# On hypergraphs with every four points spanning at most two triples

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Submitted: Jan 10, 2003; Accepted: Aug 25, 2003; Published: Sep 8, 2003

MR Subject Classifications: 05C35, 05C65, 05D05
Keywords: Hypergraph Turán numbers

**Abstract**

Let $\mathcal{F}$ be a triple system on an $n$ element set. Suppose that $\mathcal{F}$ contains more than $(1/3-\epsilon)\binom{n}{3}$ triples, where $\epsilon > 10^{-6}$ is explicitly defined and $n$ is sufficiently large. Then there is a set of four points containing at least three triples of $\mathcal{F}$. This improves previous bounds of de Caen [1] and Matthias [7] .

Given an $r$-graph $\mathcal{F}$, the Turán number $\mathrm{ex}(n,\mathcal{F})$ is the maximum number of edges in an $n$ vertex $r$-graph containing no member of $\mathcal{F}$. The Turán density $\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{r}}$. When $\pi(\mathcal{F}) \neq 0$, and $r > 2$, determining $\pi(\mathcal{F})$ is a notoriously hard problem, even for very simple $r$-graphs $\mathcal{F}$ (see [5] for a survey of results). Determining the Turán density of complete $r$-graphs is a fundamental question about set-systems. In fact, this is not known in any nontrivial case when $r \geq 3$.

Perhaps the most well-known problem in this area is to determine $\pi(\mathcal{K})$, where $\mathcal{K}$ is the complete 3-graph on four vertices (the smallest nontrivial complete $r$-graph). It is known that $5/9 \leq \pi(\mathcal{K}) \leq (3+\sqrt{17}\ )/12 = 0.59359..$, where the lower bound is due to Turán and the recent upper bound is due to Chung and Lu [2]. However, even the Turán density of $H(4,3)$, the 3-graph on four vertices with three edges, is not known. One could argue that this problem is even more basic, since $H(4,3)$ is the smallest (in the sense of both vertices and edges) 3-graph with positive Turán density (for applications of $\pi(H(4,3))$ to computer science, see [9, 6]).

*Research supported in part by the National Science Foundation under grant DMS-9970325

The upper bound $\pi(H(4,3)) \leq 1/3$ was proved by de Caen [1], and this was improved to $1/3 - 10^{-10}$ by Matthias [7]. Frankl and Füredi [4] gave a fairly complicated recursive construction yielding $\pi(H(4,3)) \geq 2/7$. In an attempt to improve de Caen's bound, the author and Rödl [8] proved that $\pi(\{C_5, H(4,3)\}) \leq 10/31$, where $C_5$ is the 3-graph $123, 234, 345, 451, 512$.

In this note, we present a short argument that improves the best upper bound slightly.

**Theorem 1** $\pi(H(4,3)) \leq 1/3 - (0.45305 \times 10^{-5})$.

**Proof:** Let $\mathcal{H}$ be a triple system on $n$ vertices containing no copy of $H(4,3)$. Suppose that $\mathcal{H}$ has $\alpha\binom{n}{3}$ edges. We will prove that $\alpha \leq 1/3 - (0.45305 \times 10^{-5}) + o(1)$. The result then follows by taking the limit as $n \to \infty$.

Let $d_{x,y}$ denote the number of triples containing both $x$ and $y$. For $i = 1, 2$, let $q_i$ denote the number of sets of four vertices that induce exactly $i$ edges. Then

$$\alpha\binom{n}{3}(n-3) = |\mathcal{H}|(n-3) = q_1 + 2q_2 \qquad \text{and} \qquad \sum_{x,y}\binom{d_{x,y}}{2} = q_2.$$

Using these equalities, $\sum_{x,y} d_{x,y} = 3\alpha\binom{n}{3}$, and convexity of binomial coefficients, we obtain

$$\alpha\binom{n}{3}(n-3) \geq q_1 + 3\alpha^2\binom{n}{3}(n-2) - 3\alpha\binom{n}{3}. \tag{1}$$

Since $q_1 \geq 0$, dividing (1) by $n^4$ and taking the limit as $n \to \infty$ gives de Caen's bound $\alpha \leq 1/3$.

The improvement arises by proving that a positive proportion of quadruples contribute to $q_1$. By (1) this immediately lowers the bound of $1/3$. Our primary tool is a result of Frankl and Füredi [4] stating that every $m$ vertex triple system, $m \equiv 0 \pmod 6$, such that every four points span 0 or 2 edges, has at most $10(m/6)^3$ edges (their result is quite a bit stronger, but this version suffices for our purposes). We will use this on subhypergraphs of $\mathcal{H}$ to lower bound $q_1$. This technique, called supersaturation, was developed by Erdős and Simonovits [3] (although frequently used in earlier papers as well).

**Claim:** Suppose that $\delta > 0$ and $12 \leq m \equiv 0 \pmod 6$ satisfy

$$\frac{\delta(m^2 - 6m)}{18(m-1)(m-2)} + \frac{5m^2}{18(m-1)(m-2)} \leq \alpha\,. \tag{2}$$

Then at least $\delta\binom{n}{m}$ sets of $m$ vertices of $\mathcal{H}$ have greater than $10(m/6)^3$ edges.

**Proof of Claim:** Otherwise, using the precise upper bound of [1] which states that $\mathrm{ex}(m, H(4,3)) \leq (m/(3(m-2)))\binom{m}{3}$ ($< 10(m/6)^3$ for $m \geq 12$), we obtain

$$|\mathcal{H}| < \frac{\delta\binom{n}{m}\frac{m}{3(m-2)}\binom{m}{3} + (1-\delta)\binom{n}{m}10(m/6)^3}{\binom{n-3}{m-3}} \leq \alpha\binom{n}{3}.$$

This contradiction proves the Claim.

For each $m$-set $S$ to which the Claim applies, [4] implies that $S$ contains a 4-element set with precisely one edge. Consequently,

$$q_1 \geq \frac{\delta\binom{n}{m}}{\binom{n-4}{m-4}} = \frac{\delta}{\binom{m}{4}}\binom{n}{4}.$$

Using this lower bound in (1) yields

$$\alpha\binom{n}{3}(n-3) \geq \frac{\delta}{\binom{m}{4}}\binom{n}{4} + 3\alpha^2\binom{n}{3}(n-2) - 3\alpha\binom{n}{3}.$$

Dividing by $n\binom{n}{3}$ and taking the limit as $n \to \infty$ we get

$$\alpha \geq \frac{\delta}{4\binom{m}{4}} + 3\alpha^2.$$

Choose $m = 18$ and $\delta = 68\alpha/3 - 15/2$. Then (2) is satisfied (with equality) and therefore

$$\alpha \geq \frac{136\alpha}{(18)_4} - \frac{45}{(18)_4} + 3\alpha^2.$$

Solving this quadratic, we obtain $\alpha \leq 0.3333288028 = 1/3 - (0.45305 \times 10^{-5})$. $\square$

**Remarks:**

• In order to simplify the presentation, we have not optimized the constants in the proof. Moreover, the upper bound is certainly far from being sharp. The value of Theorem 1 lies only in presenting a short proof that improves the previous best upper bound for this basic problem.

• It is mentioned in [4] that Erdős and Sós made the following conjecture: if $\mathcal{H}$ is an $n$ vertex 3-graph where $N(x) = \{yz : xyz \in \mathcal{H}\}$ is bipartite for every vertex $x$, then $|\mathcal{H}| < n^3/24$. There exist triple systems $\mathcal{H}$ satisfying this property with $|\mathcal{H}| > (1/4 - o(1))\binom{n}{3}$, so Erdős and Sós' conjecture, if true, would be asymptotically sharp. Since $H(4,3)$ has a vertex $x$ where $N(x)$ is a triangle, $|\mathcal{H}| \leq \mathrm{ex}(n, H(4,3))$. As far as we know, this is the best known upper bound for $\mathcal{H}$. Thus Theorem 1 improves the upper bound for this problem as well.

**Conjecture 2** *For infinitely many $n$, the construction from [4] has the most edges among $n$ vertex triple systems with no copy of $H(4,3)$. In particular, $\pi(H(4,3)) = 2/7$.*

# Acknowledgments

I thank Z. Füredi and a referee for informing me about [7].

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