Descents in Noncrossing Trees.

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Submitted: Jun 18, 2002, Accepted: Oct 6, 2003; Published: Oct 13, 2003
MR Subject Classifications: 05C30, 05A15

Abstract

The generating function for descents in noncrossing trees is found. A bijection shows combinatorially why the descent generating function with descents set equal to 2 is the generating function for connected noncrossing graphs.

1 Introduction

A **noncrossing tree** is a tree drawn on n points numbered in counterclockwise order on a circle such that the edges lie entirely within the circle and do not cross. A **descent** is an edge from a higher to a lower label along a path from the root 1. We frequently use "g.f." to stand for "generating function". In Section 2, we find the descent g.f. for noncrossing trees. In Section 3, we show a bijection between noncrossing trees and connected noncrossing graphs that explains combinatorially why the g.f. for connected noncrossing graphs equals the descent g.f. evaluated at 2.

2 Descents in Noncrossing Trees

Consider, as in [2], a noncrossing tree as a sequence of butterflies. Let

$$T(z,u,v) = \sum_{\tau} z^{|\tau|} u^{d(\tau)} v^{a(\tau)}, \tag{1}$$

where the sum is over all trees τ, the size $|\tau|$ is the number of edges, $d(\tau)$ is the number of descents, and $a(\tau)$ is the number of ascents. A **butterfly** is an ordered pair of subtrees T and T^* that lie on either side of an edge $(1,i)$ from the root to point i, T on points

*Thanks to Louis W. Shapiro and Frank W. Schmidt for useful discussions, and to the anonymous referee for an alternative proof that also proved a conjecture in the original paper.

greater than or equal to i and T^* on points less than or equal to i, not including 1. Then we have

$$T(z,u,v) = \frac{1}{1 - vzT(z,u,v)T(z,v,u)} \tag{2}$$

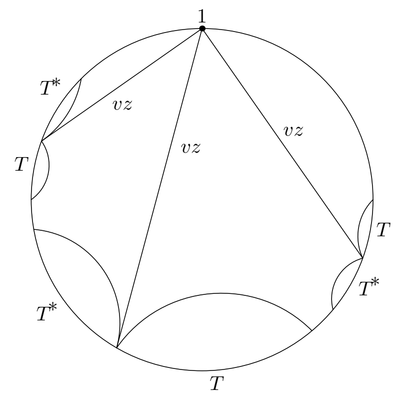

Figure 1: A butterfly diagram of noncrossing trees, where T stands for $T(z,u,v)$ and T^* for $T(z,v,u)$.

This is because when the tree is rooted at 1, every edge from 1 contributes one to the size and one ascent; and because in one of the two wings of a butterfly the role of ascents and descents is exchanged (see Figure 1). That is, $T = T(z,u,v)$ and $T^* = T(z,v,u)$. Similarly, $T(z,v,u) = 1/[1 - uzT(z,u,v)T(z,v,u)]$, and eliminating we have that $D(z,u) := T(z,u,1)$, the bivariate g.f. in which z marks edges and u descents, satisfies the cubic equation

$$zD^3 + (u-1)D^2 + (1-2u)D + u = 0 \tag{3}$$

Theorem 2.1. The descent g.f. $D(z,u)$ evaluated at u equal to -1, 0, 1, and 2 gives the g.f.'s for symmetric ternary trees, Catalan numbers, ternary numbers, and connected noncrossing graphs, respectively.

Proof. If $u = 0$ or $u = 1$, then equation (3) reduces to $D = 1 + zD^2$ or $D = 1 + zD^3$, the g.f.'s for the Catalan and ternary numbers, respectively.

If $u = 2$, then equation (3) becomes $zD^3 + D^2 - 3D + 2$, the g.f. for connected noncrossing graphs as in equation (17) of [2] (let $C = zD$).

If $u = -1$, then equation (3) becomes $zD^3 - 2D^2 + 3D - 1 = 0$. To see that this is the g.f. for symmetric ternary trees $R(z)$, note that a symmetric ternary tree can be decomposed into a ternary left subtree, a central symmetric ternary subtree, and a right subtree that is a reflection of the left subtree, as shown in Figure 2. Thus $R(z) = 1 + zT(z^2)R(z)$, and from this and the g.f. $T(z) = 1 + zT(z)^3$ for ternary trees, one obtains $zD^3 - 2D^2 + 3D - 1 = 0$. □

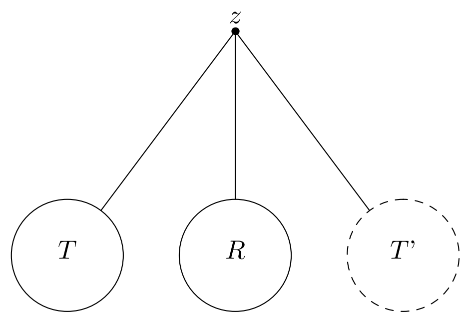

Figure 2: The g.f. for symmetric ternary trees R is related to the g.f. for ternary trees $T(z) = 1 + zT(z)^3$ by $R(z) = 1 + zR(z)T(z^2)$.

Now set $D = B + 1$ and apply Lagrange inversion to obtain the coefficients for the bivariate g.f. for noncrossing trees by edges and descents. That is, we get $B = z(1 + B)^3/[1+(1-u)B]$, an equation of the form $B = z\phi(B)$, so the Lagrange Inversion formula gives the generating function $B = \sum_{n=1}^{\infty} b_n z^n$ where $b_n = \frac{1}{n!}\frac{d^{n-1}}{dB^{n-1}}[\phi(B)^n]_{B=0}$.

Theorem 2.2. The coefficients of the g.f. $D(z,u) = \sum_{n\geq 0} d_n z^n$ are given by $d_n = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n-1+k}{n-1}\binom{2n-k}{n+1}u^k$.

Proof. The Lagrange Inversion formula gives the coefficient of u^k as a sum of the product of three binomial terms; the terms of the sum are terms of a hypergeometric series that can be evaluated by Gauss's ${}_2F_1$ identity. $\square$

Edges n	Descent g.f.
1	1
2	$2+u$
3	$5+5u+2u^2$
4	$14+21u+15u^2+5u^3$
5	$42+84u+84u^2+49u^3+14u^4$
6	$132+330u+420u^2+336u^3+168u^4+42u^5$
7	$429+1287u+1980u^2+1980u^3+1350u^4+594u^5+132u^6$
8	$1430+5005u+9009u^2+10725u^3+9075u^4+5445u^5+$ $2145u^6+429u^7$
9	$4862+19448u+40040u^2+55055u^3+55055u^4+40898u^5+$ $22022u^6+7865u^7+1430u^8$

Table 1: Descent g.f. for small numbers of edges.

The descent g.f.'s for small numbers of edges n are given in Table 1. From Theorem 2 we find that the coefficients of the n^{th} descent g.f. are log concave and so unimodal. The two equal maximal coefficients for odd n raise the question of finding a bijection on trees with $2k+1$ edges between those with $k-1$ descents and those with k descents.

3 A bijection between noncrossing trees weighted by descents and connected noncrossing graphs

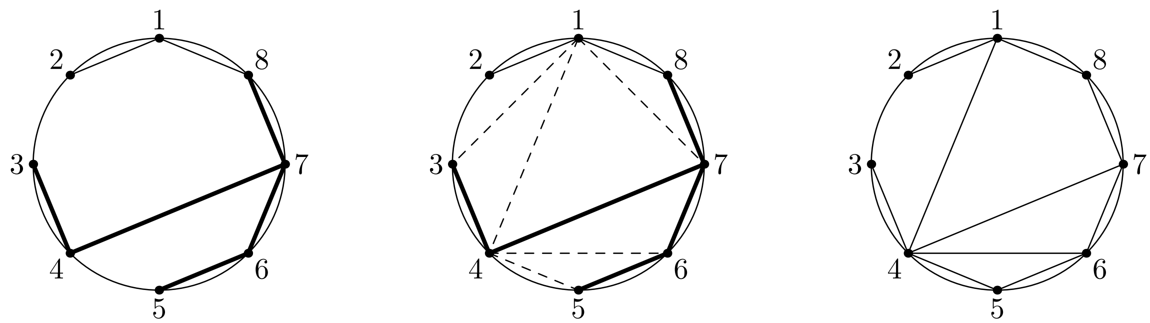

Figure 3: Each descent edge (s,t) in a noncrossing tree corresponds to an additional edge (b,t), where b is the highest-labeled neighbor less than t of a point in a chain of consecutive descents ending at t on the path from the root to t.

Theorem 3.1. There is a bijection between noncrossing trees and subsets of their descents and connected noncrossing graphs, so the descent g.f. $D(z,u)$ evaluated at 2 is the g.f. for connected noncrossing graphs.

Proof. For every descent (s,t), find the maximal path of consecutive descents from t back to the root, and let the first point on this path be a. From the neighbors of points on the path from a to s, not including points on the path, choose the neighbor b that is the largest point less than t. The new edge corresponding to descent (s,t) is (b,t).

Edge (b,t) forms a circuit, so it is a new edge. Since b is the highest neighbor on the path from a to s, (b,t) does not cross any edge in the tree or any other new edge. By adding arbitrary subsets of the new edges to a tree with d descents, one obtains 2^d distinct connected noncrossing graphs. An example of this construction is shown in Figure 3.

To complete the proof, we need to show that any connected noncrossing graph may be constructed in this way. Given a connected noncrossing graph, we construct a tree whose new edges contain the remaining edges of the graph, as follows. Starting from the root, we select the edge to the highest non-visited vertex except that no vertex b is allowed to be selected from point t that is a neighbor of another point is a path of consecutive descents from t back to the root. The exception ensures that no "new edge" corresponding to a descent is selected during the traversal, so the spanning tree created by traversing the graph is the same as the tree from which the graph was created.

$\square$

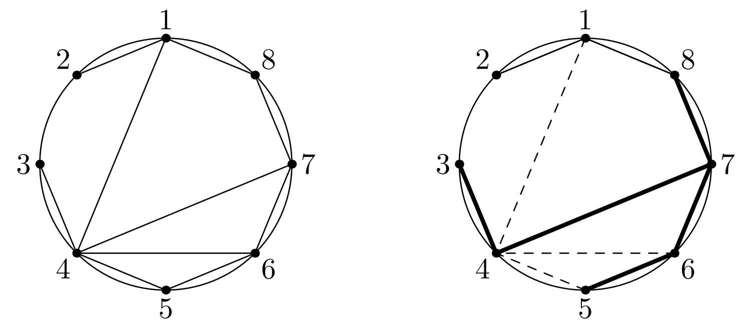

Figure 4: Any connected noncrossing graph can be decomposed into a noncrossing tree and additional edges corresponding to a subset of the descent edges.

4 Conclusions and questions

Section 3 was inspired by the construction in [3] that creates a new edge for every inversion in a labeled tree rooted at 1, leading to a correspondence between a tree with i inversions and 2^i connected graphs. In [3], a similar correspondence is found between acyclic basic digraphs and the g.f. $\bar{I}_n$ obtained by reversing the order of the coefficients of the inversions g.f. I_n; that is, $\bar{I}_n(t) = t^{\binom{n-1}{2}} I_n(1/t)$. The direct analogy in the noncrossing case does not hold, and in fact the total number of noncrossing acyclic basic digraphs does not equal the reversed g.f. evaluated at 2, but perhaps a variation might be found to work.

Question 4.1. Find a bijection between the difference between trees with an even and an odd number of descents $D(z, -1)$ and symmetric ternary trees[1].

Question 4.2. Find a bijection on trees with $2k + 1$ edges between those with $k - 1$ descents and those with k descents.

Question 4.3. Find a combinatorial correspondence between noncrossing acyclic basic digraphs and the ascents of noncrossing trees.

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