An Apéry-like difference equation for Catalan's constant

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Abstract

Applying Zeilberger's algorithm of creative telescoping to a family of certain very-well-poised hypergeometric series involving linear forms in Catalan's constant with rational coefficients, we obtain a second-order difference equation for these forms and their coefficients. As a consequence we derive a new way of fast calculation of Catalan's constant as well as a new continued-fraction expansion for it. Similar arguments are put forward to deduce a second-order difference equation and a new continued fraction for $\zeta(4) = \pi^4/90$.

1 Introduction

One of the most crucial and quite mysterious ingredients in Apéry's proof [1], [8] of the irrationality of $\zeta(2)$ and $\zeta(3)$ is the existence of the difference equations

$$\begin{gathered}(n+1)^2 u_{n+1} - (11n^2 + 11n + 3)u_n - n^2 u_{n-1} = 0, \\ u'_0 = 1, \quad u'_1 = 3, \qquad v'_0 = 0, \quad v'_1 = 5,\end{gathered} \tag{1}$$

and

$$\begin{gathered}(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5)u_n + n^3 u_n = 0, \\ u''_0 = 1, \quad u''_1 = 5, \qquad v''_0 = 0, \quad v''_1 = 6,\end{gathered}$$

with the following properties of their solutions:

$$\lim_{n\to\infty} \frac{v'_n}{u'_n} = \zeta(2), \qquad \lim_{n\to\infty} \frac{v''_n}{u''_n} = \zeta(3).$$

Unexpected inclusions $u_n', D_n^2 v_n' \in \mathbb{Z}$ and $u_n'', D_n^3 v_n'' \in \mathbb{Z}$, where D_n denotes the least common multiple of the numbers $1, 2, \dots, n$ (and $D_0 = 1$ for completeness), together with the prime number theorem ($D_n^{1/n} \to e$ as $n \to \infty$) and Poincaré's theorem, then yield the following asymptotic behaviour of the linear forms $D_n^2 u_n' \zeta(2) - D_n^2 v_n'$ and $D_n^3 u_n'' \zeta(3) - D_n^3 v_n''$ with *integral* coefficients:

$$\lim_{n\to\infty} |D_n^2 u_n' \zeta(2) - D_n^2 v_n'|^{1/n} = e^2 \left(\frac{\sqrt{5}-1}{2}\right)^5 < 1,$$
$$\lim_{n\to\infty} |D_n^3 u_n'' \zeta(3) - D_n^3 v_n''|^{1/n} = e^3 (\sqrt{2}-1)^4 < 1,$$

and thus one obtains that both $\zeta(2)$ and $\zeta(3)$ are irrational.

The two following decades after [1] were full of attempts to indicate the total list of the second-order recursions with integral solutions and to show their 'geometric' (i.e., Picard–Fuchs differential equations) origin. We do not pretend to be so heroic in this paper, and we apply quite elementary arguments to get new recurrence equations with 'almost-integral' solutions.

In our general, joint with T. Rivoal, study [9] of arithmetic properties for values of Dirichlet's beta function

$$\beta(s) := \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^s}$$

at positive integers s, we have discovered a construction of $\mathbb{Q}$-linear forms in 1 and Catalan's constant

$$G := \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^2} = \beta(2)$$

similar to the one considered by Apéry in his proof of the irrationality of $\zeta(2)$. The analogy is far from proving the desired irrationality of G, but it allows to indicate the following second-order difference equation

$$(2n+1)^2(2n+2)^2 p(n) u_{n+1} - q(n) u_n - (2n-1)^2(2n)^2 p(n+1) u_{n-1} = 0, \tag{2}$$

where

$$\begin{aligned} p(n) &= 20n^2 - 8n + 1, \\ q(n) &= 3520n^6 + 5632n^5 + 2064n^4 - 384n^3 - 156n^2 + 16n + 7, \end{aligned} \tag{3}$$

with the initial data

$$u_0 = 1, \quad u_1 = \frac{7}{4}, \qquad v_0 = 0, \quad v_1 = \frac{13}{8}. \tag{4}$$

Theorem 1. *For each $n = 0, 1, 2, \dots$, the numbers u_n and v_n produced by the recursion* (2), (4) *are positive rationals satisfying the inclusions*

$$2^{4n+3} D_n u_n \in \mathbb{Z}, \qquad 2^{4n+3} D_{2n-1}^3 v_n \in \mathbb{Z}, \tag{5}$$

and the following limit relation holds:

$$\lim_{n\to\infty} \frac{v_n}{u_n} = G.$$

The positivity and rationality of u_n and v_n follows immediately from (2)–(4). The characteristic polynomial $\lambda^2-11\lambda-1$ with zeros $\bigl((1\pm\sqrt5\,)/2\bigr)^5$ of the difference equation (2) coincides with the corresponding one for Apéry's equation (1). Therefore application of Poincaré's theorem (see also [12], Proposition 2) yields the limit relations

$$\lim_{n\to\infty}u_n^{1/n}=\lim_{n\to\infty}v_n^{1/n}=\biggl(\frac{1+\sqrt5}{2}\biggr)^5=\exp(2.40605912\dots),$$
$$\lim_{n\to\infty}|u_nG-v_n|^{1/n}=\biggl|\frac{1-\sqrt5}{2}\biggr|^5=\exp(-2.40605912\dots),$$

while the inclusions (5) and the prime number theorem imply that the linear forms $2^{4n+3}D_{2n-1}^3(u_nG-v_n)$ with integral coefficients do not tend to 0 as $n\to\infty$. However, the rational approximations v_n/u_n to Catalan's constant converge quite rapidly (for instance, $|v_{10}/u_{10}-G|<10^{-20}$) and one can use the recursion (2), (4) for fast evaluating G. Another consequence of Theorem 1 is a new continued-fraction expansion for Catalan's constant. Namely, considering v_n/u_n as convergents of a continued fraction for G and making the equivalent transform of the fraction ([6], Theorems 2.2 and 2.6) we arrive at

Theorem 2. *The following expansion holds:*

$$G=\frac{13/2|}{|q(0)}+\frac{1^4\cdot2^4\cdot p(0)p(2)|}{|q(1)}+\dots+\frac{(2n-1)^4(2n)^4p(n-1)p(n+1)|}{|q(n)}+\dotsb,$$

where the polynomials $p(n)$ and $q(n)$ are given in (3).

The multiple-integral representation for the linear forms u_nG-v_n similar to those obtained by F. Beukers in [4], formula (5), for the linear forms $u_n'\zeta(2)-v_n'$ is given by

Theorem 3. *For each $n=0,1,2,\dots,$ the identity*

$$u_nG-v_n=\frac{(-1)^n}{4}\int_0^1\int_0^1\frac{x^{n-1/2}(1-x)^ny^n(1-y)^{n-1/2}}{(1-xy)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

holds.

2 Difference equation for Catalan's constant

Consider the rational function

$$R_n(t):=n!(2t+n+1)\frac{t(t-1)\cdots(t-n+1)\cdot(t+n+1)\cdots(t+2n)}{\bigl((t+\frac12)(t+\frac32)\cdots(t+n+\frac12)\bigr)^3} \tag{7}$$

and the corresponding (very-well-poised) hypergeometric series

$$F_n := \sum_{t=0}^{\infty}(-1)^t R_n(t)$$
$$= (-1)^n n!\,\frac{\Gamma(3n+2)\,\Gamma(n+\frac12)^3\Gamma(n+1)}{\Gamma(2n+\frac32)^3\Gamma(2n+1)}$$
$$\times\,{}_6F_5\left(\begin{matrix}3n+1,\ \frac32 n+\frac32,\ n+\frac12,\ n+\frac12,\ n+\frac12,\ n+1\\ \frac32 n+\frac12,\ 2n+\frac32,\ 2n+\frac32,\ 2n+\frac32,\ 2n+1\end{matrix}\,\middle|\,-1\right). \tag{8}$$

Lemma 1. *The following equality holds:*

$$F_n = U_n\beta(3) + U_n'\beta(2) + U_n''\beta(1) - V_n, \tag{9}$$

where $U_n, D_nU_n', D_n^2U_n'', D_{2n-1}^3V_n \in 2^{-4n}\mathbb{Z}$.

Proof. We start with mentioning that

$$P_n^{(1)}(t) := \frac{t(t-1)\cdots(t-n+1)}{n!} \qquad \text{and} \qquad P_n^{(2)}(t) := \frac{(t+n+1)\cdots(t+2n)}{n!} \tag{10}$$

are integral-valued polynomials and, as it is known (see, e.g., [13], Lemma 7),

$$2^{2n}\cdot P_n(-k-\tfrac12)\in\mathbb{Z} \qquad \text{for} \quad k\in\mathbb{Z} \tag{11}$$

and, moreover,

$$2^{2n}D_n^j\cdot\frac{1}{j!}\,\frac{\mathrm{d}^jP_n(t)}{\mathrm{d}t^j}\bigg|_{t=-k-1/2}\in\mathbb{Z} \qquad \text{for} \quad k\in\mathbb{Z} \quad \text{and} \quad j=1,2,\dots, \tag{12}$$

where $P_n(t)$ is any of the polynomials (10).

The rational function

$$Q_n(t) := \frac{n!}{(t+\frac12)(t+\frac32)\cdots(t+n+\frac12)}$$

has also 'nice' arithmetic properties. Namely,

$$a_k := Q_n(t)(t+k+\tfrac12)\big|_{t=-k-1/2} = \begin{cases}(-1)^k\binom{n}{k}\in\mathbb{Z} & \text{if } k=0,1,\dots,n,\\ 0 & \text{for other } k\in\mathbb{Z},\end{cases} \tag{13}$$

that allow to write the following partial-fraction expansion:

$$Q_n(t) = \sum_{l=0}^{n}\frac{a_l}{t+l+\frac12}.$$

Hence, for $j = 1, 2, \ldots$ we obtain

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(Q_n(t)(t+k+\tfrac12)\big)\Big|_{t=-k-1/2} = \frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\sum_{l=0}^{n} a_l\bigg(1-\frac{l-k}{t+l+\frac12}\bigg)\bigg|_{t=-k-1/2}$$
$$= (-1)^{j-1}D_n^j\sum_{\substack{l=0\\ l\neq k}}^{n}\frac{1}{(l-k)^j} \in \mathbb{Z}. \qquad (14)$$

Therefore the inclusions (11)–(14) and the Leibniz rule for differentiating a product imply that the numbers

$$\begin{aligned} A_{jk} = A_{jk}(n) &:= \frac{1}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(R_n(t)(t+k+\tfrac12)^3\big)\Big|_{t=-k-1/2} \\ &= \frac{1}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big((2t+n+1)\cdot P_n^{(1)}(t)\cdot P_n^{(2)}(t)\cdot (Q_n(t)(t+k+\tfrac12))^3\big)\Big|_{t=-k-1/2} \end{aligned} \qquad (15)$$

satisfy the inclusions

$$2^{4n}D_n^j\cdot A_{jk}(n)\in\mathbb{Z} \qquad \text{for} \quad k=0,1,\ldots,n \quad \text{and} \quad j=0,1,2,\ldots. \qquad (16)$$

Mention now that the numbers (15) are coefficients in the partial-fraction expansion of the rational function (7),

$$R_n(t) = \sum_{j=0}^{2}\sum_{k=0}^{n}\frac{A_{jk}}{(t+k+\frac12)^{3-j}}. \qquad (17)$$

Substituting this expansion into the definition (8) of the quantity F_n we obtain the desired representaion (9):

$$\begin{aligned} F_n &= \sum_{j=0}^{2}\sum_{k=0}^{n}(-1)^k A_{jk}\sum_{t=0}^{\infty}\frac{(-1)^{t+k}}{(t+k+\frac12)^{3-j}} \\ &= \sum_{j=0}^{2}\sum_{k=0}^{n}(-1)^k A_{jk}\bigg(\sum_{l=0}^{\infty}-\sum_{l=0}^{k-1}\bigg)\frac{(-1)^l}{(l+\frac12)^{3-j}} \\ &= U_n\beta(3)+U_n'\beta(2)+U_n''\beta(1)-V_n, \end{aligned}$$

where

$$U_n = 2^3\sum_{k=0}^{n}(-1)^kA_{0k}(n), \quad U_n' = 2^2\sum_{k=0}^{n}(-1)^kA_{1k}(n), \quad U_n'' = 2\sum_{k=0}^{n}(-1)^kA_{2k}(n), \qquad (18)$$

$$V_n = \sum_{j=0}^{2}2^{3-j}\sum_{k=0}^{n}(-1)^kA_{jk}(n)\sum_{l=0}^{k-1}\frac{(-1)^l}{(2l+1)^{3-j}}. \qquad (19)$$

Finally, using the inclusions (16) and

$$D_{2n-1}^{3-j} \sum_{l=0}^{k-1} \frac{(-1)^l}{(2l+1)^{3-j}} \in \mathbb{Z} \qquad \text{for} \quad k=0,1,\ldots,n \quad \text{and} \quad j=0,1,2,$$

we deduce that $U_n, D_n U_n', D_n^2 U_n'', D_{2n-1}^3 V_n \in 2^{-4n}\mathbb{Z}$ as required. $\square$

Using Zeilberger's algorithm of creative telescoping ([7], Section 6) for the rational function (7), we obtain the certificate $S_n(t) := s_n(t)R_n(t)$, where

$$\begin{aligned} s_n(t) := \frac{1}{2(2t+n+1)(t+2n-1)(t+2n)} \cdot \big(&8n(2n-1)^2(20n^2+32n+13)t^4 \\ &+ 2(5440n^6+7104n^5+912n^4-1088n^3+76n^2+68n+7)t^3 \\ &+ (44800n^7+65600n^6+17568n^5-7056n^4-1088n^3+372n^2+146n-1)t^2 \\ &+ (2n+1)(34880n^7+39328n^6-2176n^5-8416n^4+964n^3+154n^2+58n-13)t \\ &+ n(2n-1)(2n+1)^2(4720n^5+6192n^4+816n^3-864n^2+69n+13)\big) \end{aligned} \tag{20}$$

satisfying the following property.

Lemma 2. *For each $n=1,2,\ldots,$ we have the identity*

$$\begin{aligned} &(2n+1)^2(2n+2)^2p(n)R_{n+1}(t) - q(n)R_n(t) - (2n-1)^2(2n)^2p(n+1)R_{n-1}(t) \\ &\qquad = -S_n(t+1) - S_n(t), \end{aligned} \tag{21}$$

where the polynomials $p(n)$ and $q(n)$ are given in (3).

Proof. Divide both sides of (21) by $R_n(t)$ and verify the identity

$$\begin{aligned} &(2n+1)^2(2n+2)^2p(n)\cdot(n+1)\frac{(2t+n+2)(t-n)(t+2n+1)(t+2n+2)}{(2t+n+1)(t+n+1)(t+n+\frac{3}{2})^3} - q(n) \\ &\qquad - (2n-1)^2(2n)^2p(n+1)\cdot\frac{(2t+n)(t+n)(t+n+\frac{1}{2})^3}{n(2t+n+1)(t-n+1)(t+2n-1)(t+2n)} \\ &= -s_n(t+1)\frac{(2t+n+3)(t+\frac{1}{2})^3(t+1)(t+2n+1)}{(2t+n+1)(t-n+1)(t+n+1)(t+n+\frac{3}{2})^3} - s_n(t), \end{aligned}$$

where $s_n(t)$ is given in (20). $\square$

Lemma 3. *The quantity* (8) *satisfies the difference equation* (2) *for $n=1,2,\ldots$.*

Proof. Multiplying both sides of the equality (21) by $(-1)^t$ and summing the result over $t=0,1,2,\ldots$ we obtain

$$(2n+1)^2(2n+2)^2p(n)F_{n+1} - b(n)F_n - (2n-1)^2(2n)^2p(n+1)F_{n-1} = -S_n(0).$$

It remains to note that, for $n \geq 1$, both functions $R_n(t)$ and $S_n(t) = s_n(t)R_n(t)$ have zero at $t=0$. Thus $S_n(0)=0$ for $n=1,2,\ldots$ and we obtain the desired recurrence (2) for the quantity (8). $\square$

Lemma 4. *The coefficients U_n, U'_n, U''_n, V_n in the representation* (9) *satisfy the difference equation* (2) *for $n = 1, 2, \ldots$.*

Proof. We can write down the partial-fraction expansion (17) in the form

$$R_n(t) = \sum_{j=1}^{4} \sum_{k=-\infty}^{+\infty} \frac{A_{jk}(n)}{(t + k + \frac{1}{2})^{3-j}},$$

where the formulae (15) remain true for *all* $k \in \mathbb{Z}$ (not for $k = 0, 1, \ldots, n$ only). Now, multiply both sides of (21) by $(-1)^k (t + k + \frac{1}{2})^3$, take the jth derivative, where $j = 0, 1, 2$, substitute $t = -k - \frac{1}{2}$ in the result, and sum over all integers k; this procedure implies that the numbers

$$U_n = 8 \sum_{k=-\infty}^{+\infty} (-1)^k A_{0k}(n), \quad U'_n = 4 \sum_{k=-\infty}^{+\infty} (-1)^k A_{1k}(n), \quad U''_n = 2 \sum_{k=-\infty}^{+\infty} (-1)^k A_{2k}(n)$$

(cf. (18)) satisfy the difference equation (2). Finally, the sequence

$$V_n = U_n \beta(3) + U'_n \beta(2) + U''_n \beta(1) - F_n$$

also satisfies the recursion (2) by Lemma 3 and the above. $\square$

Since

$$R_0(t) = \frac{2}{(t + \frac{1}{2})^2}, \qquad R_1(t) = -\frac{3/4}{(t + \frac{1}{2})^3} - \frac{3/4}{(t + \frac{3}{2})^3} + \frac{7/4}{(t + \frac{1}{2})^2} - \frac{7/4}{(t + \frac{3}{2})^2},$$

in accordance with (18), (19) we obtain

$$U'_0 = 8, \quad U_0 = U''_0 = V_0 = 0, \qquad \text{and} \qquad U'_1 = 14, \quad V_1 = 13, \quad U_1 = U''_1 = 0,$$

hence as a consequence of Lemma 4 we deduce that $U_n = U''_n = 0$ for $n = 0, 1, 2, \ldots$.

Lemma 5. *The following equality holds:*

$$F_n = U'_n G - V_n,$$

where $2^{4n} D_n U'_n \in \mathbb{Z}$ and $2^{4n} D_{2n-1}^3 V_n \in \mathbb{Z}$.

The sequences $u_n := U'_n/8$ and $v_n := V_n/8$ satisfy the difference equation (2) and initial conditions (4); the fact that $F_n \neq 0$ and $|F_n| \to 0$ as $n \to \infty$ follows from Theorem 3 and asymptotics of the multiple integral (6) proved in [4]. This completes the proof of Theorem 1.

3 Connection with ${}_3F_2$-hypergeometric series

The corresponding very-well-poised hypergeometric series (8) at $z = -1$ can be reduced to a simpler series with the help of Whipple's transform ([2], Section 4.4, formula (2)):

$$
{}_3F_2\left(\begin{matrix} 1+a-b-c,\, d,\, e \\ 1+a-b,\, 1+a-c \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(1+a)\,\Gamma(1+a-d-e)}{\Gamma(1+a-d)\,\Gamma(1+a-e)}
$$
$$
\times\, {}_6F_5\left(\begin{matrix} a,\, 1+\frac{1}{2}a, & b, & c, & d, & e \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} \,\middle|\, -1\right),
$$

if $\mathrm{Re}(1+a-d-e) > 0$. Namely, in the case $a = 3n+1$, $b = c = d = n+\frac{1}{2}$, and $e = n+1$, we obtain

$$
F_n = U_n' G - V_n = (-1)^n \cdot 2 \int_0^1 \int_0^1 \frac{x^{n-1/2}(1-x)^n y^n (1-y)^{n-1/2}}{(1-xy)^{n+1}}\, \mathrm{d}x\, \mathrm{d}y,
$$

where the Euler-type integral representation for the ${}_3F_2$-series can be derived as in [10], Section 4.1, and [3], the proof of Lemma 2.

This completes the proof of Theorem 3.

4 Concluding remarks

The conclusion (5) of Theorem 1 is far from being precise; in fact, using (2), (4) one gets experimentally (up to $n = 1000$, say) the stronger inclusions[1]

$$
2^{4n} u_n \in \mathbb{Z}, \qquad 2^{4n} D_{2n-1}^2 v_n \in \mathbb{Z}.
$$

Unfortunately, they also give no chance to prove that Catalan's constant is irrational since linear forms $2^{4n} D_{2n-1}^2 (u_n G - v_n)$ do not tend to 0 as $n \to \infty$.

In the same vein, using another very-well-poised series of hypergeometric type

$$
\begin{aligned}
\widetilde{F}_n &:= \frac{(-1)^{n+1}}{6} \sum_{t=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left((2t+n)\frac{((t-1)\cdots(t-n))^2 \cdot ((t+n+1)\cdots(t+2n))^2}{(t(t+1)\cdots(t+n))^4}\right) \\
&= \widetilde{u}_n \zeta(4) - \widetilde{v}_n
\end{aligned}
$$

and the arguments of Section 2, we deduce the difference equation

$$
(n+1)^5 u_{n+1} - r(n) u_n - 3n^3(3n-1)(3n+1) u_{n-1} = 0, \tag{22}
$$

where

$$
\begin{aligned}
r(n) &= 3(2n+1)(3n^2+3n+1)(15n^2+15n+4) \\
&= 270n^5 + 675n^4 + 702n^3 + 378n^2 + 105n + 12,
\end{aligned} \tag{23}
$$

[1] A slightly weakened form of the inclusions is proved in [15].

with the initial data

$$\widetilde{u}_0 = 1, \quad \widetilde{u}_1 = 12, \qquad \widetilde{v}_0 = 0, \quad \widetilde{v}_1 = 13$$

for its two independent solutions $\widetilde{u}_n$ and $\widetilde{v}_n$,

Theorem 4 ([14])**.** *For each $n = 0, 1, 2, \ldots$, the numbers $\widetilde{u}_n$ and $\widetilde{v}_n$ are positive rationals satisfying the inclusions*

$$6D_n \widetilde{u}_n \in \mathbb{Z}, \qquad 6D_n^5 \widetilde{v}_n \in \mathbb{Z}, \tag{24}$$

and the following limit relation holds:

$$\lim_{n\to\infty} \frac{\widetilde{v}_n}{\widetilde{u}_n} = \frac{\pi^4}{90} = \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}. \tag{25}$$

Remark. During the preparation of this article, we have known that the difference equation (22), in slightly different normalization, and the limit relation (25) without the inclusions (24) had been stated independently by H. Cohen and G. Rhin [5] using Apéry's 'accélération de la convergence' approach, and by V. N. Sorokin [11] by means of certain explicit Hermite–Padé-type approximations. We underline that our approach differs from that of [5] and [11].

Application of Poincaré's theorem yields the asymptotic relations

$$\lim_{n\to\infty} |\widetilde{u}_n|^{1/n} = \lim_{n\to\infty} |\widetilde{v}_n|^{1/n} = (3 + 2\sqrt{3}\,)^3 = \exp(5.59879212\ldots)$$

and

$$\lim_{n\to\infty} |\widetilde{u}_n \zeta(4) - \widetilde{v}_n|^{1/n} = |3 - 2\sqrt{3}\,|^3 = \exp(-2.30295525\ldots),$$

since the characteristic polynomial $\lambda^2 - 270\lambda - 27$ of the equation (22) has zeros $135 \pm 78\sqrt{3} = (3 \pm 2\sqrt{3}\,)^3$. Thus, we can consider $\widetilde{v}_n/\widetilde{u}_n$ as convergents of a continued fraction for $\zeta(4)$ and making the equivalent transform of the fraction we obtain

Theorem 5. *The following expansion holds:*

$$\zeta(4) = \frac{13|}{|r(0)} + \frac{1^7 \cdot 2 \cdot 3 \cdot 4|}{|r(1)} + \frac{2^7 \cdot 5 \cdot 6 \cdot 7|}{|r(2)} + \cdots + \frac{n^7(3n-1)(3n)(3n+1)|}{|r(n)} + \cdots,$$

where the polynomial $r(n)$ is given in (23)*.*

The mystery of the $\zeta(4)$-case consists in the fact that experimental calculations give us the better inclusions

$$\widetilde{u}_n \in \mathbb{Z}, \qquad D_n^4 \widetilde{v}_n \in \mathbb{Z}$$

(cf. (24)); unfortunately, the linear forms $D_n^4(\widetilde{u}_n \zeta(4) - \widetilde{v}_n)$ do not tend to 0 as $n \to \infty$.

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