Domination, packing and excluded minors

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Abstract

Let $\gamma(G)$ be the domination number of a graph G, and let $\alpha_k(G)$ be the maximum number of vertices in G, no two of which are at distance $\le k$ in G. It is easy to see that $\gamma(G) \ge \alpha_2(G)$. In this note it is proved that $\gamma(G)$ is bounded from above by a linear function in $\alpha_2(G)$ if G has no large complete bipartite graph minors. Extensions to other parameters $\alpha_k(G)$ are also derived.

1 Introduction and main results

Let G be a finite undirected graph. A graph H is a *minor* of G if it can be obtained from a subgraph of G by contracting edges. The *distance* $\mathrm{dist}_G(x, y)$ in G of two vertices $x, y \in V(G)$ is the length of a shortest (x, y)-path in G. The distance of a vertex x from a set $A \subseteq V(G)$ is $\min\{\mathrm{dist}_G(x, a) \mid a \in A\}$.

For a set $A \subseteq V(G)$, $G(A)$ denotes the subgraph of G induced by A. If k is a nonnegative integer, we denote by $N_k(A)$ the set of all vertices of G which are at distance $\le k$ from A. The set A is a *k-dominating set* in G if $N_k(A) = V(G)$. The cardinality of a smallest k-dominating set of G is denoted by $\gamma_k(G)$. A vertex set $X_0 \subseteq V(G)$ is an *α_k-set* if no two vertices in X_0 are at distance $\le k$ in G. Let $\alpha_k(G)$ denote the cardinality of a largest α_k-set of G. Observe that $\gamma(G) = \gamma_1(G)$ and $\alpha(G) = \alpha_1(G)$ are the usual *domination number* and the *independence* (or *stability*) *number* of G. We refer to [3] for further details on domination in graphs.

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It is clear that $\gamma_k(G) \geq \alpha_{2k}(G)$. On the other hand, for any r there is a graph such that $\alpha_{k+1}(G) = 1$ and $\gamma_k(G) \geq r$. In order to see this, let H_n be the Cartesian product of $k+1$ copies of the complete graph K_n. Then any two vertices of H_n have distance at most $k+1$ in H_n. Therefore, $\alpha_{k+1}(H_n) = 1$. Since $\deg_{H_n}(x) = (k+1)(n-1)$ and $|V(H_n)| = n^{k+1}$, it follows that $\gamma_k(H_n) \geq n/(k+1)^k$. So, $\gamma_k(H_n) \geq r$ if $n \geq r(k+1)^k$.

The main result of the present note is the following theorem which gives a linear upper bound on $\gamma_k(G)$ in terms of $\alpha_m(G)$, $k \leq m < \frac{5}{4}(k+1)$, in any set of graphs with a fixed excluded minor.

Theorem 1.1 *Let $k \geq 0$ and $m \geq 1$ be integers such that $k \leq m < \frac{5}{4}(k+1)$. If $\gamma_k(G) \geq (2mr + (q-1)(mr - r + 1))\alpha_m(G) - 2mr + r + 1$, then G has a $K_{q,r}$-minor.*

Our original motivation was the case when $k = 1$ and $m = 2$.

Corollary 1.2 *If $\gamma(G) \geq (4r + (q-1)(r+1))\alpha_2(G) - 3r + 1$, then G has a $K_{q,r}$-minor.*

By excluding $K_{3,3}$-minors, we get:

Corollary 1.3 *If G is a planar graph, then $\gamma(G) \leq 20\alpha_2(G) - 9$.*

The existence of a linear bound $\gamma(G) \leq c_1\alpha_2(G) + c_2$ for planar graphs was conjectured by F. Göring (private communication) who proved such a bound for plane triangulations.

An improvement of a very special case of Corollary 1.3 was obtained by MacGillivray and Seyffarth [4] who proved that a planar graph of diameter at most 2 has domination number at most three. Observe that a graph G has diameter at most 2 if and only if $\alpha_2(G) = 1$. They extend this result to planar graphs of diameter 3 by using an observation that in every planar graph of diameter 3, $\alpha_2(G) \leq 4$. See also [2] for further results in this direction.

Corollary 1.3 can be generalized to graphs on any surface. Since the graph $K_{3,k}$ cannot be embedded in a surface of Euler genus $g \leq (k-3)/2$ the following bound holds:

Corollary 1.4 *Suppose that G is a graph embedded in a surface of Euler genus g. Then $\gamma(G) \leq 4(2g+5)\alpha_2(G) - 9$.*

The special case of Theorem 1.1 when $k = 0$ and $m = 1$ is also interesting. The proof of Theorem 1.1 in this special case yields an even stronger statement since the sets $A_1, \ldots, A_r$ in that proof are mutually at distance 1 and hence, in the constructed minor $K_{q,r}$, any two of the r vertices in the second bipartition class are adjacent. Since $\gamma_0(G) = |V(G)|$, the following result is obtained:

Corollary 1.5 *Let $K_{q,r}^+$ be the graph obtained from $K_{q,r}$ by adding the r-clique on the vertex set of the bipartition class of cardinality r. Suppose that $K_{q,r}^+$ is not a minor of G. Then*

$$\alpha(G) \geq \frac{|V(G)| + r}{2r + q - 1}.$$

Duchet and Meyniel [1] obtained a special case of Corollary 1.5 when $q \le 1$. (Note that $K^+_{1,r-1} = K^+_{0,r} = K_r$.) They proved that in a graph G without K_r minor

$$\alpha(G) \ge \frac{|V(G)| + r - 1}{2r - 2}. \tag{1}$$

As it turns out, our proof of Theorem 1.1 restricted to this special case is quite similar to Duchet and Meyniel's proof.

Although Theorem 1.1 does not work for the case $k = 1$ and $m = 3$, the following result can be used to get such an extension:

Corollary 1.6 *Let $k \ge 0$ be an integer and let G be a graph. Let r be the largest integer such that K_r is a minor of G. Then*

$$\alpha_{2k}(G) \le r(2\alpha_{2k+1}(G) - 1).$$

Proof. Let S be a maximum α_{2k}-set in G. Define a graph H with $V(H) = S$ in which two vertices x, y are adjacent if and only if $\mathrm{dist}_G(x, y) = 2k + 1$. Suppose that K is a subgraph of H. Let K' be a subgraph of G obtained by taking vertices in $V(K)$ and, for each edge xy of K, adding a path of length $2k + 1$ in G joining x and y. Since all such paths are geodesics of odd length $2k + 1$, they cannot intersect each other. This implies that K' is a subdivision of K. In particular, if H has a K_r minor, so does G.

Clearly, $\alpha(H) \le \alpha_{2k+1}(G)$. Since $|V(H)| = \alpha_{2k}(G)$, (1) implies that H contains K_r minor, where $r \ge \alpha_{2k}(G)/(2\alpha_{2k+1}(G) - 1)$. Then also G contains a K_r minor, and this completes the proof. □

The relation between α_{2k} and α_{2k+1} in Corollary 1.6 cannot be extended to α_{2k+1} and α_{2k+2} as shown by the following examples (which are all planar and hence $K_{3,3}$ minor free). Let T_k be the tree obtained from the star $K_{1,p}$ $(p \ge 1)$ by replacing each edge by a path of length $k + 1$. Then $\gamma_k(T_k) = p$ (if $k \ge 1$), $\alpha_{2k+1}(T_k) = p$, and $\alpha_{2k+2}(T_k) = 1$. This example also shows that Theorem 1.1 cannot be extended to the value $m = 2k + 2$ if $k \ge 1$.

2 Proof of Theorem 1.1

In this section, k and m will denote fixed nonnegative integers such that $k \le m \le 2k+1$. Let G be a graph, and $A \subseteq V(G)$. Let $Q = Q^m_k(A)$ be the subgraph of G which is obtained from the vertex set $U = U_k(A) := V(G) \setminus N_k(A)$ by adding vertices and edges of all paths of length $\le m$ in G which connect two vertices in U. Since $m \le 2k + 1$, $V(Q) \cap A = \emptyset$. Observe that $U = \emptyset$ if and only if A is a k-dominating set of G.

An *extended α_m-pair* with respect to A and k is a pair (X, X_0) where $X_0 \subseteq X \subseteq V(G)$ such that:

(a) $X_0 \subseteq U_k(A)$ is an α_m-set in G and every vertex in $U_k(A)$ is at distance $\le m$ from X_0.

(b) Every vertex of $X \setminus X_0$ lies on an (X_0, X_0)-path in $Q = Q_k^m(A)$ which is of length $\leq 2m$.

(c) Every component of Q contains precisely one connected component of $Q(X)$.

Observe that by (a), $X_0 \neq \emptyset$ if A is not k-dominating.

Lemma 2.1 *If $k \leq m \leq 2k+1$ and $A \subseteq V(G)$, then there exists an extended α_m-pair (X, X_0) with respect to A and k. If $m \geq 1$ and A is not k-dominating, then $|X| \leq 2m|X_0| - 2m + 1$.*

Proof. If A is k-dominating, then $X_0 = X = \emptyset$ will do. If $m = 0$, then $X_0 = X = U_k(A)$. Suppose now that A is not k-dominating and that $m \geq 1$. Let B be a component of Q. Let $B_0 = B \cap G(U)$ and $V_0 = V(B_0)$. Let us build a set $X \subseteq V(B)$ and the corresponding α_m-set $X_0 \subseteq V_0$ as follows. Start with $X = X_0 = \{v\}$, where $v \in V_0$. If there exists a vertex of V_0 at distance in B at least $m+1$ from the current set X_0, let $u \in V_0$ be one of such vertices chosen such that its distance in B from X_0 is minimum possible. Observe that $\mathrm{dist}_G(u, X_0) \geq m+1$ although the distance in G may be smaller than the distance in B.

Let $u_0 u_1 \ldots u_r$ be a shortest path in B from X_0 (so $u_0 \in X_0$) to $u = u_r \in V_0$. Then $\mathrm{dist}_B(u_i, X_0) = i$ for $i = 0, \ldots, r$. Suppose that $r > 2m$. The vertices $u_{m+1}, \ldots, u_{r-1}$ do not belong to V_0 since their distance from X_0 is $\geq m+1$ but smaller than the distance between u and X_0. Let $p = r - \lfloor \frac{m}{2} \rfloor - 1$. By the definition of B, the edge $u_p u_{p+1}$ lies on a path of length $\leq m$ joining two vertices of V_0. In particular, an end u' of this edge is at distance $\leq \lceil \frac{m}{2} \rceil - 1$ from a vertex $u'' \in V_0$. If $\mathrm{dist}_B(u'', X_0) \leq m$, then $\mathrm{dist}_B(u, X_0) \leq \mathrm{dist}_B(u, u') + \mathrm{dist}_B(u', u'') + \mathrm{dist}_B(u'', X_0) \leq (\lfloor \frac{m}{2} \rfloor + 1) + (\lceil \frac{m}{2} \rceil - 1) + m < r$. This contradiction shows that $\mathrm{dist}_B(u'', X_0) \geq m+1$. However, $\mathrm{dist}_B(u'', X_0) \leq \mathrm{dist}_B(u'', u') + \mathrm{dist}_B(u', X_0)$. If m is even, this implies that $\mathrm{dist}_B(u'', X_0) < r$. If m is odd, then we may assume that $u' = u_p$, and then the same conclusion holds. This contradiction to the choice of u implies that $\mathrm{dist}_B(u, X_0) = r \leq 2m$.

Let us add u into X_0 and add the vertices $u_0, u_1, \ldots, u_r$ into the set X. This procedure gives rise to an extended α_m-pair inside B. Clearly, $|X| \leq 2m|X_0| - 2m + 1$.

By taking the union of such sets constructed in all components of Q, an appropriate extended α_m-pair is obtained. □

Proof of Theorem 1.1. By Lemma 2.1, there are pairwise disjoint vertex sets $A_1, A_2, \ldots, A_r$ such that (A_1, A_1^0) is an extended α_m-pair with respect to k and $A^{(1)} = \emptyset$, and (A_i, A_i^0) is an extended α_m-pair with respect to k and the set $A^{(i)} := A_1 \cup \cdots \cup A_{i-1}$, for $i = 2, \ldots, r$. Moreover, $|A_i| \leq 2m\alpha_m - 2m + 1$, where $\alpha_m = \alpha_m(G)$. Suppose that $\gamma_k(G) \geq (2mr + (q-1)(mr - r + 1))\alpha_m - 2mr + r + 1$. Then $\gamma_k(G) > (2m\alpha_m - 2m + 1)(r-1)$, so $A^{(r)}$ is not a k-dominating set. Therefore, $A_1, \ldots, A_r$ are all nonempty.

For $i = 1, \ldots, r$, let $H_i = Q_k^m(A^{(i)})$. Let $H_r^1, \ldots, H_r^t$ be the connected components of H_r. If $i \geq 2$, then $H_i \subseteq H_{i-1}$. This implies that each component of H_i is contained in some component of H_{i-1}. For $j = 1, \ldots, t$, let H_i^j be the component of H_i containing H_r^j.

By (c), each H_i^j contains a component C_i^j of $H_i(A_i)$. Each C_r^j contains at least one vertex from the α_m-set A_r^0. Therefore, $t \le \alpha_m$.

Let $B_1 = A_1 \cup \cdots \cup A_r$. Since $\gamma_k(G) > r(2m\alpha_m - 2m + 1)$, B_1 is not k-dominating. Hence, there is a vertex $v_1 \in U_k(B_1)$. By (a), v_1 is at distance $\le m$ from some component C_r^j $(1 \le j \le t)$ of $H_r(A_r)$. Then $H_r^j, H_{r-1}^j, \ldots, H_1^j$ are the components of $H_r, H_{r-1}, \ldots, H_1$ (respectively) containing C_r^j. For any of the components H_i^j $(1 \le i \le r)$, there is a path P_i^1 in G of length $\le m$ connecting v_1 with $C_i^j \subseteq H_i^j$. Let B_2 be the union of B_1 with $\{v_1\}$ and the internal vertices of the paths $P_1^1, P_2^1, \ldots, P_r^1$. Let us repeat the process with B_2 instead of B_1 to obtain a vertex $v_2 \in U_k(B_2)$ and linking paths $P_1^2, P_2^2, \ldots, P_r^2$ of length $\le m$ joining v_2 with $A_1, A_2, \ldots, A_r$, respectively.

Now, repeat the process by constructing B_3, obtaining v_3 and paths $P_1^3, P_2^3, \ldots, P_r^3$, and so on, as long as possible. This way we get a sequence of vertices $v_1, v_2, \ldots, v_s$ and paths of length $\le m$ joining these vertices with $A_1, \ldots, A_r$. The only requirement which guarantees the existence of $v_1, \ldots, v_s$ and the corresponding paths is that $\gamma_k(G) > r(2m\alpha_m - 2m + 1) + (s-1)(1 + r(m-1))$. Since $\gamma_k(G) > (2mr + (q-1)(mr - r + 1))\alpha_m - 2mr + r$, we may take $s > (q-1)\alpha_m \ge (q-1)t$. Then q of the vertices among $v_1, \ldots, v_s$ correspond to the same component C_r^j, say to C_r^1. Suppose that these vertices are $v_1, \ldots, v_q$.

Let us now consider two vertices v_i, v_j $(1 \le i < j \le q)$ and two of their paths P_a^i and P_b^j where $a \ne b$. Suppose that they intersect in a vertex v. Denote by $x = \mathrm{dist}_G(v_i, v)$, $y = \mathrm{dist}_G(v, A_a)$, $z = \mathrm{dist}_G(v_j, v)$, and $w = \mathrm{dist}_G(v, A_b)$. Then $x + y \le m$ and $z + w \le m$. This implies that

$$x + y + z + w \le 2m. \tag{2}$$

The choice of v_i and v_j was made in such a way that $z \ge k+1$, $x + v \ge k+1$, and $x + y \ge k+1$. Moreover, $y + w \ge \mathrm{dist}_G(A_a, A_b) \ge k+1$.

Suppose that $x \ge \frac{1}{2}(k+1)$. Then (2) and the inequalities after that imply that $2m \ge x + 2(k+1) \ge \frac{5}{2}(k+1)$. Similarly, if $x \le \frac{1}{2}(k+1)$, then $2m \ge 3(k+1) - x \ge \frac{5}{2}(k+1)$.

Consequently, P_a^i and P_b^j cannot intersect if $2m < \frac{5}{2}(k+1)$. In such a case it is easy to verify that vertices $v_1, \ldots, v_q$, the connected subgraphs $C_1^1, C_2^1, \ldots, C_r^1$ and the linking paths P_a^i $(1 \le i \le q,\ 1 \le a \le r)$ give rise to a $K_{q,r}$-minor in G. This completes the proof of Theorem 1.1. $\square$

References

[1] P. Duchet, H. Meyniel, On Hadwiger's number and the stability number, in "Graph theory (Cambridge, 1981)," pp. 71–73, North-Holland Math. Stud. 62, North-Holland, 1982.

[2] W. Goddard, M. A. Henning, Domination in planar graphs with small diameter, J. Graph Theory 40 (2002) 1–25.

[3] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[4] G. MacGillivray, K. Seyffarth, Domination numbers of planar graphs, J. Graph Theory 22 (1996) 213–229.