Ramsey Theory Applications

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Abstract

There are many interesting applications of Ramsey theory, these include results in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and theoretical computer science. Relations of Ramsey-type theorems to various fields in mathematics are well documented in published books and monographs. The main objective of this survey is to list applications mostly in theoretical computer science of the last two decades not contained in these.

1 Introduction

Ramsey-type theorems have roots in different branches of mathematics and the theory developed from them influenced such diverse areas as number theory, set theory, geometry, ergodic theory and theoretical computer science. Ramsey [224, 137] stated his fundamental theorem in a general setting and applied it to formal logic. The finite version says: *for all* $t, n, k \in \mathbb{N}$ *there exists* $R \in \mathbb{N}$ *so that, for* $m \geq R$*, if the* k*-tuples of a set* M *of cardinality* m *are* t*-colored, then there exists* $M' \subseteq M$ *of cardinality* n *with all the* k*-tuples of* M' *having the same color.* The infinite version is similar. Ramsey-type theorems are showing that if a large enough system is partitioned arbitrarily into finitely many subsystems, at least one subsystem has a particular property, and thus total disorder is impossible.

Earlier Schur [239] and van der Waerden [265] obtained similar results in number theory. Dilworth's classical theorem [89] for partially ordered sets is another typical example. Ramsey's theorem was rediscovered and applied to geometry by Erdős and Szekeres [95]. They also defined the Ramsey numbers and gave some upper and lower bounds for them. For the graphs $G_1, G_2, \ldots, G_t$, the *graph Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is the smallest

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integer R with the property that any complete graph of at least R vertices whose edges are partitioned into t color classes contains a monochromatic subgraph isomorphic to G_i in the i-th color for some i, $1 \leq i \leq t$. We talk about classical Ramsey numbers or simply Ramsey numbers when all G_i graphs are complete graphs, this corresponds to the original definition, later extended to any graph [66]. The probabilistic proof technique, introduced by Erdős [97] to establish a lower bound of $r(K_n, K_n)$, has been generalized and applied to combinatorics, to computer science and to various other fields. For more on Ramsey theory in general we refer to the book of Graham, Rothschild and Spencer [137], to the collection edited by Nešetřil and Rödl [202] and to the more recent survey article of Nešetřil [200]. The book of Furstenberg [124] gave ergodic theoretical and topological dynamics reformulations.

In this survey, we are concentrating on applications not contained in the above books. In the last few years, significant improvements were made in almost all areas of Ramsey theory and these improvements or previous results had been applied in many different subjects. By the early eighties, Ramsey-type theorems scattered around in different fields were put together to form Ramsey theory. Capitalizing on the maturity of the subject, theoretical computer science started to make use of it, that was perhaps initiated by the influential papers of Ajtai, Komlós, Szemerédi [3, 4, 5] and Yao [269]. Since then Ramsey theory has been applied in many different ways in theoretical computer science and these have not been documented together so far. Most of these applications are using existing theorems, but there are also papers mostly by Alon [7, 8, 9] where new Ramsey-type theorems are proved at the same time as application problems are solved.

Our original intention was to mention applications of Ramsey theory to theoretical computer science only. However, it appears to be hard to separate these from relations to mathematics. The diversity of mathematical formulations of Ramsey-type theorems provides us with possibilities for various applications in computer science. Depending on the particular problem to solve, different aspects or versions have been applied. For example to obtain lower bounds for parallel sorting, the Erdős-Rado [103] theorem was often applied. In information theory, to obtain a lower bound on the capacity of unions of channels, constructive lower bounds of Ramsey numbers were used, while density results in number theory are essential in harmonic analysis applications.

This paper is organized by grouping results according to the field of application. The second section lists new results in number theory, related to Schur's, van der Waerden's and Szemerédi's theorem. In Section 3, we describe how the new proof by Gowers of the Szemerédi theorem influenced results on basic questions in harmonic analysis, and we mention some applications to metric spaces. Section 4 surveys developments in computational geometry and in the geometry of polytopes, that are related to the Erdős-Szekeres theorem [95]. In Section 5 some examples are listed where the probabilistic method has been used, and the best constructions for the same problems are given. Information theory applications of Ramsey theory mostly involve finding maximal independent sets for various graphs, which correspond to information channels. This is the subject of Section 6. The next section shows applications of the finite Ramsey theorem to order invariant algorithms. The Erdős-Rado theorem has been applied in lower bound arguments for parallel random-access machines. Section 9 mentions lower bounds for the Boolean function computation complexity with a Ramsey theoretical lemma developed for these. Some examples of Ramsey-type theorems using automated theorem proving are listed in Section 10, where optimization techniques are often applied. Section 11 mentions approximation algorithms developed for NP-hard problems, that are related to Ramsey theory results for graphs with large independent sets. In the last section, we refer to applications of the theorems of Ramsey, van der Waerden, Hales-Jewett and Hindman in logic, complexity and games, and we mention the role of graph Ramsey theorems to obtain natural examples for classes in higher levels of the polynomial hierarchy in complexity theory.

The list of mentioned results in this survey is far from being exhaustive. Our first objective here is to show that there are many different ways Ramsey theory can be applied and related to other fields, in particular, to computer science. A regularly updated dynamic survey is perhaps a good format to make this collection as complete as possible over time.

2 Number theory

One of the earliest Ramsey-type results is Schur's theorem (1916) [239] in number theory: *if* $\mathbb{N}$ *is partitioned into a finite number of classes, at least one partition class contains a solution to the equation* $x + y = z$. There are a number of interesting results proved during the last few years concerning Schur's theorem and generalizations.

A triple x, y, z of natural numbers is called a *Schur triple* if $x \neq y$ and $x + y = z$. We denote by $S(N)$ the minimum number of monochromatic Schur triples in any 2-coloring of $[N] = \{1, 2, \ldots, N\}$. Graham, Rödl and Ruciński [136] found the lower bound $S(N) \geq (1/38)N^2 + O(N)$. They used the Ramsey multiplicity result of Goodman [129, 73], which says that in every 2-coloring of the edges of a complete graph on N vertices there are at least $N^3/24 + O(N^2)$ monochromatic triangles. Answering a question raised in [136], Robertson and Zeilberger [229], and independently Schoen [238] showed that $S(N) = (1/22)N^2 + O(N)$. Robertson and Zeilberger found a 2-coloring with $N^2/22$ monochromatic Schur triples and formulated a conjecture on the minimum number of triples through computational experiments with Maple. This was part of their project on automatic theorem proving (see also Section 10 on this subject). Schoen showed that every extremal coloring looks like the Robertson-Zeilberger construction and he used this result to find the exact number $S(N) = (1/22)N^2 - (7/22)N$. His method is general enough to be applied to arbitrary linear equations $a_1x_1 + \ldots + a_kx_k = b$.

Another way to look at Schur's theorem is in terms of sum-free sets. A set $A \subset \mathbb{N}$ is called *sum-free* if $x, y \in A$ implies $x + y \notin A$. The *Schur function* s_t is defined as the maximum $m \in \mathbb{N}$ such that $\{1, 2, \ldots, m\}$ can be partitioned into t sum-free sets. Chung and Grinstead [80, 78] showed that there is strong relation between K_3-free t-coloring and sum-free sets, namely for $t \geq l$, $r(K_3, K_3, \ldots, K_3) - 2 \geq s_t \geq c(2s_l + 1)^{t/l}$

for some constant c. Using the result $s_5 \geq 160$ of Exoo [108], this gives a lower bound on the multicolor Ramsey number $r(K_3, K_3, \ldots, K_3) \geq c(321)^{t/5}$. On the other hand, if $n \geq r(K_3, K_3, \ldots, K_3) - 1$ then any t-coloring of $[n]$ contains a monochromatic triple $x, y, x + y = z$ [137], and therefore Schur's theorem itself is a consequence of Ramsey's theorem.

The following generalizations of Schur's theorem for sum-sets follows from Rado's theorem and it was proved by Folkman and Sanders independently [137, 221, 235]. This Folkman-Rado-Sanders theorem states : *if $\mathbb{N}$ is finitely colored, there exists arbitrarily large finite set $A \subset \mathbb{N}$ such that the sum-set of A, $\mathcal{P}(A) = \{\Sigma_{a \in B}\ a : B \subseteq A, 1 \leq |B| < \infty\}$ is monochromatic.* In Schur's theorem $|A| = 2$, and Hindman's theorem [151] gives the same result when A is an infinite set. Pudlák (2003) [219] considered a complexity theory question, the communication complexity of the problem to determine if a word w is of the form $w_0 a_1 w_1 a_2 \ldots w_{k-1} a_k w_k$ for fixed letters $a_1, \ldots, a_k$. He proved that for $k = 4$ and 5, the communication complexity of the problem increases with the length of the word w, using the set-theoretical version of Hindman's theorem. Milliken [191] proved a theorem that generalizes both the infinite version of Ramsey's theorem and Hindman's theorem and using this he proved a series of results in set theory.

Sum-free sets also can be defined more generally [183], namely a set $A \subset \mathbb{N}$ is sum-free if $A \cap \mathcal{P}'(A) = \emptyset$, where $\mathcal{P}'(A) = \{\Sigma_{a \in B}\ a : B \subseteq A, 2 \leq |B| < \infty\}$. Luczak and Schoen [183] recently proved a maximal density result for sum-free sets: if $A \subset \mathbb{N}$ is sum-free, then for each n_0 there exists $n \geq n_0$ such that the density $A(n) = |A \cap \{1, 2, \ldots, n\}| \leq 403\sqrt{n \log n}$, improving earlier results of Erdős [87, 98]. They also show that this is close to the best possible. To obtain the maximal density result, they first prove that if A is a set of natural numbers with $A(n) > 402\sqrt{n \log n}$ for n large enough, then there exists d such that $\{d, 2d, 3d, \ldots\} \subseteq \mathcal{P}(A)$.

We mention here two very influential results from additive number theory using yet another sumset definition: $A + A = \{x + y : x, y \in A\}$. Balog and Szemerédi (1994) [24] proved that *"if A is a set of n integers and for some $c > 0$ there are cn numbers that have at least cn representations of the form $x, y, x + y \in A$, then there is a subset $A' \subset A$ such that $|A'| \geq c'n$ and $|A' + A'| \leq c''n$, where c', c'' are positive constants depending only on c."* This is a basic result with many possible applications. Related to this is Freiman's famous theorem (1973) [122]. A set of the form $P = P(q_1, \ldots, q_d; l_1, \ldots, l_d; a) = \{a + x_1 q_1 + \ldots + x_d q_d | 0 \leq x_i < l_i, i = 1, \ldots, d\}$ is called a *d-dimensional generalized arithmetic progression. For every α there exist constants $C = C(\alpha)$ and $d = d(\alpha)$ such that if A is any finite set of integers and $|A+A| \leq \alpha|A|$ then A is contained in a generalized arithmetic progression of dimension at most d and cardinality at most $C|A|$.* Ruzsa (1994) [234] gave an elegant, simplified, new proof for Freiman's theorem with good bounds on d and C. In most applications it is important to know the quantitative dependence of the constants $d(\alpha), C(\alpha)$ on α. A new improved bound on the Freiman-Ruzsa theorem is due to Chang [76], who shows that $d(\alpha) \leq [\alpha - 1]$ and $\log(|P|/|A|) < K\alpha^2(\log \alpha)^3$.

Another early Ramsey-type theorem in number theory was originally conjectured by Baudet and Schur independently (see [247, 248, 249]) and it became known as the van der Waerden theorem (1927) [265, 137]. It states that if the positive integers are colored with t colors, then at least one color class contains an arithmetic sequence of arbitrary length. More precisely, *for every pair of positive integers k and t, there exists a constant $M = M(k,t)$ such that if $\{1, \ldots, M\}$ is colored with t colors then some color class contains an arithmetic progression of length k.* The constants $M(k,t)$ are called *van der Waerden numbers*. No formula for $M(k,t)$ is known. The original proof of van der Waerden bounds M above by an Ackermann-type function in k. Much later, in 1988, substantially improved bound expressed as primitive recursive function was given by Shelah [243]. The real behavior of the growth of $M(k,t)$ remains unknown and the gap between Shelah's upper bound and the best known lower bound is still enormous. Van der Waerden numbers computation has been lately associated with propositional satisfiability by Dransfield et al. [90]. They show that this computational problem can be represented by propositional theories in such a way that decisions concerning their satisfiability determine the numbers. In addition to obtaining some new lower bounds for small van der Waerden numbers, Dransfield et al. used the propositional theories that arise in this research in development, testing and benchmarking of SAT solvers.

Erdős and Turán (1936) [96] conjectured that more must be true: *if a set of positive integers has positive upper density, then it contains an arithmetic sequence of arbitrary length k.* Stated differently, *for every $c > 0$ and every k positive integer there exists a positive integer $n = n(k,c)$ such that any subset S of the set $\{1, 2, \ldots, n\}$ of cardinality at least cn contains an arithmetic progression of length k.*

Roth in 1952 [231, 232] proved the conjecture for $k = 3$ using functional analysis. Bourgain (1999) [59] has sharpened Roth's theorem by giving the best bounds. He showed that a subset S of the set $\{1, 2, \ldots, n\}$ of cardinality at least $cn(\log\log n/\log n)^{1/2}$ contains an arithmetic progression of length 3. One of the most celebrated results in combinatorics is Szemerédi's theorem (1975)[256, 257], which proves the Erdős-Turán conjecture using van der Waerden's theorem and combinatorial arguments. Furstenberg (1977)[123] gave a different proof using techniques in ergodic theory. A polynomial extension of the Szemerédi theorem and of the van der Waerden theorem was established by Bergelson and Leibman, and a related quantitative result by Green [40, 138]. $r_k(n)$ denotes the largest cardinality of a subset of $[n]$ with no arithmetic sequence of length k. The lower bound of Behrend (1946)[38, 137], $ne^{-c\sqrt{\log n}} < r_3(n)$, has been applied to fast multiplication of matrices by Coppersmith and Winograd [82].

Recently, Gowers (1998, 2001) [130, 133] published a new proof of Szemerédi's theorem generalizing Roth's original method and giving new estimates for n which are substantially smaller than earlier ones. Furstenberg didn't give any bound and Szemerédi's is extremely large. Gowers main result is the following: *for every positive integer k there is a constant $c = c(k) > 0$ such that every subset of $\{1, 2, \ldots, n\}$ of size at least $n(\log\log n)^{-c}$ contains an arithmetic progression of length k. Moreover, c can be taken to be $2^{-2^{k+9}}$.* This implies an estimate for $n(k,c)$ which is doubly exponential in $1/c$ and quintuply exponential in k, that is significant improvement to the previous bounds, even for the van der Waerden's theorem. Gowers avoids using van der Waerden's theorem and Szemerédi's regularity lemma, both well known for extremely large constants and bounds. Instead he uses exponential sums like Roth, showing that Roth's proof for $k = 3$ can be generalized. In carrying out the generalization Gowers faced serious difficulties, so his strategy was to use the relevant additive number theory results of Freiman [122] and Balog-Szemerédi [24]. Ruzsa's method [234] to prove Freiman's theorem had very strong influence on this new proof, as acknowledged by Gowers.

Furstenberg and Katznelson [125] proved a multidimensional analogue to Szemerédi's theorem: *for any real number $\delta > 0$ and positive integers K, d there is a natural number $N_0 = N_0(\delta, K, d)$ such that for $N > N_0$ every subset of $[N]^d$ of size at least δN^d contains a homothetic copy of $[K]^d$.* In 1991 [126] they also proved that the Hales-Jewett theorem [146], which extends van der Waerden's result to a more combinatorial context, has a density version as conjectured by Graham. This result has only ergodic theoretic proof in general. Solymosi [252] gave a simple combinatorial proof for the generalization of Roth's theorem, showing that: *for sufficiently large N, every subset of $[N]^2$ of size at least δN^2 contains three points of the form $\{(a, b), (a + d, b), (a, b + d)\}$* . He [253] also solved a related problem of Erdős and Graham by showing that: *for sufficiently large N, every subset of $[N]^2$ of size at least δN^2 contains a square, i.e. four points with coordinates $\{(a, b), (a + d, b), (a, b + d), (a + d, b + d)\}$.* The more general theorem of Furstenberg and Katznelson [125] also implies these statements, but does not give bound on N, as it uses ergodic theory. After getting good bounds on the Szemerédi theorem, Gowers asked for quantitative proof for these theorems as well.

Bourgain [56, 57] extended the theorems of Roth and Szemerédi for sets of positive density in $\mathbb{R}^d$. He proved that for any set $A \subseteq \mathbb{R}^d$ with positive upper metric density and any set V of d points in $\mathbb{R}^d$ spanning a nondegenerate $(d-1)$-dimensional simplex, there exists a number $\lambda_0 = \lambda_0(A, V)$ such that A contains an isometric image of λV for any $\lambda > \lambda_0$. In the same paper he also gives a result on double integrals over compact abelian groups from which Roth's theorem follows.

3 Harmonic analysis, metric spaces, ergodic theory

Roth's theorem and Gower's new proof of Szemerédi's theorem strongly influenced recently the study of the Kakeya conjecture. In 1917, Besicovitch [45, 46] gave a counter-example to some basic problem on Riemann integration, by constructing a *compact set of plane Lebesgue measure zero containing a line segment in every direction.* At the same time as Besicovitch, Kakeya [157] independently raised the problem of finding sets of smallest measure inside which it is possible to rotate a segment of unit length. A *Kakeya set* is a compact set $E \subseteq \mathbb{R}^d$ containing a line segment in every direction. The *Kakeya conjecture is that a Kakeya set in $\mathbb{R}^d$ must have Hausdorff dimension d.* For $d = 2$ the conjecture was proved by Davies (1971) [85] and for $d > 2$ the Hausdorff dimension is at least $(d + 2)/2$, as established by Wolff in 1995 [267]. The motivation for studying the Kakeya conjecture comes from harmonic analysis, PDE and analytic number theory. The techniques used to solve partial results are geometrical, combinatorial and lately using additive number theory. There are many variants of this problem. A subset of $\mathbb{R}^d$ is called a (d,k)-*Besicovitch set* if it is of d-dimensional Lebesgue measure zero but contains a translate of every k-dimensional subspace of $\mathbb{R}^d$. Falconer [110] proved that (d,k)-Besicovitch sets cannot exist if $2 \leq k \leq d-1$. The original construction of Besicovitch is a $(2,1)$-Besicovitch set, and for any d the Cartesian product of such a set with $\mathbb{R}^{d-2}$ is a $(d,1)$-Besicovitch set. Much of the work on Hausdorff measures and on geometric measure theory forming the base of the geometry of fractals is due to Besicovitch. Moreover Besicovitch in 1964 [47] found a fundamental relationship between Besicovitch sets and geometric measure theory. The Kakeya set has been used lately to provide counter-examples to some major conjectures in harmonic analysis. For more on the history of the Kakeya and Besicovitch sets and relations to harmonic analysis see Falconer's book on the geometry of fractals [111].

In 1999, Bourgain [58] approached the Kakeya problem in surprisingly new ways, influenced by Gower's new proof for Szemerédi's theorem. He converted the properties of the Besicovitch set into statements about sums and differences of sets and applied additive combinatorics. He used his new bound related to Roth's theorem and Gowers new simple proof of the Balog-Szemerédi theorem to obtain a better lower bound than Wolff did for high dimension: $(13d+12)/25$. Katz and Tao [167, 168] made further improvements. Following Bourgain's ideas and converting more properties into the additive combinatorial setting Katz, Laba and Tao [166] could prove important partial results of the Kakeya conjecture in R^3 by improving slightly Wolff's bound. The relation of the Kakeya conjecture to a number of open basic questions in harmonic analysis showed its importance for this field, see for example [131, 268] for more details. There are other connections to combinatorics [268] among others to the Zarankiewicz problem [137] and to the theorem of Szemerédi-Trotter [258]. Bourgain, Katz and Tao (2004) [62] extended to finite fields some of these theorems obtaining new combinatorial and harmonic analysis results at the same time. This is a good example of the recent trend in some areas of harmonic analysis that uses influential combinatorial and additive number theoretical results of Elekes, Ruzsa, Balog and Szemerédi, Solymosi, Tardos and Trotter.

Various methods of constructing Besicovitch sets exist, for example such a set can be obtained by joining the points of a Cantor set in the x-axis to the points of a parallel Cantor-like set, as suggested by Kahane [111]. Laczkovich (2002) [178] proved the following Ramsey theorem for measurable sets: if X is a nonempty perfect Polish space and $[X]^2 = P_0 \cup \ldots \cup P_{k-1}$ is a partition with universally measurable pieces, then there is a Cantor set $C \subset X$ with $[C]^2 \subset P_i$ for some i. Earlier F. Galvin (1968) showed the same if the partition has pieces having the Blaire property.

In the geometry of Banach spaces there are several well-known Ramsey-type statements. The most influential one is Dvoretzky's Ramsey-type theorem [92, 93] proving that all symmetric convex bodies of sufficiently large dimension have a d-dimensional central cross-section which is almost ellipsoidal. Several other proofs appeared since, see [192, 193, 194]. Bourgain, Figiel and Milman [61] show an analogue of Dvoretzky's theorem for finite metric spaces. In theoretical computer science there is an extended literature on similar metric Ramsey-type theorems. These state that a given metric space contains a large subset which can be embedded with small distortion in some well-structured family of metric spaces. See for example the paper of Bartal, Linial, Mendel and Naor [28], or an application to online problems by Bartal, Bollobás and Mendel [27]. Gowers [134] proves a Ramsey-theoretic result that implies: *a separable Hilbert space is the only infinite-dimensional Banach space, up to isomorphism, which is isomorphic to every infinite-dimensional closed subspace of itself.* This solves a problem of Banach from his famous 1932 book. Bagaria and Abad [23] obtained the same result. See more on Ramsey-type results for metric spaces in [61] and for Banach spaces in [135].

The study of Furstenberg's book on recurrence in ergodic theory and combinatorial number theory [124] could be the first step to understand why these seemingly different fields interact with Ramsey theory. In this book, Furstenberg describes the evolution of ergodic theory and topological dynamics from classical dynamical systems. He defines a dynamical system as a space X together with a group of transformations of X. If additionally X is a topological space and the transformations are homeomorphisms of X then X is a *topological dynamical system*. As another possibility, X is assumed to be a measure space and the transformations preserve the given measure, to give a *measure preserving system*. Ergodic theory is the theory of measure preserving transformations. Birkhoff's (1927) recurrence theorem states: *"if X is a compact metric space and T is a continuous map of X into itself, there exists some point $x_0 \in X$ and some sequence $n_k \to \infty$ with $T^{n_k}x_0 \to x_0$."* Furstenberg shows how van der Waerden's theorem follows from a multiple recurrence version of Birkhoff's theorem. The book contains the proofs of other well known Ramsey-type theorems, like Schur's, Hindman's, Rado's, Gallai's and Hilbert's, as consequences of similar general topological dynamical theorems. It is also shown that van der Waerden's theorem can be used as a tool in diophantine approximation, and that certain diophantine approximation theorems fit well into the framework of dynamical systems. A stronger recurrence theorem of Poincaré states: *"let T be a measure preserving transformation of a measure space $(X, \mathcal{B}, \mu)$ and assume that the total measure of X is finite: $\mu(X) < \infty$. If $B \in \mathcal{B}$ is an arbitrary measurable set in X with positive measure, $\mu(B) > 0$, then there is some point $x \in B$ and integer $n \geq 1$ with $T^n x \in B$."* A multiple recurrence analogue is proved in [124], which was used to obtain as a special case Szemerédi's theorem. It was also applied by Furstenberg and Katznelson [125] to obtain the earlier mentioned multidimensional analogue of Szemerédi's theorem.

An extended exposition of related developments in ergodic Ramsey theory, and their connection with other parts of mathematics up to 1994 appeared in a survey by Bergelson [39]. More recently, Bourgain [60] has written on the mutual influence between harmonic analysis and combinatorics. Wolff [268] also indicates the relation of some combinatorial and computational geometry results to the Kakeya problem. Gowers wrote informal papers on combinatorics and connections to other fields [131, 132]. Matoušek's [185] book on discrete geometry describes relations between convex polytopes, metric spaces, Banach spaces, Dvoretzky's theorem, low-distortion embedding and theoretical computer science. The introduction of the papers of Chang, Green [76, 138] and Gowers [130, 133] are also excellent readings on the relation between these subjects. Ramsey theory on the integers is the subject of the recent textbook of Landman and Robertson [179]. Jukna [156] published a textbook on extremal combinatorics with applications in computer science and has a special section on Ramsey theory with a few of its applications. Beck's articles on combinatorial games are applying most basic results in Ramsey theory. His book on combinatorial games, announced to appear in 2005, will certainly be an interesting and valuable addition to this list on Ramsey theory.

4 Convex and computational geometry

The Erdős-Szekeres (1935) [95] paper had great impact on the development of Ramsey theory with the rediscovery of Ramsey's theorem. It influenced convex and computational geometry. Its key results have been improved, generalized and applied. There are two excellent recent surveys on related theorems, proofs and open questions by Morris and Soltan [197] and by Bárány and Károlyi [25]. Erdős and Szekeres proved that $2^{n-2}+1 \leq g(n) \leq \binom{2n-4}{n-2}+1$*, where* $g(n)$ *denotes the smallest number such that any set of at least* $g(n)$ *points in general position in the plane contains* n *points in convex position.* It was conjectured that $g(n) = 2^{n-2}+1$, which is true for $n < 6$. In 1997 three improvements were made to the upper bound, see [79, 171, 261]. The best upper bound $g(n) \leq \binom{2n-5}{n-2}+2$ was obtained by Tóth and Valtr [261]. The Erdős-Szekeres theorem is the consequence of the finite Ramsey theorem and the Morris-Soltan survey gives three Ramsey theoretical proofs for it, indicating how these theorems are related.

Bárány and Valtr [26], later Pach and Solymosi [204, 250, 251], provided systematic ways to find many convex polygons in a sufficiently large set of points in the plane. They show that every n points set in the plane in general position has r pairwise disjoint subsets, all with cardinality at least $\lfloor c_r n \rfloor$, such that no matter how we pick a point from each we obtain a set in convex position. The lower bound for the constant $c_r \geq 2^{-16r^2}$ was obtained by Solymosi [204, 251]. The constant was improved lately by Pór and Valtr [217] who also proved the following statement, answering a question of Gil Kalai. *"For every* $k \geq 4$ *there are two constants* $c = c(k), c' = c'(k)$ *such that the following holds: if* X *is a finite set of points in general position in the plane, then it has a subset* X' *of size at most* c' *such that* $X \setminus X'$ *can be partitioned into at most* c *convex* k*-clusterings."* A finite planar point set X is called *k-clustering* if it is a disjoint union of k sets $X_1, \ldots, X_k$ of equal sizes such that $x_1 x_2 \cdots x_k$ is a convex k-gon for each choice of $x_1 \in X_1, \ldots, x_k \in X_k$. The proof gives reasonable estimates on c, c', and it works also in higher dimensions. Károlyi and Tóth [163] proved for $n \geq k \geq 3$ that $\frac{(k-1)(n-1)}{2} + 2^{k/2-4} \leq g(k,n) \leq 2kn + 2^{8k}$, where $g(k,n)$ is the smallest number of points in general position in the plane that contains n points whose convex hull has at least k vertices. The Erdős-Szekeres theorem was generalized for families of convex bodies by Bisztriczky and Fejes Tóth [48]. Pach and Tóth [206, 207] study Erdős-Szekeres type problems in which the points are replaced by convex sets. For additional connection to basic theorems in discrete geometry see also the survey by Bárány and Károlyi [25].

Grünbaum's classical book on convex polytopes (1967) [142] mentions several early generalizations of the Erdős-Szekeres theorem. For example Danzer, Grünbaum and Klee [84, 142] (p.22) showed that any set of $d+3$ points in general position in $\mathbb{R}^d$ contains a subset of $d+2$ points in convex position. This is a generalization of E. Klein result for $d=2$ which was the starting point of the Erdős-Szekeres theorems [95]. Perles [142] (p.120) proved that any set of $d+3$ points in general position in $\mathbb{R}^d$ contains a subset of $d+2$ points which are the vertices of a polytope combinatorially equivalent to the cyclic polytope $C(d+2,d)$. Grünbaum [142] (p.126) has similar results for neighborly d-polytopes. For $d \geq 2$, $N_d(n)$ denotes the smallest integer with the property that every set of this many points in $\mathbb{R}^d$ in general position contains n vertices of a cyclic d-polytope. Oriented matroid methods together with Ramsey's theorem were used by Cordovil and Duchet [83] to prove its existence, but its order of magnitude is still not known; see also the book on oriented matroids by Björner et al. [49]. Morris and Soltan [197] state a general Erdős-Szekeres type theorem in another abstract geometry framework, in general convexity. This also follows from the finite Ramsey theorem.

Grünbaum [142, 143] (p.22) introduced $f(n,d)$, an analogue of $g(n)$ for $\mathbb{R}^d$, and proved its existence using Ramsey's theorem. For $n > d \geq 2$, $f(n,d)$ is the smallest integer such that any set of at least $f(n,d)$ points in general position in $\mathbb{R}^d$ contains a convex set of size n. Károlyi and Valtr [164] gave a lower bound on $f(n,d)$ that is conjectured to be asymptotically tight. Károlyi [158] gives the upper bound $f(n,d) \leq \binom{2n-2d-1}{n-d} + d$. From his proof he also obtains a stronger Erdős-Szekeres theorem: if $d \geq 3$ and n is large enough, then any set of kn points in general position in $\mathbb{R}^d$ can be partitioned into n convex subsets of size k. In the planar case this is not true even if $k=4$. He gave an $O(n \log n)$ time algorithm which decides if a given set of $4n$ points in general position in the plane can be partitioned into convex quadrilaterals. This answers a computational geometry question of Mitchell aimed at possible applications for quadrangular mesh generations. The *Ramsey-remainder* $rr(k)$ was defined by Erdős, Tuza and Valtr [106] as the smallest integer such that a large enough point set in general position in the plane can be partitioned into vertex disjoint convex sets, at least $k \geq 3$ points each, and the remaining set has at most $rr(k)$ points. The following problem is still open: is $rr(k) = 2^{k-2} - k + 1$? In higher dimensions the Ramsey-remainder is 0, see [158].

Erdős asked for the smallest number of points in the plane, no three collinear, such that a convex n-gon always exists without any point in its interior. For $n \leq 5$, Harborth [148] determined this number and Horton [152] constructed arbitrarily large sets of points which have no empty 7-gon. The remaining intriguing open question is whether a large enough set of points in the plane in general position contains the vertex set of an empty convex hexagon. Solymosi [250] relates this to a Ramsey-type problem for geometric graphs: is there an empty monochromatic triangle in any 2-coloring of the edges of a large enough complete geometric graph? (A geometric graph is drawn in the plane, the vertices represent points in general position and the edges are straight line segments.) The following generalization is by Valtr [264]: *let $h(d)$ denote the maximum number h such that any sufficiently large set of points A in general position in $\mathbb{R}^d$ contains an h-hole, i.e. h points that are vertices of a convex polytope containing no other point of A. For $d \geq 2$, $2d+1 \leq h(d) \leq 2^{d-1}[P(d-1)+1]$, where $P(d-1)$ is the product of the smallest $d-1$*

prime numbers. Valtr uses the Erdős-Szekeres theorem in his proof.

In VSLI designs or in pattern recognition it is important to find crossingfree subgraphs of geometric graphs. Deciding the existence of a crossingfree spanning tree is NP-complete [153]. There are some results for topological layouts where the edges are not restricted to be straight lines. The computational complexity of problems connected to topological layouts was studied by Kratochvil, Lubiw and Nešetřil [175]. Among the results obtained are: (1) deciding the existence of a crossingfree path between two given vertices, or of a crossingfree cycle, in a given topological layout of a 3-regular graph, are both NP-complete; (2) deciding the existence of a crossingfree k-factor in a topological layout of a $(k+1)$-regular graph is NP-complete, for $2 \leq k \leq 5$. For $k = 1$, the question is NP-complete in layouts of 3-regular graphs, while it is polynomial solvable for layouts of graphs with maximum degree too. It is easier to find crossingfree monochromatic subgraphs in 2-colorings of edges of complete geometric graphs. In [160] it is shown that if the line segments of any n points in general position in the Euclidean plane are colored by 2 colors, then at least one of the color classes contains a spanning tree without any pair of crossing edges. Károlyi et al. also proved that there exist $\lfloor (n+1)/3 \rfloor$ monochromatic pairwise-disjoint edges, as conjectured by Biaslostocki and Dierker, and they can be found in $O(n^{\log\log n+2})$ time [159, 160]. Károlyi, Pach, Tóth and Valtr [161] gave lower and upper bounds for geometric Ramsey numbers of noncrossing paths and cycles, together with an $O(n^2)$ time algorithm for finding these in any 2-coloring of the complete geometric graph of n vertices. For more on geometric Ramsey-type theorems, see [162, 149].

There are a number of Ramsey-type results in computational geometry related to intersection graphs arising from a system of simple continuous curves in the plane, where a vertex is assigned to each curve and two vertices are adjacent if and only if the corresponding curves intersect. Pach and Solymosi [205, 251] proved several statements for the intersection graph of segments, for example: any system of n segments in the plane with at least cn^2 crossings ($c > 0$) has two disjoint subsystems of cardinality at least $\frac{(2c)^A}{660}n$ each and every segments between them cross. Similar result is stated for non-crossing segments. These results, combined with Szemerédi's regularity lemma, were used to establish a fairly strong structure theorem for intersection graphs of segments. Kratochvil and Nešetřil [176] investigated the computational complexity of the independent set and the clique problems when restricted to certain intersection graphs of straight line segments in the plane. Finding a maximum independent set is NP-complete except for some very restricted cases. On the other hand a maximum clique can be found in polynomial time for intersection graphs of segments lying in at most k directions in the plane, for any fixed positive integer k, if the representation has a suitable description.

Dilworth's well known theorem [89] states that *in any partially ordered set of size* $(p-1)(q-1)+1$ *there is either a chain of size* p *or an antichain of size* q. It follows that the comparability graph of any partially ordered set of n elements contains either a clique or an independent set of size $\sqrt{n}$. Dumitrescu and Tóth [91] showed several statements on the unions of comparability graphs such as: the union of two comparability graphs on the same vertex set contains either a clique or an independent set of size at least $n^{1/3}$.

They also showed that there exist union of comparability graphs not containing clique or independent set larger than $n^{0.4118}$. Dilworth's theorem has been used lately to establish upper bounds for Ramsey numbers of noncrossing paths and cycles in geometric graphs [161, 162].

The monotone subsequence theorem of Erdős and Szekeres [95] states that *any sequence of at least* n^2+1 *real numbers contains a monotonic subsequence of at least* $n+1$ *terms*. It has been applied in computational geometry for visibility questions by Alt, Goude and Whitesides [20] and Bose et al. [55]. Visibility results have been used in graph drawing and in VSLI wire routing, see also [88]. Szabó and Tardos [255] generalized the monotone subsequence theorem, stated as a Ramsey theorem. Let $H_0,\ldots,H_d$ be a list of $d+1$ linear orderings of a finite set V. A 2^d-coloring of the edges of the complete graph on the vertex set V is defined by coloring an edge uv with the color $(c_1,\ldots,c_d)\in\{0,1\}^d$, where $c_i=0$ if H_i agrees with H_0 on $\{u,v\}$, and $c_i=1$ otherwise. For $d=1$ and $|V|=n$, the monotone subsequence theorem implies that there exists a monochromatic clique of size $\lceil\sqrt{n}\rceil$, and this result is best possible. For $d>1$ determining the size of the largest monochromatic subset can be solved with repeated applications of the monotone subsequence theorem. Instead, Szabó and Tardos considered the more challenging problem of finding the size of the largest subset that misses at least one of the 2^d colors. This problem has connection to a question of Preiss in analysis as to whether any compact set of positive Lebesgue measure in d-space admits a contraction onto a ball. There are several other ways to generalize the monotone subsequence theorem to higher dimensions, depending on the definition of monotonicity, when the real numbers in a sequence are replaced by d dimensional vectors in $\mathbb{R}^d$. For more details on these generalizations, see the paper of Kruskal [177], the survey by Steele [254] and the article of Odlyzko, Shearer and Siders [203].

5 Probabilistic method versus constructions

The first probabilistic proof is attributed to Erdős [97] who used it to find lower bounds for the classical Ramsey numbers. The idea is simple, but it became famous as it could be applied in many different situations. Let the edges of a complete graph on n vertices be colored by two colors, each with probability 1/2. It is easy to see, that if $\binom{n}{k}2^{1-\binom{k}{2}}<1$, then there is a coloring with no monochromatic complete subgraph of k vertices. This reasoning gave the lower bound $r(K_k,K_k)>ck2^{k/2}$. The Lovász local lemma [100] is a generalization of the idea of Erdős's proof and it is among the most applied techniques in combinatorics and computer science. For more on this subject see the books by Alon, Erdős and Spencer [18, 19, 105]. The probabilistic method has been applied extensively in computer science to design and analyze randomized algorithms, see for example the book of Motwani and Raghavan [198]. Also, Milman's [193] probabilistic proof to Dvoretzky's Ramsey-type theorem [92, 93] in the geometry of Banach spaces is considered revolutionary by Gowers [132], as it showed that the idea of measure concentration can be exploited in many different fields.

Although it is usually possible to get better bounds with probabilistic method than with constructions, explicit constructions are often preferable in applications. The constructive lower bound for $r(K_k, K_k)$ given by Frankl and Wilson [120] has important applications in information theory, for example. Recently Grolmusz [139, 140, 141] proved equivalent lower bounds with a method generalizable to explicit Ramsey-colorings with more than two colors. He found a relation between the ranks of codiagonal matrices, matrices with 0's in their diagonal and non zeroes elsewhere, and explicit Ramsey-graph constructions.

Probabilistic method has been used to prove the existence of some family of expanding graphs [77, 214]. Let $G = (I, O)$ be a bipartite graph, where I can be considered as the set of inputs and O as the set of outputs with $|I| = |O| = n$. If every set of at least a inputs is joined by edges to at least b different outputs, where $0 < a \leq b \leq n$, then we call the graph G (n, a, b)*-expanding*. A *superconcentrator* is an expanding graph with the additional properties of being directed, acyclic and from any set of r inputs to any set of r outputs there are r vertex disjoint paths. Expander graphs were used by Ajtai, Komlós and Szemerédi (1983)[5] to establish an $O(\log n)$ upper bound for the complexity of parallel sorting networks. Superconcentrators or expanding graphs with small number of edges are also important in the construction of graphs with special connectivity properties [77] or in the study of lower bounds [263] among other applications [215].

The explicit constructions of highly expanding graphs with small number of edges are more difficult, however for some applications these properties are essential. Alon and Milman [14, 15, 6] and independently Tanner [259] showed the relation between an expanding graph, or a regular bipartite graph $G(I, O)$ in general, and λ_2 the second eigenvalue of AA^T, where A is the adjacency matrix of the graph. Let the degree$(x) = k$ if $x \in I$ and the degree$(y) = s$ if $y \in O$. The main result of Tanner says that for $X \subset I$, $|N(X)|$, the number of neighbors of X, is at least $\frac{k^2|X|}{(ks-\lambda_2)|X|/n+\lambda_2}$. Alon [7] used this result to get the constructive lower bound $c_1 n^{4/3} < r(C_4, K_n)$. He also showed, using the result of Tanner, that the points-hyperplanes incidence graph of a finite geometry of dimension d is an $(n, x, n - n^{1+1/d}/x)$-expanding graph for all $0 < x < n$. This geometric expander is highly expanding $(b(n)/a(n) \to \infty)$, with close to the smallest possible number of edges, and it is also used to obtain results for parallel sorting in rounds.

In 1980, Ajtai, Komlós and Szemerédi [3] proved that $r(K_3, K_m) \leq O(m^2/\log m)$. This result has been applied in the construction of algorithms to find large independent sets, (see also the section on approximation algorithms). Kim (1994) [170] showed that this upper bound is tight up to a constant factor. His argument is probabilistic. The best constructive lower bound is due to Alon [8] (1994). His constructions are based on the properties of some dual error-correcting codes and Cayley graphs. The constructions give triangle-free graphs G_n on n vertices satisfying $\alpha(G_n) \leq \theta(G_n) = \Theta(n^{2/3})$, where $\alpha(G_n)$ denotes the independence number and $\theta(G_n)$ is the Lovász θ-function [182]. It also settles a geometric problem of Lovász, by proving that the maximum possible value of the Euclidean norm $\|\Sigma_{i=1}^n u_i\|$ of the sum of n unit vectors $u_1, \dots, u_n$ in $\mathbb{R}^n$, so that among any three of them some two are orthogonal, is $\Theta(n^{2/3})$.

6 Information theory, dual source codes

Ramsey theorems have been applied to information theory several times in various ways. The 1984 survey on applications of Ramsey theory by Roberts [226] explains in detail how one Ramsey-type theorem can be applied to communication channels. We mention some newer results based mostly on Alon's work and follow his definitions and notations. Let $G = (V, E)$ be a graph corresponding to an information (noisy) channel [9] where V represents the input set, i.e. all possible letters the channel can transmit in one use. In each channel use, a sender transmits an input and a receiver receives an output. The vertices corresponding to two letters are adjacent if and only if both can result in the same output and thus they could be confused. We could choose an independent set as an unambiguous code alphabet for sending messages. The maximum number of letters that can be transmitted in a single use without error is then the maximum size of an independent set, denoted by $\alpha(G)$. To obtain larger unambiguous code alphabet it is suggested [226] to introduce noisy channels whose graph is the product of graphs. Let G^n denote the graph whose vertex set is V^n and two vertices $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are adjacent if and only if for all i, $1 \le i \le n$, either $u_i = v_i$ or $u_i v_i \in E$. $\alpha(G^n)$ is the maximum number of messages that can be transmitted in n uses of the channel without confusion. Hedrlín [150, 226] proved: *if G and H are any graphs, then $\alpha(G \cdot H) \le r(\alpha(G)+1, \alpha(H)+1) - 1$.* This gives an upper bound on the size of an unambiguous code alphabet if the graph of the noisy channel is G^2. G^n can be used to get larger and larger unambiguous code alphabets, but with a cost to efficiency by using longer strings. To compensate, Shannon considered the number $(\alpha(G^n))^{1/n}$ as a measure for the capacity of the channel with an unambiguous code alphabet of strings of length n. The Shannon capacity [240] is defined as $c(G) = \lim_{n\to\infty}(\alpha(G^n))^{1/n}$, it represents the number of distinct messages per use the channel can communicate with no error while used many times.

The disjoint union of two graphs G and H, denoted by $G + H$, is defined as the graph whose vertex set is the disjoint union of the vertex sets of G and H and its edge set is the disjoint union of the edge sets of G and H. If G and H are graphs of channels, then the union of the channels corresponds to the situation that either one can be used. Shannon (1956) [240] proved that $c(G + H) \ge c(G) + c(H)$, and equality holds if the vertex set of one of the graphs can be covered by $\alpha(G)$ cliques. He conjectured that equality always holds. Alon [9] disproved this conjecture in a strong sense, proving that *for every k there is a graph G so that $c(G) \le k$, $c(\overline{G}) \le k$, while $c(G + \overline{G}) \ge k^{(1+o(1))\frac{logk}{8loglogk}}$, and the $o(1)$-term tends to zero as k tends to infinity.* For his proof he used a modified version of Frankl and Wilson's [120] well-known explicit 2-coloring that gives the constructive lower bound $k^{(1+o(1))\frac{logk}{4loglogk}} < r(K_k, K_k)$. Alon extended to $g > 2$ colors the modification of the Frankl-Wilson construction to obtain: *for every fixed integer $g \ge 2$ and $k > k_0(g)$, $k^{\frac{(1+o(1))(logk)^{g-1}}{g^g(loglogk)^{g-1}}} < r(K_k, K_k, \ldots, K_k)$.* This construction with more than 2 colors can give an example for unions of more than 2 channels with large capacities.

Alon and Orlitsky [17] study the savings afforded by repeated use in zero-error communication problems. They show that for some channels communicating one instance

requires arbitrarily many bits, but communicating multiple instances requires roughly one bit per instance. The largest number of bits a channel can communicate without error in a single use is $\gamma^{(1)} = \log \alpha(G)$. If a set I is independent in G then $I \times \ldots \times I$ is independent in G^n, and so $\alpha(G^n) \geq (\alpha(G))^n$. $\gamma^{(n)} = \log \alpha(G^n)$ is the largest number of bits the channel can communicate without confusion in n uses, hence $\gamma^{(n)} \geq n\gamma^{(1)}$.
Some channels can communicate exponentially more bits in two uses than they can in one. Let $\rho_n(l) = \max\{\alpha(G^n) : \alpha(G) \leq l\}$, and $r_n(l) = r(K_{l+1}, \ldots, K_{l+1}) - 1$, (the edges are colored with n colors). Erdős, McEliece and Taylor [101] proved that $\rho_n(l) = r_n(l)$. In [17] this is used, together with the well known upper and lower bounds for classical Ramsey numbers, (i.e. $2^{l/2} \leq r_2(l) < 2^{2l}$, [137]), to show, that $2^{\gamma^{(1)}-1} \leq \gamma^{(2)} < 2^{\gamma^{(1)}+1}$. $C^{(n)} = \gamma^{(n)}/n$ is the zero-error n-use capacity [240], its limit $C^{(\infty)}$ is also known as Shannon's zero-error capacity, the highest per-use number of bits the channel can transmit without error. $C^{(\infty)} \geq C^{(2)}$, and there are channels whose infinite-use capacity is exponentially larger than their single-use capacity: $C^{(\infty)} \geq 2^{C^{(1)}-2}$. Moreover there is an arbitrary gap between the single-use capacity and the infinite-use capacity if and only if for some constant c, $r_n(2^c)$ grows faster than any exponential in n. This is a generalization of an open question of Erdős, who asked whether the Ramsey number $r_n(2)$ grows faster than any exponential in n (see [137] page 146).

Alon and Orlitzky [17] examine dual-source codings, as well. A dual source $\mathcal{S}$ consists of a finite set X, a set Y, and a support set $S \subseteq X \times Y$. In a dual-source instance a sender is given an $x \in X$ and a receiver is given a $y \in Y$ such that (x, y) is in the support set S. What is the minimum number of bits the sender must transmit in the worst case in order for the receiver to learn x without error? The characteristic graph G of a dual source $\mathcal{S}$ has the vertex set X, and $x, x' \in X$ are connected iff there is a y jointly possible with both, i.e. there is a $y \in Y$ such that both $(x, y) \in S$ and $(x', y) \in S$. The smallest number of possible messages the sender must transmit in the worst case for a single instance of $\mathcal{S}$ is the chromatic number of G, $\chi(G)$, and the smallest number of bits the sender must transmit in the worst case for a single instance of $\mathcal{S}$ is $\sigma^{(1)} = \log \chi(G)$. In n instances of the dual source $\mathcal{S}$, the sender knows $x_1, \ldots, x_n$ while the receiver knows $y_1, \ldots, y_n$ such that each $(x_i, y_i) \in S$ and the receiver wants to learn $x_1, \ldots, x_n$. The number of bits the sender must transmit in the worst case for n instances of $\mathcal{S}$ without error is $\sigma^{(n)} = \log \chi(G^n)$. For every graph G of a dual source $\sigma^{(2)} \leq 2\sigma^{(1)}$, since if G can be colored with χ colors then G^2 can be colored with χ^2 colors. For some dual sources fewer bits are enough.

A graph is called self-complementary if it is isomorphic to its complement. A graph is called *Ramsey graph* if both its independence number and clique number are polylogarithmic in the number of vertices. Let A be a finite Abelian group and a set $K \subseteq A$ is symmetric if $-K = K$. The Cayley graph over A with respect to a symmetric set K has vertex set A and $a, b \in A$ are connected iff $a - b$ is in K. Alon and Orlitzky use probabilistic constructions of self-complementary Ramsey graphs, that are also Cayley graphs, to show that *for every prime power* $v \equiv 1 \mod 4$ *there is a v-vertex Cayley graph G with independence number at most* $(1 + o(1))16 \log^2 v$ *such that* $\chi(G) \geq \frac{v}{(1+o(1))16 \log^2 v}$, *while*

$\chi(G^2) \leq v$. This implies that for arbitrarily high values of $\sigma^{(1)}$ there are dual sources where $\sigma^{(2)} \leq \sigma^{(1)} + 2\log\sigma^{(1)} + 4 + o(1)$. These are sources where the number of bits required for a single instance is comparable to the size of the source, but two instances require only a logarithmic number of additional bits.

7 Order invariant algorithms

Yao [269] (1981), in his influential paper on searching tables, used Ramsey theory and others inspired by it followed. Subsequently, Ramsey theory was applied by Frederickson and Lynch on a problem in distributed computations [121], by Snir [246] to search on sorted tables in different parallel computation models, by Maass [184] in lower bound arguments for random-access machines, or by Moran, Snir and Manber to show properties of order invariant decision trees [196]. Yao [269] examined the following problem: "Given a set S of n distinct keys from a key space $M = \{1, 2, \ldots, m\}$, a basic information retrieval problem is to store S so that membership queries of the form "Is j in S" can be answered quickly." If the keys are stored in a sorted table, then $\lceil \log_2(n+1) \rceil$ probes are sufficient by using binary search. He first proves that, for sorted table, $\lceil \log_2(n+1) \rceil$ probes are needed with any search strategy, in the worst case, if $m \geq 2n-1, n \geq 2$. Using the finite Ramsey theorem Yao proves that, if m is sufficiently large then $\lceil \log_2(n+1) \rceil$ is needed in the worst case with any table structure. Sufficiently large here is actually extremely large, the complete n-uniform hypergraph of m vertices is colored with $n!$ colors, and m is large enough so that there is a monochromatic complete n-uniform hypergraph of $2n-1$ vertices. (He remarks that this result is not too useful in practice.) It follows from Yao's result that for sufficiently large m, using a sorted table structure is the most efficient method for information retrieval.

Moran, Snir and Manber [196] consider problems that are defined by simple inequalities between inputs, called *order invariant* problems. The input domain S^n consists of the n-tuples of elements of a totally ordered set S. Two tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are said to be order equivalent, $\mathbf{x} \equiv \mathbf{y}$, if for all $i, j = 1, \ldots, n$, $x_i < x_j$ iff $y_i < y_j$. The equivalence class of $\mathbf{x}$ is called the order type of $\mathbf{x}$. A decision problem $\mathcal{P}$ is a partition of all the n-tuples of S^n into classes $P_1, P_2, \ldots, P_q$, and the problem is to determine to which class an input belongs. $\mathcal{P}$ is said to be order invariant if order equivalent tuples are always in the same class. A query, which is a predicate defined on the set S^n, is considered order invariant if its outcome depends only on the relative order of the inputs occurring in it. A *deterministic decision tree* is a labeled binary tree, where each internal node v is labeled with a query Q_v, and each internal node has two outgoing edges labeled true or false. Each leaf of the decision tree is labeled by one of the partition classes.

The evaluation of the tree T for $\mathbf{x}$ proceeds from the root. When a node v is reached the predicate Q_v is evaluated on $\mathbf{x}$, and one of the outgoing edges is chosen according to the outcome of the evaluation. A decision tree solves a decision problem if for each input the computational path for that input reaches a leaf, labeled with the partition set this input belongs to. The decision tree is order invariant if each query occurring in it is order invariant. A decision tree is k-bounded if each query depends on at most k variables. Moran et al. prove, using Ramsey theory, that one can replace the queries of any k-bounded decision tree that solves an order invariant problem over a sufficiently large input domain with k-bounded order invariant queries. As a consequence, all existing lower bounds for comparison-based algorithms are valid for general k-bounded decision trees, where k is a constant.

Moran et al. consider that these applications and some others previously obtained follow from the following observation. If the assertion "Algorithm A solves correctly an order invariant problem $\mathcal{P}$ in t steps" can be formally expressed by a universal formula of first-order predicate calculus using the predicates $<, P_1, \ldots, P_k$, then the following formulation of Ramsey's theorem, due to Ramsey [224] can be used. *"For each j, k, m, n there is a number $N(j, k, m, n)$ such that the following holds: let F be a universal formula of size j in first-order predicate calculus, with predicates $P_1, \ldots, P_k$, and n variables. If this formula can be satisfied by a model of size $N(j, k, m, n)$, then it can be satisfied by a model of size m, where the predicates P_i are order invariant."* Ramsey's theorem thus implies that, if this formula is satisfied, with $<$ interpreted as total order, on a large enough domain, then it is satisfied on a domain of size m, where all P_i's are order invariant. Hence, lower bounds proved for algorithms represented by order invariant predicates have to be valid with no restriction on the predicates. This method has been used only for computations where each operation has a fixed number of outcomes. Another major limitation of this type of applications is again that the input domain must be large enough for Ramsey's theorem to be applied.

Moran et al. also prove an $\Omega(n \log n)$ lower bound for the element uniqueness problem for any k-bounded decision tree, such that $k = O(n^c)$ and $c < 1/2$, and this result is essentially tight. The element uniqueness problem is to decide, given n elements in S, whether they are pairwise distinct. In proving this more specific result, they use Ramsey theorem in a more direct way, so the result is valid for much smaller input domain than needed for the general result, although it is still quite large.

A more recent example for similar applications of the finite Ramsey theorem is by Naor and Stockmeyer (1995) [199], studying computations that can be done locally in a distributed parallel computing network. The question is what can be computed when algorithms must satisfy strong requirement of locality, namely, that the algorithm must run in *constant time* independent of the size of the network. A network is modeled as an undirected graph, where each vertex represents a processor, and edges represent direct connections between processors. Each processor is connected directly to at most some fixed number of others. The connections can be considered local, but there are some computations where the values computed at different nodes must fit together in some global way. Naor et al. are interested in computational problems of producing labeling of the network, more specifically locally checkable labeling within some fixed radius from the node. If an algorithm runs in constant time t, each processor v can collect information on the structure of the network, like processor id, only in the region within radius t from v.

Then the processor must choose an output based on this information. There are several advantages, for example in improved fault tolerance, a failure at processor v can only effect processors in some bounded region. Sometimes it is useful to restrict attention to order invariant algorithms that do not use the actual values of the processor id's, only their relative order. Naor and Stockmeyer show, using the finite Ramsey theorem, that when the algorithm is computed locally, in constant time, there is no loss of generality by restricting the study to order invariant algorithms.

A recent result of Fischer (2000) [117] studies property testing, introduced by Rubinfeld and Sudan [233] motivated by its connection to program checking. Fischer considers the minimum number of queries required for a randomized algorithm, called ϵ-test, to distinguish between the case of a sequence of n integers satisfying a certain property, and the case that it has to be modified in more than ϵn places to make it satisfy the property. He uses the following corollary of the Ramsey theorem [50, 137]: *"If $\mathcal{F}$ is any finite family of functions with k variables from the positive integers to a finite range, then there exists an infinite subset E of the positive integers, such that the restriction of the members of $\mathcal{F}$ to E are all order based in their variables."* He shows that an ϵ-test for any property defined in terms of the order relation cannot perform less queries in the worst case than the best ϵ-test which uses only comparisons between the queried values. As a consequence, he finds a tight lower bound of $O(\log n)$ for the number of queries required to test whether the sequence is monotone nondecreasing or ϵn-far from it. Another application of Ramsey-type theorems to property testing by Fischer and Newman [118] is considering testing certain properties of binary matrices with a fixed dimension. They also use a new restricted version of Szemerédi's regularity lemma that does not require tower like dependency on the parameters. Alon, Fischer, Krivelevich and Szegedy [10] also proved a variant of Szemerédi's regularity lemma to use in results related to testing graph properties. For more on property testing and its relation to the application of the regularity lemma, see the new survey of Fischer [116].

8 Lower bound arguments for PRAM

Valiant's [262] parallel comparison decision tree model is considered useful for studying parallel algorithms and lower bounds for order invariant problems, but this model does not deal with the problem of communicating between processors that run in parallel. Parallel random-access machine (PRAM) is perhaps a more realistic model, though the two models are not comparable in general. Concurrent-read concurrent-write parallel random-access machine (CRCW PRAM) consists of a set of processors that communicate via a shared memory, and processors can read or write simultaneously at the same memory location. There are several rules to solve writing conflicts. In the Priority rule, the value written by the processor with the highest preassigned priority, like lowest index, is accepted. In the Common rule for write conflict, all the processors writing to the cell at the same time must write the same value. These are just two of the possible rules for conflict resolutions, see [115] for a list of the other rules and their comparisons.

Fish et al. in [113] gave the first tight $\Theta(\log\log n)$ bound for finding the maximum of n integers, in the Priority PRAM model, generalizing Valiant's result for parallel comparison decision trees. The following Ramsey-type theorem [137] is used: *"let $f: W \to D$ be any function defined on an infinite domain W. Then there exists an infinite subset $W' \subset W$ such that $f|_{W'}$ is either constant or $1-1$. In particular, if D is finite, then $f|_{W'}$ is constant".* Ajtai, Komlós and Szemerédi (1983)[5] established an $O(\log n)$ upper bound for the time complexity of parallel sorting of n integers. To obtain the lower bound $\Omega(\sqrt{\log n})$ on sorting n integers in the Priority PRAM model, Meyer auf der Heide and Wigderson (1987) [190] apply the "canonical" Ramsey theorem of Erdős and Rado [103, 102, 137, 50]. They also prove that the computation of any symmetric polynomial of n integers requires exactly $\log_2 n$ steps.

Boppana [53] studied the relative powers of the Priority and Common models of parallel random-access machines. He obtained tight upper and lower bound on the cost of simulating a Priority machine by a Common machine. The motivation for studying these problems is to understand the relationship between these two models, as algorithm designers prefer the power of the Priority model, and for computer architects the Common model is easier to implement in hardware. The element distinctness problem is solvable in constant time on a Priority$[n]$ machine (using n processors). Fish, Meyer auf der Heide and Wigderson [114] showed, that a Common$[n]$ machine (using n processors) solving element distinctness$[n]$ with n integers requires $\Omega(\log\log\log n)$ time. Ragde, Steiger, Szemerédi and Wigderson [223] improved the lower bound to $\Omega(\sqrt{\log n})$. Boppana improved further these results to the best possible lower bound, to $\Theta(\log n/\log\log n)$. All these lower bounds are obtained using similar Ramsey theoretical methods, developed by Ragde et al. [223] and Meyer auf der Heide and Wigderson [190], based on the Erdős-Rado theorem. More recently Breslauer, Czumaj, Dubhashi and Meyer auf der Heide (1995, 1997) [64, 65] were again applying Ramsey theoretical methods based on the Erdős-Rado theorem, for general transformations of lower bounds in Valiant's comparison decision tree model to lower bounds in a PRAM model. The disadvantage of some of these applications using Ramsey theory is that they often require very large input domains, like for example the results of Boppana. Edmonds [94] considers that removing Ramsey theory can help solving problems on small domains. He obtains the same lower bounds as Boppana, on much smaller input domains, by better use of the partial information a processor learns about an input, and by developing some set theoretic techniques to replace Ramsey theory.

9 Lower bounds for Boolean function computation

A Boolean function f, of n Boolean variables $x_1, \ldots, x_n$, is a mapping from the set of 2^n possible $(0,1)$ input strings $b = (b_1, \ldots, b_n)$, to $\{0,1\}$. A generalization of decision trees, the branching program is a directed acyclic graph with a unique source S, some sinks (terminals) labeled 0 or 1 and some other, non-sink (non-terminal) vertices labeled each by an input variable. Each non-sink vertex has two outgoing edges labeled 0 or 1. Any assignment of values b_i to the input variables x_i defines a unique computation path from S

to a sink, leaving each vertex labeled x_i through the edge labeled b_i. The Boolean function f is computed by the branching program if for any possible $(b_1, \ldots, b_n)$ the end vertex of the path is labeled $f(b_1, \ldots, b_n)$. It is usually assumed that a vertex has a level, where the level of S is 1, and edges go from each level only to the next one. The number of levels is the length of the program, corresponding to the time of the computation. The width of the program is the maximum number of vertices on a level, its logarithm corresponds to the space of the computation. The size, or the complexity of the branching program is the number of vertices. The complexity of the Boolean function f is the complexity of an optimal branching program that computes f. A branching program is a sequential model of computation and is called input oblivious if all non-sink vertices at each level have the same label.

Most Boolean functions require exponential size branching programs. A barely nonlinear lower bound for the size of the branching programs of the majority function was proved using Ramsey theory by Pudlák [218]. For some other explicit Boolean functions nonlinear lower bounds were found, but mostly restricted to bounded width branching programs. The bounds are often for symmetric Boolean functions, where a function is symmetric if it is invariant under permutations of the variables, i.e. $f(b)$ depends only on the number of 1's among b_i's. The first nonlinear lower bound was obtained using Ramsey-type argument by Chandra, Furst and Lipton [75], for the length of any bounded width branching program that computes the symmetric function of n Boolean variables $x_1, \ldots, x_n$, whose value is 1 iff $\Sigma_{i=1}^n x_i = n/2$. It is barely nonlinear, $\Omega(nw(n))$, where $w(n)$ is the inverse of the van der Waerden numbers. Better nonlinear bound, $\Omega(n \log\log n / \log\log\log n)$ was found for threshold functions by Pudlák [218], using a different Ramsey argument. He also obtains the same lower bound, for bounded branching programs, for all but a bounded number of symmetric Boolean functions. Computing any member of a large class of symmetric Boolean functions, the $\Omega(n \log n / \log\log n)$ lower bound was obtained by Ajtai et al. [2], for bounded width branching programs, but for this they could not use the help of Ramsey-type methods. This bound was further improved to $\Omega(n \log n)$ by Babai et al. [22] and by Alon and Maass [12] independently, the later again using Ramsey-type methods. Cai and Lipton [74] obtains an $n \log\log n$ lower bound for a permutation branching program that computes the logical AND function of n Boolean variables by applying the Erdős-Szekeres monotone subsequence theorem.

Alon and Maass [12, 13] have several other results with lower bounds for the length of branching programs of various symmetric functions. They also present a new technique to obtain lower bounds for the time versus space complexity of certain functions in a general input oblivious sequential model of computation. They use the following Ramsey theoretic lemma, stated and proved for these applications. Let $X = (x_1, x_2, \ldots, x_m)$ be a sequence of elements of $N = \{1, 2, \ldots, n\}$, where the same $a \in N$ may appear several times. For an ordered pair (a, b) of distinct elements of N, $v_X(a, b)$, the order type vector of (a, b) in X, is defined as the binary vector obtained from X by replacing each a with 0, each b by 1, and omitting all other elements of N in X. The lemma states that *if each $a \in N$ appears exactly k times in X, and $N = N_1 \cup N_2$ is a partition of N into two disjoint sets, then there are two subsets $S \subseteq N_1$, $T \subseteq N_2$, $|S| \geq |N_1|/2^{2k-1}$ and $|T| \geq |N_2|/2^{2k-1}$,* *such that all the order type vectors* $\{v_X(s,t) : s \in S, t \in T\}$ *are identical.* This lemma is not far from being best possible.

Alon and Maass introduce the concept of *meanders* to describe superconcentrator type properties of sequences. A sequence $M = x_1x_2\dots x_m$ of numbers $x_i \in \{1,\dots,n\} = N$ is called a $g(x)$-*bipartite-meander* (over n numbers, of length m), if for any disjoint sets $S, T \subseteq N$, $|S| = |T|$ with $S \subseteq \{1, 2, \dots, n/2\}$ and $T \subseteq \{n/2+1, \dots, n\}$ there are in M at least $g(|S|)$ links between S and T. The interval $x_ix_{i+1}\dots x_{i+j}$ is a link between S and T if $x_{i+1}, \dots, x_{i+j-1} \notin S \cup T$ and $x_i \in S$, $x_{i+j} \in T$ or $x_i \in T$, $x_{i+j} \in S$. From the above Ramsey theoretical lemma they prove lower bounds for the length of meanders. *If* M *is a* $g(x)$*-bipartite-meander over* n *numbers of length* nf*, then* $f \geq 1/8g(n/2^{8f+1})$*. In particular, every* $\log x$*-meander has length* $\Omega(n \log n)$*, and for every* $g(x) \to \infty$ *the length of a* $g(x)$*-meander is superlinear.* As a corollary: If $s(n)$ is an arbitrary function, M is a sequence over $\{1,\dots,n\}$, and for any two sets $S \subseteq \{1, 2, \dots, n/2\}$ and $T \subseteq \{n/2+1, \dots, n\}$ of size k, for some $k \leq n/2^{s(n)}$, there are in M at least $s(n)$ links between S and T, then $|M| = \Omega(ns(n))$. These statements can be used to prove lower bounds for various branching programs, like the $\Omega(n \log n)$ lower bound for the length of bounded width branching programs. Using these same statements they also prove several sharp results for the length of R-way input oblivious branching programs, where the assigned value of b_i can be $0, 1, 2, \dots, R-1$. The Ramsey theory applications of Alon and Maass have the advantage that the derived lower bounds are optimal or close to optimal and they do not require large input numbers.

10 Automated theorem proving

The idea of automated theorem proving goes back to at least Hilbert's decision problem. He asked if there exists an algorithm to decide whether a statement in mathematics is true or not, or an algorithm which would find a proof of any mathematical statement that has a proof. Turing proved that there is no such algorithm in general. Ramsey's theorem was actually a lemma [224] to a theorem that shows that in a certain special class of first order logic the statements are decidable. Interestingly, using an extension of the finite Ramsey theorem, Paris and Harrington [208] gave the first natural statement unprovable in finite set theory, or equivalently in Peano Arithmetic. Gödel's incompleteness theorem implies the existence of such statements. The proof is using techniques of mathematical logic. A combinatorial proof was presented by Ketonen and Solovay [169], then another shorter one by Loebl and Nešetřil [181]. See more on this and on some more recent related results in [137, 180, 200].

In computer science automated proof theory is a growing field, see for example automated theorem proving in support of symbolic definite integration by Adams, Gottliebsen, Linton and Martin [1]. There are also many mathematicians in different areas who are using computer to do part of proofs. For example "A=B" of Petkovsek, Wilf and Zeilberger [212] is about identities in general, with emphasis on computer methods of discovery and proofs. Automatic theorem proving in plane geometry is related to realizability of matroids or oriented matroids. The interactive geometry software Cinderella of Kortenkamp and Richter-Gebert [225] is a good illustration. More generally the computer has been helpful, for example, in formulating conjectures, producing constructions used for lower bounds or for counterexamples, in enumerations, solving recursive relations, or finding Gröebner basis.

Robertson and Zielberger solve some Ramsey theory problems using computer. As mentioned in the number theory section, they have found (asymptotically) the smallest number of monochromatic Schur triples [229], by experimenting with Maple and by using some optimization ideas. They reformulate the problem as a discrete calculus optimization problem to find the minimal value, over the n-dimensional (discrete) unit cube $\{(x_1, \ldots, x_n) | x_i = 0, 1\}$, of

$F(x_1, \ldots, x_n) := \sum_{\substack{1 \le i < j \le n \\ i+j \le n}} [\ x_i x_j x_{i+j} + (1-x_i)(1-x_j)(1-x_{i+j})\ ]$. They determine all local minima with respect to the Hamming metric, then determine the global minimum. A computer program [228] was used to help determine the exact values of the generalized Schur numbers, called also Issai numbers. Robertson [227] (1999) also wrote programs to generate new lower bounds for classical multicolored small Ramsey numbers. He could obtain previously known lower bounds for 2 colors and found new ones with 2 or more colors.

Previously others have used computer to find Ramsey graphs. Often the methods are heuristics finding local optimums for non-linear optimization problems or approximation heuristics for combinatorial optimization. Piwakowski (1996)[216] used an adaptation of some heuristic tabu search algorithm for finding Ramsey graphs. As a result, seven new lower bounds for classical Ramsey numbers were established: $r(K_3, K_{13}) \ge 59$, $r(K_4, K_{10}) \ge 80$, $r(K_4, K_{11}) \ge 96$, $r(K_4, K_{12}) \ge 106$, $r(K_4, K_{13}) \ge 118$, $r(K_4, K_{14}) \ge 129$, and $r(K_5, K_8) \ge 95$. Similarly Exoo (1998)[109] obtains several lower bounds for classical Ramsey numbers by using well known approximation heuristics for combinatorial optimization problems, like simulated annealing, genetic algorithms, tabu search etc. But he considers that the most important is to choose the right objective function. Brandt, Brinkmann and Harmuth [63] determine the Ramsey numbers $r(K_3, G)$ for all 261080 connected graphs of order 9, and further Ramsey numbers of this type for some graphs of order up to 12, based on a program for generating maximal triangle-free graphs.

McKay and Radziszowski [187, 188] also used optimization techniques to find the classical Ramsey number $r(K_4, K_5) = 25$, and applied the same method to similar small Ramsey number computations. They used large linear and integer programming, starting with standard floating point LP codes, like LINDO, to find approximate solution, then this is rounded to a rational solution, which is verified with a separate algorithm. This approach seems to be useful to find integer optima, and the algorithms and methods can be applied more generally in searches for other Ramsey graphs or for other difficult combinatorial configurations. First they obtained the following new upper bounds for some classical Ramsey numbers, namely $r(K_4, K_5) \le 27$, $r(K_5, K_5) \le 52$, and $r(K_4, K_6) \le 43$. Then they could verify that $r(K_4, K_5) = 25$, $r(K_5, K_5) \le 49$, and $r(K_4, K_6) \le 41$, by improving one of the algorithms. Isomorphism checking or other algebraic tools are also helpful. McKay [186] developed a program called Nauty, to determine the automorphism group of a graph. This can be used as a tool to detect isomorphism amongst large families of graphs. McKay and Radziszowski [189] consider for each vertex v of a graph G the number of subgraphs of each isomorphism class that lie in the neighborhood or complementary neighborhood of v. These numbers, summed over v, are shown to satisfy a series of identities. Using these, it is proved that $r(K_5, K_5) \leq 49$ and $r(K_4, K_6) \leq 41$. They also give some experimental evidence to support their conjecture that $r(K_5, K_5) = 43$. Using the previously mentioned techniques, Radziszowski and Tse [222] has computed $r(C_4, K_7) = 22$, $r(C_4, K_8) = 26$ and related bounds. Hopefully the results obtained will be helpful in solving one of the most interesting open questions in graph Ramsey theory, the conjecture [51, 112]: $r(C_m, K_n) = (m-1)(n-1)+1$ if $m \geq n$, except $m = n = 3$.

Pikhurko [213] (2001) uses a new interesting formulation of size Ramsey numbers as the minimum of mixed integer programs (MIP). He can solve these MIP for certain complete bipartite graphs. The *size Ramsey number* $\hat{r}(G_1, \ldots, G_t)$ is the minimum number of edges a graph G can have such that, for any t coloring of its edges there is a monochromatic copy of G_i, in the i-th color for some $1 \leq i \leq t$. He proves, among other results, a conjecture of Faudree, Rousseau and Scheehan (1983)[242] that $\hat{r}(K_{2,n}, K_{2,n}) = 18n - 15$, and answers a question of Erdős, Faudree, Rousseau and Schelp [99] about the asymptotics of $\hat{r}(K_{s,n}, K_{s,n})$ for fixed s and large n. Pikhurko made a C program using a free linear programming software, lp_solve maintained by Berkelaar [42], that uses floating point arithmetics. Avis rewrote Pikhurko's program to be linked with his exact arithmetic LP code, contained in the vertex enumeration package, lrslib [21]. One of the advantages of using exact arithmetic computation is that it is much easier to prove the correctness of a result. Pikhurko remarks that his MIP is not well suited to solve other non-trivial cases, but he hopes that his method with some relaxation will give eventually good upper and lower bounds.

11 Approximation algorithms

The problem of finding an independent set of maximum size or computing $\alpha(G)$, the independence number of a graph G, is one of the earliest problem shown to be NP-hard. Boppana and Halldórsson (1992)[52, 54], have a famous polynomial time approximation algorithm, based on Ramsey theory, that finds an independent set of a guaranteed but not necessarily optimal size. The well-known upperbound for off-diagonal Ramsey numbers, stated in the classical Erdős-Szekeres paper [95], gets a new algorithmic proof in [52, 54] and it forms the basis for the algorithm, called Ramsey.

The algorithm has the first non-trivial performance guarantee for this problem. The performance guarantee is the largest ratio, over all inputs, of the size of the maximum independent set to the size of the approximation found. For the independence number problem they obtain the $O(n/(\log n)^2)$ performance guarantee, where n is the number of vertices in the graph. The same approximation algorithm can be applied for the equivalent MaxClique problem, i.e. finding the maximum size clique. Applying the maximum independent set approximation algorithm, Halldórsson [147] gives an approximation algorithm with $O(n(\log\log n)^2/(\log n)^3)$ performance guarantee for the related graph coloring problem, improving on the results of Johnson, Wigderson and Berger-Rompel [154, 266, 41]. The graph coloring problem is to find an assignment of as few colors as possible to the vertices so that no adjacent vertices have the same color. Since the coloring induces a partition of the graph into independent sets, these two problems are closely related. The dual problem to graph coloring is finding a clique cover, which is a partition of the graphs into disjoint cliques.

There are other approximation algorithms for finding large independent set, for instance when there are no triangles, or the girth of the graph is large. In 1980-81 Ajtai, Komlós and Szemerédi [3] proved that $r(K_3, K_m) \le O(m^2/\log m)$. They also provided a polynomial algorithm [4], that finds an independent set of size at least $\alpha(G) > \frac{nlogd}{100d}$ in a triangle free graph G of order n with average degree d. Later Denley (1994)[86], improving on the lower bound of Monien and Speckenmeyer (1985) [195], showed that if G is r-regular of order n and has odd girth $2k+3$, then $\alpha(G) \ge n^{1-1/k}r^{1/k}$, with similar results for graphs which are not regular. These have been further simplified and improved by Shearer (1995)[241]. Boppana and Halldórson use the technique of Ajtai, Komlós and Szemerédi [3] to improve on the algorithm Ramsey. If k is fixed and the graph does not contain odd cycles of length $2k+1$ or less, then it is possible to find an independent set of size $\Omega(n^{k/(k+1)}(\log n)^{1/(k+1)})$, in polynomial time. Other subgraph excluding algorithms are presented, using graph Ramsey numbers of the type of $r(G, K_m)$, to improve the algorithm Ramsey for graphs containing large independent set. They show that among subgraph excluding algorithms the ones they present achieve the optimal asymptotic performance guarantees.

Peinado [210, 209] adapted the Boppana-Halldórsson approximation algorithm to random graphs. Finding maximal clique, or just large clique, seems to remain hard even in random graphs. Karp [165] raised the question 20 years ago and since then many attempts to find polynomial algorithm failed, therefore it is widely believed to be a hard problem. It is conjectured that *no polynomial-time algorithm exists which finds a clique of size* $(1+\epsilon)\log_2 n$ *with significant probability, for any constant* $\epsilon > 0$ [155]. One application of the assumption of hardness of finding a maximal clique or a large clique in random graphs could be in cryptography. The objective here is to find a large clique hidden (placed randomly) in a random graph. If it is hard to find a clique, and it remains hard to find a hidden clique, then it could be used in cryptography. Alon, Krivelevich and Sudakov (1998)[11] study the hardness of finding a large hidden clique in a random graph. They present an efficient (polynomial) algorithm to find almost surely, for all $k > cn^{0.5}$, for any fixed $c > 0$, a hidden fixed clique of size k in a random graph $G(n, 1/2)$. Therefore Juels and Peinado (1998)[155] are studying smaller hidden cliques which are more difficult to find. The largest clique in a random graph is very likely to be of size about $2\log_2 n$. They show that if the above conjecture is true, then when a clique of size $(1+2\epsilon)\log_2 n$ is randomly inserted in a random graph, finding a clique of size $(1+\epsilon)\log_2 n$ remains hard. They say that it would be interesting to create a public key cryptosystem based on cliques, but its practicality remains an open question.

12 Complexity, logic and games

On the borderline of mathematical logic and theoretical computer science, connections between Ramsey theory and complexity were established by several researchers. Various extensions of first-order logic were studied from the perspective of relations to complexity classes. It has been realized that finite structures are relevant for computer science and so finite model theory became an active research area. First-order logic on finite structures lacks a recursive mechanism hence extended logics were introduced, augmented with some generalized quantifiers that are meaningful in finite structures. Van der Waerden's theorem and the Folkman-Rado-Sanders theorem were used by Kolaitis and Väänänen [173] to investigate the scope and limits of generalized quantifiers in finite model theory. Rosen [230] looks at the computational complexity of definable classes of finite structures, in his paper on existential fragment of second order logic, and uses a generalization of Ramsey's theorem, first proved by Nešetřil and Rödl [201], to prove various decidability and satisfiability results.

The Extended Markup Language (XML) emerged as the likely standard for representing and exchanging data on the Web, in which data is represented as a labeled ordered tree, not as a table. Major data processing is done by robust relational database systems and there are tools offered for exporting relational data as XML, thus helping businesses to share data with their partners over the web. The XML allows users to use types, tree languages. Given a mapping of relational data into tree data, the *typechecking* problem is to check automatically whether every database is mapped to a tree of a desired output type, i.e. verifying whether the strings generated by the ordered sets of tuples satisfying a sequence of logical formulas belong to some regular language. Techniques from finite-model theory and combinatorics are applied. Typechecking is undecidable when arbitrary first-order logic formulas are allowed in the mapping. The finite Ramsey theorem is used by Alon et al. [16] to investigate whether the *typechecking* problem for XML queries with data values is decidable. They also consider the complexity of typechecking in the decidable cases.

Lately complexity theorists got interested in graph Ramsey theory as it gives natural examples for problems complete for a higher level of the polynomial hierarchy of complexity classes, thus justifying their existence. One of the usual ways to formulate Ramsey-type statements is by using arrowing notation. $F \to (G, H)$ means that every edge coloring of F with red and blue, contains either a red G or a blue H. For the decision problem *Arrowing* the inputs are the finite graphs F, G, and H, and the question is, does $F \to (G, H)$? $F \rightarrowtail (G, H)$ means that for every edge coloring of F with red and blue, F contains either a red G or a blue H as an *induced* subgraph. For the decision problem *Strong Arrowing* the inputs are the finite graphs F, G, and H and the question is, does $F \rightarrowtail (G, H)$? For example, *Arrowing* is in coNP for fixed G and H, in this case the input is F. $F \nrightarrow (G, H)$ means, that there is a coloring which does not contain a monochromatic G or H. Then this coloring can be a certificate for the NO answer, as it can be checked in polynomial time, for any fixed coloring of F, whether it contains or not a monochromatic copy of the fixed subgraphs. $F \to (P_2, K_n)$ decides whether F has a clique of size n and so it is known to be NP-hard. There are many complexity results related to Ramsey theory, mostly by Burr. He showed that the problem of determining the value of $r(G, H)$ for arbitrary G or H is NP-hard [68]. He also proved [71, 128] that *Arrowing* for arbitrary F with any fixed 3-connected graphs G and H, or when G and H are both triangles, is coNP-complete. The proof requires the construction of particular graphs [68, 72]. If G and H are not fixed, though his constructions remain effective, the output might become exponential in the input size, and thus cannot terminate in polynomial time in the input.

A problem is in $\Pi_2^p := \mathrm{co}NP^{NP}$ if there is a certificate for the NO answer that allows a verification of the correctness of the NO answer in NP-oracle (subroutine) polynomial time. Burr (1990) [71] proved that *Arrowing* lies in Π_2^p and it is coNP-hard. He conjectured [70, 71] that it is Π_2^p-complete and Schaefer [236] proved it in 2000. To avoid exponential outputs, Schaefer restricted the graph G to be a fixed tree and H to be a complete graph. He used a well known graph Ramsey theoretical result of Chvatal [81, 66, 137] that states: $r(T, K_n) = (n-1)(k-1)+1$ for every fixed tree T on k vertices. Schaefer proved, that even with this restriction, *deciding* $F \to (T, K_n)$ *is* Π_2^p*-complete for any fixed tree* T *of size at least two.* Earlier Yannakakis [128] proved that finding induced paths is NP-complete, or equivalently, $F \rightarrowtail (P_2, P_n)$ is NP-complete. Schaefer [236] also got results for *Strong Arrowing*: *deciding* $F \rightarrowtail (P_3, P_n)$ *is* Π_2^p*-complete*. These are natural examples for problems complete for a higher level of the polynomial hierarchy. Though in complexity theory this higher level was defined previously, there were not too many natural examples. A complexity class is considered justified if it has natural complete problems. For more on complexity results and open questions related to Ramsey Theory see [68, 70, 71, 67, 69, 236, 237].

Hales and Jewett (1963) [146], in their paper on regularity and positional games, applied Ramsey type arguments in game theory. Following the terminology of Rado [220], they say that $\mathcal{S}$, a collection of sets, is N*-regular* in the set X, if for any partition of X into N parts, some part has as a subset a member of $\mathcal{S}$. If $\mathcal{S}$ is n-regular in X for each integer n then $\mathcal{S}$ is called *regular* in X. It is remarked that the main Ramsey-type theorems can be stated using this terminology. Let $X^{(m)}$ denote the set of all m element subsets of X. The finite Ramsey theorem states that, given integers k, m, n, there exists an integer r such that, if $A = \{1, 2, \ldots, r\}$, then $\{B^{(k)} : B \in A^{(m)}\}$ is n-regular in $A^{(k)}$. Van der Waerden's theorem states that given integers m and n, there exists an integer p such that the set of all arithmetic progressions of length m is n-regular in $\{1, 2, \ldots, p\}$. Hales and Jewett prove several fundamental general results on regularity, and they apply them to the analysis of certain positional games. They mean by a *positional game*, a "game played by n players on a "board" (finite set) X with which is associated a collection $\mathcal{S}$ of subsets of X. The rules are that each player, in turn, claims as his own a previously unclaimed "square" (element) of X. The game proceeds either until one player has claimed every element of some $S \in \mathcal{S}$, in which case he wins, or until every element has been claimed, but no one has yet won, in which case the game is a tie." In game theory it is known that in a finite two-player perfect information game one player has a forced win or each player can force a tie. Hales and Jewett prove for example, that in a positional game involving 2 players, where $\mathcal{S}$ is 2-regular in X, the first player has a forced win. The

k^n-game is played like tick-tack-toe on a $k \times k \times \cdots \times k$ (n times) array points in n-space, the winner being the player who first chooses k points in a straight line. They prove, that if $k \geq 3^n - 1$ (k odd) or if $k \geq 2^{n+1} - 2$ (k even), then the second player can force a tie, and for each k there exists a number n_k such that the first player can force a win in the k^n-game if $n \geq n_k$.

Erdős and Selfridge (1973) [104] studied combinatorial games. Let $\{A_k\}$ be a family of sets, they define a game in which two players alternately pick elements of $S = \cup A_k$, the winner being the first to pick all elements of one of the A_k. $m^*(n)$ is the least integer so that there are $m^*(n)$ sets of cardinality n and the first player has a winning strategy. Erdős and Selfridge prove that $m^*(n) = 2^{n-1}$ and show that for smaller collection the second player can prevent the win. This improved Hales-Jewett's result for the k^n game, by showing that the second player can force a draw if $k > cn \log n$. There are combinatorial games based on graph Ramsey theory, like the achievement game, where the players take turns in coloring still uncolored edges of a fixed graph G, each player being assigned a distinct color, choosing one edge per move, and the first player that completes a monochromatic subgraph isomorphic to H wins. Erdős and Selfridge's result also implies that the Ramsey game, restricted to the case where both graphs $G = K_n$ and $H = K_k$ are complete, is a draw if $2^l > \binom{n}{k}$, where $l = \binom{k}{2} - 1$. Ramsey's theorem implies a win for the first player if $k \leq [\log_4 n]$.

Beck [34] defined positional games, as generalized tic-tac-toe-like games, for arbitrary (finite) hypergraphs and studied the connection with Ramsey theory. He also considered the Erdős-Selfridge theorem as a game-theoretic first moment method and developed a game-theoretic second moment method. He mentions strong connections between the games and the behavior of random graphs. Beck, in his papers on the foundations of positional games [30, 31, 32], develops a new quasiprobabilistic theory for these games. He considers that the algebraic methods of combinatorial game theory can be considered as exact local theory and it can be complemented by an efficient global approach, such as the quasiprobabilistic method, which evaluates loss probabilities. Even simple games are too complicated to analyze completely, but to describe the typical behavior is possible using probability theory. He also converts the probabilistic intuition to deterministic greedy algorithms. In his paper on positional games and the second moment method [35, 36] he studies the fair Maker-Breaker graph Ramsey game $MB(n; q)$. The board is K_n, the players alternately occupy one edge a move, and the Maker wants a clique K_q. Beck shows that Maker has winning strategy if $q = 2\log_2 n - 2\log_2 \log_2 n + O(1)$, which is exactly the clique number of the random graph on n vertices with edge-probability $1/2$. It follows from the Erdős-Selfridge theorem that this is best possible. Slany [245] studied in general the complexity of graph Ramsey games and proved that the achievement game and several variants are PSPACE-complete [245, 244], where PSPACE is the class of problems that can be solved using memory space bounded by a polynomial in the size of the problem description. These games are equivalent, from the complexity theory point of view, to well-known games like GO. They are also as hard as stochastic scheduling, and they could be used to study competitive situations. Other studies on winning strategies can be found for example in [107, 145, 172, 174, 211, 29, 33, 37], see also the survey on combinatorial games in general by Guy [144], the books on games by Berlekamp, Conway and Guy [43, 44] and the dynamic survey of Fraenkel [119].

Galvin and Scheepers investigate the relation of an infinite game and a Ramseyan theorem [127]. Games are also appearing in logic, in model checking or satisfiability. To verify whether a given propositional formula is true under a given valuation, the solution can be formulated in terms of *model-checking game* between two players. To answer satisfiability questions, *satisfiability-checking games* are considered. Chandra, Furst and Lipton (1983) [75] introduced a model of multiparty communication complexity that also uses game terminology. Let $X = X_1 \times X_2 \times \cdots \times X_k$, where the X_i's are n-element sets, and k players $P_1, \ldots, P_k$ collaborate to compute a function $f : X \to \{0,1\}$ on every input $x \in X$. Each participant P_i knows the values of all inputs except x_i. The players exchange bits, by writing the bit 0 or 1 on a "board" according to a previously agreed upon protocol. The protocol specifies whose turn is to write a bit and what this bit should be according to a function of the communication history and the input the player has access to. The communication complexity of a k-party game for f is the minimal number of bits needed to communicate to compute f on the worst-case input. Chandra et al. gave the first applications of Ramsey-type arguments to prove bounds on the multiparty communication complexity, followed by Pudlak using Hindman's theorem [219] and Tesson [260] applying Hales-Jewett's theorem [146].

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