One Pile Nim with Arbitrary Move Function

Arthur Holshouser
3600 Bullard St.
Charlotte, NC, USA

Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@email.uncc.edu

Submitted: Feb 8, 2002; Accepted: Jun 11, 2003; Published: Jul 27, 2003
MR Subject Classifications: 91A46, 11B37

Abstract

This paper solves a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each successive move equals $f(t)$, where t is the previous move size and f is an arbitrary function.

The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each successive move changes during the play of the game.

Two players alternate removing a positive number of counters from the pile. An ordered pair (N, x) of positive integers is called a position. The number N represents the size of the pile of counters, and x represents the greatest number of counters that can be removed on the next move. A function $f : Z^+ \to Z^+$ is given which determines the maximum size of the next move in terms of the current move size. Thus a move in a game is an ordered pair of positions $(N, x) \mapsto (N - k, f(k))$, where $1 \le k \le \min(N, x)$.

The game ends when there are no counters left, and the winner is the last player to move in a game. In this paper we will consider $f : Z^+ \to Z^+$ to be completely arbitrary. That is, we place no restrictions on f. This paper extends a previous paper by the authors [2], which in turn extended two other papers, [1] and [3]. The paper by Epp and Ferguson [1] assumed f is non-decreasing, and the paper [3] assumed f is non-decreasing and $f(n) \ge n$. Our previous paper [2] assumed more restrictive conditions on f including as a special case all $f : Z^+ \to Z^+$ that satisfy $f(n+1) - f(n) \ge -1$.

The main theorem of this paper will also allow the information concerning the strategy of a game to be stored very efficiently. We now proceed to develop the theory.

Generalized Bases: An infinite strictly increasing sequence $B = (b_0 = 1, b_1, b_{2,} \cdots)$ of positive integers is called an infinite g-base if for each $k \geq 0, b_{k+1} \leq 2b_k$. This 'slow growth' of B's members guarantees lemma 1.

Finite g-bases. A finite strictly increasing sequence $B = (b_0 = 1, b_1, b_2, \cdots, b_t)$ of positive integers is called a *finite g-base* if for each $0 \leq k < t, b_{k+1} \leq 2b_k$.

Lemma 1. *Let B be an infinite g-base. Then each positive integer N can be represented as $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B.*

Proof. The proof is given by the following recursive algorithm. Note that $b_0 = 1 \in B$. Suppose all integers $1, 2, 3, \cdots, m-1$ have been represented as a sum of distinct members of B by the algorithm. Suppose $b_k \leq m < b_{k+1}$. Then $m = (m - b_k) + b_k$. Now $m - b_k < b_k$, for otherwise, $2b_k \leq m$. Since $m < b_{k+1}$, we have $2b_k < b_{k+1}$, which contradicts the definition of a g-base. Since $m - b_k$ is less than m, it follows that $m - b_k$ has been represented by the algorithm as a sum of distinct members of B that are less than b_k. Thus we may assume that $m - b_k = b_{i_1} + b_{i_2} + \cdots + b_{i_{t-1}}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_{t-1}}$ and each b_{i_j} belongs to B. Then $m = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$ where $b_{i_t} = b_k, b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B. □

Lemma 2. *Let $B = (b_0 = 1, b_1, b_2 \cdots b_t)$ be a finite g-base. For any positive integer N, let $\theta \geq 0$ be the unique integer such that $0 \leq N - \theta b_t < b_t$. Then the same algorithm used in the proof of lemma 1 can be used to uniquely represent $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ and each b_{i_j} belongs to B.*

In this paper we always use the algorithms used in the proofs of lemmas 1 and 2 to uniquely represent any positive integer N as the sum of distinct members of the g-base that we are dealing, whether this g-base is finite or infinite.

Definition 3. *Representation of a positive integer. Suppose $B = (b_0 = 1, b_1, b_2, \cdots)$, where $b_0 < b_1 < b_2 < \cdots$, is an infinite g-base. Let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$, be the representation of a positive integer N that is specified by the algorithm used in the proof of lemma 1. Then we define $\overline{g}(N) = b_{i_1}$.*

Suppose $B = (b_0 = 1, b_1, b_2, \cdots b_t)$, where $b_0 < b_1 < \cdots < b_t$, is a finite g-base. Let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ be the representation of N in B that is specified by the algorithm used in lemma 2. Then $\overline{g}(N) = b_{i_1}$ unless $N = \theta b_t$ in which case $\overline{g}(N) = b_t$.

Generating g-bases: For every function $f : Z^+ \rightarrow Z^+$, we generate a g-base B_f and a function $g' : B_f \rightarrow Z^+$ as follows.

Let $b_0 = 1$, $g'(b_0) = 1, b_1 = 2$, $g'(b_1) = 2$. Suppose $b_0, b_1, b_2, \cdots, b_k$ and $g'(b_0)$, $g'(b_1)$, $g'(b_2), \cdots, g'(b_k)$, where $k \geq 1$, have been generated. Then $b_{k+1} = b_k + b_i$ where b_i is the smallest member of $\{b_0, b_1, \cdots, b_k\}$ such that $g'(b_i) = b_i$ and $f(b_i) \geq g'(b_k)$ if such a b_i exists. If no such b_i exists for some k, the g-base B_f is finite. Also, $g'(b_{k+1}) =$

$\min[\{b_{k+1}\} \cup \{b_{k+1} - b_k + \overline{x} : 1 \le \overline{x} < b_k \text{ and } f(b_{k+1} - b_k + \overline{x}) < g'(\overline{g}(b_k - \overline{x}))\}]$. Of course, $\overline{g}(b_k - \overline{x})$ is computed using definition 3 with $(b_0 = 1, b_1, b_2, \cdots, b_k)$, and $\min S$ means the smallest member of S. We will explain later why B_f and g' are defined this way. Also, we note that since $b_{k+1} = b_k + b_i$ and $b_0 = 1 \le b_i \le b_k$, it follows that B_f is indeed a g-base.

Definition 4. *Suppose $f : Z^+ \to Z^+$ generates the g-base B_f and the function $g' : B_f \to Z^+$. Then for every $N \in Z^+$, we define $g'(N) = g'(\overline{g}(N))$, where $\overline{g}(N)$ is computed using B_f . Thus in the definition of $g'(b_{k+1})$, we could substitute $g'(b_k - \overline{x})$ for $g'(\overline{g}(b_k - \overline{x}))$.*

Before we state the main theorem, we need a few more definitions. We recall that for our game, we are given some arbitrary function $f : Z^+ \to Z^+$. That is, we place no restrictions on f. Also, a position in the game is an ordered pair of positive integers (N, x), and a move is $(N, x) \mapsto (N - k, f(k)), 1 \le k \le \min(N, x)$. A position (N, x) is called *unsafe* if it is unsafe to move to it. Since the player who moves to an unsafe position loses with best play, the player who moves from an unsafe position can always win. Similarly, a position is *safe* if it is safe to move to it. For each ordered pair (N, x), define $F(N, x) = 0$ if (N, x) is a safe position, and $F(N, x) = 1$ if (N, x) is an unsafe position. Note that $F(N, x) = 1$ when the list $F(N-1, f(1)), F(N-2, f(2)), \cdots, F(N-x, f(x))$ contains at least one 0. Note that $F(0, x) = 0$ for all $x \in Z^+$. $F(N, x) = 0$ when this list contains no 0's. We imagine that $F(1, x), x = 1, 2, \cdots$, is computed first. Then $F(2, x), x = 1, 2, 3, \cdots$, is computed. Then $F(3, x), x = 1, 2, 3, \cdots$, is computed, etc. Note that $F(N, x) = 1$ when $N \le x$. Note also that for a fixed $N \in Z^+$ and a variable $x \in Z^+$, the infinite sequence $F(N, x), x = 1, 2, 3, \cdots$, always consists of a finite string (possibly empty) of consecutive 0's followed by an infinite string of consecutive 1's. This is because $F(N, x) = 1$ when $x \ge N$ and also once the sequence first switches from 0 to 1 it must retain the value of 1 thereafter. For each $N \in Z^+$, define $g(N)$ to be the smallest $x \in Z^+$ such that $F(N, x) = 1$. Of course $1 \le g(N) \le N$. For every $N \in Z^+, g(N)$ is the *position* of the first 0 in the sequence $F(N-1, f(1)), F(N-2, f(2)), \cdots, F(N - g(N), f(g(N))$. This means that $F(N - g(N), f(g(N))) = 0$, but all preceding members of this sequence have a value of 1. It is obvious that $F(N, x) = 0$ when $x \le g(N) - 1$ and $f(N, x) = 1$ when $x \ge g(N)$. We can now state the main theorem.

Main theorem: Suppose we play our game with an arbitrary but fixed move function $f : Z^+ \to Z^+$. Suppose f generates the g-base B_f and the function $g' : B_f \to Z^+$. Then for every $N \in Z^+, g(N) = g'(N)$ where $g'(N)$ is defined in definition 4 using the g-base B_f and the function $g' : B_f \to Z^+$.

The main theorem implies that a position (N, x) is unsafe if $x \ge g'(N)$ and safe if $x < g'(N)$. This means a winning move must be $(N, x) \to (N - g'(N), f(g'(N)))$. The reader will note that the theorem is true whether B_f is finite or infinite. In a moment we will write a detailed proof for the case where B_f is infinite. The proof of the finite case, which involves a slight modification of the infinite case, is left to the reader.

Before we begin the proof, we would like to point out that for an enormous number of functions f it is very easy to compute B_f and g'. For example, if f is non-decreasing or if f satisfies $f(n + 1) - f(n) \ge -1$, then $g' : B_f \to Z^+$ is just the identity function on B_f, and B_f is generated by the following very simple algorithm. $b_0 = 1, b_1 = 2$ and if

$b_0, b_1, b_2, \cdots, b_k$ have been generated, then $b_{k+1} = b_k + b_i$ where b_i is the smallest member of $\{b_0, b_i, \cdots b_k\}$ such that $f(b_i) \geq b_k$. There are also many other functions f for which B_f and g' are very easy to compute. However, for many functions f, the best way to compute B_f and g' is to go ahead and compute $g(N)$ first. Note that $g(N), \quad N = 1, 2, 3, \cdots$, can be computed directly by the following algorithm: $g(1) = 1, g(2) = 2$. If $g(1), g(2), \cdots, g(k-1), k \geq 3$, have been computed, then $g(k)$ is the smallest $x \in \{1, 2, 3, \cdots, k\}$ such that $f(x) < g(k-x)$, where we agree that $g(0) = \infty$.

For those functions where $g(N)$ is computed first, it may appear that this paper has no advantages whatsoever since we already know $g(N)$. However, once $g(N)$ is computed, we know that $g' = g$, and we can then easily compute B_f.

Once we know B_f and $g' : B_f \to Z^+$, we know that B_f and g' by themselves store the complete strategy of the game. Quite often the members of B_f grow exponentially. In these cases we have an efficient way of storing the strategy of the game. We conjecture that if g is unbounded, then the members of B_t *always* grow exponentially.

At the end of this paper, we give two examples in which B_f, g' provide efficient storage. We will also give an example when B_f, g' provide inefficient storage. In the first two examples, g is unbounded and in the third, g is bounded.

Proof. The main theorem follows easily if we can prove the following statements. We assume B_f is infinite.

1. $\forall b_i \in B_f, g(b_i) = g'(b_i)$,

2. $\forall N \in Z^+ \backslash B_f, g(N) < N$,

3. $\forall N \in Z^+ \backslash B_f$, if $b_i < N < b_{i+1}$, then $N = b_i + (N - b_i)$, where $1 \leq N - bi < b_i$, and $g(N) = g(N - b_i)$. Of course, $1 \leq N - b_i < b_i$ is obvious since B_f is a g-base.

Note: At the end of the proof, we will use property 3 to explain why we defined B_f the way that we did.

The main theorem follows because if $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$, is the representation of N in the infinite g-base B_f that is computed by the algorithm used in the proof of lemma 1, we have the following.

$$\begin{aligned}
g(N) &= g((b_{i_1} + \cdots + b_{i_{k-1}}) + b_{i_k}) = g((b_{i_1} + \cdots + b_{i_{k-2}}) + b_{i_{k-1}}) \\
&= g((b_{i_1} + \cdots + b_{i_{k-3}}) + b_{i_{k-2}}) = \cdots = \\
&= g(b_{i_1}) = g'(b_{i_1}) = g'(N),
\end{aligned}$$

by the definition of $g'(N)$ since $g(b_{i_1}) = g'(\overline{g}(N))$.

Note that once we have proved statements $1, 2, 3$ for all members of $\{1, 2, 3, \cdots, \overline{k}\}$, then $\forall N \in \{1, 2, 3, \cdots \overline{k}\}$, $g(N) = g'(N)$ will also be true.

We prove statements $1, 2, 3$ by mathematical induction. First, note that no matter what f is $g(1) = g'(1) = 1$ and $g(2) = g'(2) = 2$. Conditions $2, 3$ do not apply for integers $1, 2$ since $\{1, 2\} \subseteq B_f$.

Let us suppose that condition 1 is true for all $b_i \in \{b_0, b_1, b_2, \cdots, b_k\}$, where $k \geq 1$, and conditions 2, 3 are true for all $N \in \{1, 2, 3, 4, \cdots, b_k\} \backslash B_f$. We show that conditions 2, 3 are true for all $N \in \{b_k+1, b_k+2, \cdots, b_{k+1}-1\}$, and condition 1 is true for $b_i = b_{k+1}$. Define $b_{\theta(k)}$ by $b_{k+1} = b_k + b_{\theta(k)}$, where $b_{\theta(k)} \in \{b_0, b_1, b_2, \cdots, b_k\}$.

In the following argument, we omit the first part when $b_{k+1} - b_k = 1$. So let us imagine that $b_{k+1} - b_k \geq 2$. We will prove that conditions 2, 3 are true for $N \in \{b_k + 1, b_k + 2, \cdots, b_{k+1} - 1\}$ by proving this sequentially with N starting at $N = b_k + 1$ and ending at $N = b_{k+1} - 1$.

Note that once we prove condition 3 for any $N \in \{b_k+1, \cdots, b_{k+1}-1\}$, condition 2 will follow for this N as well. This is because if $N = b_k+(N-b_k)$, where $1 \leq N-b_k < b_k$, and $g(N) = g(N - b_k)$, then $g(N) = g(N - b_k) \leq N - b_k < N$. Note that $g(N - b_k) \leq N - b_k$ is always true. So let us now prove condition 3 is true for N as N varies sequentially over $b_k + 1, \cdots, b_{k+1} - 1$.

Since we are assuming that $b_{k+1} - b_k \geq 2$, this means that $f(1) < g'(b_k) = g(b_k)$ is assumed as well since $g'(1) = 1$. Therefore, $g(b_k + 1) = g(1) = 1$ is obvious. Therefore, suppose we have proved condition 3 for all $N \in \{b_k + 1, b_k + 2, \cdots, b_k + t - 1\}$ where $b_k + t - 1 \leq b_{k+1} - 2$. This implies for all $N \in \{1, 2, 3, \cdots, b_k + t - 1\}, g(N) = g'(N)$. We now prove condition 3 for $N = b_k + t$. This means we know that $g(i) = g(b_k + i), i = 1, 2, 3, \cdots, t - 1$ and we wish to prove $g(t) = g(b_k + t)$. Recall that $g(t)$ is the smallest positive integer x such that the list (1) $F(t-1, f(1)), F(t-2, f(2)), \cdots, F(t-x, f(x))$ contains exactly one 0 (which comes at the end of the list).

Also, $g(b_k + t)$ is the smallest positive integer x such that the list (2) $F(b_k + t - 1, f(1)), F(b_k+t-2, f(2)), \cdots, F(b_k+t-x, f(x))$ contains exactly one 0 (which comes at the end). Since we are assuming that $g(i) = g(b_k+i), i = 1, 2, \cdots, t-1$, we know that the above two lists must be identical as long as $1 \leq x \leq t-1$. This follows from the definition of g since $g(N)$ tells us that $F(N, x) = 0$ when $1 \leq x \leq g(N) - 1$ and $f(N, x) = 1$ when $g(N) \leq x$. Now if $t \notin \{b_0 = 1, b_1, b_2 \cdots, b_{\theta(k)-1}\}$, we know from condition 2 that $g(t) < t$. This tells us that for list (1) the smallest x such that list (1) contains exactly one 0 satisfies $1 \leq x \leq t - 1$. Therefore, since the two lists (1),(2) are identical when $1 \leq x \leq t - 1$, this tells us that $g(t) = g(b_k + t)$.

Next, suppose $t \in \{b_0 = 1, b_1, b_2 \cdots, b_{\theta(k)-1}\}$. Of course, $g(t) = g'(t)$ since $t < b_k+t-1$. Now if $g(t) = g'(t) < t$, the same argument used above holds to show that $g(t) = g(b_k+t)$. Now if $g(t) = g'(t) = t$, we know from the definition of how $b_{k+1} = b_k + b_{\theta(k)}$ is generated that $f(t) < g'(b_k) = g(b_k)$. Since $g'(t) = g(t) = t$, we know that the first $t-1$ members of each of the lists (1), (2) consists of all 1's since they are identical up this point and $g(t) = t$. Now in list (1), $F(t - t, f(t)) = F(0, f(t)) = 0$. Also, in list (2), $F(b_k + t - t, f(t)) = F(b_k, f(t)) = 0$ since $f(t) < g(b_k) = g'(b_k)$. This tells us that $g(t) = g(b_k + t) = t$ when $t \in \{b_0, b_1, \cdots, b_{\theta(k)-1}\}$, and $g(t) = g'(t) = t$. Of course, $g(t) = g(b_k+t)$ is what we wished to show.

Finally, we show that $g(b_{k+1}) = g'(b_{k+1})$. Recall that $b_{k+1} = b_k+b_{\theta(k)}$ where $g'(b_{\theta(k)}) = g(b_{\theta(k)}) = b_{\theta(k)}$ and $f(b_{\theta(k)}) \geq g'(b_k) = g(b_k)$. We now know that $\forall N \in \{1, 2, 3, 4, \cdots, b_{k+1}-1\}, g(N) = g'(N)$. Also, we know that $g(i) = g(b_k + i), i = 1, 2, 3, \cdots, b_{\theta(k)} - 1$. Since $g(b_{\theta(k)}) = b_{\theta(k)}$ we know that all terms in the following sequence are 1's except the final term which is 0.

(3) $F(b_{\theta(k)}-1, f(1)), F(b_{\theta(k)}-2, f(2)), F(b_{\theta(k)}-3, f(3)), \cdots, F(1, f(b_{\theta(k)}-1))$, $F(0, f(b_{\theta(k)}))=0$. Now $g(b_{k+1})=g(b_k+b_{\theta(k)})$ is the *position* of the first 0 in the following sequence where we note that the *position* of a term $F(b_k+b_{\theta(k)}-i, f(i))$ is i.

(4) $F(b_k+b_{\theta(k)}-1, f(1)), F(b_k+b_{\theta(k)}-2, f(2)), \cdots, F(b_k+1, f(b_{\theta(k)}-1)), F(b_k, f(b_{\theta(k)}))$, $F(b_k-1, f(b_{\theta(k)}+1)),\quad F(b_k-2, f(b_{\theta(k)}+2)), \cdots, F(1, f(b_{\theta(k)}+b_k-1)), F(0, f(b_{\theta(k)}+b_k))=0$.

Since $g(b_k+i)=g(i), i=1,2,\cdots,b_{\theta(k)}-1$, we know that the first $b_{\theta(k)}-1$ terms of (4) are 1's since the first $b_{\theta(k)}-1$ terms of (3) are 1's. Now $F(b_k, f(b_{\theta(k)}))=1$ from the definition of g since $f(b_{\theta(k)}) \geq g'(b_k)=g(b_k)$. Note that the last term in (4) is 0. Recall that for all $N \in \{1,2,3,\ldots,b_{k+1}-1\}, \quad g(N)=g'(N)$.

From (4), we see that $g(b_{k+1})$ is the smallest $b_{\theta(k)}+\overline{x}, 1 \leq \overline{x} \leq b_k$ such that $F(b_k-\overline{x}, f(b_{\theta(k)}+\overline{x}))=0$. Since $F(b_k-b_k, f(b_{\theta(k)}+b_k))=0$, we see that $g(b_{k+1})$ is the *smaller* of $b_{\theta(k)}+b_k=b_{k+1}$ and the smallest $b_{\theta(k)}+\overline{x}, 1 \leq \overline{x} < b_k$, such that $F(b_k-\overline{x}, f(b_{\theta(k)}+\overline{x}))=0$ if such a $b_{\theta(k)}+\overline{x}$ exists. Now $F(b_k-\overline{x}, f(b_{\theta(k)}+\overline{x}))=0$, when $1 \leq \overline{x} < b_k$, if and only if $f(b_{\theta(k)}+\overline{x}) < g(b_k-\overline{x})$. Since $b_{\theta(k)}=b_{k+1}-b_k$ and since $g(b_k-\overline{x})=g'(b_k-\overline{x})=g'(\overline{g}(b_k-\overline{x}))$, if we compare the above definition of $g(b_{k+1})$ with the definition of $g'(b_{k+1})$ given earlier in this paper, we see that $g(b_{k+1})=g'(b_{k+1})$. $\square$

Observation: We now explain why we defined B_f the way that we did. Assuming that B_f is infinite, we know from the definition of B_f that $b_{i+1}-b_i \leq b_i$.

Also, from the definition of b_{i+1}, we know that $g'(b_{i+1}-b_i)=b_{i+1}-b_i$.

Also, from the definition of $g'(b_{i+1})$ (since $\overline{x} \geq 1$ in the definition), we know that $g'(b_{i+1}) > b_{i+1}-b_i = g'(b_{i+1}-b_i)$. Thus $g(b_{i+1}) > g(b_{i+1}-b_i)$.

However, from statement 3 (at the beginning of the proof) we know that for all N satisfying $b_i < N < b_{i+1}$ it is true that $g(N)=g(N-b_i)$. This *change* at b_{i+1} is precisely why we defined B_f the way that we did.

The misère version: To win at the misère version (N,x) of dynamic nim, simply use the theory to win the game $(N-1,x)$, so that your opponent is forced to take the last counter. The reader may like to figure out the strategy for the following variation. Suppose $S \subseteq Z^+ \cup \{0\}$, and the game is over as soon as $N \in S$, N being the pile size. In this paper $S=\{0\}$.

A difficult problem is to find (with proof) functions $f: Z^+ \rightarrow Z^+$ such that B_f and g' satisfy the following. $\{b_i : b_i \in B_f, g'(b_i) < b_i\}$ is infinite and $\{b_i : b_i \in B_f, g'(b_i)=b_i\}$ is infinite. We now give two examples of such functions. In both examples, B_f and g' will store the strategy extremely efficiently since the members of B_f grow exponentially.

Example 1: Define $f(n)=n, n \neq 8^k, k=0,1,2,3,\cdots, f(8^k)=4\cdot 8^k$.

Then $B_f=\{a\cdot 8^b : a,b \text{ integers}, 1 \leq a \leq 7, 0 \leq b\}$, and $g'(a\cdot 8^b)=\phi(a)\cdot 8^b$, where $\phi(1)=1, \phi(2)=2, \phi(3)=3, \phi(4)=4, \phi(5)=2, \phi(6)=2, \phi(7)=3$.

Noting that $g'(a\cdot 8^b) \leq 4\cdot 8^b$, we leave the proof as an exercise for the reader. The proof is by induction in blocks of seven starting with proving that $\{1,2,3,4,5,6,7\} \subseteq B_f$ and $g'(1)=1, g'(2)=2, g'(3)=3, g'(4)=4, g'(5)=2, g'(6)=2, g'(7)=3$.

Example 2: Define $f(n) = n$, if n is even and $f(n) = 4n$, if n is odd.

Instead of calling $B_f = (b_0 = 1 < b_1 < b_2 < \cdots)$, it is more convenient to call $B_f = (a_1 = 1 < b_1 < c_1 < a_2 < b_2 < c_2 < d_2 < a_3 < b_3 < c_3 < d_3 < \cdots < a_i < b_i < c_i < d_i < \cdots)$. The first few terms of B_f and the corresponding g' are computed by the following recursion. First, we define a strictly increasing sequence $\Delta_2, \Delta_3, \ldots$ recursively as follows: $\Delta_2 = 1, \Delta_3 = 3$ and for all $i \geq 4, \Delta_i = \Delta_{i-1} + 4\Delta_{i-2}$. Also, define $a_1 = 1, g'(a_1) = 1, b_1 = 2, g'(b_1) = 2, c_1 = 3, g'(c_1) = 3, a_2 = 4, g'(a_2) = 4, b_2 = 5, g'(b_2) = 2, c_2 = 6, g'(c_2) = 2, d_2 = 7, g'(d_2) = 7$. For all $i \geq 3$, define $a_i, g'(a_i), b_i, g'(b_i), c_i, g'(c_i), d_i, g'(d_i)$ recursively as follows: $a_i = d_{i-1} + \Delta_i, g'(a_i) = a_i, b_i = a_i + \Delta_i, g'(b_i) = 2\Delta_i, c_i = b_i + \Delta_i, g'(c_i) = 2\Delta_i, d_i = c_i + \Delta_i, g'(d_i) = d_i$.

In the third example, we give an example of a function f such that $B_f = Z^+$. If $B_f = Z^+$, it is easy to prove that g must be bounded. The reader can show that if either g or f is bounded, then eventually g must become periodic.

Example 3: Let $f(1) = 4, f(n) = 2, n \geq 2$. Then $g(1) = 1, g(2 + 4k) = 2$, $g(3 + 4k) = 3, g(4 + 4k) = 4, g(5 + 4k) = 2, k = 0, 1, 2, 3, \cdots$. Also, it is easy to see that $B_f = Z^+$. Of course, $g' = g$ on B_f.

Appendix. The functions f, g, g' and the base B_f described in this paper have a great many properties, a few of which are listed below. Recall that f is given, and it determines the other three.

1. g is periodic on Z^+ if and only if B_f is finite.

2. If f is bounded, there exists a positive integer a such that g is periodic on $[a, \infty)$.

3. If g is bounded, then there exists a positive integer a such that g is periodic on $[a, \infty)$.

4. Suppose the positive integer a satisfies $f(i) < g(a)$ for all $i = 1, 2, 3, \ldots, a$. Then g is periodic on Z^+ and $[1, a]$ is a period.

5. g is bounded if and only if $\{b_i : \ b_i \in B_f, g(b_i) = b_i\}$ is finite.

6. There exists a positive integer a such that g is periodic on $[a, \infty)$ if and only if g is bounded on B_f.

Acknowledgement. The authors also acknowledge the referee who made some timely and valuable suggestions.

References

[1] Richard Epp and Thomas Ferguson, A Note on Takeaway Games, *The Fibonacci Quarterly*, **18** (1980), 300–303.

[2] A. Holshouser, J. Rudzinski and H. Reiter, Dynamic One-Pile Nim, *The Fibonacci Quarterly*, to appear.

[3] A. J. Schwenk, Take-Away Games, *The Fibonacci Quarterly*, **8** (1970), 225–234.