A note on the edge-connectivity of cages

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Abstract

A $(k;g)$-graph is a k-regular graph with girth g. A $(k;g)$-cage is a $(k;g)$-graph with the smallest possible number of vertices. In this paper we prove that $(k;g)$-cages are k-edge-connected if $k \geq 3$ and g is odd.

1 Introduction

Any undefined notation follows Bondy and Murty [1]. We consider only finite simple graphs, and refer to them as graphs.

Suppose that V' (or E') is a nonempty subset of V (or E). The subgraph (or the edge-induced subgraph) of G induced by V' is denoted by $G[V']$ (or $G[E']$). An induced subgraph (or edge-induced subgraph) is one that is induced by some subset of vertices (or edges). The subgraph obtained from G by deleting the vertices in V' together with their incident edges is denoted by $G - V'$. The graph obtained from G by adding a set of edges $E' \subseteq \overline{E}$ is denoted by $G + E'$.

For a vertex v of G and a set of vertices $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of vertices in S that are adjacent to v. For two vertices $uv \in S \subseteq V(G)$, let $d_S(u,v)$ denote the distance between u and v in $G[S]$. The distance between a vertex u and a set of vertices X in $S \subseteq V(G)$, denoted by $d_S(u,X)$, is the minimum distance between u and a vertex in X. When $S = V(G)$, we simply use $N(v), d(u,v)$ and $d(u,X)$.

The length of a shortest cycle in a graph G is called the girth of G. Clearly, adding edges to a graph G might decrease the girth of G. If G' is obtained from G by adding edges, we use the term *smaller cycle of* G' to denote any cycle of G' having length less than g.

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A k-regular graph with girth g is called a $(k;g)$-graph and a $(k;g)$-cage is a $(k;g)$-graph with the least possible number of vertices. We use $f(k;g)$ to denote the number of vertices of any $(k;g)$-cage.

Cages were introduced by Tutte in 1947[7], and since then have been widely studied. The problem of finding cages has a prominent place in both extremal graph theory and algebraic graph theory. A survey paper by P. K. Wong [8] in 1982 refers to 70 publications. The study of cages has led to interesting applications of algebra to graph theory. Recently, it also attracted some attention from researchers in computer science (see [2], [5]). In these papers, new computer search algorithms are used to find new cages or provide better bounds of $f(k;g)$.

Some fundamental properties of cages were established by H.L. Fu, K.C. Huang and C.A. Rodger in 1997 [4]. They first proved that all cages are 2-connected, and then subsequently showed that all cubic cages are 3-edge-connected. It follows from this theorem that all cubic cages are 3-connected. They then conjectured that all simple $(k;g)$-cages are k-connected. Recently, M. Daven and C.A. Rodger [3], and independently T. Jiang and D. Mubayi [6], proved that all $(k;g)$-cages are 3-connected for $k \geq 3$. This implies that all $(k;g)$-cages are 3-edge connected for $k \geq 3$. We will prove a much stronger result: all $(k;g)$-cages are k-edge-connected if g is odd. Our proofs involve a new method, which we are also able to use to prove that all $(4;g)$-cages are 4-connected [9].

We shall often use the following theorem of Fu, Huang and Rodger.

Monotonicity Theorem. [4] If $k \geq 3$ and $3 \leq g_1 < g_2$, then $f(k;g_1) < f(k;g_2)$.

If a $(k;g)$-graph has two vertices at distance $g+1$, then one can delete these two vertices and add a perfect matching of k edges between their neighbors so as to obtain a new k-regular graph with girth at least g. By the Monotonicity Theorem, this $(k;g)$-graph will not be a cage. We use this type of argument to prove results concerning the connectivity of cages. First, we focus on finding two vertices at greatest possible distance. We then delete these two vertices and add some carefully chosen edges in such a way that k-regularity is maintained and the girth of the resulting graph is at least g.

2 Edge-Connectivity of $(k;g)$-Cages

Since the $(k;3)$-cage, K_{k+1} and the $(k;4)$-cage, the complete bipartite graph $K_{k,k}$, are k-edge-connected, we assume in what follows that $g \geq 5$.

Let G be a $(k;g)$-cage and S is a minimal edge cut of G. Since G is a k-regular graph, $|S| < k$. We may assume that $|S| = k-1$ because all the following arguments will be easier if $|S| < k-1$. Let $S = \{e_1, e_2, \cdots, e_{k-1}\}$ be an edge-cut of G such that $G - S$ has only two components, G_1 and G_2. Let $X = V(G_1) \cap V(G[S])$, $Y = V(G_2) \cap V(G[S])$, $m_1 = |X|$ and $m_2 = |Y|$. We first prove two lemmas on finding two vertices, one in G_1 and one in G_2, at large enough distance from each other.

Lemma 2.1. *Suppose that G is an h-edge-connected graph of girth g, where $h \leq k-1$ and g is odd. Then, there exists $u \in V(G_1)$ such that $d_{G_1}(u, X) \geq \lfloor g/2 \rfloor$, and there exists $v \in V(G_2)$ such that $d_{G_2}(v, Y) \geq \lfloor g/2 \rfloor$.*

Proof. Form a path $(u_0, u_1, ..., u_{\lfloor g/2 \rfloor})$ in G_1 such that $u_0 \in X$ and $d(u_i, X) = i$ for $i = 0, 1.., \lfloor g/2 \rfloor$. Such a sequence can be constructed recursively for the following reason: If each of the k neighbors of u_i (for $i \geq 1$) is distance at most i from X in G_1 then, since $|S| \leq k-1$, at least two of these vertices have shortest paths to X that end at the same vertex in S. The union of these paths, together with the two edges joining the neighbors, must contain a cycle of length at most $2i + 2 \leq g - 1$, a contradiction. So, let u_{i+1} be a neighbor of u_i distance $i+1$ from X. Then, there exists a vertex $u_{\lfloor g/2 \rfloor} = u \in G_1$ such that $d_{G_1}(u, X) \geq \lfloor g/2 \rfloor$. Similarly, there also exists a vertex $v_{\lfloor g/2 \rfloor} = v \in G_2$ such that $d_{G_2}(v, Y) \geq \lfloor g/2 \rfloor$. $\square$

Theorem 2.1. *Let G be a $(k; g)$-cage, where $k \geq 3$ and g is odd. Then, G is k-edge-connected.*

Proof. As noted at the start of this section, we may assume that $g \geq 5$. Suppose G is a $(k-1)$-edge-connected graph, and let S be an edge-cut of size $k-1$ in G. We shall use X, and Y as they were defined in the beginning of this section. By Lemma 2.1, there a vertex u in G_1 at distance at least $\lfloor g/2 \rfloor$ from X and there a vertex v in G_2 at distance at least $\lfloor g/2 \rfloor$ from Y. Let $U = N_{G_1}(u)$ and $W = N_{G_2}(v)$. Clearly, $d(\bar{u}, \bar{v}) \geq g - 2$ for $\bar{u} \in U$ and $\bar{v} \in W$.

We shall prove that there are k pairs of vertices $(\bar{u}, \bar{v})$ where $\bar{u} \in U$ and $\bar{v} \in W$ such that $d(\bar{u}, \bar{v}) \geq g-1$. Consider the bipartite graph B with bipartition (U, W) and edge-set $\{\bar{u}\bar{v} \mid \bar{u} \in U, \bar{v} \in W, d(\bar{u}, \bar{v}) \geq g-1\}$. Let $A = \{\bar{u}\bar{v} \mid \bar{u} \in U, \bar{v} \in W, d(\bar{u}, \bar{v}) < g-1\}$. Clearly, $B = K_{k,k} - A$. We claim that each edge $xy \in S$ gives rise to at most one element of A. Otherwise, without loss of generality, there must be $\bar{u_1}$ and $\bar{u_2} \in U$ which both have distance at most $\lfloor g/2 \rfloor - 1$ to x. On the other hand, $d_{G_1}(\bar{u_1}, x) \geq \lfloor g/2 \rfloor - 1$ and $d_{G_1}(\bar{u_2}, x) \geq \lfloor g/2 \rfloor - 1$ by Lemma 2.1. Hence, $d_{G_1}(\bar{u_1}, x) = d_{G_1}(\bar{u_2}, x) = \lfloor g/2 \rfloor - 1$. It is easy to see that there is a cycle of length $g-1$ containing the vertices in $\{u, \bar{u_1}, \bar{u_2}, x\}$, a contradiction. This proves the claim, and hence $|A| \leq |S| = k-1$. It can be easily verified using Hall's Theorem that B has a 1-factor, M. By the definition of the bipartite graph B, the distance between two end vertices of each edge in M is at least $g-1$.

Now consider the k-regular graph $G' = G - \{u, v\} + M$. Since the distance in G between the two ends of each edge in M is at least $g-1$, the girth of G' is at least g. This contradicts with G being a $(k; g)$-cage. Therefore, G must be k-edge-connected. This completes the proof. $\square$

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