A Schröder Generalization of Haglund's Statistic on Catalan Paths

E. S. Egge*

Department of Mathematics
Gettysburg College, Gettysburg, PA 17325
eggee@member.ams.org

J. Haglund

Department of Mathematics
University of Pennsylvania, Philadelphia, PA 19104
jhaglund@math.upenn.edu

K. Killpatrick

Mathematics Department
Pepperdine University, Malibu, CA 90263-4321
Kendra.Killpatrick@pepperdine.edu

D. Kremer†

Department of Mathematics
Gettysburg College, Gettysburg, PA 17325
dkremer@gettysburg.edu

Submitted: May 6, 2002; Accepted: Apr 17, 2003; Published: Apr 23, 2003
MR Subject Classifications: 05A15, 05E05

Abstract

Garsia and Haiman (*J. Algebraic. Combin.* **5** (1996), 191 − 244) conjectured that a certain sum $C_n(q,t)$ of rational functions in q, t reduces to a polynomial in q, t with nonnegative integral coefficients. Haglund later discovered (*Adv. Math.*, in press), and with Garsia proved (*Proc. Nat. Acad. Sci.* **98** (2001), 4313 − 4316) the refined conjecture $C_n(q,t) = \sum q^{\mathrm{area}} t^{\mathrm{bounce}}$. Here the sum is over all Catalan lattice paths and area and bounce have simple descriptions in terms of

*Partially supported by a Gettysburg College Professional Development Grant.
†Partially supported by a Gettysburg College Professional Development Grant.

the path. In this article we give an extension of (area, bounce) to Schröder lattice paths, and introduce polynomials defined by summing $q^{\mathrm{area}}t^{\mathrm{bounce}}$ over certain sets of Schröder paths. We derive recurrences and special values for these polynomials, and conjecture they are symmetric in q, t. We also describe a much stronger conjecture involving rational functions in q, t and the ∇ operator from the theory of Macdonald symmetric functions.

1 Introduction

In the early 1990's Garsia and Haiman introduced an important sum $C_n(q,t)$ of rational functions in q, t which has since been shown to have interpretations in terms of algebraic geometry and representation theory. This rational function is defined explicitly in section 4; for now we wish to note that it follows easily from this definition that $C_n(q,t)$ is symmetric in q and t. Garsia and Haiman conjectured $C_n(q,t)$ reduces to a polynomial in q, t with nonnegative integral coefficients [GH96], and called $C_n(q,t)$ the q,t-Catalan polynomial since $C_n(1,1)$ equals the nth Catalan number. The special cases $C_n(q,1)$ and $C_n(q,1/q)$ yield two different q-analogs of the Catalan numbers, introduced by Carlitz and Riordan, and MacMahon, respectively [CR64],[Mac01]. Haglund [Hag] introduced the refined conjecture $C_n(q,t) = \sum q^{\mathrm{area}}t^{\mathrm{bounce}}$, where area and bounce are simple statistics on lattice paths described below. Garsia and Haglund later proved this conjecture by an intricate argument involving plethystic symmetric function identities [GH01],[GH02].

A natural question to consider is whether the lattice path statistics for $C_n(q,t)$ can be extended, in a way which preserves the rich combinatorial structure, to related combinatorial objects. In this article we show that many of the important properties of $C_n(q,t)$ appear to extend to a more general family of polynomials related to the *Schröder numbers*, which are close combinatorial cousins of the Catalan numbers.

A *Schröder path* is a lattice path from $(0,0)$ to (n,n) consisting of north N $(0,1)$, east E $(1,0)$, and diagonal D $(1,1)$ steps, which never goes below the line $y = x$. We let $\mathcal{S}_{n,d}$ denote the set of such paths consisting of d D steps, $n-d$ N steps and $n-d$ E steps. Throughout the remainder of this article, Π will denote a Schröder path. A Schröder path with no D steps is a *Catalan path*. We call a $45-90-45$ degree triangle with vertices (i,j), $(i+1,j)$ and $(i+1,j+1)$ for some i, j a "lower triangle" and the lower triangles below a path Π and above the line $y = x$ "area triangles". Define the area of Π, denoted $area(\Pi)$, to be the number of such triangles.

For $\Pi \in \mathcal{S}_{n,d}$, let pword($\Pi$) denote the sequence $\sigma_1 \cdots \sigma_{2n-d}$ where the ith letter σ_i is either an N, D, or E depending on whether the ith step (starting at $(0,0)$) of Π is an N, D, or E step, respectively. Furthermore let word(Π) denote the word of 2's, 1's and 0's obtained by replacing all N's, D's and E's in pword(Π) by 2's, 1's and 0's, respectively. By a *row* of Π we mean the region to the right of an N or D step and to the left of the line $y = x$. We let $\mathrm{row}_i(\Pi)$ denote the ith row, from the top, of Π. We call the number of area triangles in this row the *length* of the row, denoted $\mathrm{area}_i(\Pi)$. For example, the path on the left side of Figure 1 has pword $= NDNENDDENENEE$ and word $= 2120211020200$, with $\mathrm{area}_1(\Pi) = 1$, $\mathrm{area}_2(\Pi) = 1$, $\mathrm{area}_3(\Pi) = 2$, etc. Note $\mathrm{area}_n(\Pi) = 0$ for all $\Pi \in \mathcal{S}_{n,d}$.

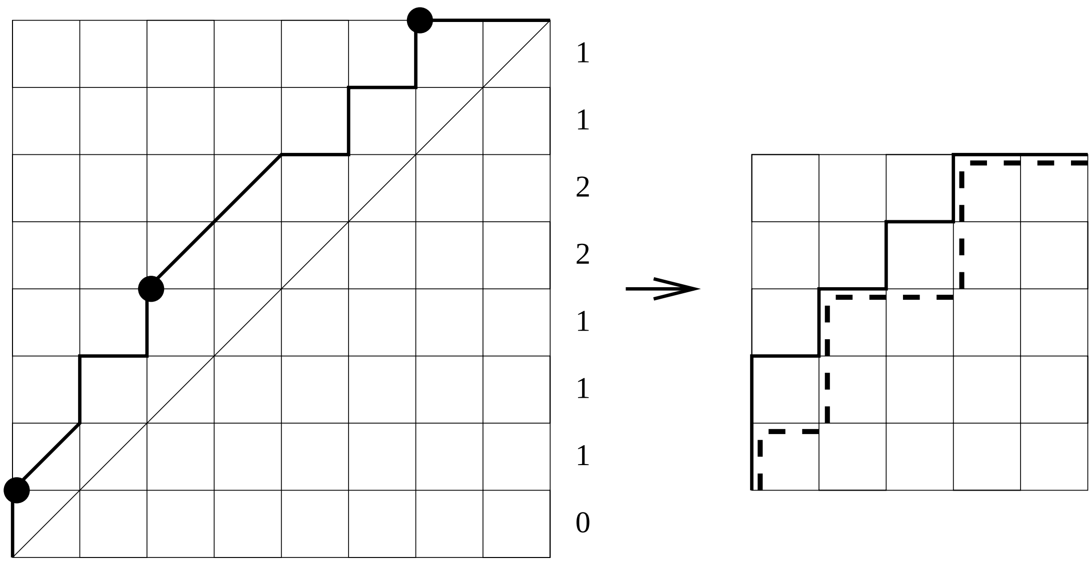

Figure 1: On the left, a Schröder path Π, with the top of each peak marked by a dot. To the right of each row is the length of the row. On the right is the Catalan path $C(\Pi)$ and its bounce path (the dotted path).

We now introduce what we call the *bounce statistic* for a path Π, denoted *bounce*(Π). To calculate this, we first form an associated Catalan path $C(\Pi)$ by deleting all D steps and collapsing the remaining path, so pword$(C(\Pi))$ is the same as pword(Π) with all D's removed. See Figure 1. Then we form the "bounce path" for $C(\Pi)$ (the dotted path in Figure 1) by starting at $(n-d, n-d)$, going left until we reach the top of an N step of $C(\Pi)$, then "bouncing" down to the line $y = x$, then iterating: left to the path, down to the line $y = x$, and so on until we reach $(0, 0)$. As we travel from $(n-d, n-d)$ to $(0, 0)$ our bounce path hits the line $y = x$ at various points, say at $(j_1, j_1), (j_2, j_2), \ldots, (j_k, j_k)$ ($(3, 3), (1, 1), (0, 0)$ in Figure 1) with $n - d > j_1 > \cdots > j_k = 0$.

We call the vector $(n-d-j_1, j_1-j_2, \ldots, j_{k-1})$ the *bounce vector* of Π. Geometrically, the ith coordinate of this vector is the length of the ith "bounce step" of our path. Note that the N steps of the bounce path which occur immediately after the bounce path changes from going west to south will also be N steps of $C(\Pi)$. The N steps of Π which correspond to these N steps of $C(\Pi)$ are called the *peaks* of Π. Specifically, for $1 \leq i \leq k$ we call the j_{i-1}th N step of Π peak i, with the convention that $j_0 = n - d$. Say Π has β_0 D steps above peak 1, β_k D steps below peak k, and for $1 \leq i \leq k-1$ has β_i D steps between peaks i and $i+1$. We call $(\beta_0, \beta_1, \ldots, \beta_k)$ the *shift vector* of Π. For example, the path of Figure 1 has bounce vector $(2, 2, 1)$ and shift vector $(0, 2, 1, 0)$.

Given the above definitions, our bounce statistic for Π is given by

$$\text{bounce}(\Pi) = \sum_{i=1}^{k-1} j_i + \sum_{i=1}^{k} i\beta_i. \tag{1}$$

(N. Loehr has observed that bounce(Π) also equals the sum, over all peaks p, of the number of squares to the left of p and to the right of the y axis). For the path on the left in Figure 1, area = 9 and bounce = 8. Note $\sum_i i\beta_i$ can be viewed as the sum, over all D steps g, of the number of peaks above g.

For $n, d \in \mathbb{N}$ let

$$S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{area(\Pi)} t^{bounce(\Pi)}. \tag{2}$$

Conjecture 1 *For all n, d,*

$$S_{n,d}(q,t) = S_{n,d}(t,q). \tag{3}$$

Conjecture 1 has been verified using Maple for all n, d such that $n + d \leq 10$.

If Π has no D steps, the area(Π) and bounce(Π) statistics reduce to their counterparts for Catalan paths. Thus Garsia and Haglund's result can be phrased as $C_n(q,t) = S_{n,0}(q,t)$, and since $C_n(q,t) = C_n(t,q)$ this implies Conjecture 1 is true when $d = 0$. It is an open problem to find a bijective proof of this case. We don't know how to prove Conjecture 1 for any value of $d > 0$ by any method. (Unless you let d depend on n; for example, the cases $d = n$ and $d = n - 1$ are simple to prove.)

When $t = 1$, $S_{n,d}(q,1)$ reduces to an "inversion based" q-analog of $S_{n,d}(1,1)$ studied by Bonin, Shapiro and Simion [BSS93] (See also [BLPP99]). In section 2 we derive a formula for $S_{n,d}(q,t)$ in terms of sums of products of q-trinomial coefficients, and obtain recurrences for the sum of $q^{area}t^{bounce}$ over subsets of Schröder paths satisfying various constraints. We then use these to prove inductively that when $t = 1/q$,

$$q^{\binom{n}{2} - \binom{d}{2}} S_{n,d}(q, 1/q) = \frac{1}{[n-d+1]_q} \begin{bmatrix} 2n-d \\ n-d, n-d \end{bmatrix}_q. \tag{4}$$

Here $\begin{bmatrix} m \\ a,b \end{bmatrix}_q := [m]!/([a]![b]![m-a-b]!)$ is the q-trinomial coefficient. Bonin, et. al. showed that [BSS93]

$$\sum_{\Pi \in \mathcal{S}_{n,d}} q^{\mathrm{maj}(\mathrm{word}(\Pi))} = \frac{q^{n-d}}{[n-d+1]_q} \begin{bmatrix} 2n-d \\ n-d, n-d \end{bmatrix}_q, \tag{5}$$

where $\mathrm{maj}(\tau_1, \ldots, \tau_m) = \sum_{\tau_i > \tau_{i+1}} i$ is the usual major index statistic. Thus by (4) this natural "descent based" q-analog of $S_{n,d}(1,1)$ can be obtained from $S_{n,d}(q,t)$ by setting $t = 1/q$.

In [HL], Haglund and Loehr describe an alternate pair of statistics (dinv, area) on Catalan paths, originally studied by Haiman, which also generate $C_n(q,t)$. They also include a simple, invertible transformation on Catalan paths which sends (dinv, area) to (area, bounce). In section 3 we show how the dinv statistic, as well as this simple transformation, can be extended to Schröder paths. As a corollary we obtain the result $S_{n,d}(q,1) = S_{n,d}(1,q)$, which further supports Conjecture 1.

$C_n(q,t)$ is part of a broader family of rational functions which Garsia and Haiman defined as the coefficients obtained by expanding a complicated sum of Macdonald symmetric functions in terms of Schur functions. They defined $C_n(q,t)$ as the coefficient of the Schur function s_{1^n} in this sum, and ideally we hoped to find a related rational function expression for $S_{n,d}(q,t)$. We are indebted to the referee for suggesting that $S_{n,d}(q,t)$ should equal the sum of the rational functions corresponding to the coefficients of the Schur functions for the two hook shapes $s_{d,1^{n-d}}$ and $s_{d+1,1^{n-d-1}}$. Independently of this suggestion, A. Ulyanov and the second author noticed that $q^{\mathrm{dinv}}t^{\mathrm{area}}$ summed over a subset of Schröder paths (counted by the "little" Schröder numbers) seems to generate the rational function corresponding to an individual hook shape. These conjectures, which turn out to be equivalent, are described in detail in section 4.

2 Recurrence Relations and Explicit Formulae

We begin with a simple lemma involving area and Schröder paths. Throughout this section we use the q-notation $[m]_q = (1-q^m)/(1-q)$, $[m]_q! := \prod_{i=1}^m [i]_q$ and

$$\begin{bmatrix} m \\ a,b \end{bmatrix}_q = \begin{cases} 1, & \text{if } a=b=0 \\ 0, & \text{if } a<0 \text{ or } b<0 \\ 0, & \text{if } m<a+b \text{ and either } a>0 \text{ or } b>0 \\ \frac{[m]_q!}{[a]_q![b]_q![m-a-b]_q!}, & \text{else} \end{cases} \tag{6}$$

For a given vector (u,v,w) of three nonnegative integers, let $bdy(u,v,w)$ denote the "boundary" lattice path from $(0,0)$ to $(v+w,u+v)$ consisting of w E steps, followed by v D steps, followed by u N steps. Let $T_{u,v,w}$ denote the set of lattice paths from from $(0,0)$ to $(v+w,u+v)$ consisting of u N, v D and w E steps (in any order). For $\tau \in T_{u,v,w}$, let $A(\tau,u,v,w)$ denote the number of lower triangles between τ and $bdy(u,v,w)$.

Lemma 1

$$\sum_{\tau\in T_{u,v,w}} q^{A(\tau,u,v,w)} = \begin{bmatrix} u+v+w \\ u,v \end{bmatrix}_q. \tag{7}$$

Proof. We claim the number of inversions of word(τ) equals $A(\tau,u,v,w)$ (where as usual two letters w_i, w_j of word(τ) form an inversion if $i<j$ and $w_i > w_j$). To see why, note that if we interchange two consecutive steps of τ, the number of lower triangles between τ and $bdy(u,v,w)$ changes by either 1 or 0 in exactly the same way that the number of inversions of word(τ) changes upon interchanging the corresponding letters in word(τ). The lemma now follows from the well-known fact that the q-multinomial coefficient is the generating function for the number of inversions of permutations of a multiset [Sta86, p. 26]. □

We now obtain an expression for $S_{n,d}(q,t)$ in closed form which doesn't reference the bounce or area statistics. This and other results in this section are for the most part generalizations of arguments and results in [Hag] (corresponding to the $d=0$ case).

Theorem 1 *For all $n > d \geq 0$,*

$$S_{n,d}(q,t) = \sum_{k=1}^{n-d} \sum_{\substack{\alpha_1+\ldots+\alpha_k=n-d,\ \alpha_i>0 \\ \beta_0+\ldots+\beta_k=d,\ \beta_i\geq 0}} \begin{bmatrix}\beta_0+\alpha_1 \\ \beta_0\end{bmatrix}_q \begin{bmatrix}\beta_k+\alpha_k-1 \\ \beta_k\end{bmatrix}_q q^{\binom{\alpha_1}{2}+\ldots+\binom{\alpha_k}{2}}$$

$$t^{\beta_1+2\beta_2+\ldots+k\beta_k+\alpha_2+2\alpha_3+\ldots+(k-1)\alpha_k} \prod_{i=1}^{k-1} \begin{bmatrix}\beta_i+\alpha_{i+1}+\alpha_i-1 \\ \beta_i, \alpha_{i+1}\end{bmatrix}_q \tag{8}$$

Proof. The sum over α and β above is over all possible bounce vectors $(\alpha_1,\ldots,\alpha_k)$ and shift vectors $(\beta_0,\ldots,\beta_k)$. The power of t is the bounce statistic evaluated at any Π with these bounce and shift vectors. It remains to show that when we sum over all such Π, $q^{\mathrm{area}(\Pi)}$ generates the terms involving q.

Let Π_0 be the portion of Π above peak 1 of Π, Π_k the portion below peak k, and for $1 \leq i \leq k-1$, Π_i the portion between peaks i and $i+1$. We call Π_i *section* i of Π, and let $\mathrm{word}(\Pi_i)$ be the portion of $\mathrm{word}(\Pi)$ corresponding to Π_i. We begin by breaking the area below Π into regions as in Figure 2. There will be triangular regions immediately below and to the right of each peak, whose area triangles are counted by the sum of $\binom{\alpha_i}{2}$. The remaining regions are between some Π_i and a boundary path as in Lemma 1. Note the conditions on Π_i for $1 \leq i \leq k-1$ require that it begin at the top of peak $i+1$, travel to the bottom of peak i using α_{i+1} E steps, β_i D steps and $\alpha_i - 1$ N steps (in any order), then use an N step to arrive at the top of peak i. Thus when we sum over all such Π_i the area of these regions will be counted by the product of q-trinomial coefficients above by Lemma 1. At first glance it may seem we need to use a different idea to calculate the area below Π_0 and Π_k, but these cases are also covered by Lemma 1, corresponding to the cases $w = 0$ and $u = 0$ of either no N steps or no E steps, in which case the q-trinomial coefficients reduce to the q-binomial coefficients above. $\square$

Let $\mathcal{S}_{n,d,p,b}$ denote the set of Schröder paths which are elements of $\mathcal{S}_{n,d}$ and in addition contain exactly p E steps and b D steps after the last N step (i.e. after peak 1). Furthermore let $S_{n,d,p,b}(q,t)$ denote the sum of $q^{\mathrm{area}}t^{\mathrm{bounce}}$ over all such paths. In the identities for $S_{n,d,p,b}(q,t)$ in the remainder of this section we will assume $n,d,p,b \geq 0$, $n-d \geq p$ and $d \geq b$ (otherwise $S_{n,d,b,p}(q,t)$ is zero).

Theorem 1 can be stated in terms of the following recurrence.

Theorem 2 *For all $n > d$,*

$$S_{n,d,p,b}(q,t) = q^{\binom{p}{2}} t^{n-p-b} \begin{bmatrix}p+b \\ p\end{bmatrix}_q \sum_{r=0}^{n-p-d} \sum_{m=0}^{d-b} \begin{bmatrix}m+r+p-1 \\ m+r\end{bmatrix}_q S_{n-p-b,d-b,r,m}(q,t), \tag{9}$$

with the initial conditions

$$S_{n,d,p,b}(q,t) = \begin{cases} 0, & \text{if } n > d \text{ and } p = 0 \\ 0, & \text{if } n = d,\ p = 0 \text{ and } b < d \\ 1, & \text{if } n = d,\ p = 0 \text{ and } b = d \end{cases} \tag{10}$$

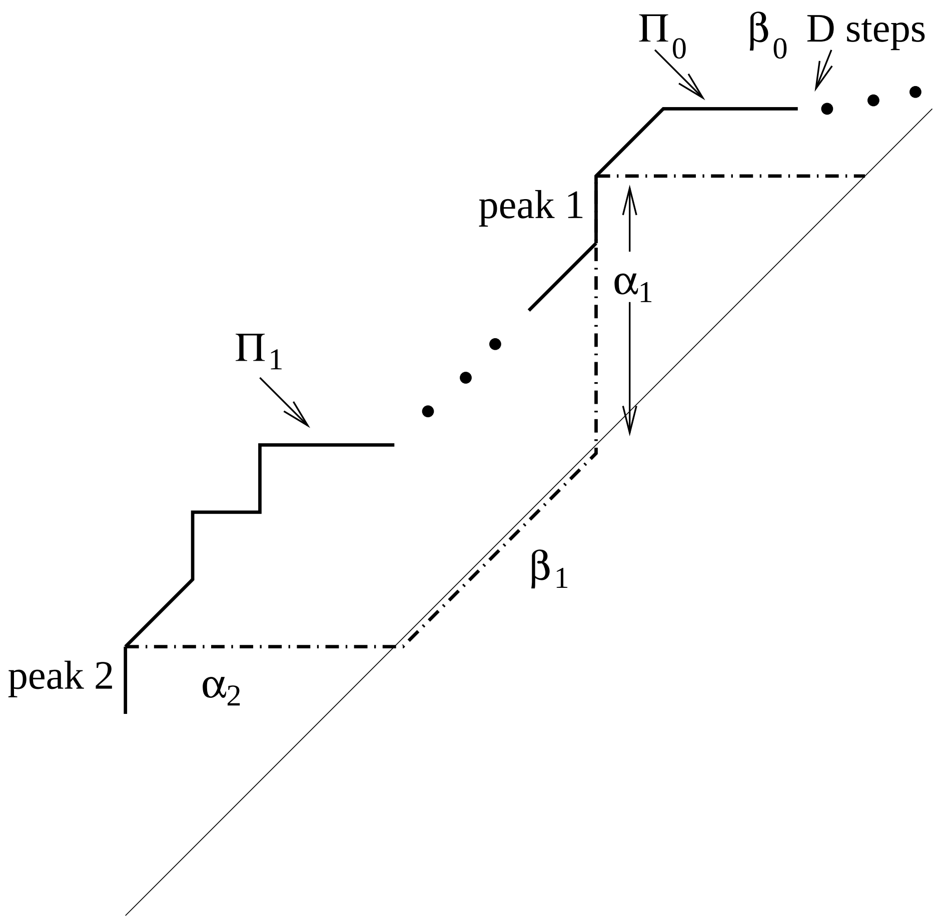

Figure 2: The sections of a Schröder path

Proof. We give a proof based on a geometric argument. An alternative proof by induction can be obtained by expressing $S_{n-p-b,d-b,r,m}(q,t)$ as an explicit sum, as in Theorem 1 , and plugging this in above.

In Figure 2 replace α_1 by p, β_0 by b, α_2 by r and β_1 by m. Since $n > d$ we know Π has at least one peak and so $p \geq 1$. If we remove the $p-1$ N steps from Π_1 and collapse in the obvious way, the part of Π consisting of $\Pi_i, 2 \leq i \leq k$ and the collapsed Π_1 is in $\mathcal{S}_{n-p-b,d-b,r,m}$. The bounce statistic for this truncated version of Π is bounce$(\Pi) - (n - p - d) - (d-b)$, since by removing peak 1 we decrease the shift contribution by $d-b$ (the number of D steps below peak 1) and we decrease the bounce contribution by $n-p-d$. The area changes in three ways. First of all there is the $\binom{p}{2}$ contribution from the triangle of length p below and to the right of peak 1. Second there is the area below Π_0 and above the first step of the bounce path, which generates the $\begin{bmatrix} p+b \\ p \end{bmatrix}_q$ factor. Third there is the area between Π_1 and the boundary path. View this area as equal to the number of inversions of word(Π_1), and group the inversions involving 1's and 0's separately from the inversions involving 2's and 1's or 2's and 0's. When we remove the 2's (i.e. N steps) from the word the inversions involving 1's and 0's still remain, and become part of the area count of the truncated Π. The number of inversions involving 2's and 1's or 2's and 0's is independent of the how the 1's and 0's are arranged with respect to each other, and so when we sum over all possible ways of inserting the 2's into the fixed sequence of 1's and 0's, we generate the q-binomial coefficient $\begin{bmatrix} m+r+p-1 \\ m+r \end{bmatrix}_q$. $\square$

Since the D steps above peak 1 of Π don't affect bounce(Π), it follows that

$$S_{n,d,p,b}(q,t) = \begin{bmatrix} p+b \\ p \end{bmatrix}_q S_{n-b,d-b,p,0}(q,t). \tag{11}$$

Setting $b=0$ in (9), then applying (11) in the inner sum on the right-hand-side we get the following recurrence for $S_{n,d,p,0}(q,t)$.

Theorem 3 *For all $n>d$,*

$$S_{n,d,p,0}(q,t) = q^{\binom{p}{2}} t^{n-p} \sum_{r=0}^{n-p-d} \sum_{m=0}^{d} \begin{bmatrix} m+r+p-1 \\ m, r \end{bmatrix}_q S_{n-p-m,d-m,r,0}(q,t) \tag{12}$$

with the initial conditions

$$S_{n,d,p,0}(q,t) = \begin{cases} 0, & \text{if } n>d \text{ and } p=0 \\ 1, & \text{if } n=d \text{ and } p=0 \end{cases} \tag{13}$$

We now use our recurrences to evaluate $S_{n,d,p,b}(q,1/q)$ and $S_{n,d}(q,1/q)$. We sometimes abbreviate $[m]_q$ by $[m]$ and $[m]_q!$ by $[m]!$.

Theorem 4 *For all $n>d$ and $p \geq 1$,*

$$q^{\binom{n}{2}-\binom{d}{2}} S_{n,d,p,0}(q,1/q) = q^{n(p-1)} \frac{[p]_q}{[2n-d-p]_q} \begin{bmatrix} 2n-d-p \\ n-d-p, n-d \end{bmatrix}_q. \tag{14}$$

Proof. We argue by induction on n. If $p=n-d$ then $C(\Pi)$ has only one peak, and we easily obtain

$$S_{n,d,n-d,0}(q,1/q) = q^{\binom{n-d}{2}} q^{-d} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q, \tag{15}$$

so (14) holds in this case. In particular when $n=1$ we must have $p=1$ and $d=0$, so we may assume $n>d+p$ and $n>1$.

Multiply both sides of (12) by $q^{\binom{n}{2}-\binom{d}{2}}$, set $t=1/q$, and interchange the order of summation to find

$$q^{\binom{n}{2}-\binom{d}{2}} S_{n,d,p,0}(q,1/q) = \sum_{m=0}^{d} \sum_{r=0}^{n-p-d} \begin{bmatrix} r+m+p-1 \\ r, m \end{bmatrix}_q q^{\binom{p}{2}} q^{-n+p} q^{\binom{n}{2}-\binom{d}{2}} S_{n-p-m,d-m,r,0}(q,1/q). \tag{16}$$

Since we are assuming $n-d>p$, in (16) we must have $n-p-m>0$ so by (13) the $r=0$ terms in this last line are zero. Using induction we then obtain

$$\begin{aligned} & q^{\binom{n}{2}-\binom{d}{2}} S_{n,d,p,0}(q,1/q) \\ = & \sum_{m=0}^{d} \sum_{r=1}^{n-p-d} \frac{[r+p+m-1]![r][2n-2p-m-d-r-1]!q^{pow} q^{(r-1)(n-m-p)}}{[r]![m]![p-1]![n-p-d-r]![n-p-d]![d-m]!} \\ = & \frac{1}{[p-1]![n-p-d]!} \sum_{m=0}^{d} \frac{q^{pow}}{[m]![d-m]!} \\ & \times \sum_{r \geq 1} \frac{[p+m+(r-1)]![2n-2p-m-d-2-(r-1)]!}{[r-1]![n-p-d-1-(r-1)]!} q^{(r-1)(n-m-p)}, \quad (17) \end{aligned}$$

where $pow := \binom{p}{2} - n + p + \binom{n}{2} - \binom{d}{2} - \binom{n-p-m}{2} + \binom{d-m}{2}$. We now phrase (17) in the language of basic hypergeometric series using the standard notation

$$(x;q)_m := (x)_m = (1-x)(1-xq)\cdots(1-xq^{m-1}) \text{ and } {}_2\phi_1\left(\begin{matrix} x, y \\ w \end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{(x)_m (y)_m}{(q)_m (w)_m} z^m. \quad (18)$$

First set $u = r-1$ and employ the simple identities

$$[m]!_q = \frac{(q)_m}{(1-q)^m} \quad \text{and} \quad (q^a)_{m-r} = \frac{(q^a)_m q^{\binom{r+1}{2}-(m+a)r}(-1)^r}{(q^{1-m-a})_r} \quad (19)$$

in (17) to obtain

$$\begin{aligned} q^{\binom{n}{2}-\binom{d}{2}} S_{n,d,p,0}(q,1/q) = & \frac{1}{[p-1]![n-p-d]!} \sum_{m=0}^{d} \frac{[p+m]![2n-2p-d-m-2]!q^{pow}}{[m]![d-m]![n-p-d-1]!} \\ & \times \sum_{u \geq 0} \frac{(q^{p+m+1})_u (q^{-(n-p-d-1)})_u}{(q)_u (q^{-(2n-2p-d-m-2)})_u} q^{u((n-p-d-(2n-2p-d-m-1))} q^{u(n-m-p)}. \quad (20) \end{aligned}$$

Note the exponent $u((n-p-d-1)-(2n-2p-d-m-2)) + u(n-m-p)$ is equal to u. We now use the well-known *q-Vandermonde convolution*, i.e. [GR90, p. 236]

$${}_2\phi_1\left(\begin{matrix} x, q^{-m} \\ w \end{matrix}; q\right) = \frac{(w/x)_m}{(w)_m} x^m, \quad (21)$$

to simplify the inner sum in (20) as follows.

$$\begin{aligned}
&\sum_{u\geq 0}\frac{(q^{p+m+1})_u(q^{-(n-p-d-1)})_u}{(q)_u(q^{-(2n-2p-d-m-2)})_u}q^u\\
&= \frac{(q^{-(2n-p-d-1)})_{n-p-d-1}}{(q^{-(2n-2p-d-m-2)})_{n-p-d-1}}q^{(p+m+1)(n-p-d-1)}\\
&= \frac{(q^{2n-p-d-1}-1)\cdots(q^{n+1}-1)q^{-(p+m+1)(n-p-d-1)+(p+m+1)(n-p-d-1)}}{(q^{2n-2p-d-m-2}-1)\cdots(q^{n-p-m}-1)}\\
&= \frac{[2n-p-d-1]![n-p-m-1]!}{[n]![2n-2p-d-m-2]!}. \qquad (22)
\end{aligned}$$

Using this, the right-hand-side of (20) reduces to

$$\begin{aligned}
&\frac{[2n-p-d-1]!}{[p-1]![n-p-d]![n-p-d-1]![n]!}\sum_{m=0}^{d}q^{pow}\frac{[p+m]![n-p-m-1]!}{[m]![d-m]!}\\
&= \frac{[2n-p-d-1]![n-p-1]![p]!}{[p-1]![n-p-d]![n-p-d-1]![n]![d]!}\sum_{m=0}^{d}\frac{(q^{-d})_m(q^{p+1})_m}{(q)_m(q^{-n+p+1})_m}q^{m(d+1-(n-p))+pow}. \qquad (23)
\end{aligned}$$

Note $pow+m(d+1-n+p)=n(p-1)+m$, and again by (21),

$$\begin{aligned}
\sum_{m=0}^{d}\frac{(q^{-d})_m(q^{p+1})_m}{(q)_m(q^{-n+p+1})_m}q^m &= \frac{(q^{-n})_d}{(q^{1+p-n})_d}q^{(p+1)d}\\
&= \frac{(q^n-1)\cdots(q^{n-d+1}-1)}{(q^{n-1-p}-1)\cdots(q^{n-1-p-d+1}-1)}q^{-(p+1)d+(p+1)d}\\
&= \frac{[n]![n-p-d-1]!}{[n-d]![n-1-p]!}. \qquad (24)
\end{aligned}$$

Plugging this into the inner sum in (23) completes the proof. □

Theorem 4 and (11) imply

Corollary 1 *For all $n>d\geq 0$ and $p,b\geq 0$,*

$$q^{\binom{n-b}{2}-\binom{d-b}{2}}S_{n,d,p,b}(q,1/q)=q^{(n-b)(p-1)}\begin{bmatrix}p+b\\ b\end{bmatrix}_q\frac{[p]_q}{[2n-p-d-b]_q}\begin{bmatrix}2n-p-d-b\\ n-d,n-p-d\end{bmatrix}_q. \qquad (25)$$

We should mention that although $S_{n,d,p,b}(q,t)$ has a nice recursive structure and a compact expression when $t=1/q$, it is *not* in general symmetric in q,t.

We now use Corollary 1 to evaluate $q^{\binom{n}{2}-\binom{d}{2}}S_{n,d}(q,1/q)$.

Theorem 5 *For all $n\geq d\geq 0$ we have*

$$q^{\binom{n}{2}-\binom{d}{2}}S_{n,d}(q,1/q)=\frac{1}{[n-d+1]_q}\begin{bmatrix}2n-d\\ n-d,n-d\end{bmatrix}_q. \qquad (26)$$

Proof. For $n \le 1$ or $n = d$ both sides reduce to 1. Suppose $n > 1$ and $n > d$ and abbreviate $q^{\binom{n}{2}-\binom{d}{2}} S_{n,d}(q, 1/q)$ by S. Since $S_{n,d,0,k}(q,t) = 0$ if $n > d$,

$$S_{n,d}(q,t) = \sum_{p=1}^{n-d} \sum_{k=0}^{d} S_{n,d,p,k}(q,t). \tag{27}$$

Using (27) and Corollary 1 yields

$$S = \sum_{k=0}^{d} \sum_{p=1}^{n-d} \frac{[p+k]!}{[p]![k]!} \frac{[p][2n-k-d-p-1]!}{[n-d-p]![n-d]![d-k]!} q^{\left(\binom{n}{2}-\binom{d}{2}-\binom{n-k}{2}+\binom{d-k}{2}+(n-k)(p-1)\right)}$$

$$= \sum_{k=0}^{d} \frac{q^{\left(\binom{n}{2}-\binom{d}{2}-\binom{n-k}{2}+\binom{d-k}{2}\right)}}{[k]![n-d]![d-k]!} \sum_{p\ge 1} \frac{[k+1+(p-1)]![2n-d-k-2-(p-1)]!}{[p-1]![n-d-1-(p-1)]!} q^{(p-1)(n-k)}. \tag{28}$$

Setting $u = p-1$ and writing the inner sum as a basic hypergeometric series we obtain

$$\begin{aligned} S \quad &= \quad \frac{1}{[n-d]!} \sum_{k=0}^{d} \frac{q^{\left(\binom{n}{2}-\binom{d}{2}-\binom{n-k}{2}+\binom{d-k}{2}\right)}[k+1]![2n-d-k-2]!}{[k]![d-k]![n-d-1]!} \\ &\times \sum_{u\ge 0} \frac{(q^{k+2})_u (q^{-(n-d-1)})_u}{(q)_u (q^{-(2n-d-k-2)})_u} q^{u(n-k)+u(n-d-1-2n+d+k+2)}. \end{aligned} \tag{29}$$

Note the exponent of q in the inner sum is u. By (21),

$$\begin{aligned} \sum_{u\ge 0} \frac{(q^{k+2})_u (q^{-(n-d-1)})_u}{(q)_u (q^{-(2n-d-k-2)})_u} q^u \quad &= \quad \frac{(q^{-(2n-d)})_{n-d-1}}{(q^{-(2n-d-k-2)})_{n-d-1}} q^{(k+2)(n-d-1)} \\ &= \quad \frac{(q^{2n-d}-1)\cdots(q^{n+2}-1)}{(q^{2n-d-k-2}-1)\cdots(q^{n-k}-1)} \\ &= \quad \frac{[2n-d]![n-k-1]!}{[n+1]![2n-d-k-2]!}. \end{aligned} \tag{30}$$

Plugging this back into (29) we see

$$S = \frac{[2n-d]!}{[n+1]![n-d]![n-d-1]!} \sum_{k=0}^{d} \frac{q^{\left(\binom{n}{2}-\binom{d}{2}-\binom{n-k}{2}+\binom{d-k}{2}\right)}[k+1]![n-1-k]!}{[k]![d-k]!}$$

$$= \frac{[2n-d]![n-1]!}{[n+1]![n-d]![n-d-1]![d]!} \sum_{k=0}^{d} \frac{(q^2)_k (q^{-d})_k}{(q)_k (q^{1-n})_k} q^{\left(\binom{n}{2}-\binom{d}{2}-\binom{n-k}{2}+\binom{d-k}{2}\right)} q^{k(1-n+d)}. \tag{31}$$

The exponent $\left(\binom{n}{2} - \binom{d}{2} - \binom{n-k}{2} + \binom{d-k}{2}\right) + k(1-n+d)$ is equal to k, so by (21) the inner sum is

$$\begin{aligned}\sum_{k=0}^{d} \frac{(q^2)_k (q^{-d})_k}{(q)_k (q^{1-n})_k} q^k &= \frac{(q^{-1-n})_d}{(q^{1-n})_d} q^{2d} \\ &= \frac{(q^{n+1}-1)\cdots(q^{n+2-d}-1)}{(q^{n-1}-1)\cdots(q^{n-d}-1)} \\ &= \frac{[n+1]![n-d-1]!}{[n+1-d]![n-1]!}, \end{aligned} \tag{32}$$

and the theorem follows. □

By replacing q by $1/q$ in Theorem 5 we easily obtain the fact that Conjecture 1 is true when $t = 1/q$.

Corollary 2 *For all n, d, $S_{n,d}(q, 1/q) = S_{n,d}(1/q, q)$.*

3 A Schröder Generalization of Haiman's Statistic

The left border of each row of a path Π will consist of either an N step or a D step, and we call it an N row or a D row, accordingly. Recalling the notations $\mathrm{row}_i(\Pi)$ and $\mathrm{area}_i(\Pi)$ from the introduction, we define a statistic $\mathrm{dinv}(\Pi)$ to be the number of "inversion pairs" of Π, which are pairs (i, j), $1 \le i < j \le n$, such that either

$$\mathrm{area}_i(\Pi) = \mathrm{area}_j(\Pi) \text{ and } \mathrm{row}_j \text{ is an } N \text{ row} \tag{33}$$

or

$$\mathrm{area}_i(\Pi) = \mathrm{area}_j(\Pi) - 1 \text{ and } \mathrm{row}_i \text{ is an } N \text{ row.} \tag{34}$$

For example, for the path on the left side of Figure 3, the inversion pairs are $(1,2), (1,4)$, $(1,5), (1,6), (2,4), (2,5), (2,6), (3,5), (3,6), (3,7), (3,8)$ and $(5,6)$ so $\mathrm{dinv}(\Pi) = 12$.

We call the length of the longest N-row of Π the *height* of Π. For Π of height h, we call the vector $(n_0, n_1, \ldots, n_h)$ whose ith coordinate is the number of N rows of Π of length i the N-area vector of Π. Similarly we call the vector $(d_0, d_1, \ldots, d_{h+1})$ whose ith coordinate is the number of D rows of Π of length i the D-area vector of Π.

Theorem 6 *There exists a bijection $\phi : \mathcal{S}_{n,d} \to \mathcal{S}_{n,d}$ such that*

$$\mathit{dinv}(\Pi) = \mathit{area}(\phi(\Pi)) \text{ and } \mathit{area}(\Pi) = \mathit{bounce}(\phi(\Pi)). \tag{35}$$

Moreover, the N-area and D-area vectors of Π equal the bounce and shift vectors of $\phi(\Pi)$, respectively.

Corollary 3 *For all n, d,*

$$S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\mathit{dinv}(\Pi)} t^{\mathit{area}(\Pi)}. \tag{36}$$

Corollary 4 *For all n, d,*

$$S_{n,d}(q,1) = S_{n,d}(1,q). \tag{37}$$

Proof. (of Theorem 6). Given $\Pi \in \mathcal{S}_{n,d}$, we will argue that the following algorithm creates a path $\phi(\Pi)$ with the requisite properties.

Algorithm ϕ $[(\mathrm{dinv}, \mathrm{area}) \to (\mathrm{area}, \mathrm{bounce})]$:
Initialize to $(0, 0)$.
Input: the area sequence $(area_i(\Pi), R_i)_{i=1}^{n}$ with Π of height h.
Here R_i is either N or D indicating whether row_i of Π is an N or D row.
Output: a path $\phi(\Pi) \in \mathcal{S}_{n,d}$.
For $v = h$ to -1;
    For $w = 1$ to n;
        If $area_w(\Pi) = v$ and row_w is an N row then take a N step;
        If $area_w(\Pi) = v + 1$ and row_w is a D row then take a D step;
        If $area_w(\Pi) = v + 1$ and row_w is an N row then take a E step;
    repeat;
repeat;

In Figure 3 we have a path Π and its image $\phi(\Pi)$ under the map described above.

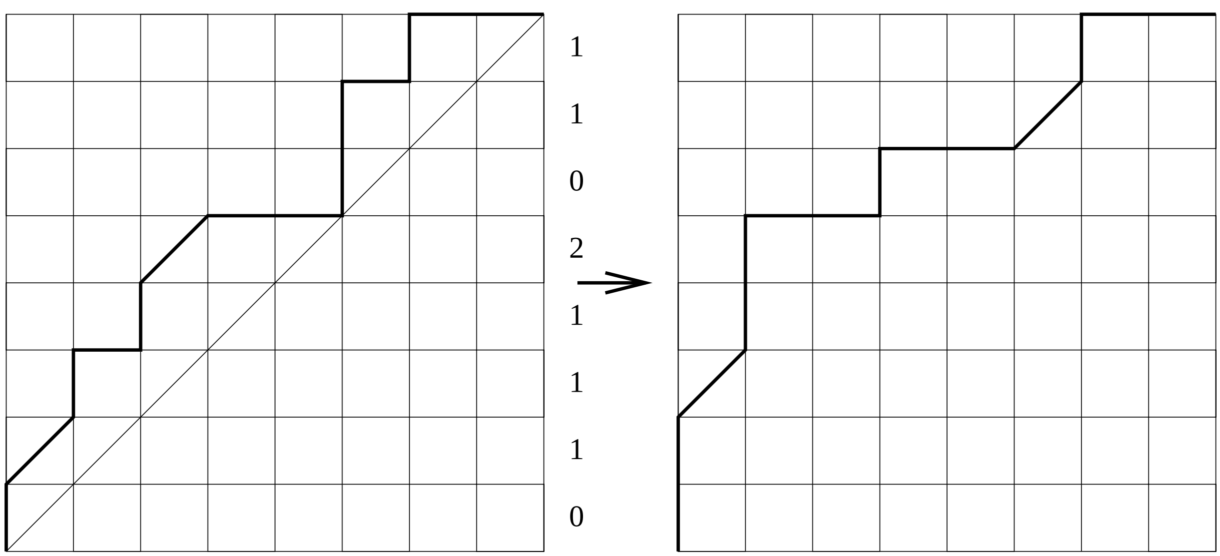

Figure 3: A path Π and its image $\phi(\Pi)$

First note that if row_i is an N row with length j, then it will first contribute an N step to $\phi(\Pi)$ during the $v = j$ loop, then a E step to $\phi(\Pi)$ during the $v = j - 1$ loop. If row_i is a D row of length j, then it will contribute a D step to $\phi(\Pi)$ during the $v = j - 1$ loop. It follows that $\phi(\Pi) \in \mathcal{S}_{n,d}$.

We now prove by induction on i that for $0 \leq i \leq h$, the last step created during the $v = i$ loop is peak $i+1$ of $\phi(\Pi)$, and also that the $i+1$st coordinates of the bounce and shift vectors of $\phi(\Pi)$ equal n_i and d_i, respectively, the $i+1$st coordinates of the N-area and D-area vectors of Π. The $v = -1$ loop ceates n_0 E steps and d_0 D steps. Now it is clear geometrically that for any Π, if $\text{area}_j = i+1$ for $i > -1$ then there exists $l > j$ such that $\text{area}_l = i$ and row_l is an N row. Thus for $i > -1$ the last step of $\phi(\Pi)$ created during the $v = i$ loop will be an N step. In particular, the last step created during the $v = 0$ loop is peak 1 of $\phi(\Pi)$. Now assume by induction that the last step created during the $v = i-1$ loop is peak i, and that the length of the ith bounce step of $\phi(\Pi)$ is n_{i-1}. The fact that the last step created during the $v = i$ loop is an N step, together with the fact that the number of N steps created during the $v = i-1$ loop is n_{i-1}, imply that the length of the $i+1$st bounce step of $\phi(\Pi)$ equals the number of E steps created during the $v = i-1$ loop, which is n_i. Also, the last N step created during the $v = i$ loop will become peak $i+1$. Furthermore, the number of D steps created during the $v = i-1$ loop is d_i, and these are exactly the set of D steps of $\phi(\Pi)$ which end up between peaks i and $i+1$ (if $1 \leq i-1 < h$) or below peak $h+1$ if $i-1 = h$. This completes the induction.

Now $\text{area}(\phi(\Pi)) = \sum_i \binom{n_i}{2}$, plus the area between each of the sections $\phi(\Pi)_i$ and the corresponding boundary path. Note $\sum_i \binom{n_i}{2}$ counts the number of inversion pairs of Π between rows of the same length, neither of which is a D row. Fix i, $0 \leq i \leq k$, and let $\text{row}_{j_1}, \text{row}_{j_2}, \ldots, \text{row}_{j_p}$, $j_1 < j_2 < \cdots < j_p$ be the sequence of rows of Π affecting the $v = i$ loop, i.e. rows which are either N rows of length i or $i+1$, or D rows of length $i+1$ (so $p = n_i + n_{i+1} + d_{i+1}$). Let τ be the word of 2's, 1's and 0's corresponding to the portion of Π affecting the $v = i$ loop. One verifies from the definition of dinv that the number of inversion pairs of Π of the form (x, y) with row_x of length i, row_y of length $i+1$ and x not a D row equals the number of inversion pairs of τ involving 2's and 1's or 2's and 0's. Similarly, the number of inversion pairs of Π of the form (x, y) with row_x of length i, row_y of length $i+1$ and y not a D row equals the number of inversion pairs of τ involving 1's and 0's. It follows from Lemma 1 that $\text{dinv}(\Pi) = \text{area}(\phi(\Pi))$.

It remains to show the algorithm is invertible. The bounce and shift vectors of $\phi(\Pi)$ yield the N-area and D-area vectors of Π, respectively. In particular they tell us how many rows of Π there are of length 0. From section 0 of $\phi(\Pi)$ we can determine which subset of these are D rows, since the $v = -1$ iteration of the ϕ algorithm produces an E step in section 0 of $\phi(\Pi)$ for each N row of length 0 in Π, and a D step in section 0 of $\phi(\Pi)$ for each D row of length 0 in Π. From section 1 of $\phi(\Pi)$ we can determine how the rows of length 1 of Π are interleaved with the rows of length 0 of Π, and also which rows of length 1 are D rows, since the $v = 0$ iteration of ϕ creates an N step in section 1 of $\phi(\Pi)$ for every N row of Π of length 0, and a D or E step of $\phi(\Pi)$ for every N-row or D row, respectively, of length 1 in Π. When considering how the rows of length 2 of Π are related to the rows of length 1, we can ignore the rows of length 0, since no row of length 0 can be directly below a row of length 2 and still be the row sequence of a Schröder path. Hence from section 2 of $\phi(\Pi)$ we can determine how the rows of length 2 of Π are interleaved with the rows of length 1, and which ones are D rows. Continuing in this way we can completely determine Π. An explicit algorithm for the inverse is described below.

Algorithm ϕ^{-1} [$(\text{area}, \text{bounce}) \to (\text{dinv}, \text{area})$]:
Initialize to $(a) = (-1, N)$.
Input: a path $\phi(\Pi) \in \mathcal{S}_{n,d}$ with k peaks, where the top of peak i has coordinates (x_i, y_i) for $1 \le i \le k$. Define $x_{k+1} = 0, y_{k+1} = 0$ and $x_0 = n, y_0 = n$.
Let M_i denote the number of steps in the ith section of $\phi(\Pi)$, $0 \le i \le k$.
Output: a sequence of pairs $(area_i(\Pi), R_i)_{i=1}^n$, where R_i is either N or D indicating whether row$_i$ of Π is an N row or a D row.
For $i = 1$ to $k + 1$;
    Number the steps of $\phi(\Pi)$ beginning at (x_i, y_i), moving up the path until reaching (x_{i-1}, y_{i-1}).
    Given the sequence (a) created thus far, we insert a new subsequence of $(i-1, N)$'s and $(i-1, D)$'s starting to the left of the first element of (a) and moving to the right.
    For $j = 1$ to M_{i-1};
        If step j is a N step then move past the next $(i-2, N)$ in (a);
        If step j is a D step then insert a $(i-1, D)$ immediately to the left of the next $(i-2, N)$ in (a);
        If step j is a E step then insert a $(i-1, N)$ immediately to the left of the next $(i-2, N)$ in (a);
    repeat;
repeat;
remove the $(-1, N)$ from (a).

□

A consequence of Corollary 3 is that the right-hand-side of (8) equals $\sum q^{\text{dinv}(\Pi)} t^{\text{area}(\Pi)}$. We give a brief argument as to how this can be shown directly, which builds on the corresponding argument for the $d = 0$ case in [Hag]. Let $(n_0, \ldots, n_{k-1})$ and $(d_0, \ldots, d_k)$ be two N-area and D-area vectors, and consider the sum of q^{dinv} over all Π with these N-area and D-area vectors. We construct such a Π by starting with a sequence $\text{row}_1, \text{row}_2, \ldots$ of n_0 N rows of length zero, then insert d_0 D rows of length zero into this sequence. The resulting sequence will be the sequence $\text{row}_1, \text{row}_2, \ldots$ of a corresponding path Π, and consider the value of $\text{dinv}(\Pi)$. All the D rows will create an inversion pair with the N rows after it, and any pair of N rows will create an inversion pair. Thus as we sum over all ways to insert the D rows we generate a factor of

$$q^{\binom{n_0}{2}} \begin{bmatrix} d_0 + n_0 \\ d_0 \end{bmatrix}_q . \tag{38}$$

Next we wish to insert the $n_1 + d_1$ rows of length one. For simplicity consider the case where after inserting these rows all the N rows of length one occur before all the D rows of length one. We have the constraint that we cannot insert a row of length one just before a D row of length zero and still have the row sequence of an actual Schröder path. In particular we must have an N row of length zero immediately following the last row of length one. Now each of the rows of length one will create an inversion pair with each N

row of length zero before it, but will not create an inversion pair with any of the D rows of length zero. It follows that we can essentially ignore the D rows of length zero, and when summing over all possible insertions we generate a factor of

$$q^{\binom{n_1}{2}} \begin{bmatrix} n_1 + d_1 + n_0 - 1 \\ n_1 + d_1 \end{bmatrix}_q, \tag{39}$$

since each pair of N rows of length one will generate an inversion pair, but none of the D rows of length one will occur in an inversion pair with any row of length one. In fact, (39) gives the (weighted) count of the inversion pairs between rows of length zero and of length one, and between N rows of length one, no matter how the N rows and D rows of length one are interleaved with each other. Thus when we sum over all such possible arrangements, we generate an extra factor of

$$\begin{bmatrix} n_1 + d_1 \\ n_1 \end{bmatrix}_q, \tag{40}$$

and so the total contribution is

$$q^{\binom{n_1}{2}} \begin{bmatrix} n_1 + d_1 + n_0 - 1 \\ n_1 + d_1 \end{bmatrix}_q \begin{bmatrix} n_1 + d_1 \\ n_1 \end{bmatrix}_q = q^{\binom{n_1}{2}} \begin{bmatrix} n_1 + d_1 + n_0 - 1 \\ n_1, d_1 \end{bmatrix}_q. \tag{41}$$

When inserting the rows of length 2, we cannot insert before any row of length 0 and still correspond to a Schröder path. Also, none of the rows of length 2 will create inversion pairs with any row of length 0. Thus by the argument above we get a factor of

$$q^{\binom{n_2}{2}} \begin{bmatrix} n_2 + d_2 + n_1 - 1 \\ n_2, d_2 \end{bmatrix}_q. \tag{42}$$

It is now clear how the right-hand-side of (8) is obtained.

4 A Conjecture Involving the Nabla Operator

For λ a partition of n, denoted $\lambda \vdash n$, let λ' denote the conjugate partition, s_λ the Schur function, and e_n the nth elementary symmetric function. If μ is another partition of n, let $K_{\lambda,\mu}(q,t)$ be Macdonald's q,t-Kostka polynomial [Mac95, p. 354], and let

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} t^{\eta(\mu)} K_{\lambda,\mu}(q, 1/t) s_\lambda \tag{43}$$

be the modified Macdonald polynomial, where $\eta(\mu) = \sum_i (i-1)\mu_i$. One of the ways of defining $C_n(q,t)$ is by the relation

$$C_n(q,t) = < \nabla e_n, s_{1^n} >, \tag{44}$$

where ∇ is the linear operator defined on the $\tilde{H}_\mu$ basis via

$$\nabla \tilde{H}_\mu = t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu, \tag{45}$$

and $<,>$ is the usual Hall scalar product with respect to which the Schur functions form an orthonormal system. In other words, $C_n(q,t)$ is the coefficient of s_{1^n} when expanding ∇e_n in terms of Schur functions.

Haiman has proven the $n!$ conjecture, which implies $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ [Hai01]. In a subsequent work he has shown that $< \nabla e_n, s_\lambda >$ equals the coefficient of s_λ in the Frobenius series of the character of the module $\mathcal{H}_n$ of diagonal harmonics under the diagonal action of the symmetric group [Hai02]. In particular this implies that $C_n(q,t)$ is the bigraded Hilbert series of $\mathcal{H}_n^\epsilon$, the subspace of antisymmetric elements of $\mathcal{H}_n$. Both this and Garsia and Haglund's result imply $C_n(q,t) \in \mathbb{N}[q,t]$, but neither one contains the other.

The referee has suggested the following conjecture, which gives a beautiful interpretation for $S_{n,d}(q,t)$ in terms of diagonal harmonics.

Conjecture 2 *For all n, d,*

$$S_{n,d}(q,t) = < \nabla s_{1^n}, e_{n-d} s_d > . \tag{46}$$

By the Pieri rule, an equivalent formulation of Conjecture 2 is that (for $0 < d < n$) $S_{n,d}(q,t)$ is the sum of the coefficients of the two hook shapes $s_{d+1,1^{n-d-1}}$ and $s_{d,1^{n-d}}$ in ∇e_n. If $d = 0$ or $d = n$, it reduces to the known formulas for the coefficients of s_{1^n} and s_n in ∇e_n, respectively.

Independently of this, the second author and A. Ulyanov formulated the next conjecture, which describes how $< \nabla e_n, s_{d+1,1^{n-d-1}} >$ can be obtained by summing $q^{\mathrm{dinv}} t^{\mathrm{area}}$ over certain special paths. To be precise, say $\mathrm{row}_i(\Pi)$ is the *leading row* of Π if for all $j \neq i$ either

(1) $\mathrm{area}_i(\Pi) > \mathrm{area}_j(\Pi)$
    or
(2) $\mathrm{area}_i(\Pi) = \mathrm{area}_j(\Pi)$ and $i < j$.

Let $\tilde{\mathcal{S}}_{n,d}$ denote the set of all Π whose leading row is an N row, and set

$$\tilde{S}_{n,d}(q,t) = \sum_{\Pi \in \tilde{\mathcal{S}}_{n,d}} q^{\mathrm{dinv}(\Pi)} t^{\mathrm{area}(\Pi)}. \tag{47}$$

(It follows from algorithm ϕ that for $0 \le d < n$, $\tilde{S}_{n,d}(q,t)$ also equals the sum of $q^{\mathrm{area}} t^{\mathrm{bounce}}$ over all $\Pi \in \mathcal{S}_{n,d}$ for which $\mathrm{pword}(\Pi)$ has no D before the first E.)

Conjecture 3 *For all $n, d \ge 0$,*

$$\tilde{S}_{n,d}(q,t) = < \nabla e_n, s_{d+1,1^{n-d-1}} > . \tag{48}$$

The Schröder numbers r_n count the number of Schröder paths from $(0,0)$ to (n,n) with any number of diagonals. Conjecture 2 implies that if we define a two parameter Schröder number via

$$r_n(q,t) = \sum_{d=0}^{n} S_{n,d}(q,t) \tag{49}$$

then $r_n(q,t)$ is twice the sum of the coefficients of all the hook shapes in ∇e_n. The study of the numbers r_n goes hand in hand with the study of the *little Schröder numbers*, which we denote by $\tilde{r}_n$. These numbers count many objects [Sta99, p. 178], including the number of Schröder paths from $(0,0)$ to (n,n) which have no D steps on the line $y = x$. It is well-known, although not combinatorially obvious, that $2\tilde{r}_n = r_n$. If we define

$$\tilde{r}_n(q,t) = \sum_{d=0}^{n-1} \tilde{S}_{n,d}(q,t) \tag{50}$$

then we have a two-parameter little Schröder number which, assuming Conjecture 3, is the sum of the coefficients of all the hook shapes in ∇e_n. Thus we must have $2\tilde{r}_n(q,t) = r_n(q,t)$. In fact, it is easy to show this combinatorially, since the N step in the leading row of any path in $\tilde{\mathcal{S}}_{n,d}$ must be followed by an E step, and replacing this NE pair by a D step doesn't change either area or dinv. Thus for $0 \le d \le n$,

$$S_{n,d}(q,t) = \tilde{S}_{n,d-1}(q,t) + \tilde{S}_{n,d}(q,t) \tag{51}$$

with $\tilde{S}_{n,n}(q,t) = \tilde{S}_{n,-1}(q,t) = 0$. Hence Conjecture 3 implies Conjecture 2. On the other hand, since $\tilde{S}_{n,0}(q,t) = C_n(q,t)$, if

$$S_{n,1}(q,t) = \tilde{S}_{n,0}(q,t) + \tilde{S}_{n,1}(q,t) = \nabla e_n|_{s_{1^n}} + \nabla e_n|_{s_{2,1^{n-2}}}, \tag{52}$$

we must have

$$\tilde{S}_{n,1}(q,t) = \nabla e_n|_{s_{2,1^{n-2}}}. \tag{53}$$

Continuing in this way we find Conjecture 2 implies Conjecture 3 so they are equivalent.

Conjecture 2 is part of a broader conjecture involving several parameters currently being researched by N. Loehr, J. Remmel, A. Ulyanov and the second author. Using Maple programs created in connection with this A. Ulyanov has confirmed that Conjecture 2 holds for all n, d satisfying $n + d \le 10$. By previous remarks, it is also true when $d = 0$ or $d = n$. In addition we can show it holds in the following special cases.

Proposition 1 *Conjecture 3 is true when either q or t is either 0 or 1, and also when $t = 1/q$.*

Proof. $\tilde{S}_{n,d}(q,0)$ is the sum of q^{dinv} over all Schröder paths in $\tilde{\mathcal{S}}_{n,d}$ all of whose rows are of length 0 and whose first (i.e. top) row is an N row. We get an inversion pair for each pair (i,j) with $i < j$ and row_j an N row. This implies

$$\tilde{S}_{n,d}(q,0) = e_{n-d-1}(q, q^2, \ldots, q^n) = q^{\binom{n-d}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q. \tag{54}$$

The case $t = 0$ of Conjecture 3 now follows from the known formula for the Frobenius series of the space of harmonics in one set of variables (see [Hai94, eq. (13)]). By the comment following (47), $\tilde{S}_{n,d}(0,t)$ equals the sum of t^{bounce} over all paths in $\mathcal{S}_{n,d}$ whose pword begins with NE and all of whose rows are of length 0. The case $q = 0$ follows easily. J. Remmel [Rem] has a bijective proof of the case $t = 1$, which also implies the case $q = 1$ by Corollary 4.

[GH96, Eq. 88] states that when $t = 1/q$,

$$q^{\binom{n}{2}} \nabla e_n|_{s_\lambda} = \frac{s_{\lambda'}(1, q, \ldots, q^n)}{[n+1]_q}. \tag{55}$$

Using the known formula for $s_\lambda(1, \ldots, q^n)$ [Sta99, p. 374] and Theorem 5 we obtain the case $t = 1/q$ of Conjecture 3. $\square$

We now describe a rational function identity which is equivalent to Conjecture 3. For $\mu \vdash n$ let s denote a cell in the Ferrers diagram of μ. By the *arm* (respectively, *co-arm*) of s we mean the set of cells in the same row as s and strictly to the right (respectively, left) of s. By the *leg* (respectively, *co-leg*) of s we mean the set of cells in the same column as s and strictly below (respectively, above) s. When s has been specified, we let a (respectively, a') denote the number of cells in the arm (respectively, co-arm) of s, and l (respectively, l') denote the number of cells in the leg (respectively, co-leg) of s. For example, for the cell labeled s in Figure 4 we have $a = 5$, $a' = 4$, $l = 3$ and $l' = 2$.

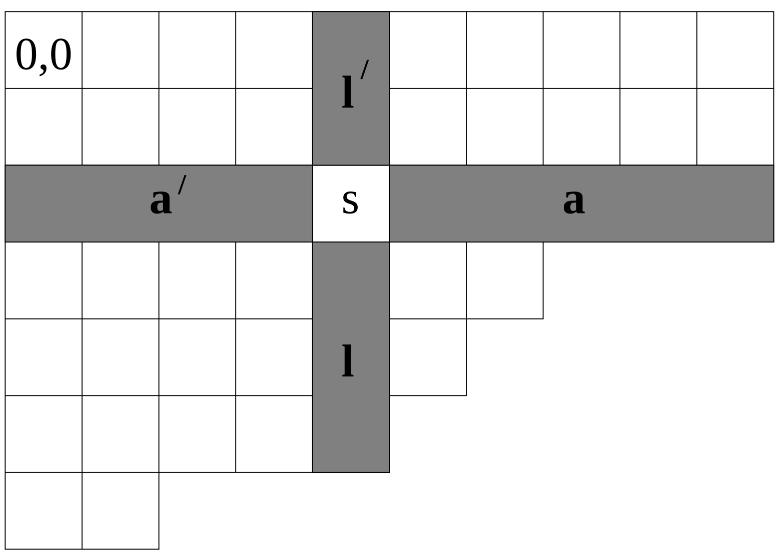

Figure 4: Arm, leg, co-arm and co-leg of s

Garsia and Haiman obtained the following [GH96, Eq. (83)]

$$\nabla e_n = \sum_{\mu \vdash n} \frac{t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu[X; q, t](1-t)(1-q) \prod^{(0,0)} (1 - q^{a'} t^{l'}) \left(\sum q^{a'} t^{l'}\right)}{\prod (q^a - t^{l+1})(t^l - q^{a+1})}, \tag{56}$$

where the sums and products are over the cells of μ, and the symbol $\prod^{(0,0)}$ indicates the the upper left-hand corner cell of μ is not included. In general the coefficients of the s_λ in $\tilde{H}_\mu[X;q,t]$ have no known combinatorial description, although a result of Macdonald [Mac95, ex. 2, p. 362] implies that for $0 \le d \le n-1$ the coefficient of the hook shape $s_{d+1,1^{n-d-1}}$ in $\tilde{H}_\mu[X;q,t]$ equals the coefficient of z^d in $\prod^{(0,0)}(z+q^{a'}t^{l'})$. Here the product is over the non-$(0,0)$ cells of μ as above. Thus Conjectures 2 and 3 are equivalent to the following.

Conjecture 4 *For $n \ge 1$,*

$$\sum_{d=0}^{n-1} z^d \tilde{S}_{n,d}(q,t) = \sum_{\mu \vdash n} \frac{t^{\eta(\mu)} q^{\eta(\mu')} \prod^{(0,0)}(z+q^{a'}t^{l'}) \prod^{(0,0)}(1-q^{a'}t^{l'})(1-t)(1-q) \sum_{s\in\mu} q^{a'}t^{l'}}{\prod (q^a - t^{l+1})(t^l - q^{a+1})}. \tag{57}$$

and

$$\sum_{d=0}^{n} z^d S_{n,d}(q,t) = \sum_{\mu \vdash n} \frac{t^{\eta(\mu)} q^{\eta(\mu')} \prod (z+q^{a'}t^{l'}) \prod^{(0,0)}(1-q^{a'}t^{l'})(1-t)(1-q) \sum_{s\in\mu} q^{a'}t^{l'}}{\prod (q^a - t^{l+1})(t^l - q^{a+1})}. \tag{58}$$

It is interesting to note that the case $z=1$ of (57) reduces to

$$\tilde{r}_n(q,t) = \sum_{\mu \vdash n} \frac{t^{\eta(\mu)} q^{\eta(\mu')} \prod^{(0,0)}(1-q^{2a'}t^{2l'})(1-t)(1-q) \sum_{s\in\mu} q^{a'}t^{l'}}{\prod (q^a - t^{l+1})(t^l - q^{a+1})}. \tag{59}$$

5 Acknowledgments

The authors would like to thank the referee not only for pointing out Conjecture 2 but also for making many helpful remarks about the exposition. In addition they would like to thank N. Loehr, J. Remmel and A. Ulyanov for useful comments involving Conjecture 2.

References

[BLPP99] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Some combinatorial interpretations of q-analogs of Schröder numbers. *Ann. Comb.*, 3:171–190, 1999.

[BSS93] J. Bonin, L. Shapiro, and R. Simion. Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths. *J. Statist. Plann. Inference*, 34:35–55, 1993.

[CR64] L. Carlitz and J. Riordan. Two element lattice permutation numbers and their q-generalization. *Duke J. Math.*, 31:371–388, 1964.

[GH96] A. Garsia and M. Haiman. A remarkable q, t-Catalan sequence and q-Lagrange inversion. *J. Algebraic Combin.*, 5(3):191–244, 1996.

[GH01] A. Garsia and J. Haglund. A positivity result in the theory of Macdonald polynomials. *Proc. Nat. Acad. Sci. U.S.A.*, 98:4313–4316, 2001.

[GH02] A. Garsia and J. Haglund. A proof of the q, t-Catalan positivity conjecture. *Discrete Math.*, 256:677–717, 2002.

[GR90] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, 1990.

[Hag] J. Haglund. Conjectured statistics for the q, t-Catalan numbers. *Adv. in Math.* in press; preprint available at www.math.upenn.edu/~jhaglund/.

[Hai94] M. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3:17–76, 1994.

[Hai01] M. Haiman. Hilbert schemes, polygraphs, and the Macdonald positivity conjecture. *Amer. Math. Soc.*, 14:941–1006, 2001.

[Hai02] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 149:371–407, 2002.

[HL] J. Haglund and N. Loehr. A conjectured combinatorial formula for the Hilbert series for Diagonal Harmonics. *Proceedings of the FPSAC 2002 Conference*. preprint available at www.math.upenn.edu/~jhaglund/.

[Mac95] I.G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs, second ed., Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1995.

[Mac01] P. A. MacMahon. *Combinatory Analysis*, volume 1. Chelsea Pub. Co., 3rd edition, 2001.

[Rem] J. B. Remmel. Private communication.

[Sta86] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Wadsworth & Brooks/Cole, Monterey, California, 1986.

[Sta99] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, United Kingdom, 1999.