The Rank of a Cograph

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Abstract

The rank of the adjacency matrix of a graph is bounded above by the number of distinct non-zero rows of that matrix. In general, the rank is lower than this number because there may be some non-trivial linear combination of the rows equal to zero. We show the somewhat surprising result that this never occurs for the class of cographs. Therefore, the rank of a cograph is equal to the number of distinct non-zero rows of its adjacency matrix.

1 Introduction and Motivation

The *rank* of a graph X is the rank of its adjacency matrix $A(X)$. As the rank is such a fundamental algebraic concept, the relationship between the structure of a graph and its rank is a natural topic of study for algebraic graph theorists. Perhaps the most well-known of such investigations is the study of the relationship between the rank and the chromatic number of a graph (see Alon & Seymour [1]). Despite this and other efforts, there is surprisingly little that can be said about the rank of the adjacency matrix of a graph. In general, the known results are limited to calculations of the rank in special cases, or how the rank varies under certain operations (for example, see Bevis et.al [2]). In fact, considerably more is known about the rank of other matrices associated with a graph, or about the ranks of certain graphs over finite fields (for examples, see Godsil & Royle [7]).

In calculating the rank of a graph, the maximum value that it can take is the number of distinct non-zero rows of the adjacency matrix. In general, these will not be linearly independent, and the rank will be somewhat lower. While experimenting on the rank-chromatic number question by computer, Torsten Sillke [8] observed that for all the *cographs* he checked, the rank turned out to be precisely equal to the number of distinct non-zero rows of the adjacency matrix, and he conjectured that this was true for all cographs.

The purpose of the current paper is to provide a proof for this conjecture, thereby demonstrating that cographs always have "maximal rank" in this sense. The result is curious in several ways — firstly, there are so few existing results connecting graph structure and rank, and secondly because there is no clear reason why cographs should crop up in this fashion. There does not appear to be any other natural class of graphs for which an analogous result is true.

For general background regarding graphs, we recommend Diestel [6], and for the algebraic graph theory concepts and notation Godsil & Royle [7].

2 Cographs

Complement-reducible graphs, or *cographs*, are the graphs belonging to the following recursively defined family:

1. K_1 is a cograph,
2. If X is a cograph, then so is its complement $\overline{X}$, and
3. If X and Y are cographs, then so is their union $X \cup Y$.

This class of graphs has been extensively studied for many years, both from the computational viewpoint and the graph-theoretical viewpoint. Studies in the former area have led to fast algorithms for the recognition of cographs, and for the restricted versions of many standard graph-theoretic problems such as colouring, matching, isomorphism etc. Studies in the latter area have led to many different characterizations of cographs, some of which have become so well known that they are now equally often used as the definiton of cographs. The most well-known of these characterizations is probably the following:

Theorem 2.1 *A graph is a cograph if and only if it does not have the path P_4 as an induced subgraph.* ∎

The classic reference for cographs is the paper by Corneil et. al [4], and we refer the reader to the book *Graph Classes* by Brandstädt, Le and Spinrad [3] for further details and more recent references.

We will use a slightly different characterization of cographs as the foundation for later inductive proofs. This uses the concept of the *join* of two graphs X and Y, which is the graph $X \bigtriangledown Y$ obtained by joining every vertex of X to Y. It is easy to see that

$$X \bigtriangledown Y = \overline{\overline{X} \cup \overline{Y}}.$$

Lemma 2.1 *Let $\mathcal{C}$ be the class of graphs defined as follows:*

1. K_1 is in $\mathcal{C}$,

2. *If X and Y are in $\mathcal{C}$, then so is their union $X \cup Y$, and*

3. *If X and Y are in $\mathcal{C}$, then so is their join $X \bigtriangledown Y$.*

Then $\mathcal{C}$ is the class of cographs. ■

3 Spectrum of a cograph

The characteristic polynomial $\varphi(X)$ of the graph X is the characteristic polynomial of its adjacency matrix $A(X)$. The *spectrum* of X is the collection of eigenvalues of $A(X)$, or equivalently, the collection of zeros of $\varphi(X)$. The rank of X is the number of non-zero values occurring in the spectrum.

In order to study the spectra of cographs, we need to know how the characteristic polynomial behaves under the operations of union and join. The following result is from Cvetkovic, Doob and Sachs [5].

Theorem 3.1 *Let X and Y be graphs on x and y vertices respectively. The characteristic polynomial of the graph $X \cup Y$ is given by*

$$\varphi(X \cup Y) = \varphi(X)\varphi(Y),$$

and the characteristic polynomial of $X \bigtriangledown Y$ is given by

$$\begin{aligned}\varphi(X \bigtriangledown Y, \lambda) \;&=\; (-1)^y\ \varphi(X,\lambda)\ \varphi(\overline{Y}, -\lambda-1)\\ &\quad +(-1)^x\ \varphi(Y,\lambda)\ \varphi(\overline{X}, -\lambda-1)\\ &\quad -(-1)^{x+y}\ \varphi(\overline{X}, -\lambda-1)\ \varphi(\overline{Y}, -\lambda-1).\end{aligned}$$

■

We will be using this specifically for the eigenvalues 0 and -1 and so reproduce the relevant expressions in those cases:

$$\begin{aligned}\varphi(X \bigtriangledown Y, 0) \;&=\; (-1)^y\ \varphi(X,0)\ \varphi(\overline{Y}, -1)\\ &\quad +(-1)^x\ \varphi(Y,0)\ \varphi(\overline{X}, -1)\\ &\quad -(-1)^{x+y}\ \varphi(\overline{X}, -1)\ \varphi(\overline{Y}, -1).\end{aligned}$$

$$\begin{aligned}\varphi(X \bigtriangledown Y, -1) \;&=\; (-1)^y\ \varphi(X,-1)\ \varphi(\overline{Y}, 0)\\ &\quad +(-1)^x\ \varphi(Y,-1)\ \varphi(\overline{X}, 0)\\ &\quad -(-1)^{x+y}\ \varphi(\overline{X}, 0)\ \varphi(\overline{Y}, 0).\end{aligned}$$

Lemma 3.1 *The graph $K_1 \bigtriangledown X$ has -1 as an eigenvalue if and only if $\overline{X}$ has 0 as an eigenvalue.*

Proof. Recall that $\varphi(K_1, \lambda) = \lambda$, and so substituting $X = K_1$, $x = 1$, $Y = X$ and $y = x$ into the above expression we get

$$\varphi(K_1 \bigtriangledown X, -1) \quad = \quad (-1)^x(-1)\varphi(\overline{X}, 0)$$

which is 0 if and only if $\varphi(\overline{X}, 0) = 0$. ■

Our next result is the key observation, relating the eigenvalues 0 and -1 for a cograph and its complement. It implies that the value

$$p(X) := (-1)^{|V(X)|}\varphi(X, 0)\varphi(\overline{X}, -1)$$

is always zero or negative when X is a cograph.

Theorem 3.2 *If Z is a cograph then*

$$(-1)^{|V(Z)|}\varphi(Z, 0)\varphi(\overline{Z}, -1) \leq 0.$$

Proof. We prove this by induction on the number of vertices of Z. It is true for $Z = K_1$; if Z has more than one vertex, then it is either the union or join of two smaller cographs X and Y.

When $Z = X \cup Y$, we have

$$\begin{aligned}
p(Z) &= (-1)^{x+y}\ \varphi(X \cup Y, 0)\ \varphi(\overline{X} \bigtriangledown \overline{Y}, -1)\\
&= (-1)^{x+y}\ \varphi(X, 0)\ \varphi(Y, 0)\ \times\\
&\quad \Big((-1)^y\ \varphi(\overline{X}, -1)\ \varphi(Y, 0) + (-1)^x\ \varphi(\overline{Y}, -1)\ \varphi(X, 0)\\
&\quad -(-1)^{x+y}\ \varphi(X, 0)\ \varphi(Y, 0)\Big)\\
&= (-1)^x\ \varphi(X, 0)\ \varphi(\overline{X}, -1)\ \varphi(Y, 0)^2\\
&\quad +(-1)^y\ \varphi(Y, 0)\ \varphi(\overline{Y}, -1)\ \varphi(X, 0)^2\\
&\quad -\varphi(X, 0)^2\ \varphi(Y, 0)^2.
\end{aligned}$$

By the inductive hypothesis, the first two terms of this sum are non-positive, and clearly the final term is non-positive, and so $p(Z) \leq 0$.

When $Z = X \bigtriangledown Y$, we have

$$\begin{aligned}
p(Z) &= (-1)^{x+y}\ \varphi(X \bigtriangledown Y, 0)\ \varphi(\overline{X} \cup \overline{Y}, -1)\\
&= (-1)^{x+y}\ \varphi(\overline{X}, -1)\ \varphi(\overline{Y}, -1)\ \times\\
&\quad \Big((-1)^y\ \varphi(X, 0)\ \varphi(\overline{Y}, -1) + (-1)^x\ \varphi(Y, 0)\ \varphi(\overline{X}, -1)\\
&\quad -(-1)^{x+y}\ \varphi(\overline{X}, -1)\ \varphi(\overline{Y}, -1)\Big)\\
&= (-1)^x\ \varphi(X, 0)\ \varphi(\overline{X}, -1)\ \varphi(\overline{Y}, -1)^2\\
&\quad +(-1)^y\ \varphi(Y, 0)\ \varphi(\overline{Y}, -1)\ \varphi(\overline{X}, -1)^2\\
&\quad -\varphi(\overline{X}, -1)^2\ \varphi(\overline{Y}, -1)^2.
\end{aligned}$$

Once again, the inductive hypothesis yields that the first two terms are non-positive, and the final term is the negation of a square. ■

This proof can be examined more carefully to give additional information about equality; notably we can see that a cograph can only have -1 as an eigenvalue if its complement has 0 as an eigenvalue.

Corollary 3.1 *If Z is a cograph then*

$$(-1)^{|V(Z)|}\varphi(Z,0)\varphi(\overline{Z},-1)=0$$

if and only if $\varphi(Z,0)=0$.

Proof. This is true for the graph K_1, and so it suffices to prove it for the union and join of two smaller cographs. If $p(X \bigtriangledown Y)=0$, then all three terms in the expression for $p(X \bigtriangledown Y)$ are equal to 0, and exchanging X and Y if necessary, we may assume that $\varphi(\overline{X},-1)=0$. By the inductive hypothesis, this shows that $\varphi(X,0)=0$ and substituting these values into the expression for $\varphi(X \bigtriangledown Y)$ yields that $\varphi(X \bigtriangledown Y)=0$ as desired.

If $p(X \cup Y)=0$, then similarly we may assume that $\varphi(X,0)=0$ and it follows immediately that $\varphi(X \cup Y)=0$. ■

Finally we finish with a simple restatement of the expression given earlier for $\varphi(X \bigtriangledown Y,0)$

$$\varphi(X,0)\varphi(Y,0)-\varphi(X \bigtriangledown Y,0)= \\ \Big(\varphi(X,0)-(-1)^x\varphi(\overline{X},-1)\Big)\Big(\varphi(Y,0)-(-1)^y\varphi(\overline{Y},-1)\Big).$$

4 Rank of a cograph

This section proves the main result of the paper first for the case where every row of $A(Z)$ is distinct and non-zero, and then extending this to the general case.

Theorem 4.1 *Let Z be a cograph where every row of $A(Z)$ is distinct and non-zero. Then Z has full rank (that is, rank equal to $|V(Z)|$).*

Proof. We will prove this by induction on $|V(Z)|$. Clearly it is true if $|V(Z)|=2$ because the only cograph of full rank on two vertices is K_2. Therefore we consider the cases where $Z=X\cup Y$, and $Z=X \bigtriangledown Y$. In each case we show that X has full rank by showing that it does not have zero as an eigenvalue.

In the case where $X=X\cup Y$, it is immediate that the rows of $A(X)$ and $A(Y)$ are distinct and non-zero, so by the inductive hypothesis X and Y have full rank, and hence do not have zero as an eigenvalue. The spectrum of $X\cup Y$ is the union of the spectra of X and Y, and so $X\cup Y$ does not have zero as an eigenvalue.

Therefore, consider the case where $Z = X \bigtriangledown Y$, and recall that

$$\varphi(X,0)\varphi(Y,0) - \varphi(X \bigtriangledown Y, 0) = \\ \left(\varphi(X,0) - (-1)^x\varphi(\overline{X},-1)\right)\left(\varphi(Y,0) - (-1)^y\varphi(\overline{Y},-1))\right).$$

We consider each factor in the expression, starting with

$$\left(\varphi(X,0) - (-1)^x\varphi(\overline{X},-1)\right).$$

If X has full rank, then $\varphi(X,0) \neq 0$ and by Corollary 3.1, $\varphi(\overline{X},-1) \neq 0$. By the main theorem of the previous section, $\varphi(X,0)$ and $-(-1)^x\varphi(\overline{X},-1)$ have the same sign, and so

$$\left|\varphi(X,0) - (-1)^x\varphi(\overline{X},-1)\right| > |\varphi(X,0)|.$$

If X does not have full rank, then because $A(Z)$ has no repeated rows, it must be the case that $A(X)$ has a single zero row, and that $X = K_1 \cup F$ and $A(F)$ has all rows distinct and non-zero (or F might be empty; this causes no problems). By the inductive hypothesis, F has full rank and so by Lemma 3.1, $K_1 \bigtriangledown \overline{F}$ does not have -1 as an eigenvalue and so $(-1)^x\varphi(\overline{X},-1) \neq 0$. Therefore, again we have

$$\left|\varphi(X,0) - (-1)^x\varphi(\overline{X},-1)\right| > |\varphi(X,0)|$$

An identical argument with Y in place of X holds for the other factor, and we conclude that

$$|\varphi(X,0)\varphi(Y,0) - \varphi(X \bigtriangledown Y,0)| > |\varphi(X,0)\varphi(Y,0)|$$

Therefore, 0 is not an eigenvalue of $X \bigtriangledown Y$ and so Z has full rank. ∎

Corollary 4.1 *The rank of a cograph X is equal to the number of distinct non-zero rows of its adjacency matrix $A(X)$.*

Proof. We prove this by induction on $|V(X)|$; it is clearly true for $X = K_1$. If X has no repeated rows then the result holds by the theorem. If X does have a repeated row, then it has two vertices, say u and v, with the same neighbourhood. The graph $X - u$ is a cograph, and so by the inductive hypothesis it has rank equal to the number of distinct non-zero rows of $A(X - u)$. It is clear that the rank of $X - u$ is equal to the rank of X, and that $A(X)$ has the same number of distinct non-zero rows as $A(X - u)$. Therefore the result follows. ∎

We finish with two questions:

Question 4.1 *Are there any other natural classes of graphs for which this rank property holds?*

Question 4.2 *We used an inductive characterization of cographs to provide an inductive proof of the result. Is it possible to prove this result directly using one of the many alternative characterizations of cographs?*

Note added in proof: An entirely different proof of this result, using cotrees and threshold graphs has recently been discovered by Türker Bıyıkoğlu.

References

[1] N. Alon and P. D. Seymour. A counterexample to the rank-coloring conjecture. *J. Graph Theory*, 13(4):523–525, 1989.

[2] Jean H. Bevis, Kevin K. Blount, George J. Davis, Gayla S. Domke, and Valerie A. Miller. The rank of a graph after vertex addition. *Linear Algebra Appl.*, 265:55–69, 1997.

[3] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. *Graph Classes: a survey*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[4] D. G. Corneil, H. Lerchs, and L. Stewart Burlingham. Complement reducible graphs. *Discrete Appl. Math.*, 3(3):163–174, 1981.

[5] Dragoš M. Cvetković, Michael Doob, and Horst Sachs. *Spectra of Graphs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Theory and application.

[6] Reinhard Diestel. *Graph Theory*. Springer-Verlag, New York, second edition, 2000.

[7] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[8] Torsten Sillke. Graphs with maximal rank. http://www.mathematik.uni-bielefeld.de/~sillke/PROBLEMS/cograph, 2001.