (K_4,K_k-e) for $k \le 5$ and (K_5,K_k-e) for $k \le 4$.

(j) $R(K_k-e,K_k-e) \le 4R(K_{k-2},K_k-e) - 2$ [LiShen].
For a similar inequality for complete graphs see 2.3.b.

(k) The upper bounds from [ShZ1, ShZ2] are subsumed by a later article [Shi2].

(l) The upper bounds in [HZ2] were obtained by a reasoning generalizing the bounds for classical numbers in [HZ1]. Several other results from section 2.3 apply, though checking in which situation they do may require looking inside the proofs whether they still hold for K_n-e.

3.2. Triangle versus other graphs

(a) $R(3,k) = \Theta(k^2/\log k)$ [Kim].

(b) Explicit construction for $R(3,3k+1) \geq 4R(3,k+1) - 3$, for all $k \geq 2$ [CleDa], explicit construction for $R(3,4k+1) \geq 6R(3,k+1) - 5$, for all $k \geq 1$ [ChCD].

(c) Explicit triangle-free graphs with independence k on $\Omega(k^{3/2})$ vertices [Alon2, CPR].

(d) $R(K_3, K_7 - 2P_2) = R(K_3, K_7 - 3P_2) = 18$ [SchSch2].

(e) $R(K_3, K_3 + \bar{K}_m) = R(K_3, K_3 + C_m) = 2m+5$ for $m \geq 212$ [Zhou1].

(f) $R(K_3, K_2 + T_n) = 2n+3$ for n-vertex trees T_n, for $n \geq 4$ [SonGQ].

(g) $R(K_3, G) = 2n(G) - 1$ for any connected G on at least 4 vertices and with at most $(17n(G)+1)/15$ edges, in particular for $G = P_i$ and $G = C_i$, for all $i \geq 4$ [BEFRS1].

(h) Relations between $R(3,k)$ and graphs with large $\chi(G)$ [Für], further detailed study of the relation between $R(3,k)$ and the chromatic gap [GySeT].

(i) $R(K_3, G) \leq 2e(G) + 1$ for any graph G without isolated vertices [Sid3, GK].

(j) $R(K_3, G) \leq n(G) + e(G)$ for all G, a conjecture [Sid2].

(k) $R(K_3, G)$ for all connected G up to 9 vertices [BBH1, BBH2], see also section 8.1.

(l) For every positive constant c, Δ, and n large enough, there exists graph G with $\Delta(G) \leq \Delta$ for which $R(K_3, G) > cn$ [Bra3].

(m) For $R(K_3, K_n)$ see section 2, and for $R(K_3, K_n - e)$ see section 3.1.

(n) Formulas for $R(nK_3, mG)$ for all G of order 4 without isolates [Zeng].

(o) Since $B_1 = F_1 = C_3 = W_3 = K_3$, other sections apply. See also [Boh, AKS, BBH1, BBH2, FrLo, Fra1, Fra2, Für, Gri, GySeT, Loc, KlaM1, LiZa1, RK3, RK4, She2, Spe2, Stat, Yu1].

3.3. Complete bipartite graphs

NOTE: This subsection gathers information on Ramsey numbers where specific bipartite graphs are avoided in edge colorings of K_n (as everywhere in this survey), in contrast to often studied bipartite Ramsey numbers (not covered in this survey) where the edges of complete bipartite graphs $K_{n,m}$ are colored.

3.3.1. Numbers

The following Tables IVa and IVb gather information mostly from the surveys by Lortz and Mengersen [LoM3, LoM4]. All cases involving $K_{1,2} = P_3$ are solved by a formula for $R(P_3, G)$, holding for all isolate-free graphs G, derived in [CH2]. All star versus star numbers are given below in the item 3.3.2.a and in section 5.5.

p,q / m,n	1, 2	1, 3	1, 4	1, 5	1, 6	2, 2	2, 3	2, 4	2, 5	3, 3	3, 4
2, 2	4 CH2	6 CH2	7 Par3	8 Par3	9 FRS4	6 CH1					
2, 3	5 CH2	7 FRS4	9 Stev	10 FRS4	11 FRS4	8 HaMe4	10 Bu4				
2, 4	6 CH2	8 HaMe3	9 Stev	11 HaMe4	13 LoM4	9 HaMe4	12 ExRe	14 EHM2			
2, 5	7 CH2	9 HaMe3	11 Stev	13 Stev	14 LoM4	11 HaMe4	13 LoM3	16 LoM1	18 EHM2		
2, 6	8 CH2	10 HaMe3	11 Stev	14 Stev	15^* Shao	12 HaMe4	14 LoM3	17 LoM3	20 LoM1		
3, 3	7 CH2	8 HaMe3	11 LoM4	12 LoM4	13 LoM4	11 Lortz	13 HaMe3	16 LoM4	18 LoM4	18 HaMe3	
3, 4	7 CH2	9 HaMe3	11 LoM4	13 LoM4	14 LoM4	11 Lortz	14 LoM4	17 Sh+	≤ 21 LoM4	≤ 25 LoM2	≤ 30 LoM2
3, 5	9 CH2	10 HaMe3	13 Sh+	15 Sh+		14 HaMe4	$\geq 15^*$ Shao	$\geq 16^*$ Shao	$\geq 21^*$ Shao	≤ 28 LoM2	≤ 33 LoM2

Table IVa. Ramsey numbers $R(K_{m,n}, K_{p,q})$.
(unpublished results are marked with a *, Sh+ abbreviates ShaXBP)

m / n	2	3	4	5	6	7	8	9	10	11
6	12 HaMe4	14 LoM3	17 LoM3	20 LoM1	21 EHM2					
7	14 HaMe4	17 LoM3	19 LoM3	21 LoM3	24 LoM1	26 EMH2				
8	15 HaMe4	18 LoM3	20 LoM3	22^*-23 LoM3	24-25 LoM3	28 LoM1	30 EMH2			
9	16 HaMe4	19 LoM3	22 LoM3	25^* Shao	27^* Shao	29^* Shao	32 LoM1	33 EHM2		
10	17 HaMe4	21 LoM3	24 LoM3	27 LoM3	27-29 LoM3	28-31 LoM3	32-33 LoM3	36 LoM1	38 EHM2	
11	18 HaMe4						≤ 35 LoM3	36-37 LoM3	40 LoM1	42 EHM2

Table IVb. Known Ramsey numbers $R(K_{2,n}, K_{2,m})$, for $6 \leq n \leq 11$, $2 \leq m \leq 11$.
(unpublished results are marked with a *)

(a) The next few easily computed values of $R(K_{1,n}, K_{2,2})$, extending data in the first row of Table IVa, are 13, 14, 21 and 22 for n equal to 9, 10, 16 and 17, respectively. See function $f(n)$ in 3.3.2.c of the next subsection below.

(b) Formula for $R(K_{1,n}, K_{k_1,k_2,\ldots,k_t,m})$ for m large enough, in particular for $t=1, k_1=2$ with $n \le 5, m \ge 3$ and $n=6, m \ge 11$, for example $R(K_{1,5}, K_{2,7}) = 15$ [Stev].

(c) The values and bounds for higher cases of $R(K_{2,2}, K_{2,n})$ are 20, 22, 22/23, 22/24, 25, 26, 27/28, 28/29, 30 and 32 for $12 \le n \le 21$, respectively. More exact values can be found for prime powers $\lceil \sqrt{n}\, \rceil$ and $\lceil \sqrt{n}\, \rceil + 1$ [HaMe4].

(d) The known values of $R(K_{2,2}, K_{3,n})$ are 15, 16, 17, 20 and 22 for $6 \le n \le 10$ [Lortz], and $R(K_{2,2}, K_{3,11}) = 24$ [Shao]. See Tables IVa and IVb for the smaller cases, and [HaMe4] for upper bounds and values for some prime powers $\lceil \sqrt{n}\, \rceil$.

(e) $R(K_{2,n}, K_{2,n})$ is equal to 46, 50, 54, 57 and 62 for $12 \le n \le 16$, respectively.
The first open diagonal case is $65 \le R(K_{2,17}, K_{2,17}) \le 66$ [EHM2].
The status of all higher cases for $n < 30$ is listed in [LoM1].

(f) $R(K_{1,4}, K_{4,4}) = R(K_{1,5}, K_{4,4}) = 13$ [ShaXPB]
$R(K_{1,4}, K_{1,2,3}) = R(K_{1,4}, K_{2,2,2}) = 11$ [GuSL]
$R(K_{1,7}, K_{2,3}) = 13$ [Par4, Par6]
$R(K_{1,15}, K_{2,2}) = 20$ [La2]
$R(K_{2,2}, K_{4,4}) = 14$ [HaMe4]
$R(K_{2,2}, K_{4,5}) = 15$ [Shao]
$R(K_{2,2}, K_{4,6}) = 16$ [Shao]
$R(K_{2,2}, K_{5,5}) = R(K_{2,3}, K_{3,5}) = 17$ [Shao]
$R(K_{3,5}, K_{3,5}) \le 38$ [LoM2]
$R(K_{4,4}, K_{4,4}) \le 62$ [LoM2]

3.3.2. General results

(a) $R(K_{1,n}, K_{1,m}) = n + m - \varepsilon$, where $\varepsilon = 1$ if both n and m are even and $\varepsilon = 0$ otherwise [Har1]. It is also a special case of multicolor numbers for stars obtained in [BuRo1].

(b) $R(K_{1,3}, K_{m,n}) = m + n + 2$ for $m, n \ge 1$ [HaMe3].

(c) $R(K_{1,n}, K_{2,2}) = f(n) \le n + \sqrt{n} + 1$, with $f(q^2) = q^2 + q + 1$ and $f(q^2+1) = q^2 + q + 2$ for every q which is a prime power [Par3]. Furthermore, $f(n) \ge n + \sqrt{n} - 6n^{11/40}$ [BEFRS4]. For more bounds and values of $f(n)$ see [Par5, Chen, ChenJ, MoCa].

(d) $R(K_{1,n+1}, K_{2,2}) \le R(K_{1,n}, K_{2,2}) + 2$ [Chen].

(e) $R(K_{2,\lambda+1}, K_{1,v-k+1})$ is either $v+1$ or $v+2$ if there exists a (v, k, λ)-difference set. This and other related results are presented in [Par4, Par5]. See also [GoCM, GuLi].

(f) Formulas and bounds on $R(K_{2,2}, K_{2,n})$, and bounds on $R(K_{2,2}, K_{m,n})$. In particular, $R(K_{2,2}, K_{2,k}) = n + k\sqrt{n} + c$, for $k = 2, 3, 4$ and some prime powers $\lceil \sqrt{n}\, \rceil$ and $\lceil \sqrt{n}\, \rceil + 1$, for some $-1 \le c \le 3$ [HaMe4].

(g) $R(K_{2,n}, K_{2,n}) \leq 4n - 2$ for all $n \geq 2$, and the equality holds iff there exists a strongly regular $(4n-3, 2n-2, n-2, n-1)$-graph [EHM2].

(h) Conjecture that $4n - 3 \leq R(K_{2,n}, K_{2,n}) \leq 4n - 2$ for all $n \geq 2$. Many special cases are solved and several others are discussed in [LoM1].

(i) $R(K_{2,n-1}, K_{2,n}) \leq 4n - 4$ for all $n \geq 3$, with the equality if there exists a symmetric Hadamard matrix of order $4n - 4$. There are only 4 cases in which the equality does not hold for $3 \leq n \leq 58$, namely 30, 40, 44 and 48 [LoM1].

(j) $R(K_{2,n-s}, K_{2,n}) \leq 4n - 2s - 3$ for $s \geq 2$ and $n \geq s + 2$, with the equality in many cases involving Hadamard matrices or strongly regular graphs. Asymptotics of $R(K_{2,n}, K_{2,m})$ for $m \gg n$ [LoM3].

(k) Some algebraic lower and upper bounds on $R(K_{s,n}, K_{t,m})$ for various combinations of n, m and $1 \leq t, s \leq 3$ [BaiLi, BaLX]. A general lower bound $R(K_{m,n}) \geq 2^m (n - n^{0.525})$ for large n [Dong].

(l) Upper bounds for $R(K_{2,2}, K_{m,n})$ for $m, n \geq 2$, with several cases identified for which the equality holds. Special focus on the cases for $m = 2$ [HaMe4].

(m) Bounds for the numbers of the form $R(K_{k,n}, K_{k,m})$, specially for fixed k and close to the diagonal cases. Asymptotics of $R(K_{3,n}, K_{3,m})$ for $m \gg n$ [LoM2].

(n) $R(nK_{1,3}, mK_{1,3}) = 4n + m - 1$ for $n \geq m \geq 1$, $n \geq 2$ [BES].

(o) Asymptotics for $K_{2,m}$ versus K_n [CLRZ]. Upper bound asymptotics for $K_{k,m}$ versus K_n [LiZa1] and for some bipartite graphs K_n [JiSa].

(p) Special two-color cases apply in the study of asymptotics for multicolor Ramsey numbers for complete bipartite graphs [ChGra1].

4. Two Colors: Numbers Involving Cycles

4.1. Cycles, cycles versus paths and stars

The paper *Ramsey Numbers Involving Cycles* [Ra4] is based on the revision #12 of this survey. It collects and comments on the results involving cycles versus any graphs, in two or more colors. It contains some more details than this survey, but only until 2009.

Cycles

$R(C_3, C_3) = 6$ [GG, Bush]
$R(C_4, C_4) = 6$ [CH1]

$R(C_3, C_n) = 2n-1$ for $n \geq 4$, $R(C_4, C_n) = n+1$ for $n \geq 6$,
$R(C_5, C_n) = 2n-1$ for $n \geq 5$, and $R(C_6, C_6) = 8$ [ChaS]

Result obtained independently in [Ros1] and [FS1], a new simpler proof in [KáRos]:

$$R(C_m, C_n) = \begin{cases} 2n-1 & \text{for } 3 \leq m \leq n,\ m \text{ odd, } (m,n) \neq (3,3), \\ n-1+m/2 & \text{for } 4 \leq m \leq n,\ m \text{ and } n \text{ even, } (m,n) \neq (4,4), \\ \max\{n-1+m/2, 2m-1\} & \text{for } 4 \leq m < n,\ m \text{ even and } n \text{ odd.} \end{cases}$$

$R(mC_3, nC_3) = 3n+2m$ for $n \geq m \geq 1$, $n \geq 2$ [BES]

$R(mC_4, nC_4) = 2n+4m-1$ for $m \geq n \geq 1$, $(n,m) \neq (1,1)$ [LiWa1]

Formulas for $R(mC_4, nC_5)$ [LiWa2]

Formulas and bounds for $R(nC_m, nC_m)$ [Den, Biel1]

Unions of cycles, formulas and bounds for various cases including diagonal, different lengths, different multiplicities [MiSa, Den], and their relation to 2-local Ramsey numbers [Biel1].

Cycles versus paths

Result obtained by Faudree, Lawrence, Parsons and Schelp in 1974 [FLPS]:

$$R(C_m, P_n) = \begin{cases} 2n-1 & \text{for } 3 \leq m \leq n,\ m \text{ odd,} \\ n-1+m/2 & \text{for } 4 \leq m \leq n,\ m \text{ even,} \\ \max\{m-1+\lfloor n/2 \rfloor, 2n-1\} & \text{for } 2 \leq n \leq m,\ m \text{ odd,} \\ m-1+\lfloor n/2 \rfloor & \text{for } 2 \leq n \leq m,\ m \text{ even.} \end{cases}$$

For all n and m it holds that $R(P_m, P_n) \leq R(C_m, P_n) \leq R(C_m, C_n)$. Each of the two inequalities can become an equality, and, as derived in [FLPS], all four possible combinations of $<$ and $=$ hold for an infinite number of pairs (m,n). For example, if both m and n are even, and at least one of them is greater than 4, then $R(P_m, P_n) = R(C_m, P_n) = R(C_m, C_n)$. For related generalizations see [BEFRS2].

Cycles versus stars

Only partial results for C_m versus stars are known. Lawrence [La1] settled the cases for odd m and for long cycles (see also [Clark, Par6]). The case for short even cycles is open, it is related in particular to bipartite graphs. Partial results for $C_4 = K_{2,2}$ are pointed to in subsections 3.3.1 and 3.3.2.

$$R(C_m, K_{1,n}) = \begin{cases} 2n+1 & \text{for odd } m \le 2n+1, \\ m & \text{for } m \ge 2n. \end{cases}$$

4.2. Cycles versus complete graphs

Since 1976, it was conjectured that $R(C_n, K_m) = (n-1)(m-1) + 1$ for all $n \ge m \ge 3$, except $n = m = 3$ [FS4, EFRS2]. The parts of this conjecture were proved as follows: for $n \ge m^2 - 2$ [BoEr], for $n > 3 = m$ [ChaS], for $n \ge 4 = m$ [YHZ1], for $n \ge 5 = m$ [BJYHRZ], for $n \ge 6 = m$ [Schi1], for $n \ge m \ge 7$ with $n \ge m(m-2)$ [Schi1], for $n \ge 7 = m$ [ChenCZ1], and for $n \ge 4m+2$, $m \ge 3$ [Nik]. Open conjectured cases are marked in Table V by "conj."

	C_3	C_4	C_5	C_6	C_7	C_8	C_9	...	C_n for $n \ge m$
K_3	6 GG-Bush	7 ChaS	9 ...	11	13	15	17	... ...	$2n-1$ ChaS
K_4	9 GG	10 CH2	13 He2/JR4	16 JR2	19 YHZ1	22 ...	25	... ...	$3n-2$ YHZ1
K_5	14 GG	14 Clan	17 He2/JR4	21 JR2	25 YHZ2	29 BJYHRZ	33 ...	... ...	$4n-3$ BJYHRZ
K_6	18 Kéry	18 Ex2-RoJa1	21 JR5	26 Schi1	31 ...	36	41	... ...	$5n-4$ Schi1
K_7	23 Ka2-GrY	22 RT-JR1	25 Schi2	31 CheCZN	37 CheCZN	43 JaBa/Ch+	49 Ch+	... ...	$6n-5$ Ch+
K_8	28 GR-MZ	26 RT	29-33 JaAl2	36 ChenCX	43 ChenCZ1	50 JaAl1/ZZ3	57 BatJA	... ...	$7n-6$ conj.
K_9	36 Ka2-GR	30-32 RT-XSR1					65 conj.	... ...	$8n-7$ conj.
K_{10}	40-43 Ex5-RK2	34-39 RT-XSR1						...	$9n-8$ conj.

Table V. Known Ramsey numbers $R(C_n, K_m)$.
(Ch+ abbreviates ChenCZ1, for comments on joint credits see 4.2.b)

(a) The first column in Table V gives data from the first row in Table I.

(b) Joint credit [He2/JR4] in Table V refers to two cases in which Hendry [He2] announced the values without presenting the proofs, which later were given in [JR4]. The special cases of $R(C_6, K_5) = 21$ [JR2] and $R(C_7, K_5) = 25$ were solved independently in [YHZ2] and [BJYHRZ]. The double pointer [JaBa/ChenCZ1] refers to two independent papers, similarly as [JaAl1/ZZ3], except that in the latter case [ZZ3] refers to an unpublished manuscript. For joint credits marked in Table V with "-", the first reference is for the lower bound and the second for the upper bound.

(c) Erdős et al. [EFRS2] asked what is the minimum value of $R(C_n, K_m)$ for fixed m, and they suggested that it might be possible that $R(C_n, K_m)$ first decreases monotonically, then attains a unique minimum, then increases monotonically with n.

(d) There exist constants $c_1, c_2 > 0$ such that $c_1(m/\log m)^{3/2} \le R(C_4, K_m) \le c_2(m/\log m)^2$. The lower bound was obtained by Spencer [Spe2] using the probabilistic method. The upper bound is in a paper by Caro, Li, Rousseau and Zhang [CRLZ], who in turn give the credit to an unpublished work by Szemerédi from 1980.

(e) Erdős, in 1981, in the Ramsey problems section of the paper [Erd2] formulated a challenge by asking for a proof of $R(C_4, K_m) < m^{2-\varepsilon}$, for some $\varepsilon > 0$. No such proof is known to date.

(f) Lower bound asymptotics [Spe2, FS4, AlRö].

(g) Upper bound asymptotics [BoEr, FS4, EFRS2, CLRZ, Sud1, LiZa2, AlRö, DoLL2].

4.3. Cycles versus wheels

Note: In this survey the wheel graph $W_n = K_1 + C_{n-1}$ has n vertices, while some authors use the definition $W_n = K_1 + C_n$ with $n+1$ vertices. For the cases involving $W_3 = C_3$ versus C_m see sections 3.2 and 4.2.

	C_3	C_4	C_5	C_6	C_7	C_8	C_m	for
W_4	9 GG	10 CH2	13 He2	16 JR2	19 YHZ1	22 ...	$3m-2$ ...	$m \geq 4$ YHZ1
W_5	11 Clan	9 Clan	9 He4	11 JR2	13 SuBB2	15 ...	$2m-1$ ...	$m \geq 5$ SuBB2
W_6	11 BE3	10 JR3	13 ChvS	16 SuBB2	19 ...	22 ...	$3m-2$ ...	$m \geq 4$ SuBB2
W_7	13 BE3	9 Tse1					$2m-1$ ...	$m \geq 10$ ChenCMN
W_8	15 BE3	11 Tse1			19* ChenCN	22* ...	$3m-2$* ...	$m \geq 7$ ChenCN
W_9	17 BE3	12 Tse1					$2m-1$	$m \geq 13$ ChenCMN
W_n for	... $2n-1$ $n \geq 6$ BE3		$2n-1$ $n \geq 19$ Zhou2		$2n-1$ $n \geq 29$ Zhou2		large wheels	cycles

Table VI. Ramsey numbers $R(W_n, C_m)$, for $n \leq 9$, $m \leq 8$.
(results from unpublished manuscript are marked with a *)

(a) $R(C_3, W_n) = 2n-1$ for $n \geq 6$ [BE3]. All critical graphs have been enumerated. The critical graphs are unique for $n = 3, 5,$ and for no other n [RaJi].

(b) $R(C_4, W_n) = 13, 14, 16, 17$ for $n = 10, 11, 12, 13$, respectively [Tse1]. $R(C_4, W_n) \leq n + \lceil (n-1)/3 \rceil$ for $n \geq 7$ [SuBUB].

(c) $R(W_n, C_m) = 2n-1$ for odd m with $n \geq 5m-6$ [Zhou2].

(d) $R(W_n, C_m) = 3m-2$ for even $n \geq 4$ with $m \geq n-1$, $m \neq 3$, was conjectured by Surahmat et al. [SuBT1, SuBT2, Sur]. Parts of this conjecture were proved in [SuBT1, ZhaCC1, Shi5], and the proof was completed in [ChenCN].

(e) Conjecture that $R(W_n, C_m) = 2m-1$ for odd $n \geq 3$ and all $m \geq 5$ with $m > n$ [Sur]. It was proved for $2m \geq 5n-7$ [SuBT1], and further for $2m \geq 3n-1$ [ChenCMN]. See also [Shi5].

(f) Observe apparently four distinct situations with respect to parity of m and n.

(g) Cycles are Ramsey unsaturated for some wheels [AliSur], see also comments on [BaLS] in subsection 5.16.

(h) Study of cycles versus generalized wheels $W_{k,n}$ [Sur, SuBTB, Shi5].

4.4. Cycles versus books

	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_m	for
B_2	7 RS1	7 F*al*6	9 C*al*	11 F*al*8	13 ...	15	17	19	21	$2m-1$ ...	$m \geq 4$ F*al*8
B_3	9 RS1	9 F*al*6	10 F*al*8	11 JR2	13 Shi5	15 F*al*8	17 ...	19	21	$2m-1$ ...	$m \geq 6$ F*al*8
B_4	11 RS1	11 F*al*6	11 F*al*8	12 S*al*1	13 S*al*1	15 Shi5	17 Shi5	19 F*al*8	21 ...	$2m-1$ ...	$m \geq 7$ F*al*8
B_5	13 RS1	12 F*al*6	13 F*al*8	14 S*al*1	15 S*al*1	15 S*al*2	17 S*al*2	19 Shi5	21 Shi5	$2m-1$ ...	$m \geq 8$ F*al*8
B_6	15 RS1	13 F*al*6	15 F*al*8	16 S*al*2	17 S*al*2	18 S*al*2	18 S*al*2		21 Shi5	$2m-1$ ...	$m \geq 11$ Shi5
B_7	17 RS1	16 F*al*6	17 F*al*8	16 S*al*2	19 S*al*2	20 S*al*2	21 S*al*2			$2m-1$	$m \geq 13$ Shi5
B_8	19 RS1	17 Tse1	19 F*al*8	17 S*al*2	19 S*al*2	22 S*al*2	≥ 23 S*al*2			$2m-1$	$m \geq 14$ Shi5
B_9	21 RS1	18 Tse1	21 F*al*8	18 S*al*2			≥ 25 S*al*2	≥ 26 S*al*2		$2m-1$	$m \geq 16$ Shi5
B_{10}	23 RS1	19 Tse1	23 F*al*8	19 S*al*2				≥ 28 S*al*2		$2m-1$	$m \geq 17$ Shi5
B_{11}	25 RS1	20 Tse1	25 F*al*8							$2m-1$	$m \geq 19$ Shi5
B_n for	... $2n+3$ $n \geq 2$ RS1	$\approx n$ some (c)	... $2n+3$ $n \geq 4$ F*al*8		$2n+3$ $n \geq 15$ F*al*8		$2n+3$ $n \geq 23$ F*al*8		$2n+3$ $n \geq 31$ F*al*8	large books	cycles

Table VII. Ramsey numbers $R(B_n, C_m)$ for $n, m \leq 11$.
(*et al.* abbreviations: F*al*/FRS, C*al*/CRSPS, S*al*1/ShaXBP, S*al*2/ShaXB)

(a) For the cases of $B_1 = K_3$ versus C_m see section 4.2.
The exact values for the cases (3,7), (4,8), (4,9), (5,10), (5,11) were obtained independently in [S*al*1, S*al*2]/[ShaXBP, ShaXB] using computer algorithms.

(b) $R(C_4, B_{12}) = 21$ [Tse1], $R(C_4, B_{13}) = 22$, $R(C_4, B_{14}) = 24$ [Tse2].
$R(C_4, B_8) = 17$ [Tse2] (it was reported incorrectly in [FRS6] to be 16).

(c) $q^2 + q + 2 \leq R(C_4, B_{q^2-q+1}) \leq q^2 + q + 4$ for prime power q [FRS6]. B_n is a subgraph of B_{n+1}, hence likely $R(C_4, B_n) = n + O(\sqrt{n})$ (compare to $R(C_4, K_{2,n})$ in section 3.3).

(d) $R(B_n, C_m) = 2n + 3$ for odd $m \geq 5$ with $n \geq 4m - 13$ [FRS8].

(e) $R(B_n, C_m) = 2m - 1$ for $n \geq 1$, $m \geq 2n + 2$ [FRS8]. The range of m was extended to $m \geq 2n - 1 \geq 7$ in [ShaXB], and to $m > (6n + 7)/4$ in [Shi5].

(f) $R(B_n, C_n) \geq 3n - 2$ and $R(B_{n-1}, C_n) \geq 3n - 4$ for $n \geq 3$ [ShaXB].

(g) More theorems on $R(B_n, C_m)$ in [FRS6, FRS8, NiRo4, Zhou1]

(h) Cycles versus some generalized books [Shi5].

4.5. Cycles versus other graphs

(a) C_4 versus stars [Par3, Par4, Par5, BEFRS4, Chen, ChenJ, GoMC, MoCa]. For several exact results see $K_{2,2}$ in Tables IVa and IVb, and for general results see items 3.3.1.a, 3.3.2.c and 3.3.2.d.

(b) C_4 versus unions of stars [HaABS, Has]

(c) C_4 versus trees [EFRS4, Bu7, BEFRS4, Chen]

(d) C_4 versus all graphs on six vertices [JR3]

(e) C_4 versus various types of complete bipartite graphs, see section 3.3

(f) $R(C_4, G) \leq 2q + 1$ for any isolate-free graph G with q edges [RoJa2]

(g) $R(C_4, G) \leq p + q - 1$ for any connected graph G on p vertices and q edges [RoJa2]

(h) $R(C_5, K_6 - e) = 17$ [JR4]

(i) $R(C_5, K_4 - e) = 9$ [CRSPS]

(j) C_5 versus all graphs on six vertices [JR4]

(k) $R(C_6, K_5 - e) = 17$ [JR2]

(l) C_6 versus all graphs on five vertices [JR2]

(m) $R(C_{2m+1}, G) = 2n - 1$ for sufficiently large sparse graphs G on n vertices, in particular $R(C_{2m+1}, T_n) = 2n - 1$ for all $n > 1512m + 756$, for n-vertex trees T_n [BEFRS2].

(n) $R(C_n, G) \leq 2q + \lfloor n/2 \rfloor - 1$, for $3 \leq n \leq 5$, for any isolate-free graph G with $q > 3$ edges. It is conjectured that it also holds for other n [RoJa2].

(o) Cycles versus trees [BEFRS2, FSS1]

(p) Monotone paths and cycles [Lef]

(q) Cycles versus $K_{n,m}$ and multipartite complete graphs [BoEr]

(r) Cycles versus generalized books and wheels [Shi5, Sur, SuBTB], and versus other special graphs of the form $K_n + G$ with small $n \leq 3$ and sparse G [Shi5].

5. General Graph Numbers in Two Colors

This section includes data with respect to general graph results. We tried to include all nontrivial values and identities regarding exact results (or references to them), but only those out of general bounds and other results which, in our opinion, may have a direct connection to the evaluation of specific numbers. If some small value cannot be found below, it may be covered by the cumulative data gathered in section 8, or be a special case of a general result listed in this section. Note that $P_2 = K_2$, $B_1 = F_1 = C_3 = W_3 = K_3$, $B_2 = K_4 - e$, $P_3 = K_3 - e$, $W_4 = K_4$ and $C_4 = K_{2,2}$ imply other identities not mentioned explicitly.

5.1. Paths

$R(P_m, P_n) = n + \lfloor m/2 \rfloor - 1$ for all $n \geq m \geq 2$ [GeGy]

Stripes mP_2 [CocL1, CocL2, Lor]

Disjoint unions of paths (also called linear forests) [BuRo2, FS2]

5.2. Wheels

Note: In this survey the wheel graph $W_n = K_1 + C_{n-1}$ has n vertices, while some authors use the definition $W_n = K_1 + C_n$ with $n+1$ vertices.

m \ n	3	4	5	6	7
3	6	9 GG	11 Clan	11 BE3	13 BE3
4		18 GG	17 He3	19 FM	
5			15 He2	17 FM	
6				17 FM	

Table VIII. Ramsey numbers $R(W_m, W_n)$, for $m \leq n \leq 7$.

(a) $R(W_3, W_n) = 2n - 1$ for all $n \geq 6$ [BE3]
All critical colorings for $R(W_3, W_n)$ for all $n \geq 3$ [RaJi]

(b) The value $R(W_5, W_5) = 15$ was given in the Hendry's table [He2] without a proof. Later the proof was published in [HaMe2].

(c) All critical colorings (2, 1 and 2) for $R(W_n, W_6)$ for $n = 4, 5, 6$ [FM]

(d) $R(W_6, W_6) = 17$, $R(4,4) = 18$ and $\chi(W_6) = 4$ give a counterexample $G = W_6$ to the Erdős conjecture (see [GRS]) that $R(G, G) \geq R(K_{\chi(G)}, K_{\chi(G)})$.

5.3. Books

m \ n	1	2	3	4	5	6	7
1	6	7 CH2	9 Clan	11 RS1	13 RS1	15 RS1	17 RS1
2		10 CH1	11 Clan	13 Rou	16 RS1	17 Rou/BLR	18 BLR
3			14 RS1	15 Sh+	17 RS1		
4				18 RS1	≤20 RS1	22 RS1	
5					21 RS1		
6						26 RS1	

Table IX. Ramsey numbers $R(B_m, B_n)$, for $m, n \le 7$.
(Sh+ abbreviates ShaXBP)

(a) $254 \le R(B_{37}, B_{88}) \le 255$ [Par6]

(b) Unpublished result $R(B_2, B_3) = 17$ [Rou] was later confirmed in [BLR].

(c) There are 4 Ramsey-critical graphs for $R(B_2, B_3)$, unique graph for $R(B_3, B_4)$ [ShaXBP], 3 for $R(B_2, B_6)$ and 65 for $R(B_2, B_7)$ [BLR].

(d) $R(B_1, B_n) = 2n+3$ for all $n > 1$ [RS1]

(e) $R(B_n, B_m) = 2n+3$ for all $n \ge cm$ for some $c < 10^6$ [NiRo2, NiRo3]

(f) $R(B_n, B_n) = (4+o(1))n$ [RS1, NiRS]

(g) In general, $R(B_n, B_n) = 4n+2$ for $4n+1$ a prime power. Several other specific values (like $R(B_{62}, B_{65}) = 256$) and general equalities and bounds for $R(B_n, B_m)$ can be found in [RS1, FRS7, Par6, NiRS, LiRZ2].

5.4. Trees and forests

In this subsection T_n and F_n denote n-vertex tree and forest, respectively.

(a) $R(T_n, T_n) \le 4n+1$ [EG]

(b) $R(T_n, T_n) \ge \lfloor (4n-1)/3 \rfloor$ [BE2], see also section 5.15

(c) Conjecture that $R(T_n, T_n) \le 2n-2$, note that this is almost the same as asking if $R(T_n, T_n) \le R(K_{1,n-1}, K_{1,n-1})$ [BE2], see also [Bu7, FSS1, ChGra2]. Discussion of the conjecture that $R(T_m, T_n) \le n+m-2$ holds for all trees [FSS1].

(d) If $\Delta(T_m) = m-2$ and $\Delta(T_n) = n-2$ then the exact values of $R(T_m, T_n)$ are known, and they are between $n+m-5$ and $n+m-3$ depending on n and m. In particular, for $n = 2k+1$ we have $R(T_{2k+1}, T_{2k+1}) = 2n-5$ [GuoV].

(e) Examples of families T_m and T_n (including P_n) for which $R(T_m, T_n) = n+m-c$, $c = 3, 4, 5$ [SunZ], extending the results in [GuoV].

(f) View tree T as a bipartite graph with parts t_1 and t_2, $t_2 \geq t_1$. Define $b(T) = \max\{2t_1+t_2-1, 2t_2-1\}$. Then the bound $R(T,T) \geq b(T)$ holds always, $R(T,T) = b(T)$ holds for many classes of trees [EFRS3, GeGy], and asymptotically [HaŁT], but cases for nonequality have been found [GHK].

(g) Comments in [BaLS] about some conjectures on Ramsey saturation of non-star trees, which would imply that $R(T_n, T_n) \leq 2n-2$ holds for sufficiently large n.

(h) $R(T_m, K_{1,n}) \leq m+n-1$, with equality for $(m-1) \mid (n-1)$ [Bu1].

(i) $R(T_m, K_{1,n}) = m+n-1$ for sufficiently large n for almost all trees T_m [Bu1]. Many cases were identified for which $R(T_m, K_{1,n}) = m+n-2$ [Coc, ZZ1], see also [Bu1].

(j) $R(T_m, K_{1,n}) \leq m+n$ if T_n is not a star and $(m-1) \nmid (n-1)$, some classes of trees and stars for which the equality holds [GuoV].

(k) $R(F_n, F_n) > n + \log_2 n - O(\log\log n)$ [BE2], forests are tight for this bound [CsKo].

(l) Forests, linear forests (unions of paths) [BuRo2, FS3, CsKo].

(m) Paths versus trees [FSS1], see also other parts of this survey involving special graphs, in particular sections 5.5, 5.6, 5.10, 5.12 and 5.15.

5.5. Stars, stars versus other graphs

$R(K_{1,n}, K_{1,m}) = n+m-\varepsilon$, where $\varepsilon = 1$ for even n and m, and $\varepsilon = 0$ otherwise [Har1]. This is also a special case of multicolor numbers for stars 6.6.e obtained in [BuRo1].

$R(K_{1,n}, K_m) = n(m-1)+1$ by Chvátal's theorem [Chv].

Stars versus C_4 [Par3, Par4, Par5, BEFRS4, Chen, ChenJ, GoMC, MoCa]
Stars versus $K_{2,n}$ [Par4, GoMC]
Stars versus $K_{n,m}$ [Stev, Par3]
Stars versus complete bipartite graphs [Par4, Stev]
See also section 3.3

$R(K_{1,4}, B_4) = 11$ [RS2]
$R(K_{1,4}, K_{1,2,3}) = R(K_{1,4}, K_{2,2,2}) = 11$ [GuSL]
$nK_{1,m}$ versus W_5 [BaHA]
Stars versus W_5 and W_6 [SuBa1]
Stars versus W_9 [Zhang2, ZhaCZ1]
Stars versus wheels [HaBA1, ChenZZ2, Kor]
Stars versus paths [Par2, BEFRS2]
Stars versus cycles [La1, Clark], see also [Par6] and section 4.1
Stars versus books [CRSPS, RS2]

Stars versus trees [Bu1, Cheng, Coc, GuoV, SunZ, ZZ1]
Stars versus stripes mP_2 [CocL1, CocL2, Lor]
Stars versus $K_n - tK_2$ [Hua1, Hua2]
Stars versus $2K_2$ [MeO]
Union of two stars [Gros2]
Unions of stars versus C_4 and W_5 [HaABS, Has]
Unions of stars versus wheels [BaHA, HaBA2, SuBAU1]

5.6. Paths versus other graphs

Note: for cycles versus P_n see section 4.1.

P_3 versus all isolate-free graphs [CH2]
Paths versus stars [Par2, BEFRS2]
Paths versus trees [FS4, FSS1, SunZ]
Paths versus books [RS2]
Paths versus K_n [Par1]
Paths versus $2K_n$ [SuAM]
Paths versus $K_{n,m}$ [Häg]
Paths versus W_5 and W_6 [SuBa1]
Paths versus W_7 and W_8 [Bas]
Paths versus wheels [BaSu, ChenZZ1, SaBr3, Zhang1]
Paths versus beaded wheels [AliBT2]
Paths versus fans [SaBr2]
Paths versus $K_1 + P_m$ [SaBr1, SaBr4]
Paths and cycles versus trees [FSS1]

Unions of paths [BuRo2]
Paths and unions of paths versus Jahangir graphs [AliBas, AliBT1, AliSur]
Paths and unions of paths versus $K_{2m} - mK_2$ [AliBB]

Sparse graphs versus paths and cycles [BEFRS2]
Graphs with long tails [Bu2, BG]
Monotone paths and cycles [Lef]

5.7. Fans, fans versus other graphs

$R(F_1, F_n) = R(K_3, F_n) = 4n + 1$ for $n \geq 2$, and bounds for $R(F_m, F_n)$ [LR2, GGS]

$R(F_2, F_n) = 4n + 1$ for $n \geq 2$ and $R(F_m, F_n) \leq 4n + 2m$ for $n \geq m \geq 2$ [LinLi]

$R(K_4, F_n) = 6n + 1$ for $n \geq 3$ [SuBB3]

Fans versus paths, formulas for a number of cases including $R(P_6, F_n)$ [SaBr2].
Missing case $R(P_6, F_4) = 12$ solved in [Shao].

Fans versus cycles [Shi5]
Fans versus K_n [LR2]
Lower bounds on $R(F_2, K_n)$ from cyclic graphs for $n \leq 9$ [Shao]

5.8. Wheels versus other graphs

Notes: In this survey the wheel graph $W_n = K_1 + C_{n-1}$ has n vertices, while some authors use the definition $W_n = K_1 + C_n$ with $n+1$ vertices. For cycles versus W_n see section 4.3.

$R(W_5, K_5 - e) = 17$ [He2][YH]
$R(W_5, K_5) = 27$ [He2][RST]
$R(W_5, K_6) \geq 33$, $R(W_5, K_7) \geq 43$ [Shao]
W_5 and W_6 versus stars and paths [SuBa1]
W_5 versus $nK_{1,m}$ [BaHA]
W_5 versus unions of stars [Has]
W_5 and W_6 versus trees [BSNM]
W_7 and W_8 versus paths [Bas]

W_7 versus trees T_n with $\Delta(T_n) \geq n-3$, other special trees T, and for $n \leq 8$ [ChenZZ3, ChenZZ5, ChenZZ6]

W_7 and W_8 versus trees [ChenZZ4, ChenZZ5]
W_9 versus stars [Zhang2, ZhaCZ1, ZhaCC2]
W_9 versus trees of high degree [ZhaCZ2]

Wheels versus stars [HaBA1, ChenZZ2, Kor]
Wheels W_n, for even n, versus star-like trees [SuBB1]
Wheels versus paths [BaSu, ChenZZ1, SaBr3, Zhang1]
Wheels versus books [Zhou3]
Wheels versus unions of stars [BaHA, HaBA2, SuBAU1]
Wheels versus linear forests (disjoint unions of paths) [SuBa2]
Generalized wheels versus cycles [Shi5]
Upper bound asymptotics for $R(W_n, K_m)$ [Song5, SonBL]

5.9. Books versus other graphs

Note: for cycles versus B_n see section 4.4.

$R(B_3, K_4) = 14$ [He3]
$R(B_3, K_5) = 20$ [He2][BaRT]
$R(B_4, K_{1,4}) = 11$ [RS2]

Cyclic lower bounds for $R(B_m, K_n)$ for $m \leq 7$, $n \leq 9$ and for $R(B_3, K_n - e)$ for $n \leq 7$ [Shao]

Books versus paths [RS2]
Books versus stars [CRSPS, RS2]
Books versus trees [EFRS7]
Books versus K_n [LR1, Sud2]
Books versus wheels [Zhou3]
Books versus $K_2 + C_n$ [Zhou3]
Books and $(K_1 + \mathit{tree})$ versus K_n [LR1]

Generalized books $K_3 + qK_1$ versus cycles [Shi5]
Generalized books $K_r + qK_1$ versus K_n [NiRo1, NiRo4]

5.10. Trees and forests versus other graphs

In this subsection T_n and F_n denote n-vertex tree and forest, respectively.

$R(T_n, K_m) = (n-1)(m-1)+1$ [Chv]

$R(T_n, C_{2m+1}) = 2n-1$ for all $n > 1512m + 756$ [BEFRS2]

$R(T_n, B_m) = 2n-1$ for all $n \geq 3m-3$ [EFRS7]

$R(F_{nk}, K_m) = (n-1)(m-2)+nk$ for all forests F_{nk} consisting of k trees with n vertices each, also exact formula for all other cases of forests versus K_m [Stahl]

Exact results for almost all small ($n(G) \leq 5$) connected graphs G versus all trees [FRS4]

Trees versus C_4 [EFRS4, Bu7, BEFRSS5, Chen]
Trees versus paths [FS4, FSS1]
Trees versus cycles [FSS1, EFRS6]
Trees versus stars [Bu1, Cheng, Coc, GuoV, ZZ1]
Trees versus books [EFRS7]
Trees versus W_5 and W_6 [BSNM]
Trees versus W_7 and W_8 [ChenZZ4, ChenZZ5]

Trees T_n with $\Delta(T_n) \geq n-3$, other special trees T,
and for $n \leq 8$ versus W_7 [ChenZZ3, ChenZZ5, ChenZZ6]
Trees T_n with $\Delta(T_n) \geq n-4$ versus W_9 [ZhaCZ2]

Star-like trees versus odd wheels [SuBB1, ChenZZ3]
Trees versus $K_n + \bar{K}_m$ [RS2, FSR]
Trees versus bipartite graphs [BEFRS4, EFRS6]
Trees versus almost complete graphs [GoJa2]
Trees versus multipartite complete graphs [EFRS8, BEFRSGJ]

Linear forests versus $3K_3$ and $2K_4$ [SuBAU2]
Linear forests versus wheels [SuBa2]
Forests versus almost complete graphs [CGP]
Forests versus complete graphs [BE1, Stahl, BaHA]

Study of graphs G for which all or almost all trees are G-good [BF, BEFRSGJ], see also section 5.15 and 5.16, item [Bu2], for the definition and more pointers.

See also various parts of this survey for special trees and section 5.4.

5.11. Cases for $n(G)$, $n(H) \leq 5$

Clancy [Clan], in 1977, presented a table of $R(G, H)$ for all isolate-free graphs G with $n(G) = 4$ and H with $n(H) = 4$, except 5 entries. All five of the open entries have been solved as follows:

$R(B_3, K_4) = 14$	[He3]
$R(K_4 - e, K_5) = 16$	[BH]
$R(W_5, K_4) = 17$	[He2]
$R(K_5 - e, K_4) = 19$	[EHM1]
$R(K_5, K_4) = R(4,5) = 25$	[MR4]

An interesting case in [Clan] is

$$R(K_4, K_5 - P_3) = R(K_4, K_4 + e) = R(4, 4) = 18.$$

Hendry [He2], in 1989, presented a table of $R(G, H)$ for all graphs G and H on 5 vertices without isolates, except 7 entries. Five of the open entries have been solved:

$R(K_5, K_4 + e) = R(4, 5) = 25$	[Ka1][MR4]
$R(K_5, K_5 - P_3) = 25$	[Ka1][Boza2, CalSR]
$R(K_5, B_3) = 20$	[He2][BaRT]
$R(K_5, W_5) = 27$	[He2][RST]
$R(W_5, K_5 - e) = 17$	[He2][YH]

The still open cases for K_5 versus $K_5 - e$ and K_5 are:

$30 \leq R(K_5, K_5 - e) \leq 34$	[Ex6][Ex8]
$43 \leq R(K_5, K_5) \leq 49$	[Ex4][MR5]

All critical colorings for the case $R(C_5 + e, K_5) = 17$ were found in [He5].

5.12. Mixed cases

$26 \leq R(K_{2,2,2}, K_{2,2,2})$, $K_{2,2,2}$ is an octahedron [Ex8]
Unicyclic graphs [Gros1, Köh, KrRod]
$K_{2,m}$ and C_{2m} versus K_n [CLRZ]
$K_{2,n}$ versus any graph [RoJa2]
Union of two stars [Gros2]
Double stars* [GHK, BahS]
Graphs with bridge versus K_n [Li1]
Multipartite complete graphs [BFRS, FRS3, Stev]
Multipartite complete graphs versus sparse graphs [EFRS4]
Multipartite complete graphs versus trees [EFRS8, BEFRSGJ]
Graphs with long tails [Bu2, BG]
Brooms+ [EFRS3]

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* double star is a union of two stars with their centers joined by an edge

+ broom is a star with a path attached to its center

5.13. Multiple copies of graphs, disconnected graphs

(a) $2K_2$ versus all isolate-free graphs [CH2]

(b) nK_2 versus mK_2, in particular $R(nK_2, nK_2) = 3n - 1$ for $n \geq 1$ [CocL1, CocL2, Lor]

(c) nK_3 versus mK_3, in particular $R(nK_3, nK_3) = 5n$ for $n \geq 2$ [BES], see also section 4.1

(d) nK_3 versus mK_4 [LorMu]

(e) $nK_{1,m}$ versus W_5 [BaHA]

(f) $R(nK_4, nK_4) = 7n + 4$ for large n [Bu8]

(g) Stripes mP_2 [CocL1, CocL2, Lor]

(h) $R(G, H)$ for all disconnected isolate-free graphs H on at most 6 vertices versus all G on at most 5 vertices, except 3 cases [LoM5]. Missing cases were completed in [KroMe].

(i) $R(F, G \cup H) \leq \max\{R(F, G) + n(H), R(F, G)\}$ [Par6]

(j) $R(mG, nH) \leq (m-1)n(G) + (n-1)n(H) + R(G, H)$ [BES]
Formulas for $R(nK_3, mG)$ for all isolate-free graphs G on 4 vertices [Zeng]
Variety of results for numbers $R(nG, mH)$ [Bu1, BES, HaBA2, SuBAU1]

(k) Disjoint unions of paths (linear forests) [BuRo2, FS2]
Linear forests versus $3K_3 \cup 2K_4$ [SuBAU2]

(l) Forests versus K_n [Stahl, BaHA] and W_n [BaHA]. Generalizations to forests versus other graphs G in terms of $\chi(G)$ and the chromatic surplus of G [Biel4], and for linear forests versus $2K_n$ [SuAM].

(m) Disconnected graphs versus other graphs [BE1, GoJa1]

(n) See section 4.1 for cases involving unions of cycles

(o) See also [Bu9, BE1, LorMu, MiSa, Den, Biel1, Biel2]

5.14. General results for special graphs

[BEFS] $R(K_m^p, K_n^q) = R(K_m, K_n)$ for $m, n \geq 3$, $m + n \geq 8$, $p \leq m/(n-1)$ and $q \leq n/(m-1)$, where K_s^t is a K_s with additional vertex connected to it by t edges. Some applications can be found in [BLR].

[RoJa2] $R(K_{2,k}, G) \leq kq + 1$, for $k \geq 2$, for isolate-free graphs G with $q \geq 2$ edges.

[FM] $R(W_6, W_6) = 17$ and $\chi(W_6) = 4$. This gives a counterexample $G = W_6$ to the Erdős conjecture (see [GRS]) $R(G, G) \geq R(K_{\chi(G)}, K_{\chi(G)})$, since $R(4,4) = 18$.

[BE1] $R(G + K_1, H) \leq R(K_{1, R(G,H)}, H)$.

[LiShen] $R(\overline{K}_2 + G, \overline{K}_2 + G) \leq 4R(G, \overline{K}_2 + G) - 2$.

[LinLD] Study of $R(G + K_1, nH + K_1)$.

[NiRo1] $R(K_{p+1}, B_q^r) = p(q + r - 1) + 1$ for generalized books $B_q^r = K_r + qK_1$, for all sufficiently large q.

[LR1] Study of $R(T+K_1, K_n)$ for trees T. Asymptotic upper bounds for $R(T+K_2, K_n)$ [Song7], see also [SonGQ].

[LR3] Bounds on $R(H+\bar{K}_n, K_n)$ for general H. Also, for fixed k and m, as $n \to \infty$, $R(K_k+\bar{K}_m, K_n) \le (m+o(1))\, n^k/(\log n)^{k-1}$ [LiRZ1].

[LiTZ] Asymptotics of $R(H+\bar{K}_n, K_n)$. In particular, the order of magnitude of $R(K_{m,n}, K_n)$ is $n^{m+1}/(\log n)^m$.

[HoIs] Study of the largest k such that if the star $K_{1,k}$ is removed from K_r, $r = R(G,H)$, any edge 2-coloring of the remaining part still contains monochromatic G or H, as for K_r, for various special G and H [HoIs].

[LiRZ2] Let G'' be a graph obtained from G by deleting two vertices. Then $R(G,H) \le A+B+2+2\sqrt{(A^2+AB+B^2)/3}$, where $A = R(G'',H)$ and $B = R(G,H'')$.

5.15. General results for sparse graphs

[Chv] $R(K_n, T_m) = (n-1)(m-1)+1$ for any tree T_m on m vertices.

[BE3] Graphs yielding $R(K_n, G) = (n-1)(n(G)-1)+1$, called Ramsey n-good, and related results (see also [EFRS5]). An extensive survey and further study of n-goodness appeared in [NiRo4].

[BEFRS2] $R(C_{2m+1}, G) = 2n-1$ for sufficiently large sparse graphs G on n vertices, little more complicated formulas for P_{2m+1} instead of C_{2m+1}.

[CRST] $R(G,G) \le c_d\, n(G)$ for all G, where constant c_d depends only on the maximum degree d in G. The constant was improved in [GRR1, FoxSu1]. Tight lower and upper bounds for bipartite G [GRR2, Con2]. Further improvements of the constant c_d in general were obtained in [ConFS5], and for graphs with bounded bandwidth in [AllBS].

[BE1] Study of L-sets, which are sets of pairs of graphs whose Ramsey numbers are linear in the number of vertices. Conjecture that Ramsey numbers grow linearly for d-degenerate graphs (graph is d-degenerate if all its subgraphs have minimum degree at most d). Progress towards this conjecture was obtained by several authors, including [KoRö1, KoRö2, KoSu, FoxSu1, FoxSu2].

[ChenS] $R(G,G) \le c_d n$ for all d-arrangeable graphs G on n vertices, in particular with the same constant for all planar graphs. The constant c_d was improved in [Eaton]. An extension to graphs not containing a subdivision of K_d [RöTh].

[AllBS] Conjecture that $R(G,G) \le 12n(G)$ for all planar G, for large n.

[Shi3] Ramsey numbers grow linearly for degenerate graphs versus some sparser graphs, arrangeable graphs, crowns, graphs with bounded maximum degree, planar graphs, and graphs without any topological minor of a fixed clique.

[NeOs] Discussion of various old and new classes of Ramsey linear graphs.

[EFRS9] Study of graphs G, called *Ramsey size linear*, for which there exists a constant c_G such that for all H with no isolates $R(G,H) \le c_G e(H)$. An overview and further results were given in [BaSS].

[LRS] $R(G,G) < 6n$ for all n-vertex graphs G, in which no two vertices of degree at least 3 are adjacent. This improves the result $R(G,G) \le 12n$ in [Alon1]. In an early paper [BE1] it was proved that if any two points of degree at least 3 are at distance at least 3 then $R(G,G) \le 18n$.

[Shi1] $R(Q_n, Q_n) \le 2^{(3+\sqrt{5})n/2+o(n)}$, for the n-dimensional cube Q_n with 2^n vertices. This bound can also be derived from a theorem in [KoRö1]. An improvement was obtained in [Shi4], and a further one to $R(Q_n, Q_n) \le 2^{2n+5}n$ in [FoxSu1].

[Gros1] Conjecture that $R(G,G) = 2n(G)-1$ if G is unicyclic of odd girth. Further support for the conjecture was given in [Köh, KrRod].

[-] See also earlier subsections 5.* for various specific sparse graphs.

5.16. General results

[CH2] $R(G,H) \ge (\chi(G)-1)(c(H)-1)+1$, where $\chi(G)$ is the chromatic number of G, and $c(H)$ is the size of the largest connected component of H.

[CH3] $R(G,G) > (s\, 2^{e(G)-1})^{1/n(G)}$, where s is the number of automorphisms of G. Hence $R(K_{n,n}, K_{n,n}) > 2^n$, see also item 6.7.i.

[BE2] $R(G,G) \ge \lfloor (4n(G)-1)/3 \rfloor$ for any connected G, and $R(G,G) \ge 2n-1$ for any connected nonbipartite G. These bounds can be achieved for all $n \ge 4$.

[Bu2] Graphs H yielding $R(G,H) = (\chi(G)-1)(n(H)-1)+s(G)$, where $s(G)$ is a chromatic surplus of G, defined as the minimum number of vertices in some color class under all vertex colorings in $\chi(G)$ colors (such H's are called G-good). This idea, initiated in [Bu2], is a basis of a number of exact results for $R(G,H)$ for large and sparse graphs H [BG, BEFRS2, BEFRS3, Bu5, FS, EFRS4, FRS3, BEFSRGJ, BF, LR4, Biel2, SuBAU3, Song6, AllBS]. Surveys of this area appeared in [FRS5, NiRo4].

[BaLS] Graph G is Ramsey saturated if $R(G+e, G+e) > R(G,G)$ for every edge e in $\overline{G}$. This paper contains several theorems involving cycles, cycles with chords and trees on Ramsey saturated and unsaturated graphs, and also seven conjectures including one stating that almost all graphs are Ramsey unsaturated. Some classes of graphs were proved to be Ramsey unsaturated [Ho]. Special cases involving cycles and Jahangir graphs were studied in [AliSur].

[Für] Relations between $R(3,k)$ and graphs with large $\chi(G)$. Further detailed study of the relation between $R(3,k)$ and the chromatic gap [GySeT].

[Bra3] $R(G,H) > h(G,d)\, n(H)$ for all nonbipartite G and almost every d-regular H, for some h unbounded in d.

[DoLL1] Lower asymptotics of $R(G, H)$ depending on the average degree of G and the size of H. This continues the study initiated in [EFRS5], later much enhanced for both lower and upper bounds in [Sud3].

[LiZa1] Lower bound asymptotics of $R(G, H)$ for large dense H.

[AlKS] Discussion of a conjecture by Erdős that there exists a constant c such that $R(G, G) \leq 2^{c\sqrt{e(G)}}$ for all isolate-free graphs G. Proof for bipartite graphs and progress in other cases. In 2011, Sudakov [Sud4] completed the proof of this conjecture.

[Kriv] Lower bound on $R(G, K_n)$ depending on the density of subgraphs of G. This construction for $G = K_m$ produces a bound similar to the best known probabilistic lower bound by Spencer [Spe2]. Further lower and upper bounds on $R(G, K_n)$ in terms of n and $e(G)$ can be found in [Sud3].

[Con3] Upper bounds on $R(G, K_n)$ for dense graphs G.

[BE1] Relations between the cases of G or $G + K_1$ versus H or $H + K_1$.

[HaKr] Study of cyclic graphs yielding lower bounds for Ramsey numbers. Exact formulas for paths and cycles, and values for small complete graphs and for graphs with up to five vertices.

[Par3] Relations between some Ramsey graphs and block designs. See also [Par4].

[Li2] Relations between the Shannon capacity of noisy communication channels and graph Ramsey numbers. See also section 6 in [Ros2].

[Bu6] Given integer m and graphs G and H, determining whether $R(G, H) \leq m$ holds is NP−hard. Further complexity results related to Ramsey theory were presented in [Bu10].

[Scha] Ramsey arrowing is Π_2^p−complete, a rare natural example of a problem higher than NP in the polynomial hierarchy of computational complexity theory.

[-] Special cases of multicolor results listed in section 6.

[-] See also surveys listed in section 8.

6. Multicolor Ramsey Numbers

The only known value of a multicolor classical Ramsey number:

$R_3(3) = R(3,3,3) = R(3,3,3\,;2) = 17$	[GG]
2 critical colorings (on 16 vertices)	[KaSt, LayMa]
2 colorings on 15 vertices	[Hein]
115 colorings on 14 vertices	[PR1]

6.1. Bounds for classical numbers

General upper bound, implicit in [GG]:

$$R(k_1, \ldots, k_r) \le 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r) \tag{a}$$

Inequality in (a) is strict if the right hand side is even, and at least one of the terms in the summation is even. It is suspected that this upper bound is never tight for $r \ge 3$ and $k_i \ge 3$, except for $r = k_1 = k_2 = k_3 = 3$. However, only two cases are known to improve over (a), namely $R_4(3) \le 62$ [FKR] and $R(3,3,4) \le 31$ [PR1, PR2], for which (a) produces the bounds of 66 and 34, respectively.

Diagonal Cases

r \ m	3	4	5	6	7	8	9
3	17 GG	128 HiIr	417 Ex17	1070 Mat	3214 XuR1	6079 XSR2	13761 XXER
4	51 Chu1	634 XXER	3049 Xu	15202 XXER	62017 XXER		
5	162 Ex10	3416 XXER	26912 Xu				
6	538 FreSw						
7	1682 FreSw						

Table X. Known nontrivial lower bounds for diagonal multicolor Ramsey numbers $R_r(m)$, with references.

The best published bounds corresponding to the entries in Table X marked as personal communications [Ex17] and [Xu] are $415 \le R_3(5)$, $2721 \le R_4(5)$ and $26082 \le R_5(5)$ [XXER].

The most studied and intriguing open case is

$$\text{[Chu1]} \quad 51 \le R_4(3) = R(3,3,3,3) \le 62 \quad \text{[FKR]}$$

The construction for $51 \le R_4(3)$ as described in [Chu1] is correct, but be warned of a typo found by Christopher Frederick in 2003 (there is a triangle (31,7,28) in color 1 in the displayed matrix). The inequality 6.1.a implies $R_4(3) \le 66$, Folkman [Fol] in 1974 improved this bound to 65, and Sánchez-Flores [San] in 1995 proved $R_4(3) \le 64$.

The upper bounds in $162 \le R_5(3) \le 307$, $538 \le R_6(3) \le 1838$, $1682 \le R_7(3) \le 12861$, $128 \le R_3(4) \le 236$ and $634 \le R_4(4) \le 6474$ are implied by 6.1.a (we repeat lower bounds from Table X just to see easily the ranges). All the latter and other upper bounds obtainable from known smaller bounds and 6.1.a can be computed with the help of a LISP program written by Kerber and Rowat [KerRo].

Off-Diagonal Cases

Three colors:

m / k	4	5	6	7	8	9	10	11	12	13	14
3	30 Ka2	45 Ex2	60 Rob3	81 Ex16	101 Ex17	118 Gerb	142 Gerb	158 Gerb	182 LSS2	212 LSS2	233 6.2.f
4	55 KLR	89 Ex18	117 Ex18	145 Ex18	193 6.2.f						
5	89 Ex18	139 Ex18	181 Ex18								

Table XI. Known nontrivial lower bounds for 3-color Ramsey numbers of the form $R(3,k,m)$, with references.

In addition, the bounds $303 \le R(3,6,6)$, $609 \le R(3,7,7)$ and $1689 \le R(3,9,9)$ were derived in [XXER] (used there for building other lower bounds for some diagonal cases).

The other most studied, and perhaps the only open case of a classical multicolor Ramsey number, for which we can anticipate exact evaluation in the not-too-distance future is

$$\text{[Ka2]} \quad 30 \le R(3,3,4) \le 31 \quad \text{[PR1, PR2]}$$

In [PR1] it is conjectured that $R(3,3,4) = 30$, and the results in [PR2] eliminate some cases which could give $R(3,3,4) = 31$. The upper bounds in $45 \le R(3,3,5) \le 57$, $55 \le R(3,4,4) \le 79$, and $89 \le R(3,4,5) \le 160$ are implied by 6.1.a (we repeat lower bounds from the Table XI to show explicitly the current ranges).

Four colors:

$97 \le R(3,3,3,4) \le 153$	[Ex18], 6.1.a
$171 \le R(3,3,4,4) \le 462$	[Ex16, XXER], 6.1.a
$381 \le R(3,4,4,4) \le 1619$	6.2.j, 6.1.a
$162 \le R(3,3,3,5)$	[XXER]
$565 \le R(3,3,3,11)$	6.2.f
$681 \le R(3,4,5,5)$	[XXER]

Lower bounds for higher numbers can be obtained by using general constructive results from section 6.2 below. For example, the bounds $261 \le R(3,3,15)$ and $247 \le R(3,3,3,7)$ were not published explicitly but are implied by 6.2.f and 6.2.g, respectively.

6.2. General results for complete graphs

(a) $R(k_1, \ldots, k_r) \le 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$ [GG]

(b) $R_r(3) \ge 3R_{r-1}(3) + R_{r-3}(3) - 3$ [Chu1]

(c) $R_r(m) \ge c_m (2m-3)^r$, and some slight improvements of this bound for small values of m were described in [AbbH, Gi1, Gi2, Song2]. For $m = 3$, the best known lower bound is $R_r(3) \ge (3.199...)^r$ [XXER].

(d) $R_r(3) \le r!(e - e^{-1} + 3)/2 \approx 2.67 r!$ [Wan], which improves the classical $3r!$ [GRS].

(e) The limit $L = \lim_{r \to \infty} R_r(3)^{1/r}$ exists, though it can be infinite [ChGri].
It is known that $3.199 < L$, as implied by (c) above. For more related results, mostly on the asymptotics of $R_r(3)$, see [AbbH, Fre, Chu2, GRS, GrRö].

(f) $R(3, k, l) \ge 4R(k, l-1) - 3$, and in general for $r \ge 2$ and $k_i \ge 2$,
$R(3, k_1, \ldots, k_r) \ge 4R(k_1 - 1, k_2, \ldots, k_r) - 3$ for $k_1 \ge 5$, and
$R(k_1, 2k_2 - 1, k_3, \ldots, k_r) \ge 4R(k_1 - 1, k_2, \ldots, k_r) - 3$ for $k_1 \ge 5$ [XX2, XXER].

(g) $R(3, 3, 3, k_1, \ldots, k_r) \ge 3R(3, 3, k_1, \ldots, k_r) + R(k_1, \ldots, k_r) - 3$ [Rob2]

(h) For $r+1$ colors, avoiding K_3 in the first r colors and avoiding K_m in the last color, $R(3, \ldots, 3, m) \le r!\, m^{r+1}$ [Sár].

(i) $R(k_1, \ldots, k_r) \ge S(k_1, \ldots, k_r) + 2$, where $S(k_1, \ldots, k_r)$ is the generalized Schur number [AbbH, Gi1, Gi2]. In particular, the special case $k_1 = \ldots = k_r = 3$ has been widely studied [Fre, FreSw, Ex10, Rob3].

(j) $R(k_1, \ldots, k_r) \ge L(k_1, \ldots, k_r) + 1$, where $L(k_1, \ldots, k_r)$ is the maximal order of any cyclic $(k_1, \ldots, k_r)$–coloring, which can be considered a special case of Schur partitions defining (symmetric) Schur numbers. Many lower bounds for Ramsey numbers were established by cyclic colorings. The following recurrence can be used to derive lower bounds for higher parameters. For $k_i \geq 3$ [Gi2],

$$L(k_1, \dots, k_r, k_{r+1}) \geq (2k_{r+1} - 3)L(k_1, \dots, k_r) - k_{r+1} + 2.$$

(k) $R_r(m) \geq p+1$ and $R_r(m+1) \geq r(p+1)+1$ if there exists a K_m–free cyclotomic r–class association scheme of order p [Mat].

(l) If the quadratic residues Paley graph Q_p of prime order $p = 4t+1$ contains no K_k, then $R(s, k+1, k+1) \geq 4ps - 6p + 3$ [XXER].

(m) $R_r(pq+1) > (R_r(p+1)-1)(R_r(q+1)-1)$ [Abb1]

(n) $R_r(pq+1) > R_r(p+1)(R_r(q+1)-1)$ for $p \geq q$ [XXER]

(o) $R(p_1q_1+1, \dots, p_rq_r+1) > (R(p_1+1, \dots, p_r+1)-1)(R(q_1+1, \dots, q_r+1)-1)$ [Song3]

(p) $R_{r+s}(m) > (R_r(m)-1)(R_s(m)-1)$ [Song2]

(q) $R(k_1, k_2, \dots, k_r) > (R(k_1, \dots, k_i) - 1)(R(k_{i+1}, \dots, k_r) - 1)$ in [Song1], see [XXER].

(r) $R(k_1, k_2, \dots, k_r) > (k_1 + 1)(R(k_2 - k_1 + 1, k_3, \dots, k_r) - 1)$ [Rob4]

(s) Further lower bound constructions, though with more complicated assumptions, were presented in [XX2, XXER].

(t) Grolmusz [Grol1] generalized the classical constructive lower bound by Frankl and Wilson [FraWi] (item 2.3.t) to more colors and to hypergraphs [Grol3] (item 7.3.i).

(u) Exact asymptotics of a very special but important case is known, namely $R(3, 3, n) = \Theta(n^3 \text{poly–log}\, n)$ [AlRö]. For general upper bounds and more asymptotics see in particular [Chu4, ChGra2, ChGri, GRS, GrRö].

All lower bounds in (b) through (t) above are constructive. (g) generalizes (b), (o) generalizes both (m) and (q), and (q) generalizes (p). (n) is stronger than (m). Finally, we note that the construction in (o) with $q_1 = \dots = q_i = 1 = p_{i+1} = \dots = p_r$ is the same as (q).

6.3. Cycles

The paper *Ramsey Numbers Involving Cycles* [Ra4] is based on the revision #12 of this survey. It collects and comments on the results involving cycles versus any graphs, in two or more colors. It contains some more details than this survey, but only until 2009.

6.3.1. Three colors

(a) One long cycle.

The first larger paper in this area by Erdős, Faudree, Rousseau and Schelp [EFRS1] appeared in 1976. It gives several formulas and bounds for $R(C_m, C_n, C_k)$ and $R(C_m, C_n, C_k, C_l)$ for large m. For three colors [EFRS1] includes:

$R(C_m, C_{2p+1}, C_{2q+1}) = 4m - 3$ for $p \geq 2$, $q \geq 1$,
$R(C_m, C_{2p}, C_{2q+1}) = 2(m+p) - 3$ and
$R(C_m, C_{2p}, C_{2q}) = m + p + q - 2$ for p, $q \geq 1$ and large m.

$m\ n\ k$	$R(C_m, C_n, C_k)$	references	general results
3 3 3	**17**	GG	page 33
3 3 4	17	ExRe	
3 3 5	21	Sun1+/Tse3	$5k-4$ for $k \geq 5$, $m=n=3$ [Sun1+]
3 3 6	26	Sun1+	
3 3 7	31	Sun1+	
3 4 4	12	Schu	
3 4 5	13	Sun1+/Rao/Tse3	
3 4 6	13	Sun1+/Tse3	
3 4 7	15	Sun1+/Tse3	
3 5 5	≥ 17	Tse3	
3 5 6	21	Sun1+	
3 5 7	25	Sun1+	
3 6 6			
3 6 7	21	Sun1+	
3 7 7			
4 4 4	**11**	BS	
4 4 5	12	Sun2+/Tse3	
4 4 6	12	Sun2+/Tse3	$k+2$ for $k \geq 11$, $m=n=4$ [Sun2+]
4 4 7	12	Sun2+/Tse3	values for $k=8, 9, 10$ are 12, 13, 13 [Sun2+]
4 5 5	13	Tse3	
4 5 6	13	Sun1+	
4 5 7	15	Sun1+	
4 6 6	11	Tse3	
4 6 7	13	Sun1+/Tse3	
4 7 7			
5 5 5	**17**	YR1	
5 5 6	21	Sun1+	
5 5 7	25	Sun1+	
5 6 6			
5 6 7	21	Sun1+	
5 7 7			
6 6 6	**12**	YR2	$R_3(C_{2q}) \geq 4q$ for $q \geq 2$ [DzNS]
6 6 7	15	Sun1+	see 6.3.1.a for larger parameters
6 7 7			see 6.3.1.a for larger parameters
7 7 7	**25**	FSS2	$R_3(C_{2q+1}) = 8q+1$ for large q [KoSS]
8 8 8	**16**	Sun/SunY	$R_3(C_{2q}) = 4q$ for large q [BenSk]

Table XII. Ramsey numbers $R(C_m, C_n, C_k)$ for $m, n, k \leq 7$ and $m=n=k=8$.
(Sun1+ abbreviates SunYWLX, Sun2+ abbreviates SunYLZ2,
the work in [SunYWLX] and [SunYLZ2] is independent from [Tse3])

(b) Triple even cycles.

$R_3(C_{2m}) \geq 4m$ for all $m \geq 2$ [DzNS], see also 6.3.2.c/d/e.

In 2005, Dzido [Dzi1] conjectured that $R_3(C_{2m}) = 4m$ for all $m \geq 3$. It is known that $R(C_n, C_n, C_n) = (2+o(1))n$ for even n [FiŁu1, GyRSS]. Next, the diagonal case was improved to exactly $2n$ for large n [BenSk]. The first open case is for $R_3(C_{10})$, known to be at least 20. A more general result holds for slightly off-diagonal cases [FiŁu1]:

$$R(C_{2\lfloor \alpha_1 n \rfloor}, C_{2\lfloor \alpha_2 n \rfloor}, C_{2\lfloor \alpha_3 n \rfloor}) = (\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))\, n, \quad \textit{for all} \quad \alpha_1, \alpha_2, \alpha_3 > 0.$$

(c) Triple odd cycles.

$R_3(C_{2m+1}) = 8m+1$ for all sufficiently large m, or equivalently
$R(C_n, C_n, C_n) = 4n-3$ for all sufficiently large odd n [KoSS].

$R(C_n, C_n, C_n) \le (4+o(1))\,n$, with equality for odd n [Łuc]. In 1981, it was conjectured by Bondy and Erdős, see [Erd2], that $R(C_n, C_n, C_n) \le 4n-3$ for $n \ge 4$. If true, then for all odd $n \ge 5$ we have $R(C_n, C_n, C_n) = 4n-3$. The first open case is for $R_3(C_9)$, known to be at least 33.

(d) $R(C_3, C_3, C_k) = 5k-4$ for $k \ge 5$ [SunYWLX], and $R(C_4, C_4, C_k) = k+2$ for $k \ge 11$ [SunYLZ2]. All exceptions to these formulas for small k are listed in Table XII.

(e) Asymptotics for triples of cycles of mixed parity similar in form to (b) [FiŁu2].

(f) Almost all of the off-diagonal cases in Table XII required the use of computers.

6.3.2. More colors

For results on $R_k(C_3) = R_k(K_3)$ see sections 6.1, 6.2.

$R_4(C_4) = 18$	[Ex2] [SunYLZ1]
$18 \le R_4(C_6)$	[SunYJLS]
$27 \le R_5(C_4) \le 29$	[LaWo1]
$R_5(C_6) = 26$	[SunYJLS] [SunYW]
$24 \le R(C_3, C_4, C_4, C_4) \le 27$	[DyDz] [XuR2]
$30 \le R(C_3, C_3, C_4, C_4) \le 36$	[DyDz] [XuR2]
$49 \le R(C_3, C_3, C_3, C_4)$	6.7.e

(a) Formulas for $R(C_m, C_n, C_k, C_l)$ for large m [EFRS1].

(b) $R_k(C_4) \le k^2 + k + 1$ for all $k \ge 1$, $R_k(C_4) \ge k^2 - k + 2$ for all $k-1$ which is a prime power [Ir, Chu2, ChGra1], and $R_k(C_4) \ge k^2 + 2$ for odd prime power k [LaWo1]. The latter was extended to any prime power k in [Ling, LaMu].

Bounds in (c) through (g) below cover different situations and each is best in some respect.

(c) $R_k(C_{2m}) \ge (k+1)m$ for odd k and $m \ge 2$, and
$R_k(C_{2m}) \ge (k+1)m - 1$ for even k and $m \ge 2$ [DzNS].

(d) $R_k(C_{2m}) \ge 2(k-1)(m-1) + 2$ [SunYXL].

(e) $R_k(C_{2m}) \ge k^2 + 2m - k$ for $2m \ge k+1$ and prime power k [SunYJLS].

(f) $R_k(C_{2m}) = \Theta(k^{m/(m-1)})$ for fixed $m = 2, 3$ and 5 [LiLih].

(g) $R_k(C_{2m}) \le 201km$ for $k \le 10^m/201m$ [EG].

(h) $R_k(C_{2m}) \le 2km + o(m)$ for all fixed $k \ge 2$ [ŁucSS].

(i) $R_k(C_5) < \sqrt{18^k k!}/10$ [Li4].

(j) $2^k m < R_k(C_{2m+1}) \le (k+2)!(2m+1)$ [BoEr].
Better upper bound $R_k(C_{2m+1}) < 2(k+2)!m$ was obtained in [EG].
Much better upper bound $R_k(C_{2m+1}) \le (c^k k!)^{1/m}$, for some positive constant c, if all Ramsey-critical graphs for C_{2m+1} are not far from regular, was obtained in [Li4].

(k) Conjecture that $R_k(C_{2m+1}) = 2^k m + 1$ for all $m \ge 2$, was credited by several authors to Bondy and Erdős [BoEr], though only lower bound not the conjecture is in this paper.

(l) $R(C_n, C_{l_1}, \ldots, C_{l_k}) = 2^k(n-1) + 1$ for all l_i's odd with $l_i > 2^i$, and every sufficiently large n, in particular we have $R_k(C_n) = 2^{k-1}(n-1) + 1$ for large odd n [AllBS].

(m) $R_k(C_{2m+1}) \le k2^k(2m+1) + o(m)$ for all fixed $k \ge 4$ [ŁucSS].

(n) Asymptotic bounds for $R_k(C_n)$ [Bu1, GRS, ChGra2, Li4, LiLih, ŁucSS].

(o) Survey of multicolor cycle cases [Li3].

6.3.3. Cycles versus other graphs

$20 \le R(C_4, C_4, K_4) \le 22$	[DyDz] [XSR1]
$27 \le R(C_3, C_4, K_4) \le 32$	[DyDz] [XSR1]
$52 \le R(C_4, K_4, K_4) \le 72$	[XSR1]
$34 \le R(C_4, C_4, C_4, K_4) \le 50$	[DyDz] [XSR1]
$43 \le R(C_3, C_4, C_4, K_4) \le 76$	[DyDz] [XSR1]
$87 \le R(C_4, C_4, K_4, K_4) \le 179$	[XSR1]
$R(K_{1,3}, C_4, K_4) = 16$	[KlaM2]
$R(C_4, C_4, K_4 - e) = 16$	[DyDz]
$R(C_4, C_4, C_4, T) = 16$ for $T = P_4$ and $T = K_{1,3}$	[ExRe]

(a) Study of $R(C_n, K_{t_1}, \ldots, K_{t_k})$ and $R(C_n, K_{t_1, s_1}, \ldots, K_{t_k, s_k})$ for large n [EFRS1].

(b) $R(C_n, K_{t_1}, \ldots, K_{t_k}) = (n-1)(r-1)$ for $n \ge 4r + 2$, where $r = R(K_{t_1}, \ldots, K_{t_k})$ [OmRa2].

(c) Study of asymptotics for $R(C_m, \ldots, C_m, K_n)$, in particular for any fixed number of colors $k \ge 4$ we have $R(C_4, C_4, \ldots, C_4, K_n) = \Theta(n^2/\log^2 n)$ [AlRö].

(d) Study of asymptotics for $R(C_{2m}, C_{2m}, K_n)$ for fixed m [AlRö, ShiuLL], in particular $R(C_4, C_4, K_n) = \Theta(n^2 \text{ poly–log } n)$ [AlRö].

(e) Monotone paths and cycles [Lef].

(f) For combinations of C_3 and K_n see sections 2.2, 3.2, 4.2, 6.1 and 6.2.

6.4. Paths, paths versus other graphs

In 2007, Gyárfás, Ruszinkó, Sárközy and Szemerédi [GyRSS] established that for all sufficiently large n we have

$$R(P_n, P_n, P_n) = 2n - 2 + n \bmod 2.$$

6.4.1. Three color path and path-cycle cases

(a) $R(P_m, P_n, P_k) = m + \lfloor n/2 \rfloor + \lfloor k/2 \rfloor - 2$ for $m \geq 6(n+k)^2$ [FS2],
the equality holds asymptotically for $m \geq n \geq k$ with an extra term $o(m)$ [FiŁu1],
extensions of the range of m, n, k for which (a) holds were obtained in [Biel3].

(b) $R_3(P_3) = 5$ [Ea1], $R_3(P_4) = 6$ [Ir],
$R(P_m, P_n, P_k) = 5$ for other $m{-}n{-}k$ combinations with $3 \leq m, n, k \leq 4$ [AKM],
$R_3(P_5) = 9$ [YR1], $R_3(P_6) = 10$ [YR1], and $R_3(P_7) = 13$ [YY].

(c) $R(P_4, P_4, P_{2n}) = 2n + 2$ for $n \geq 2$,
$R(P_5, P_5, P_5) = R(P_5, P_5, P_6) = 9$,
$R(P_5, P_5, P_n) = n + 2$ for $n \geq 7$,
$R(P_5, P_6, P_n) = R(P_4, P_6, P_n) = n + 3$ for $n \geq 6$,
$R(P_6, P_6, P_{2n}) = R(P_4, P_8, P_{2n}) = 2n + 4$ for $n \geq 14$ [OmRa1].

(d) $R(P_m, P_n, C_k) = 2n + 2\lfloor m/2 \rfloor - 3$ for large n and odd $m \geq 3$ [DzFi2],
improvements on the range of m, n, k [Biel3, Fid1].

(e) $R(P_3, P_3, C_m) = 5, 6, 6$, for $m = 3, 4$ [AKM], 5,
$R(P_3, P_3, C_m) = m$ for $m \geq 6$ [Dzi2].
$R(P_3, P_4, C_m) = 7$ for $m = 3, 4$ [AKM] and 5,
$R(P_3, P_4, C_m) = m + 1$ for $m \geq 6$ [Dzi2].
$R(P_4, P_4, C_m) = 9, 7, 9$ for $m = 3, 4$ [AKM] and 5 [Dzi2],
$R(P_4, P_4, C_m) = m + 2$ for $m \geq 6$ [DzKP].

(f) $R(P_3, P_5, C_m) = 9, 7, 9, 7, 9$ for $m = 3, 4, 5, 6, 7$ [Dzi2, DzFi2],
$R(P_3, P_5, C_m) = m + 1$ for $m \geq 8$ [DzKP].
A table of $R(P_3, P_k, C_m)$ for all $3 \leq k \leq 8$ and $3 \leq m \leq 9$ [DzFi2].

(g) $R(P_4, P_5, C_m) = 11, 7, 11, 11$, and $m + 2$ for $m = 3, 4, 5, 7$ and $m \geq 23$,
$R(P_4, P_6, C_m) = 13, 8, 13, 13$, and $m + 3$ for $m = 3, 4, 5, 7$ and $m \geq 18$ [ShaXSP].

(h) $R(P_3, P_n, C_4) = n + 1$ for $n \geq 6$ [DzFi2],
$R(P_3, P_n, C_6) = n + 2$ for $n \geq 6$,
$R(P_3, P_n, C_8) = n + 3$ for $n \geq 7$ [Fid1],
$R(P_3, P_n, C_k) = 2n - 1$, and
$R(P_4, P_n, C_k) = 2n + 1$ for odd $k \geq 3$ and $n \geq k$ [DzFi2].

(i) $R(P_3, P_6, C_m) = m + 2$ for $m \geq 23$,
$R(P_6, P_6, C_m) = R(P_4, P_8, C_m) = m + 4$ for $m \geq 27$,
$R(P_6, P_7, C_m) = m + 4$ for $m \geq 57$,
$R(P_4, P_n, C_4) = R(P_5, P_n, C_4) = n + 2$ for $n \geq 5$ [OmRa1].

(j) $R(P_3, C_3, C_3) = 11$ [BE3], $R(P_3, C_4, C_4) = 8$ [AKM], $R(P_3, C_6, C_6) = 9$ [Dzi2],
$R(P_3, C_m, C_m) = R(C_m, C_m) = 2m - 1$ for odd $m \geq 5$ [DzKP] (for $m = 5, 7$ [Dzi2]),

(k) $R(P_3, C_n, C_m) = R(C_n, C_m)$ for $n \geq 7$ and odd m, $5 \leq m \leq n$, and some values and bounds on $R(P_3, C_n, C_m)$ in other cases [Fid1].

(l) $R(P_3, C_3, C_4) = 8$ [AKM], $R(P_3, C_3, C_5) = 9$, $R(P_3, C_3, C_6) = 11$,
$R(P_3, C_3, C_7) = 13$, $R(P_3, C_4, C_5) = 8$, $R(P_3, C_4, C_6) = 8$,
$R(P_3, C_4, C_7) = 8$, $R(P_3, C_5, C_6) = 11$, $R(P_3, C_5, C_7) = 13$ and
$R(P_3, C_6, C_7) = 11$ [Dzi2].

(m) Formulas for $R(pP_3, qP_3, rP_3)$ and $R(pP_4, qP_4, rP_4)$ [Scob].

(n) $R(P_3, K_4 - e, K_4 - e) = 11$ [Ex7]. All colorings (which can be any color neighborhood for the open case $R_3(K_4 - e)$, see section 6.5) were found in [Piw2].

6.4.2. More colors

(a) $R_k(P_3) = k + 1 + (k \bmod 2)$, $R_k(2P_2) = k + 3$ for all $k \geq 1$ [Ir].

(b) $R_k(P_4) = 2k + c_k$ for all k and some $0 \leq c_k \leq 2$. If k is not divisible by 3 then $c_k = 3 - k \bmod 3$ [Ir]. Wallis [Wall] showed $R_6(P_4) = 13$, which already implied $R_{3t}(P_4) = 6t + 1$, for all $t \geq 2$. Independently, the case $R_k(P_4)$ for $k \neq 3^m$ was completed by Lindström in [Lind], and later Bierbrauer proved $R_{3^m}(P_4) = 2 \cdot 3^m + 1$ for all $m > 1$. $R_3(P_4) = 6$ [Ir].

(c) Formula for $R(P_{n_1}, \ldots, P_{n_k})$ for large n_1 [FS2], and some extensions [Biel3].
Conjectures about $R(P_{n_1}, \ldots, P_{n_k})$ when all or all but one of n_i's are even [OmRa1].

(d) Formulas for $R(P_{n_1}, \ldots, P_{n_k}, C_m)$ for some cases, for large m [OmRa1].

(e) Formula for $R(n_1 P_2, \ldots, n_k P_2)$, in particular $R(nP_2, nP_2, nP_2) = 4n - 2$ [CocL1].

(f) Cockayne and Lorimer [CocL1] found the exact formula for $R(n_1 P_2, \ldots, n_k P_2)$, and later Lorimer [Lor] extended it to a more general case of $R(K_m, n_1 P_2, \ldots, n_k P_2)$. More general cases of the latter, with multiple copies of the complete graph, stars and forests, were studied in [Stahl, LorSe, LorSo, GyRSS].

(g) Multicolor cases for one large path or cycle involving small paths, cycles, complete and complete bipartite graphs [EFRS1].

(h) See section 8.2, especially [AKM], for a number of cases for triples of small graphs.

6.5. Special cases

$R_3(K_3+e) = R_3(K_3)$ [=17]	[YR3, AKM], where $K_3+e = K_4 - P_3$
$R(K_3+e, K_3+e, K_4-e) = 17$	[ShWR]
If $R_4(K_3) = 51$ then $R_4(K_3+e) = 52$, and if $R_4(K_3) > 51$ then $R_4(K_3+e) = R_4(K_3)$	[ShWR]
$28 \le R_3(K_4-e) \le 30$	[Ex7] [Piw2]
$R(P_3, K_4-e, K_4-e) = 11$	[Ex7], all colorings [Piw2]
$21 \le R(K_3, K_4-e, K_4-e) \le 27$	[ShWR]
$33 \le R(K_4, K_4-e, K_4-e) \le 60$	[ShWR]
$R(C_4, P_4, K_4-e) = 11$	[DyDz], correcting an error in [AKM]
$R(C_4, C_4, K_4-e) = 16$	[DyDz]
$19 \le R(C_4, K_4-e, K_4-e) \le 22$	[DyDz]

6.6. General results for special graphs

(a) Formulas for $R_k(G)$, where G is one of the graphs P_3, $2K_2$ and $K_{1,3}$ for all k, and for P_4 if k is not divisible by 3 [Ir]. For some details see section 6.4.2.b.

(b) $tk^2+1 \le R_k(K_{2,t+1}) \le tk^2+k+2$, where the upper bound is general, and the lower bound holds when both t and k are prime powers [ChGra1, LaMu].

(c) $(m-1)\lfloor (k+1)/2 \rfloor < R_k(T_m) \le 2km+1$ for any tree T_m with m edges [EG], see also [GRS]. The lower bound can be improved for special large k [EG, GRS]. The upper bound was improved to $R_k(T_m) < (m-1)(k+\sqrt{k(k-1)})+2$ in [GyTu].

(d) $k(\sqrt{m}-1)/2 < R_k(F_m) < 4km$ for any forest F_m with m edges [EG], see [GRS]. See also pointers in items (l) and (m) below.

(e) $R(S_1, \ldots, S_k) = n + \varepsilon$, where S_i's are arbitrary stars, $n = n(S_1) + \ldots + n(S_k) - 2k$, and we set $\varepsilon = 1$ if n is even and some $n(S_i)$ is odd, and $\varepsilon = 2$ otherwise [BuRo1]. See also [GauST, Par6].

(f) Formula for $R(S_1, \ldots, S_k, K_n)$, where S_i's are arbitrary stars [Jac]. It was generalized to a formula for $R(S_1, \ldots, S_k, K_{k_1}, \ldots, K_{k_r})$ expressed in terms of $R(k_1, \ldots, k_r)$ and star orders [BoCGR]. A much shorter proof of the latter was presented in [OmRa2].

(g) Formula for $R(S_1, \ldots, S_k, nK_2)$, where S_i's are arbitrary stars [CocL2].

(h) Formula for $R(S_1, \ldots, S_k, T)$, where S_i's are stars and T is a tree [ZZ1].

(i) Formulas for $R(S_1, \ldots, S_k)$, where each S_i's is a star or $m_i K_2$ [ZZ2, EG],
formula for the case $R(S, mK_2, nK_2)$ [GySá2].

(j) Bounds on $R_k(G)$ for unicyclic graphs G of odd girth.
Some exact values for special graphs G, for $k = 3$ and $k = 4$ [KrRod].

(k) $R_k(K_{3,3}) = (1+o(1))\,k^3$ [AlRóS].

(l) Bounds on $R_k(K_{s,t})$, in particular for $K_{2,2} = C_4$ and $K_{2,t}$ [ChGra1, AFM]. Asymptotics of $R_k(K_{s,t})$ for fixed k and s [DoLi, LiTZ]. Upper bounds on $R_k(K_{s,t})$ [SunLi].

(m) Bounds on $R_k(G)$ for trees, forests, stars and cycles [Bu1].

(n) Bounds for trees $R_k(T)$ and forests $R_k(F)$ [EG, GRS, BB, GyTu, Bra1, Bra2, SwPr].

(o) Study of the case $R(K_m, n_1P_2, \ldots, n_kP_2)$ [Lor]. More general cases, with multiple copies of the complete graph, stars and forests, were investigated in [Stahl, LorSe, LorSo, GyRSS]. See also section 6.4.

(p) See section 8.2, especially [AKM], for a number of cases for other small graphs, similar to those listed in sections 6.3 and 6.4.

6.7. General results

(a) Szemerédi's Regularity Lemma [Szem] states that the vertices of every large graph can be partitioned into similar size parts so that the edges between these parts behave almost randomly. This lemma in various forms has been used extensively to prove the upper bounds, including [BenSk, GyRSS, GySS1, HaŁP1+, HaŁP2+, KoSS].

(b) $R(m_1G_1, \ldots, m_kG_k) \le R(G_1, \ldots, G_k) + \sum_{i=1}^{k} n(G_i)(m_i - 1)$, exercise 8.3.28 in [West].

(c) If G is connected and $R(K_k, G) = (k-1)(n(G)-1)+1$, in particular if G is any n-vertex tree, then $R(K_{k_1}, \ldots, K_{k_r}, G) = (R(k_1, \ldots, k_r) - 1)(n - 1) + 1$ [BE3]. A generalization for connected $G_1, \ldots, G_n$ in place of G appeared in [Jac].

(d) If F, G, H are connected graphs then $R(F,G,H) \ge (R(F,G)-1)(\chi(H)-1) + \min\{R(F,G), s(H)\}$, where $s(G)$ is the chromatic surplus of G (see item [Bu2] in section 5.16). This leads to several formulas and bounds for F and G being stars and/or trees when $H = K_n$ [ShiuLL].

(e) $R(K_{k_1}, \ldots, K_{k_r}, G_1, \ldots, G_s) \ge (R(k_1, \ldots, k_r) - 1)(R(G_1, \ldots, G_s) - 1) + 1$ for arbitrary graphs $G_1, \ldots, G_s$ [Bev]. This generalizes 6.2.o.

(f) Constructive bound $R(G_1, \ldots, G_{t^{n-1}}) \ge t^n + 1$ for decompositions of K_{t^n} [LaWo1, LaWo2].

(g) $R(G_1, \ldots, G_k) \le 32\Delta k^{\Delta} n$, where $n \ge n(G_i)$ and $\Delta \ge \Delta(G_i)$ for all $1 \le i \le k$ [FoxSu1].

(h) $R(G_1, \ldots, G_k) \le k^{2k\Delta q} n$, where $q \ge \chi(G_i)$ for all $1 \le i \le k$ [FoxSu1].

(i) $R_k(G) > (sk^{e(G)-1})^{1/n(G)}$, where s is the number of automorphisms of G [CH3]. Other general bounds for $R_k(G)$ [CH3, Par6].

(j) Study of $R(G_1, \ldots, G_k, G)$ for large sparse G [EFRS1, Bu3].

(k) Study of asymptotics for $R(C_n, \ldots, C_n, K_m)$ [AlRö]. See also sections 6.3.3.b/c.

(l) See surveys listed in section 8.

7. Hypergraph Numbers

7.1. Values and bounds for numbers

The only known value of a classical Ramsey number for hypergraphs:

$R(4,4;3) = 13$	
more than 200000 critical colorings	[MR1]

The computer evaluation of $R(4,4;3)$ in 1991 consisted of an improvement of the upper bound from 15 to 13. This result followed an extensive theoretical study of this number by several authors [Gi4, Isb1, Sid1].

(a)	$33 \le R(4,5;3)$	[Ex13]
	$38 \le R(4,6;3)$	[HuSo+]
	$65 \le R(5,5;3)$	[Ea1]
	$56 \le R(4,4,4;3)$	[Ex8]
	$34 \le R(5,5;4)$	[Ex11]
(b)	$R(K_4-t, K_4-t;3) = 7$	[Ea2]
	$R(K_4-t, K_4;3) = 8$	[Sob, Ex1, MR1]
	$14 \le R(K_4-t, K_5;3)$	[Ex1]
	$13 \le R(K_4-t, K_4-t, K_4-t;3) \le 16$	[Ex1] [Ea3]

(c) The first bound on $R(4,5;3) \ge 24$ was obtained by Isbell [Isb2]. Shastri [Shas] gave a weak bound $R(5,5;4) \ge 19$ (now 34 in [Ex11]), nevertheless his lemmas, the stepping-up lemmas by Erdős and Hajnal (see [GRS, GrRö], also 7.4.a below), and others in [Ka3, Abb2, GRS, GrRö, HuSo, SonYL] can be used to derive better lower bounds for higher numbers.

(d) Several lower bound constructions for 3-uniform hypergraphs were presented in [HuSo]. Study of lower bounds on $R(p,q;4)$ can be found in [Song3] and [SonYL, Song4] (the latter two papers are almost the same in contents). Most of the concrete lower bounds in these papers can be easily improved by using the same techniques, but starting with better constructions for small parameters as listed above.

(e) $R(p,q;4) \ge 2R(p-1,q;4) - 1$ for $p, q > 4$, and
$R(p,q;4) \ge (p-1)R(p-1,q;4) - p + 2$ for $p \ge 5$, $q \ge 7$ [SonYL].
Lower bound asymptotics for $R(p,q;4)$ [SonLi].

7.2. Cycles and paths

Definitions. A *loose* 3-uniform ($r = 3$) cycle C_n on $[n]$ is the set of triples $\{123, 345, 567, \ldots, (n-1)n1\}$. Note that n must be even. In 3-uniform *tight* cycles and *tight* paths consecutive edges share two points. A 3-uniform *Berge* cycle is formed by n distinct vertices, such that all consecutive pairs ($t = 2$) of vertices are in an edge of the cycle, and all of the cycle edges are distinct. Berge cycles are not determined uniquely. These definitions can be generalized to t-tight cycles and r-uniform hypergraphs.

(a) Tetrahedron, or four triples on the set of four points, can be seen as a tight 3-uniform cycle C_4. The corresponding Ramsey number is $R(4,4;3) = 13$ [MR1].

(b) For loose cycles, $R(C_3, C_3; 3) = 7$, $R(C_4, C_4; 3) = 9$, and in general for r-uniform case $R(C_3, C_3; r) = 3r - 2$ and $R(C_4, C_4; r) = 4r - 3$, for $r \geq 3$. Results and discussion of several related cases involving paths were presented in [GyRa].

(c) For 3-uniform Berge cycles and two colors, $R(C_n, C_n; 3) = n$ for $n \geq 5$ [GyLSS].

(d) For loose cycles, $R(C_{4k}, C_{4k}; 3) > 5k - 2$ and $R(C_{4k+2}, C_{4k+2}; 3) > 5k + 1$, and asymptotically these lower bounds are tight [HaŁP1+]. Generalizations to r-uniform hypergraphs and graphs other than cycles appeared in [GySS1].

(e) For loose cycles, $R(C_3, C_3, C_3; 3) = 8$, and in general for $k \geq 4$ colors Gyárfás and Raeisi established the bounds $k + 5 \leq R_k(C_3; 3) \leq 3k$ [GyRa].

(f) For tight cycles, $R(C_{3k}, C_{3k}; 3) \approx 4k$ and $R(C_{3k+i}, C_{3k+i}; 3) \approx 6k$ for $i = 1$ or 2, and for tight paths $R(P_k, P_k; 3) \approx 4k/3$ [HaŁP2+]. Some related results are discussed in [PoRRS].

(g) For 3-uniform Berge cycles, $R_3(C_n; 3) = (1 + o(1))5n/4$ [GySá1].

(h) Gyárfás, Sárközy and Szemerédi proved that, for sufficiently large n, every 2-coloring of the edges of the complete 4-uniform hypergraph K_n contains a monochromatic 3-tight Berge cycle C_n [GySS2]. Special multicolor cases for r-uniform hypergraphs were studied in [GyLSS].

7.3. General results for 3-uniform hypergraphs

(a) $2^{cn^2} < R(n, n; 3) < 2^{2^n}$ is credited to Erdős, Hajnal and Rado (see [ChGra2] p. 30).

(b) For some a, b the numbers $R(m, a, b; 3)$ are at least exponential in m [AbbS].

(c) Improved lower and upper asymptotics for $R(s, n; 3)$ for fixed s and large n, proof of related Erdős and Hajnal conjecture on the growth of $R(4, n; 3)$, and the lower bound $2^{n^{c \ln n}} < R(n, n, n; 3)$ [ConFS2].

(d) $R(G, G; 3) \leq c \cdot n(H)$ for some constant c depending only on the maximum degree of a 3-uniform hypergraph H [CooFKO1, NaORS]. Similar results were proved for r-uniform hypergraphs in [KüCFO, Ishi, CooFKO2, ConFS1], see also item 7.4.f.

(e) Upper bounds on $R_k(H;3)$ for complete multipartite 3-uniform hypergraphs H, a 4-color case, and some other general and special cases [ConFS1, ConFS2, ConFS3].

7.4. General results

(a) If $R(n,n;r) > m$ then $R(2n+r-4, 2n+r-4; r+1) > 2^m$, for $n > r \geq 3$ (see [GRS] p. 106). This is the so-called stepping-up lemma, usually credited to Erdős and Hajnal. An improvement of the stepping-up lemma implying better lower bounds for a few types of hypergraph Ramsey numbers were obtained by Conlon, Fox and Sudakov [ConFS4].

(b) Lower bounds on $R_k(n;r)$ are discussed in [AbbW, DLR].

(c) General lower bounds for large number of colors were given in an early paper by Hirschfeld [Hir], and some of them were later improved in [AbbL].

(d) Lower and upper asymptotics of $R(s,n;k)$ for fixed s [ConFS2].

(e) Exact results for large 2-loose cycles (generalizing 7.2.d above) and 2- and 3-color cases for all r-uniform diamond matchings [GySS1].

(f) $R(H,H;r) \leq c \cdot n(H)^{1+\varepsilon}$, for some constant $c = c(\Delta, r, \varepsilon)$ depending only on the maximum degree of H, r and $\varepsilon > 0$ [KoRö3]. The proofs of the linear bound $c \cdot n(H)$ were obtained independently in [KüCFO] and [Ishi], the latter including the multicolor case, and then without regularity lemma in [ConFS1]. More discussion of lower and upper bounds for various cases can be found in [ConFS1, ConFS2, ConFS3, CooFKO2].

(g) Let T_r be an r-uniform hypergraph with r edges containing a fixed $(r-1)$-vertex set S and the $(r+1)$-st edge intersecting all former edges in one vertex outside S. Then $R(T_r, K_t; r) = O(t^r/\log t)$ [KosMV].

(h) Let $H^r(s,t)$ be the complete r-partite r-uniform hypergraph with $r-2$ parts of size 1, one part of size s, and one part of size t (for example, for $r=2$ it is the same as $K_{s,t}$). For the multicolor numbers, Lazebnik and Mubayi [LaMu] proved that

$$tk^2 - k + 1 \leq R_k(H^r(2,t+1);r) \leq tk^2 + k + r,$$

where the lower bound holds when both t and k are prime powers. For the general case of $H^r(s,t)$, more bounds are presented in [LaMu].

(i) Grolmusz [Grol1] generalized the classical constructive lower bound by Frankl and Wilson [FraWi] (section 2.3.t) to more colors and to hypergraphs [Grol3].

(j) Lower and upper asymptotics, and other theoretical results on hypergraph numbers are gathered in [GrRö, GRS, ConFS1, ConFS2, ConFS3].

8. Cumulative Data and Surveys

8.1. Cumulative data for two colors

[CH1]	$R(G,G)$ for all graphs G without isolates on at most 4 vertices.
[CH2]	$R(G,H)$ for all graphs G and H without isolates on at most 4 vertices.
[Clan]	$R(G,H)$ for all graphs G on at most 4 vertices and H on 5 vertices, except five entries (now all solved, see section 5.11). All critical colorings for the isolate-free graphs G and H studied in [Clan] were found in [He4].
[Bu4]	$R(G,G)$ for all graphs G without isolates and with at most 6 edges.
[He1]	$R(G,G)$ for all graphs G without isolates and with at most 7 edges.
[HaMe2]	$R(G,G)$ for all graphs G on 5 vertices and with 7 or 8 edges.
[He2]	$R(G,H)$ for all graphs G and H on 5 vertices without isolates, except 7 entries (2 still open, see 5.11 and the paragraph at the end of this section).
[LoM5]	$R(G,H)$ for all disconnected isolate-free graphs H on at most 6 vertices versus all G on at most 5 vertices, except 3 cases. Missing cases were completed in [KroMe].
[HoMe]	$R(G,H)$ for $G=K_{1,3}+e$ and $G=K_4-e$ versus all connected graphs H on 6 vertices, except $R(K_4-e,K_6)$. The result $R(K_4-e,K_6)=21$ was claimed by McNamara [McN, unpublished].
[FRS4]	$R(G,T)$ for all connected graphs G with $n(G)\le 5$, and almost all trees T.
[FRS1]	$R(K_3,G)$ for all connected graphs G on 6 vertices.
[Jin]	$R(K_3,G)$ for all connected graphs G on 7 vertices. Some errors in [Jin] were found [SchSch1].
[Zeng]	Formulas for $R(nK_3,mG)$ for all G of order 4 without isolates.
[Brin]	$R(K_3,G)$ for all connected graphs G on at most 8 vertices. The numbers for K_3 versus sets of graphs with fixed number of edges, on at most 8 vertices, were presented in [KlaM1].
[BBH1]	$R(K_3,G)$ for all connected graphs G on 9 vertices. See also [BBH2].
[JR3]	$R(C_4,G)$ for all graphs G on at most 6 vertices.
[JR4]	$R(C_5,G)$ for all graphs G on at most 6 vertices.
[JR2]	$R(C_6,G)$ for all graphs G on at most 5 vertices.
[LoM3]	$R(K_{2,n},K_{2,m})$ for all $2\le n$, $m\le 10$ except 8 cases, for which lower and upper bounds are given. Further data for other complete bipartite graphs are gathered in section 3.3 and [LoMe4].
[HaKr]	All best lower bounds up to 102 from cyclic graphs. Formulas for best cyclic lower bounds for paths and cycles, and values for small complete graphs and for graphs with up to five vertices.

Chvátal and Harary [CH1, CH2] formulated several simple but very useful observations how to discover values of some numbers. All five missing entries in the tables of Clancy [Clan] have been solved (section 5.11). Out of 7 open cases in [He2] 5 have been solved, including $R(4,5) = R(G_{19}, G_{23}) = 25$ and other cases listed in section 5.11. The still open 2 cases are for K_5 versus K_5 (section 2.1) and K_5 versus $K_5 - e$ (section 3.1).

8.2. Cumulative data for three colors

[YR3]	$R_3(G)$ for all graphs G with at most 4 edges and no isolates.
[YR1]	$R_3(G)$ for all graphs G with 5 edges and no isolates, except $K_4 - e$. The case of $R_3(K_4 - e)$ remains open (see section 6.5).
[YY]	$R_3(G)$ for all graphs G with 6 edges and no isolates, except 10 cases.
[AKM]	$R(F, G, H)$ for most triples of isolate-free graphs with at most 4 vertices. Some of the missing cases completed in [KlaM2].
[DzFi2]	$R(P_3, P_k, C_m)$ for all $3 \le k \le 8$ and $3 \le m \le 9$.

8.3. Surveys

[Bu1]	A general survey of results in Ramsey graph theory by S. A. Burr (1974)
[Par6]	A general survey of results in Ramsey graph theory by T. D. Parsons (1978)
[BuRo3]	Survey of results and new problems on multiplicities and Ramsey multiplicities by S. A. Burr and V. Rosta (1980)
[Har2]	Summary of progress by Frank Harary (1981)
[ChGri]	A general survey of bounds and values by F. R. K. Chung and C. M. Grinstead (1983)
[JGT]	Special volume of the *Journal of Graph Theory* (1983)
[Rob1]	A review of Ramsey graph theory for newcomers by F. S. Roberts (1984)
[Bu7]	What can we hope to accomplish in generalized Ramsey Theory? (1987)
[GrRö]	Survey of asymptotic problems by R. L. Graham and V. Rödl (1987)
[GRS]	An excellent book by R. L. Graham, B. L. Rothschild and J. H. Spencer, second edition (1990)
[FRS5]	Survey by Faudree, Rousseau and Schelp of graph goodness results, i.e. conditions for the formula $R(G, H) = (\chi(G) - 1)(n(H) - 1) + s(G)$ (1991)
[Neš]	A chapter in *Handbook of Combinatorics* by J. Nešetřil (1996)
[Caro]	Survey of zero-sum Ramsey theory by Y. Caro (1996)

[Chu4] Among 114 open problems and conjectures of Paul Erdős, presented and commented by F. R. K. Chung, 31 are concerned directly with Ramsey numbers. 216 references are given (1997). An extended version of this work was prepared jointly with R. L. Graham [ChGra2]. (1998)

[West] An extensive chapter on Ramsey theory in a widely used student textbook and researcher's guide of graph theory (2001)

[GrNe] Ramsey Theory and Paul Erdős (2002)

[CoPC] Special issue of *Combinatorics, Probability and Computing* (2003)

[Ros2] Dynamic survey of Ramsey theory applications by V. Rosta (2004). A website maintained by W. Gasarch [Gas] gathers over 50 pointers to literature on applications of Ramsey theory in computer science. (2009)

[Soi1] History, results and people of Ramsey theory. The mathematical coloring book, mathematics of coloring and the colorful life of its creators. (2009)

[Soi2] Ramsey Theory. Yesterday, Today and Tomorrow, a special volume in the series *Progress in Mathematics*. A survey of Ramsey numbers involving cycles by the author is included in this volume [Ra4]. (2011)

The surveys by S. A. Burr [Bu1] and T. D. Parsons [Par6] contain extensive chapters on general exact results in graph Ramsey theory. F. Harary presented the state of the theory in 1981 in [Har2], where he also gathered many references including seven to other early surveys of this area. More than two decades ago, Chung and Grinstead in their survey paper [ChGri] gave less data than in this work, but included a broad discussion of different methods used in Ramsey computations in the classical case. S. A. Burr, one of the most experienced researchers in Ramsey graph theory, formulated in [Bu7] seven conjectures on Ramsey numbers for sufficiently large and sparse graphs, and reviewed the evidence for them found in the literature. Three of them have been refuted in [Bra3].

For newer extensive presentations see [GRS, GrRö, FRS5, Neš, Chu4, ChGra2], though these focus on asymptotic theory not on the numbers themselves. A very welcome addition is the 2004 compilation of applications of Ramsey theory by V. Rosta [Ros2]. This survey could not be complete without recommending special volumes of the *Journal of Graph Theory* [JGT, 1983] and *Combinatorics, Probability and Computing* [CoPC, 2003], which, besides a number of research papers, include historical notes and present to us Frank P. Ramsey (1903-1930) as a person. Finally, read a colorful book by A. Soifer [Soi1, 2009] on history and results in Ramsey theory, followed by a collection of essays and technical papers based on presentations from the 2009 Ramsey theory workshop at DIMACS [Soi2, 2011].

The historical perspective and, in particular, the timeline of progress on prior best bounds, can be obtained by checking all the previous versions of this survey since 1994 at http://www.cs.rit.edu/~spr/ElJC/eline.html.

9. Concluding Remarks

This compilation does not include information on numerous variations of Ramsey numbers, nor related topics, like size Ramsey numbers, zero-sum Ramsey numbers, irredundant Ramsey numbers, induced Ramsey numbers, local Ramsey numbers, connected Ramsey numbers, chromatic Ramsey numbers, avoiding sets of graphs in some colors, coloring graphs other than complete, or the so called Ramsey multiplicities. Interested readers can find such information in the surveys listed in section 8 here.

Ramsey@Home [RaHo] is a distributed computing project at the University of Wisconsin-Oshkosh designed to find new lower bounds for various Ramsey numbers. Join and help! Readers may be interested in knowing that the US patent 6965854 B2 issued on November 15, 2005 claims a method of using Ramsey numbers in "Methods, Systems and Computer Program Products for Screening Simulated Traffic for Randomness". Check the original document at http://www.uspto.gov/patft if you wish to find out whether your usage of Ramsey numbers is covered by this patent.

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The author apologizes for any omissions or other errors in reporting results belonging to the scope of this work. Suggestions for any kind of corrections or additions will be greatly appreciated and considered for inclusion in the next revision of this survey.