On the crossing number of $K_{m,n}$

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Abstract

The best lower bound known on the crossing number of the complete bipartite graph is :

$$cr(K_{m,n}) \geq (1/5)(m)(m-1)\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$$

In this paper we prove that:

$$cr(K_{m,n}) \geq (1/5)m(m-1)\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 9.9 \times 10^{-6} m^2 n^2$$

for sufficiently large m and n.

1 Introduction

Determining the crossing number of the complete bipartite graph is one of the oldest crossing number open problems. It was first posed by Turan and known as Turan's brick factory problem. In 1954 , Zarankiewicz conjectured that it is equal to

$$Z(m,n) = \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor \lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor$$

He even gave a proof and a drawing that matches the lower bound, but the proof was shown to be flawed by Richard Guy [1]. Then in 1970 Kleitman proved that Zarankiewicz conjecture holds for $Min(m,n) \leq 6$ [2]. In 1993 Woodall proved it for $m \leq 8, n \leq 10$ [3]. Previously the best known lower bound in the general case was the one proved by Kleitman [2] :

$$cr(K_{m,n}) \geq (1/5)(m)(m-1)\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor.$$

Richter and Thomassen discussed the relation between the crossing numbers of the complete and the complete bipartite graphs [4].

2 A new bound

We will start by giving definitions that will be used throughout the paper. They are taken from Woodall[3] and Kleitman [2].

Definition 1 *Two edges e_1 and e_2 are said to have a crossing in a drawing D of $K_{m,n}$ if e_1 can be closed by a curve disjoint from e_2 connecting the two endpoints of e_1 and such that there are points of e_2 both inside and outside the closed curve.*

Definition 2 *The crossing number cr_G of a graph G is the smallest crossing number of any drawing of G in the plane, where the crossing number cr_D of a drawing D is the number of non-adjacent edges that have a crossing in the drawing.*

Definition 3 *A good drawing a graph G is a drawing where the edges are non-self-intersecting where each two edges have at most one point in common, which is either a common end vertex or a crossing.*

Clearly a drawing with minimum crossing number must be a good drawing.

Let A be one partite and B the other partite. The elements of A are $a_1, a_2, a_3, \dots, a_m$, and the elements of B are $b_1, b_2, \dots, b_n$. In a drawing D, we denote by $cr_D(a_i, a_k)$ the number of crossings of arcs, one terminating at a_i, the other at a_k and by $cr_D(a_i)$ the number of crossings on arcs which terminate at a_i :

$$cr_D(a_i) = \sum_{k=1}^{n} cr_D(a_i, a_k)$$

The crossing number of the drawing D is therefore:

$$cr_D = \sum_{i=1}^{n} \sum_{k=i+1}^{n} cr_D(a_i, a_k)$$

Let us define :

$$Z(m) = \lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor.$$

Let S_n^* be the set of the $(n-1)!$ different cyclic orderings of a set V_n of n elements. (The significance of cyclic ordering is that 01234 is considered as being the same as 34012 or 12340). If z_1 and z_2 belong to S_n^* then the distance $d(z_1, z_2)$ is the minimum number of transpositions between adjacent elements in the cyclic ordering necessary to turn z_1 into z_2. If a belongs to S_n^* then $\bar{a}$ denotes the reverse ordering of a. For example, in S_7^*, if $a = 0354162$, then $\bar{a} = 0261453$. The antidistance $\bar{d}(a, b)$ between two elements a and b is the distance between $\bar{a}$ and b (or between $\bar{b}$ and a). Woodall [3] gave a detailed proof of the two following propositions:

Theorem 1 *If $a \in S_n^*$, then $\bar{d}(a, a) = Z(n)$.*

Theorem 2 *In a drawing D of $K_{2,n}$ on two sets $\{x, x'\}$ and V_n, let the clockwise orders in which the edges leave x and x' to go to V_n be the elements a and b of S_n^*. Then $cr_D \geq \bar{d}(a, b)$, and if n is odd then cr_D is of the same parity than $\bar{d}(a, b)$.*

Kleitman proved the following equalities:

Theorem 3

$$cr(K_{5,n}) = 4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor \tag{1}$$

$$cr(K_{6,n}) = 6\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor. \tag{2}$$

From this he deduced that

$$cr(K_{m,n}) \geq (1/5)(m)(m-1)\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor \tag{3}$$

in the following way: There are $\binom{m}{5}K_{5,n}$ which are subgraphs of $K_{m,n}$, with the partite with n vertices in the $K_{5,n}$ being B. Let σ be the sum over all such $K_{5,n}$ of the number of crossings that each of these $K_{5,n}$ contain in a drawing D. Obviously $\sigma \geq \binom{m}{5}Z(5,n)$. Each crossing appears in exactly $\binom{m-2}{3}K_{5,n}$. Therefore

$$cr(K_{m,n}) \geq \frac{\binom{m}{5}Z(5,n)}{\binom{m-2}{3}}$$

$$cr(K_{m,n}) \geq (1/5)(m)(m-1)\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$$

We will obtain a small improvement on this lower bound for large values of n, with $m \geq 7$, by proving that there is a number of $K_{5,n}$ subgraphs of $K_{m,n}$ which must have more than $Z(5,n)$ crossings in any drawing of $K_{m,n}$

The cyclic ordering of the edges around each $b_j \in B$ can be considered as a cyclic ordering of the elements of A, and therefore as an element of S^*_m.

Suppose we have a $K_{2,m}$ which first partite is $\{u_1, u_2\}$ and second partite is $\{u'_1, \ldots, u'_m\}$. And let $c(u_1)$ denote the cyclic ordering of the edges around u_1, and $c(u_2)$ denote the cyclic ordering of the edges around u_2. Woodall [3] proved the following theorem:

Theorem 4 *If a good drawing of $K_{2,m}$ has r crossings, there is a sequence $Seq(u_1, u_2)$ of r transpositions between adjacent elements in $c(u_1)$, such that if we apply this sequence to $c(u_1)$ we obtain $\bar{c}(u_2)$, and there is a crossing in the $K_{2,2}$ subgraph of $K_{2,m}$ on vertices $u_1, u_2, u'_i, u'_{i'}$ if and only if exactly one of the transpositions takes place between elements u'_i and $u'_{i'}$ in $Seq(u_1, u_2)$. (In a good drawing a $K_{2,2}$ can have one crossing at most.)*

We can now prove our first lemma:

Lemma 1 *In any $K_{2,7}$ subgraph of $K(m,n)$ where the two vertices with 7 edges have the same cyclic ordering of edges, there is a $K_{2,4}$ subgraph which has 6 crossings.*

Proof :Let A' be a subset of 7 elements of A, say $w_1, w_2, \ldots, w_7$ (for every l, $w_l = a_k$ for some k). Let $\{b_k, b_l\}$ be one partite of a $K_{2,7}$ subgraph G of $K_{m,n}$ and let A' be the second partite. Now suppose $c(b_k) = c(b_l)$ in G. Let $W(b_k, b_l)$ be the set of pairs $\{w_i, w_{i'}\}$ of elements of A' such that there is a transposition exchanging w_i and $w_{i'}$ in $Seq(b_k, b_l)$ in the drawing of G. Let w_y be one element of A', and let $A^* = A' \setminus \{w_y\}$. Let A_1 be the set of elements a_x of A^* such that $\{a_x, w_y\} \in W(b_k, b_l)$ and $A_2 = A^* \setminus A_1$. It is clear that every triple $\{w_y, w_z, w_{z'}\}$ reverses its ordering between $c(b_k)$ and $\bar{c}(b_l) = \bar{c}(b_k)$, which implies that either all three pairs $\{w_z, w_{z'}\}$, $\{w_y, w_z\}$, $\{w_y, w_{z'}\}$ belong to $W(b_k, b_l)$ or exactly one of these pairs belong to $W(b_k, b_l)$. Therefore, if a pair of elements $\{w_z, w_{z'}\} \subset A'$ either has both of its elements in A_1 or has both of its elements in A_2, then $\{w_z, w_{z'}\} \in W(b_k, b_l)$. Either $Card(A_2) \geq 4$ or not. If $Card(A_2) \geq 4$, every 4-subset of A_2 has 6 2subsets that belong $W(b_k, b_l)$. If $Card(A_2) < 4$, $Card(A_1) \geq 3$, and every 3-subset of A_1 has 3 2-subset belong $W(b_k, b_l)$. Let A'_1 be such a subset. Then $A'_1 U \{w_y\}$ is a 4-subset that has 6 2-subsets in $W(b_k, b_l)$. Therefore there exists a subgraph χ of G, having 6 crossings, where one partite is $\{b_k, b_l\}$ and the other partite is a 4-subset.□

There are 3 distinct $K_{2,5}$ in G that have χ as a subgraph so each of them must have at least 6 crossings.

We will also need the following lemma :

Lemma 2 *Let $D_{5,z}$ be an arbitrary drawing of some $K_{5,z}$ and let t_0 be an element of the partite F with z elements and let T be the set of all elements of F having the same cyclic ordering of edges incident on them as t_0. The elements of T are $t_0, t_1, \ldots$. Let η be the number of pairs $\{t_k \in T, t_{k'} \in T\}$ such that $cr_{D_{5,z}}(t_k, t_{k'}) \geq 6 \geq Z(3, 5) + 2$.*
Then $Z(5, z) + 2\eta \leq cr_{D_{5,z}}$.

Proof:Let $h(t_k)$ be the sum of the number of crossings of the edges of t_k with edges not incident on any vertex of T. Let t_{min} be the element of T such that $h(t_{min}) \leq h(t_k)$ for all $t_k \in T$. Using a construction used by Kleitman,[2] ("Constructive Argument " p319), we can obtain from $D_{5,z}$ another drawing $D'_{5,z}$ where the position of the edges not incident on a vertex of T remains unchanged, and $h(t_k) = h(t_{min})$ for all k, and $cr_{D'_{5,z}}(t_k, t_{k'}) = Z(3, 5)$. (The construction mainly consists at placing t_k close to t_{min} and letting the edges incident on t_k follow the path of the corresponding edges of t_{min}.) Therefore

$$Z(5, z) + 2\eta \leq cr_{D'_{5,z}} + 2\eta \leq cr_{D_{5,z}}$$

Let θ be the number of distinct $K_{2,5}$ subgraphs of $K_{m,n}$ having at least 6 crossings in D and such that the partite with 2 elements is a subset of B, and the partite with 5 elements is a subset of A, and let σ have the same definition as it had in the proof of (3). Then $\sigma \geq \binom{m}{5} Z(5, n) + 2\theta$.
In the following lemma, A' is defined in the same way as in the proof of Lemma 1.

Lemma 3 *Let λ be the number of distinct pairs of elements of B having the same cyclic ordering of edges incident on an element of A'. Then, if $n \geq 2 \times 6!$, we have*

$$\lambda \geq (6!) \binom{\lfloor n/6! \rfloor}{2}$$

Proof : Let α_1 and α_2 be two distinct elements of (S_7^*) and let n_{α_1} be the number of vertices of B having the cyclic ordering of their edges equal to α_1 and let n_{α_2} be the number of vertices of B having the cyclic ordering of their edges equal to α_2. If $n_{\alpha_1} - n_{\alpha_2} \geq 2$, it is always possible to reduce the number of pairs of vertices having the same cyclic ordering of edges by assigning α_2 to one of the vertices which cyclic ordering of edges was α_1. So the minimum possible number of pairs of vertices having the same cyclic ordering of edges can only occur if $|n_{\alpha_1} - n_{\alpha_2}| = 1$ or $|n_{\alpha_1} - n_{\alpha_2}| = 0$ for all $\{\alpha_1, \alpha_2\} \subset S_7^*$. Therefore

$$\lambda \geq (6!)\binom{\lfloor n/6! \rfloor}{2}$$

Let ν be the minimum number of $K_{2,5}$ subgraphs of a $K_{2,7}$ whose number of crossings must be at least 6 when cyclic ordering of the 7 edges around the vertices of the partite with 2 elements are identical. As noted previously $\nu = 3$.
$\binom{m-5}{2}$ is the number of $K_{2,7}$ subgraphs of $K_{n,m}$ in which a given $K_{2,5}$ appears.

$$\theta \geq \frac{\binom{m}{7}\lambda\nu}{\binom{m-5}{2}}$$

From the above we can deduce a new lower bound on the crossing number of $K_{m,n}$. We have:

$$cr(K_{m,n}) \geq \frac{\sigma}{\binom{m-2}{3}}$$

$$\sigma \geq \binom{m}{5} Z(5,n) + 2\theta$$

$$cr(K_{m,n}) \geq \frac{\binom{m}{5} Z(5,n) + 2\lambda\nu\binom{m}{7}/\binom{m-5}{2}}{\binom{m-2}{3}} \tag{4}$$

For sufficiently large m and n, we therefore have :

$$cr(K_{m,n}) \geq (1/5)m(m-1)\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 9.9 \times 10^{-6} m^2 n^2$$

3 Conclusion

In this paper, we have proved a new lower bound on the crossing number of $K_{m,n}$ by proving the existence of certain non optimal drawings of $K_{5,n}$ subgraphs in any drawing of $K_{m,n}$. By proving the existence of other non optimal drawings, we might perhaps get improvements on the current lower bound. So this method could be one possible way to progress on Zarankiewicz conjecture.

References

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