Short and Easy Computer Proofs of the Rogers-Ramanujan Identities and of Identities of Similar Type

Peter Paule*
Research Institute for Symbolic Computation
Johannes Kepler University Linz
A–4040 Linz, Austria
Peter.Paule@risc.uni-linz.ac.at

Submitted: July 15, 1994; Accepted: July 26, 1994.

Abstract

New short and easy computer proofs of finite versions of the Rogers-Ramanujan identities and of similar type are given. These include a very short proof of the first Rogers-Ramanujan identity that was missed by computers, and a new proof of the well-known quintuple product identity by creative telescoping.

AMS Subject Classification. 05A19; secondary 11B65, 05A17

1 Introduction

The celebrated Rogers-Ramanujan identities stated as series-product identities are

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2+ak}}{(1-q)(1-q^2)\cdots(1-q^k)}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+a+1})(1-q^{5j-a+4})} \tag{1}$$

where $a=0$ or $a=1$, see e.g. Andrews [6] which also contains a brief historical account.

It is well-known that number theoretic identities like these, or of similar type, can be deduced as limiting cases of q-hypergeometric finite-sum identities. Due to recent algorithmic breakthroughs, see for instance Zeilberger [24], or, Wilf and Zeilberger [23], *proving* these finite versions becomes more and more routine work that can be left to the computer.

For instance, the computers Ekhad and Tre [12] delivered a purely verification proof of the following finite version, which was stated first in this form by Andrews [3], see also Bressoud [10].

$$\sum_k \frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_k \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}, \tag{2}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geq 1$, $(a;q)_0=1$, and $1/(q;q)_n=0$ for $n<0$. In the limit $n\to\infty$ one gets identity (1) with $a=0$, after applying Jacobi's triple product identity, see section 5.1.

It is evident that the hard part of deducing, for instance, the Rogers-Ramanujan identities as a limiting case of a finite version consists in *finding* such a representation. On the other hand, it is to expect that automated *proving* will enlarge the heuristic toolbox in a significant way.

Nevertheless, in this paper we do not intend to enter the recent discussion on the rôle of computer proofs, which was stimulated by Zeilberger [27], see also Andrews' rebuttal [9]. The objective of this paper is to show that for classical Rogers-Ramanujan type identities, computers

*Supported in part by grant P7220 of the Austrian FWF

are able to produce "nice" proofs, i.e. not kind of Byzantine computer output a human usually is unwilling, or even unable, to verify. In the particular situations of this article it turns out that even the computer-constructed proof certificates are amenable to human verification.

The paper is organized as follows. In section 2 we are going to specify our use of the notion "computer proof". A computer proof of a q-hypergeometric identity will be understood as a proof by qWZ-certification, based on the method of creative telescoping.

In section 3 we present a new "very short" qWZ-certification proof of (2) that was missed by the computers Ekhad and Tre [12].

In section 4 we provide the qWZ-certificate of another finite Rogers-Ramanujan version due to Andrews [2] and which originates in the work of Schur [21], who independently discovered (1). As briefly indicated in section 4, this finite version plays an outstanding rôle for certain applications in physics.

Besides the versions given in Andrews [8], all known finite Rogers-Ramanujan type identities deliver the product-side only after applying expansion formulas like the triple or the quintuple product identities. In section 5, for sake of completeness, qWZ-certificates of corresponding finite versions are presented.

In section 6 we briefly comment on the newly developed computer program we used to find the proof certificates in this paper.

2 qWZ-certification

Analogous to the algorithm presented by Zeilberger [25], the computer program we used takes terminating q-hypergeometric sums as input. Its output is a linear recurrence that is satisfied by the input sum, together with a rational function serving as a proof certificate. As made explicit in this section, verifying the recurrence is equivalent to checking a rational function identity.

Let $f := (f_{n,k})$ be a double-indexed sequence with values in a suitable (e.g. computable) ground-field F containing the rational number field and q. In the following we consider q as an indeterminate, which could be specialized to a nonzero complex number (with $|q| < 1$ for limit considerations). We shall consider only sequences where n runs through the nonnegative integers, whereas the second parameter k might run through all integers.

The sequence f is called q-hypergeometric in both parameters if both quotients

$$\frac{f_{n,k}}{f_{n-1,k}} \quad \text{and} \quad \frac{f_{n,k}}{f_{n,k-1}}$$

are rational functions in q^n and q^k over F for all n and k where the quotients are well-defined. For instance, if $[l] := (1-q^l)/(1-q)$ for integer l, and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \displaystyle\prod_{l=1}^{k} \frac{[n-l+1]}{[l]} & \text{, if } 0 \le k \le n \\ 0 & \text{, otherwise} \end{cases}$$

are the Gaussian polynomials, then

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \Big/ \begin{bmatrix} n \\ k+1 \end{bmatrix} = \frac{1-q^{n+1}}{1-q^{n-k}} \quad \text{and} \quad \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \Big/ \begin{bmatrix} n+1 \\ k \end{bmatrix} = \frac{1-q^{n-k+1}}{1-q^{k+1}}.$$

For $q = 1$ (resp. $q \to 1$) the Gaussian polynomial $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ turns into $\binom{n}{k}$, the ordinary binomial coefficient.

We say the sequence f has finite support with respect to k, if the following is true for each n fixed: $f_{n,k} \neq 0$ for all k from a finite integer interval I_n, and $f_{n,k} = 0$ for all k outside I_n. A simple example for such a sequence is provided by $f_{n,k} := \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ where $I_n = \{0, 1, \ldots, n\}$.

Given a sequence $(f_{n,k})$ being q-hypergeometric in both parameters and having finite support with respect to k. For nonnegative integer n define $S_n := \sum_k f_{n,k}$, a finite sum due to finite support property. We use the convention that the summation parameter k runs through all the integers, in case the summation range is not specified explicitly.

Alternatively, the sequence $S := (S_n)$ can be represented in terms of a recursion of some order d. For instance, as the unique solution of the recurrence

$$c_0(n)S_n + c_1(n)S_{n-1} + \cdots + c_d(n)S_{n-d} = 0 \tag{3}$$

for $n \geq d$, with certain coefficients $c_i(n)$, $c_0(n) \neq 0$, and with respect to the initial values $S_0, S_1, \ldots, S_{d-1}$. Equivalently, the annihilating operator

$$A_S(N) := \sum_{l=0}^{d} c_l(n)N^{-l}, \tag{4}$$

where N denotes the shift-operator with respect to n, together with the d initial values represent S in a unique way.

Let f and S be as above, then the qWZ-certificate of the annihilating operator is by definition a rational function $cert_S(n,k)$, rational in q^n and q^k, such that

$$A_S(N)f_{n,k} = g_{n,k} - g_{n,k-1} \tag{5}$$

where

$$g_{n,k} := cert_S(n,k)\ f_{n,k}. \tag{6}$$

We shall use this type of certification following Wilf's and Zeilberger's paradigm of "creative telescoping", see e.g [26], [22] or [23]. Namely, given S in sum representation as above, the qWZ-certificate contains all information necessary to prove that S satisfies the homogeneous recursion described by the operator $A_S(N)$. This is true, because summing (5) over sufficiently many k, i.e. over an integer interval containing the finite support such that the bounds evaluate to zero, immediately implies (3). Thus what is left is to verify (5). But this is equivalent to checking a *rational function identity*, i.e.

$$r(n,k) = cert_S(n,k) - cert_S(n,k-1)\frac{f_{n,k-1}}{f_{n,k}} \tag{7}$$

where $r(n,k)$, rational in q^n and q^k, comes from

$$A_S(N)f_{n,k} = r(n,k)\ f_{n,k}.$$

The actual computation of $r(n,k)$ is straightforward from rewriting the N^{-l} actions as rational function multiples of $f_{n,k}$, i.e.

$$N^{-1}f_{n,k} = \frac{f_{n-1,k}}{f_{n,k}}f_{n,k}, \quad N^{-2}f_{n,k} = \frac{f_{n-2,k}}{f_{n-1,k}}\frac{f_{n-1,k}}{f_{n,k}}f_{n,k}, \text{ a.s.o.}$$

3 A Computer Proof that was Missed by Computers

Ekhad and Tre [12] gave a qWZ-certification proof of (2) by showing that both sides are solutions of a certain *fifth* order linear recurrence equation. More precisely, they showed that the left hand side is annihilated by a recurrence operator of order two, whereas the right hand side by a fifth order multiple of that operator. What Ekhad and Tre missed observing is that *both* sides of a slight variation of the original form of identity (2) satisfy indeed the same recurrence of order *two*. This variation makes use of the summand's symmetry. It consists in introducing an extra factor $1+q^k$ in the summand of the right hand side, which, because of symmetry in k and $-k$, amounts to the same as multiplying the sum by 2. This is made more transparent if we look at the decomposition of the summand $f(k)$ into its even and its odd part, in the usual way,

$$f(k) = f_e(k) + f_o(k) = \frac{f(k)+f(-k)}{2} + \frac{f(k)-f(-k)}{2}.$$

Summing over all integers the sum of the odd parts obviously vanishes and it is true that

$$\sum_k f(k) = \sum_k f_e(k) = \sum_k \frac{f(k)+f(-k)}{2}.$$

Theorem 1 *For nonnegative integers n:*

$$\sum_k \frac{2q^{k^2}}{(q;q)_k(q;q)_{n-k}} = \sum_k \frac{(-1)^k(1+q^k)q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}. \tag{8}$$

In addition, both sides are annihilated by

$$A(N) := (1-q^n)N^0 - (1+q-q^n+q^{2n-1})N^{-1} + qN^{-2}. \tag{9}$$

Proof: Denote the left hand side of (8) by $L(n)$, the right hand side by $R(n)$. Since the initial values coincide,

$$L(0) = R(0) = 2 \quad \text{and} \quad L(1) = R(1) = 2\frac{1+q}{1-q}, \tag{10}$$

it suffices to prove that both sides are annihilated by $A(N)$. But this follows, as explained in section 2, from the corresponding qWZ-certificates:

$$cert_L(n,k) = -q^{2n-1}(1-q^{n-k}) \tag{11}$$

and

$$cert_R(n,k) = \frac{q^{2n+3k}}{1+q^k}(1-q^{n-k}). \tag{12}$$

Remark: We want to point out that all computational steps needed to verify the proof are indeed of "human" size, i.e. they can be carried out by hand without too much pain in a reasonable amount of time. (The verification of this statement is left to the reader.)

From a q-hypergeometric point of view (2) is a terminating special instance of Watson's q-analogue of Whipple's theorem, see e.g. Gasper and Rahman [13], (III.18). The successful variation, i.e. the idea of introducing the extra factor $1+q^k$, stems from the (human) author's study, in interaction with the computer, of various terminating cases of Watson's q-analogue.

The introduction of the extra factor $1+q^k$ or, more generally, summing the even part turns out to be crucial for boiling down qWZ-certificates of many more finite identities of similar type. More precisely, in these instances the telescoping equation (5) finds a solution for an annihilating operator of smaller degree than for the original summand.

For example:

- Without introducing the extra symmetry factor $1+q^{-k}$, the right hand side of (1.1) in Bressoud [11], another finite version of (1, $a=0$), has minimal recursion order 5, instead of 2, with respect to qWZ-certification. Analogous observations hold for all other finite versions of Rogers-Ramanujan type identities presented in [11].
- Without introducing the extra symmetry factor $1+q^{2k}$, the left hand side of (55) in Paule [17], a finite version of a Goellnitz-Gordon identity, has minimal recursion order 4, instead of 2, with respect to qWZ-certification.
- Without introducing the extra symmetry factor $1+q^k$, the left hand side of (48) in Paule [17], a finite version of a Rogers-Selberg identity, has minimal recursion order 7 (!), instead of 3, with respect to qWZ-certification.

4 The Andrews-Schur Identity

As observed by Andrews [2], the following *polynomial* identity for $a=0$ and $a=1$ immediately implies (1) by taking $n\to\infty$ and using Jacobi's triple product identity (see the next section).

$$\sum_{k=0}^{n} q^{k^2+ak} \begin{bmatrix} 2n-k+a \\ k \end{bmatrix} = \sum_{k=-\infty}^{\infty} q^{10k^2+(4a-1)k} \begin{bmatrix} 2n+2a+2 \\ n-5k \end{bmatrix} \frac{[10k+2a+2]}{[2n+2a+2]}. \tag{13}$$

(This corresponds to $\alpha = 0$ and $n \to 2n-1$, or, $\alpha = -1$ and $n \to 2n$ in the theorem of [2], cf. also (3.10) and (3.12) in Andrews [8].)

As pointed out by Andrews [2], this finite version of (1) has its origin in the work of Schur [21]. The importance of (13) comes from the fact that it describes $q \to q^{-1}$ duality of regimes of R.J. Baxter's solution to the hard hexagon model of statistical mechanics, see e.g. Andrews [5] or [7].

With respect to creative telescoping it turns out that the following reciprocal variant of (13) finds nicer qWZ-certificates than the identity in its original form.

Theorem 2 *For $a = 0$ or $a = 1$:*

$$\sum_{k=-\lfloor a/2 \rfloor}^{n} q^{k^2+2ak} \begin{bmatrix} n+k+a \\ n-k \end{bmatrix} = \sum_{k=-\lfloor (n+2a+2)/5 \rfloor}^{\lfloor n/5 \rfloor} q^{15k^2+(6a+1)k} \begin{bmatrix} 2n+2a+2 \\ n-5k \end{bmatrix} \frac{[10k+2a+2]}{[2n+2a+2]}. \tag{14}$$

Proof: Denote the left hand side of (14) by $L^{(a)}(n)$, the right hand side by $R^{(a)}(n)$. In view of

$$L^{(a)}(0) = R^{(a)}(0) = 1 \quad \text{and} \quad L^{(a)}(1) = R^{(a)}(1) = [a+1] + q^{2a+1} \tag{15}$$

it suffices to prove that both sides are annihilated by

$$A^{(a)}(N) := N^0 - (1 + q + q^{2n+2a-1})N^{-1} + qN^{-2}. \tag{16}$$

But this follows, as explained in section 2, from the corresponding qWZ-certificates:

$$cert_{L^{(a)}}(n,k) = -q^{2n+2a-1} \frac{[n-k]}{[n+k+a]} \tag{17}$$

and

$$cert_{R^{(a)}}(n,k) = -q^{2n+20k+6(a+1)} \frac{[n-5k-4][n-5k-3][n-5k-2][n-5k-1][n-5k]}{[10k+2a+2][2n+2a-2][2n+2a-1][2n+2a][2n+2a+1]}. \tag{18}$$

Remark: (i) It is interesting to note that both sides of (14) are annihilated by the same operator (16) *for all* nonnegative integers a. But only for the specific choices $a = 0$ or $a = 1$ the initial values of the sums match.

(ii) For combinatorial interpretations of the finite sums arising in (13) and (14) see, for instance, Agarwal and Andrews [1].

Identity (14) is the reciprocal dual of (13). This can be seen, for instance, as follows. On the left hand side of (13) reverse the order of summation, i.e. $k \to n-k$. The rest follows immediately by $q \to q^{-1}$ using the well-known fact that Gaussian polynomials are reciprocal, i.e.

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q \to q^{-1}} = q^{m(m-n)} \begin{bmatrix} n \\ m \end{bmatrix}.$$

In concluding this section we note that, in contrast to other finite versions of (1), the Andrews-Schur identity finds a nontrivial specialization for $q = 1$. Namely, for $q = 1$ both identities (13) and (14) turn into the binomial identity ($a = 0$ or $a = 1$)

$$\sum_{k=-\lfloor a/2 \rfloor}^{n} \binom{n+k+a}{n-k} = \sum_{k=-\lfloor (n+2a+2)/5 \rfloor}^{\lfloor n/5 \rfloor} \binom{2n+2a+2}{n-5k} \frac{5k+a+1}{n+a+1}. \tag{19}$$

For $a=0$ (resp. $a=1$) this corresponds to two different binomial sum representations of the even (resp. odd) Fibonacci numbers. That can be seen, for instance, directly from the defining recursion (16) for $q=1$, and the first two initial values. For integer $a>1$ identity (19) results in nice Fibonacci relations which group together mod 5. More precisely, let $l(n,a)$ (resp. $r(n,a)$) denote the left (resp. right) side of (19) and let $(F(k))_{k\geq 0}$ be the Fibonacci sequence with $F(0)=F(1)=1$. Then for all nonnegative integers a,m,n we have $l(n,a)=F(2n+a)$ and $r(n,5m)=F(10m+2n)$, $r(n,5m+1)=F(10m+2n+1)$, $r(n,5m+2)=-F(10m+2n+3)$, $r(n,5m+3)=-F(10m+2n+6)$, and $r(n,5m+4)=0$.

5 Triple and Quintuple Product Expansions

Besides those ones given in Andrews [8], all known finite Rogers-Ramanujan type identities deliver the product-side only after applying expansion formulas like the triple or the quintuple product identities. For sake of completeness, we present qWZ-certificates of corresponding finite versions. Version (27) for the quintuple product identity seems to be stated the first time in this form.

5.1 Jacobi's Triple Product Identity

It is well-known, see e.g. Andrews [6], that Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} x^k = \prod_{j=1}^{\infty}(1-q^j)(1+x^{-1}q^j)(1+xq^{j-1}) \tag{20}$$

can be deduced, for instance, as a limiting case $n\to\infty$ of the following finite variant of the q-binomial theorem.

Theorem 3 *For nonnegative integers n and $x\neq 0$:*

$$\sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\binom{k}{2}} x^k = (-x^{-1}q;q)_n(-x;q)_n. \tag{21}$$

In addition, both sides are annihilated by

$$A(N) = N^0 - (1+x^{-1}q^n)(1+xq^{n-1})N^{-1}. \tag{22}$$

Proof: Denote the sum by $L(n)$. It suffices to prove that both sides of (21) evaluate to 1 for $n=0$, and are annihilated by $A(N)$. The nontrivial part follows, as explained in section 2, from the corresponding qWZ-certificate:

$$cert_L(n,k) = q^{n-1}\frac{[n-k](q^{k+1}[n-k-1]-x[n+k])}{[2n-1][2n]}. \tag{23}$$

Application: It is easy to see that in the limit $n\to\infty$ the left hand side of (13) tends to the left hand side of (1), whereas the right hand side tends to

$$\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty} q^{10k^2+(4a-1)k}(1-q^{10k+2a+2}), \tag{24}$$

where $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

This can be rewritten as

$$\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^k q^{(5k^2+(4a-1)k)/2}, \tag{25}$$

which turns into the product side of (1) after applying (20) with $q\to q^5$ and $x\to -q^{2a+2}$.

5.2 The Quintuple Product Identity

A standard form of the quintuple identity reads as ($z \neq 0$):

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2} z^{3k}(1+zq^k) = \prod_{j=1}^{\infty}(1-q^j)(1+z^{-1}q^j)(1+zq^{j-1})(1+z^{-2}q^{2j-1})(1+z^2q^{2j-1}), \tag{26}$$

see, for instance, Gasper and Rahman [13] (Ex. 5.6, together with the references).

Again we prove (26) as a limiting case.

Theorem 4 *For nonnegative integers n and $w \neq 0$:*

$$\sum_k (-1)^k q^{(3k^2+k)/2} w^{3k} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \frac{1-w^2q^{2k+1}}{(w^{-2};q)_{n-k}(w^2q^2;q)_{n+k}} = \frac{1-w^2q}{(w^{-1};q)_n(wq;q)_n}. \tag{27}$$

In addition, both sides are annihilated by

$$A(N) = N^0 - (1-w^{-1}q^{n-1})^{-1}(1-wq^n)^{-1}N^{-1}. \tag{28}$$

Proof: Denote the sum by $L(n)$, which is defined over a q-hypergeometric summand sequence having finite support with respect to k. It suffices to prove that both sides of (27) evaluate to $1-w^2q$ for $n=0$, and are annihilated by $A(N)$. The nontrivial part follows, as explained in section 2, from the corresponding qWZ-certificate:

$$\begin{aligned} cert_L(n,k) &= wq^{n+2k}\frac{(1-w^{-2}q^{n-k-1})(1-wq^{k+1})}{(1-w^{-1}q^{n-1})(1-w^2q^{2k+1})(1-wq^n)} \times \\ &\quad \frac{[n-k]([n-k-1]-wq^n[2n-1]+w^2q[n+k])}{[2n][2n-1]}. \end{aligned} \tag{29}$$

For $n \to \infty$ in (27) we get the identity, first stated by G.N. Watson,

$$\sum_k (-1)^k q^{(3k^2+k)/2} w^{3k}(1-w^2q^{2k+1}) = \frac{(q;q)_\infty(w^{-2};q)_\infty(w^2q;q)_\infty}{(w^{-1};q)_\infty(wq;q)_\infty}, \tag{30}$$

which is easily seen to be equivalent to (26).

Remark: Identity (27) is a a terminating version of a very-well-poised ${}_6\psi_6$ identity due to W.N. Bailey. We derived it following Andrews' proof of (26) given in [4].

Application: Taking the limit $n \to \infty$ in (26) and applying the quintuple product identity to the right hand side series

$$\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty} q^{15k^2+(6a+1)k}(1-q^{10k+2a+2}), \tag{31}$$

the reciprocal Andrews-Schur identity (14) becomes ($a=0$ or $a=1$)

$$\sum_{k=0}^{\infty} \frac{q^{k^2+2ak}}{(q;q)_{2k+a}} = \prod_{j=0}^{\infty}\frac{1}{(1-q^{2j+1})(1-q^{20j+4a+4})(1-q^{20j-4a+16})}. \tag{32}$$

These are identities due to L.J. Rogers, see e.g. Agarwal and Andrews [1].

6 Conclusion

All qWZ-certificates presented in this paper have been found by a computer program which is a q-analogue of Zeilberger's "fast" algorithm for proving terminating hypergeometric identities, see Zeilberger [25]. The program is written in Mathematica by Riese [20]. The algorithmic backbone of the program is a q-analogue of Gosper's algorithm [14], see also Graham, Knuth and Patashnik [15], based on recent work of the author [18]. After completion the program will be available upon request.

Riese's package is not the first implementation of a q-analogue of Zeilberger's "fast" algorithm. There are Maple programs written by Zeilberger, cf. [12], and Koornwinder [16]. In [16], one can also find a rigorous description of the algorithmic background of Koornwinder's implementation.

We want to conclude with a short example concerning the performance of Riese's program. With respect to Watson's q-analogue of Whipple's theorem, see e.g. Gasper and Rahman [13] (III.18), Ekhad and Tre [12] remarked that their memories did not suffice for coming up with a qWZ-certification proof. With Riese's program one needs approximately 10 minutes on an Apollo 4500 workstation for obtaining the certificates (of order 2) of both sides. The qWZ-certificates presented in this paper are obtained in about 60 seconds or less. A more detailed discussion of algorithmic background, performance, and examples will be given in paper by the author and Riese [19].

Acknowledgment: The author thanks Volker Strehl, Herb Wilf and Doron Zeilberger for valuable suggestions and comments.

References

[1] A.K. Agarwal and G.E. Andrews, *Hook differences and lattice paths*, J. Stat. Planning and Inference, **14** (1986), 5–14.

[2] G.E. Andrews, *A polynomial identity which implies the Rogers-Ramanujan Identities*, Scripta. Math., 28 (1970), 297–305.

[3] G.E. Andrews, *Problem 74-12*, SIAM Rev., **16** (1974), 390.

[4] G.E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev., **16** (1974), 441-484.

[5] G.E. Andrews, *The hard-hexagon model and Rogers-Ramanujan type identities*, Proc. Natl. Acad. Sci. USA, **78** (1981), 5290–5292.

[6] G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications Vol.2, G.-C. Rota ed., Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, London and New York, 1985).

[7] G.E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series, No. 66, American. Math. Soc., Providence, 1986.

[8] G.E. Andrews, *The Rogers-Ramanujan identities without Jacobi's triple product*, Rocky Mountain J. Math., **17** (1987), 659–672.

[9] G.E. Andrews, *The death of proof? Semi-rigorous mathematics? You've got to be kidding!*, Math. Intelligencer, to appear.

[10] D.M. Bressoud, *Solution of Problem 74-12*, SIAM Rev., **23** (1981), 101–104.

[11] D.M. Bressoud, *Some identities for terminating q-series*, Math. Proc. Camb. Phil. Soc., **89** (1981), 211–223.

[12] S.B. Ekhad and S. Tre, *A purely verification proof of the first Rogers-Ramanujan identity*, J. Comb. Th. (A), **54** (1990), 309–311.

[13] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications Vol.35, G.-C. Rota ed., Cambridge University Press, London and New York, 1990.

[14] R.W. Gosper, *Decision procedures for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA., **75**, 40–42.

[15] R. Graham, D. Knuth and O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science.* Addison-Wesley, 1989.

[16] T.H. Koornwinder, *On Zeilberger's Algorithm and its q-analogue*, J. of Comput. and Appl. Math., **48** (1993), 91–111.

[17] P. Paule, *On identities of the Rogers-Ramanujan type*, J. Math. Anal. Appl., **107** (1985), 255–284.

[18] P. Paule, *Greatest-Factorial Factorization and Symbolic Summation I*, RISC-Linz Report Series No. 93-02 (submitted to J. Symb. Computation).

[19] P. Paule and A. Riese, *A Mathematica version of Zeilberger's Algorithm for proving terminating q-hypergeometric series identities*, in preparation.

[20] A. Riese, *A Mathematica q-analogue of Zeilberger's algorithm for proving q-hypergeometric identities*, diploma thesis, in preparation.

[21] I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math.Kl. (1917) 302–321. (Reprinted: Gesammelte Abhandlungen, Vol.2, Springer, Berlin (1973), 117–136.

[22] H.S. Wilf and D. Zeilberger, *Rational function certification of multisum/integral/'q'identities*, Bull. Amer. Math. Soc. (N.S.), **27** (1992), 148–153.

[23] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math., **108** (1992), 575–633.

[24] D. Zeilberger, *A holonomic systems approach to special function identities*, J. Comp. Appl. Math., **32** (1990), 321–368.

[25] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discr. Math., **80** (1990), 207–211.

[26] D. Zeilberger, *The method of creative telescoping*, J. Symb. Computation, **11** (1991), 195–204.

[27] D. Zeilberger, *Theorems for a price: tomorrow's semi-rigorous mathematical culture*, Notices of the A.M.S., **40** (1993), 978–981.