Monochrome symmetric subsets in 2-colorings of groups

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Submitted: Jan 31, 2002; Accepted: Jul 11, 2003; Published: Aug 3, 2003
MR Subject Classifications: 05D10, 20B07

Abstract

A subset A of a group G is called symmetric with respect to the element $g \in G$ if $A = gA^{-1}g$. It is proved that in any 2-coloring, every infinite group G contains monochrome symmetric subsets of arbitrarily large cardinality $< |G|$.

A topological space is called *resolvable* if it can be partitioned into two dense subsets [8]. In [4] W. Comfort and J. van Mill proved that each nondiscrete topological Abelian group with finitely many elements of order 2 is resolvable. In that paper it was also posed the problem of describing of absolutely resolvable groups. A group is called *absolutely resolvable* if it can be partitioned into two subsets dense in any nondiscrete group topology. This problem turned out to be rather difficult even for rational group $\mathbb{Q}$ [11], and for real group $\mathbb{R}$ it had remained unsolved. In Abelian case this problem was finally solved by Y. Zelenyuk who proved that each infinite Abelian group with finitely many elements of order 2 is absolutely resolvable [13].

It is easy to see that an Abelian group G is absolutely resolvable if and only if it can be partitioned into two subsets not containing subsets of the form $g + U$ where U is a neighborhood of zero in some nondiscrete group topology. In [10] I. Protasov considered a question close to the above problem. He described Abelian groups which can be partitioned into two subsets not containing infinite subsets of the form $g + U$ where $U = -U$. Such subsets were called *symmetric* and groups that can be partitioned into two subsets not containing infinite symmetric subsets – *assymetrically resolvable*. More precisely, there was given the following equivalent definition of a symmetric subset. A subset A of an Abelian group G is called *symmetric* with respect to the element $g \in G$ if $2g - A = A$. Later on R. Grigorchuk extended this definition to arbitrary groups. A subset A of a group G is called *symmetric* with respect to the element $g \in G$ if $gA^{-1}g = A$. This notion turned out to be enough fruitful, especially against a background of Ramsey Theory (see surveys [1,2]).

According [10] an infinite Abelian group is assymetrically resolvable if and only if it is either a direct product of an infinite cyclic group and a finite Abelian group or a countable periodic Abelian group with finitely many elements of order 2. The problem of describing of all assymetrically resolvable groups is considerably more complicated. For example, it was open for the free group on two generators [2, problem 1.2], and also for each infinite finitely generated periodic group. In case of infinite finitely generated groups of finite torsion, it was not even known whether there exist arbitrarily large finite monochrome symmetric subsets in any 2-coloring [2, problem 1.7].

In this note, the first theorem states if the commutator subgroup G' of a group G contains a finitely generated subgroup different from an almost cyclic group, then G is not assymetrically resolvable. Recall that an almost cyclic group is a group containing a cyclic subgroup of finite index. In particular, every finite group is almost cyclic. By the first theorem, it follows that both the free group on two generators and every infinite finitely generated periodic group are not assymetrically resolvable. Next, by means of this result we prove more two theorems. One theorem states that in any 2-coloring, every infinite group G contains monochrome symmetric subsets of arbitrarily large cardinality $< G$. Another theorem concerns the problem of describing of all assymetrically resolvable groups. It states that every such group G is either almost cyclic or countable locally finite provided G' is finite or G' is infinite and G/G' is periodic.

The proof of the first theorem uses the following nontrivial fact: every group of linear growth is almost cyclic. Indeed, every group of polynomial growth contains a nilpotent subgroup G of finite index [6] and a degree d of a polynomial is evaluated by means of the lower central series

$$G = G_1 > G_2 > \cdots,\ G_{k+1} = [G, G_k],$$

from formula

$$d = \sum_{k \geq 1} k \cdot r_0(G_k/G_{k+1}),$$

where $r_0(A)$ is a free rank of Abelian group A [3]. If $d = 1$, then rank of the first section equals 1, ranks of the others equal 0. Since G is finitely generated and nilpotent, all terms of series are finitely generated. Then all sections are finitely generated Abelian groups. Hence, the first section is almost cyclic, the others are finite, consequently, G is an almost cyclic group.

Theorem 1. Let G be a group containing a finite subset $X = X^{-1} \ni 1$ such that a subgroup $[X, X] = \langle [x, y] : x, y \in X \rangle$ is different from an almost cyclic group. Then in any 2-coloring, G contains an infinite monochrome subset symmetric with respect to some element from $X^2 = \{xy : x, y \in X\}$.

Proof. Suppose the contrary. Put $F^* = \cup\{F_z : z \in X^2\}$, where $F_z = \{g \in G : g$ and $zg^{-1}z$ is monochrome$\}$. Then F^* is finite and for any $g \in G \setminus F^*$ and $z \in X^2$, g and $zg^{-1}z$ are different colored (then g and $z^{-1}g^{-1}z^{-1}$ are also different colored). Put

$$F = F^* \bigcup \{g \in G : \text{there is } z \in X^2 \text{ such that } z^{-1}g^{-1}z^{-1} \in F^*\}.$$

Then F is finite and for any $g \in G \setminus F$ and $z \in X^2$, g and $zgz = 1(z^{-1}g^{-1}z^{-1})^{-1}1$ are monochrome, because we have only two colors. The passages of the form $g \to zgz$ we shall call elementary. Note that for any finite $K \subset G$ and natural s there are only finitely many elements in G from which it can be passed to K by $\leq s$ elementary passages.

Let $U = \{[x, y] : x, y \in X\}$, $H = \langle U \rangle = [X, X]$. The idea of the proof is following. We choose some element $a \in H \setminus F$ and pass from a to a^{-1} by means of elementary passages, outside F, that will be a contrary. To do this passage outside F, we choose some element $b \in H$ and pass first from a to ab, then from ab to ba^{-1} and at last from ba^{-1} to a^{-1}. We write the element a, that has not be chosen yet, as $a = z_1 \cdots z_m$, $z_i \in X$ and the element b, that has not also be chosen yet, as $b = u_1 \cdots u_n$, $u_j \in U$, $u_j = [x_j, y_j]$, $x_j, y_j \in X$. From a to ab we pass as follows:

$$a \to x_1 a x_1 \to y_1 x_1 a x_1 y_1 \to a x_1 y_1 (y_1 x_1)^{-1} = a x_1 y_1 x_1^{-1} y_1^{-1} = a[x_1, y_1] = a u_1 \to \cdots \\ \to a u_1 \cdots u_n = ab.$$

To pass from ab to ba^{-1}, we first pass from $ab = z_1 \cdots z_m b$ to $z_m \cdots z_1 b$. We shall content ourselves with demonstrating the passage from $z_1 \cdots z_{i-1} z_i z_{i+1} z_{i+2} \cdots z_m b$ to $z_1 \cdots z_{i-1} z_{i+1} z_i z_{i+2} \cdots z_m b$:

$$z_1 \cdots z_m b \to z_2 \cdots z_m b z_1^{-1} \to \cdots \to z_{i+2} \cdots z_m b z_1^{-1} \cdots z_{i+1}^{-1} \to \\ \to z_{i+1} z_i z_{i+2} \cdots z_m b z_1^{-1} \cdots z_{i-1}^{-1} \to z_{i-1} z_{i+1} z_i z_{i+2} \cdots z_m b z_1^{-1} \cdots z_{i-2}^{-1} \to \cdots \\ \to z_1 \cdots z_{i-1} z_{i+1} z_i z_{i+2} \cdots z_m b.$$

Then we pass from $z_m \cdots z_1 b$ to $b z_m^{-1} \cdots z_1^{-1} = ba^{-1}$. As is seen, the number of elementary passages only depends on m when we pass from ab to ba^{-1} . At last, we do the passage from ba^{-1} to a^{-1}. To this end, we need to choose the elements a and b.

For each $g \in H$ define the least length of the decomposition of g in terms of elements from U by $l(g)$:

$$l(g) = \min\{n < \omega : g \in U^n\}.$$

Notice that $l(g^{-1}) = l(g)$, $l(gh) \leq l(g) + l(h)$ and then $l(gh) \geq |l(g) - l(h)|$.

We shall choose a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U by the next lemma.

Lemma 1. There exists a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U such that $l(u_1 \cdots u_n) = n$.

Proof. Given $g \in H$, $l(g) = n$, we fix the minimal decomposition $g = u_1(g) \cdots u_n(g)$. Pick the sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in H such that $l(g_n) \geq n$ and $u_1(g_n) = u_1$ for some $u_1 \in U$. Next, pick the subsequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $\langle g_n \rangle_{n \in \mathbb{N}}$ such that $u_2(h_n) = u_2$ for some $u_2 \in U$ and so forth. □

We shall choose the element b from the sequence of products $\langle u_1 \cdots u_n \rangle_{n \in \mathbb{N}}$. We need only to indicate the number n. This will be later. And now we choose a natural k so that the passage

$$g \to xgx \to yxgxy \to gxy(yx)^{-1} = g[x, y]$$

holds outside F for any $g \in H$, $l(g) > k$ and $x, y \in X$.

We shall choose the element a by the next lemma.

Lemma 2. There exists an element $a \in H$, $l(a) > k$ such that $l(au_1 \cdots u_n) > k$ and $l(u_1 \cdots u_n a^{-1}) > k$ for all n.

Proof. Suppose the contrary. Then for any $a \in H$ there exists $n \in \mathbb{N}$ such that either $au_1 \cdots u_n \in \Gamma(k)$ or $u_1 \cdots u_n a^{-1} \in \Gamma(k)$, where $\Gamma(k) = \{g \in H : l(g) \leq k\}$. Consequently, for any $a \in H$ either $a \in (u_1 \cdots u_n \Gamma(k))^{-1}$ or $a \in \Gamma(k) u_1 \cdots u_n$. Hence,

$$H = \{(u_1 \cdots u_n g)^{-1}, g u_1 \cdots u_n : n \in \mathbb{N}, g \in \Gamma(k)\}.$$

It follows from this that

$$\Gamma(n) \subseteq \{(u_1 \cdots u_n g)^{-1}, g u_1 \cdots u_n : i \leq n + k, g \in \Gamma(k)\}.$$

Put $\gamma(n) = |\Gamma(n)|$. Clearly γ is the growth function of H and $\gamma(n) \leq 2\gamma(k)(n+k)$. So the growth of H is linear, a contradiction. $\square$

We choose the number n so that the passage from $au_1 \cdots u_n = ab$ to ba^{-1} holds outside F. Then the general passage from a to a^{-1} also holds outside F. The proof of Theorem 1 is complete. $\square$

In [12] it was proved that in any 3-coloring, every uncountable Abelian group G of regular cardinality contains either a monochrome symmetric subset of cardinality $|G|$ or a monochrome coset modulo subgroup of arbitrarily large cardinality $< |G|$.

Proposition. In any 2-coloring, every uncountable group G of regular cardinality contains either a monochrome symmetric subset of cardinality $|G|$ or a monochrome coset modulo subgroup of arbitrarily large cardinality $< |G|$.

Proof. First notice that a coset is symmetric with respect to any its own element:

$$g(gH)^{-1}g = gH^{-1}g^{-1}g = gH.$$

Now consider three cases.

Case 1. $|G'| = |G|$.

Suppose that G has no symmetric subsets of cardinality $|G|$. Let $\omega \leq k < |G|$. We need to find a monochrome coset of cardinality k. Pick a set K of commutators of G that has cardinality k. Given $u \in K$, we assign the elements $x_u, y_u \in G$ such that $[x_u, y_u] = x_u y_u x_u^{-1} y_u^{-1} = u$. Form a subgroup $A = \langle x_u, y_u : u \in K \rangle$ generated by a subset $\{x_u, y_u : u \in K\}$. Then $|A| = |A'| = k$. Next, given $a \in A$, we assign a set S_a of all $x \in G$ that the elements x and $ax^{-1}a$ are monochrome. By assumption, $|S_a| < |G|$, therefore the cardinality of the subgroup $H = \langle \cup_{a \in A} S_a \cup A \rangle$ is also $< |G|$, because $|G|$ is regular. By constructing of the subgroup H, for every $g \in G \setminus H$ and $a \in A$, the elements g and $ag^{-1}a$ have the different color. Since we have only two colors, the elements g and $aga = (a^{-1}g^{-1}a^{-1})^{-1}$ are monochrome. So, for every $g \in G \setminus H$ and $a, b \in A$, the elements g and $g[a, b]$ are also monochrome:

$$g \to aga \to bagab \to gab(ba)^{-1} = g[a, b].$$

Therefore, the coset gA' is monochrome.

Case 2. $|G'| < |G|$ and $|\{g^2 : g \in G\}| = |G|$.

Suppose that G has no symmetric subsets of cardinality $|G|$. Let $\omega \leq k < |G|$. We need to find a monochrome coset of cardinality k. We may take $k \geq |G'|$. Let K be a set of all commutators of G and let P be a subset of $\{g^2 : g \in G\}$ that has cardinality k. Given $u \in K$, we assign the elements $x_u, y_u \in G$ such that $[x_u, y_u] = u$. And given $v \in P$, we assign $z_v \in G$ such that $z_v^2 = v$. Put $A = \langle x_u, y_u, z_v : u \in K, v \in P\rangle$. Then $|A| = |A^2| = k$, where $A^2 = \langle a^2 : a \in A\rangle$, and every commutator of elements of G equals some commutator of elements of A. Next, as in case 1, we pick the subgroup H, $A \subseteq H \subset G$, $|H| < |G|$ such that for every $g \in G \setminus H$ and $a \in A$, the elements g and aga are monochrome. Then the elements g and ga^2 are also monochrome. Indeed, putting $b = x_{[a,g]}$ and $c = y_{[a,g]}$ we obtain:

$$g \to aga = [a, g]ga^2 = [b, c]ga^2 = bcb^{-1}c^{-1}ga^2 \to$$
$$\to b^{-1}c^{-1}ga^2(bc)^{-1} = b^{-1}c^{-1}ga^2c^{-1}b^{-1} \to c^{-1}ga^2c^{-1} \to ga^2.$$

Therefore, the coset gA^2 is monochrome.

Case 3. $|G'| < |G|$ and $|\{g^2 : g \in G\}| < |G|$.

In this case we shall prove that there exists a monochrome symmetric subset of cardinality $|G|$. For each $a \in \{g^2 : g \in G\}$, let $C_a = \{g \in G : g^2 = a\}$. Since $G = \cup_a C_a$, $|\{g^2 : g \in G\}| < |G|$ and the cardinal $|G|$ is regular, $|C_a| = |G|$ for some a. Similarly, since $|G'| < |G|$, for fixed $c_0 \in C_a$ there exists a subset $C \subseteq C_a$ of cardinality $|G|$ such that all commutators $[c, c_0^{-1}]$, $c \in C$ are equal, say to the element b. Then a subset $B = \{g \in G : g^2 = b\}$ has also cardinality $|G|$. Indeed, for each $c \in C$, $(cc_0^{-1})^2 = [c, c_0^{-1}]c_0^{-1}c^2c_0^{-1} = [c, c_0^{-1}] = b$. Pick an arbitrary $c \in C$ and put $X = \{1, c_0, c_0^{-1}, c, c^{-1}\}$. We shall show that there is a monochrome subset of cardinality $|G|$ in G, symmetric with respect to some element of $X^2 = \{xy : x, y \in X\}$. Suppose the contrary. Then there exists a subgroup H, $X \subseteq H \subset G$, $|H| < |G|$ such that for every $g \in G \setminus H$ and $z \in X^2$ the elements g and zgz are monochrome. Thus, for every $g \in G \setminus H$ and $x, y \in X$, the elements g and $g[x, y]$ are also monochrome. So, the coset $g[X, X]$ is monochrome. Pick $g \in (G \setminus H) \cap B$. Then $g^2 = b \in [X, X]$. So, both elements g and g^{-1} belong to $g[X, X]$. Therefore, g and g^{-1} are monochrome, a contradiction. $\square$

Theorem 2. In any 2-coloring, every infinite group G contains monochrome symmetric subsets of arbitrarily large cardinality $< |G|$.

Proof. Assume first that G is uncountable. Let $\omega \leq k < |G|$. If the cardinal $|G|$ is regular then we apply Proposition to the group G, otherwise to any its subgroup of cardinality k^+.

Now assume that $|G| = \omega$. If elements orders of G are unbounded in totality, then we use van der Waerden's Theorem: there is a function $n(r,l)$ on natural numbers such that for any r-coloring of the initial set of $n(r,l)$ natural numbers there exists a length l monochrome arithmetic progression, see in [5]. If G has finite torsion and it is a locally finite group, then by Kargapolov-Hall-Kulatilaka Theorem [9], we pick an infinite Abelian subgroup and use Craham-Leeb-Rothschild Theorem: for any r-coloring of an infinite Abelian group of finite torsion there exists an arbitrarily large monochrome coset modulo finite subgroup, see in [5]. If G has finite torsion and it is not a locally finite group, then, as well, its commutator subgroup is not locally finite, so use Theorem 1. □

Question. Does every infinite group G contain monochrome symmetric subsets of arbitrarily large cardinality $< |G|$ in any finite coloring?

Theorem 3. Let G be an assymetrically resolvable group. Assume that either G' is finite or G' is infinite and G/G' is periodic. Then G is either almost cyclic or countable locally finite.

To prove Theorem 3, we need some auxiliary assertions.

Let A be a subgroup of a group G. We refer to a subset of all elements of G that are central with respect to some subgroup of A of finite index as an almost centralizer of A. Clearly, an almost centralizer is a subgroup.

Lemma 3. Let A be an infinite cyclic subgroup of G and let H be an almost centralizer of A. If number of double cosets of H modulo A is infinite, then G is not assymetrically resolvable.

In particular, a group is not assymetrically resolvable when it contains an infinite cyclic subgroup, that has an infinite index in its centralizer.

Proof. Let $A = \langle a \rangle$. We shall show that there is an infinite monochrome subset in H symmetric with respect to some element of $1, a, a^{-1}$. Suppose the contrary. Then there is a finite subset $F \subset H$ such that for any element $g \in H \setminus F$ a subset $\{g, aga, a^{-1}ga^{-1}\}$ is monochrome. Therefore, for any element $g \in H \setminus AFA$ a subset $\{a^n g a^n : n \in \mathbb{Z}\} \subseteq AgA$ is monochrome. For some natural $k \geq 1$ one has $a^k g a^k = g a^{2k}$. Consequently, the subset $\{a^n g a^n : n \in \mathbb{Z}\}$ contains a coset $g \cdot \langle a^{2k} \rangle$. □

Corollary. If a group G is assymetrically resolvable and G' is finite, then G is either locally finite or almost cyclic.

Proof. Let G is not locally finite. Then G is not periodic, since G' is finite. Let $A = \langle a \rangle$ be an infinite cyclic subgroup of G. Since $xax^{-1} = [x,a]axx^{-1} = [x,a]a$ and G' is finite, the conjugation class of the element a is finite. Thus, the centralizer $H = C_G(A) = C_G(a)$ has finite index in G. By Lemma 3, index $|H : A|$ is finite. Therefore, index $|G : A|$ is also finite. □

Lemma 4. Let G be a group containing a normal infinite almost cyclic subgroup and let G be different from an almost cyclic group. Then G is not assymetrically resolvable.

Proof. First recall that for each natural $n \geq 1$ there are only finitely many subgroups of index n in a finitely generated group. Let N be a normal subgroup of G and let $A = \langle a \rangle$ be an infinite cyclic subgroup of N of finite index. Since number of subgroups of N, that have index $|N : A|$, is finite and N is normal, the conjugation class of the element a in G is finite. Then the centralizer $H = C_G(A)$ has finite index in G. Since G is different from almost cyclic, index $|G : N|$ is infinite. Therefore, index $|H : A|$ is also infinite. To complete the proof, use Lemma 3. $\square$

Lemma 5. Let G be a non-periodic group different from an almost cyclic group such that every finitely generated subgroup of G is almost cyclic. Then G is not assymetrically resolvable.

Proof. Let $A = \langle a \rangle$ be an infinite cyclic subgroup of G and let H be its almost centralizer. For each element $g \in G$, some non-identity subgroup A is normal in the almost cyclic subgroup $\langle A \cup \{g\} \rangle$. Consequently, for each $g \in G$ there is a natural number $n \geq 1$ such that either $ga^n g^{-1} = a^n$ or $ga^n g^{-1} = a^{-n}$. The first case is equivalent to $g \in H$ and the second case is equivalent to $g \in G \setminus H$. To use lemma 3 we need to verify that number of double cosets of H modulo A is infinite. Suppose the contrary. Then $H = AFA$ for some finite $F \subset G$ and therefore, H is finitely generated. Since G is not finitely generated, we can choose $b \in G \setminus H$ and $c \in G \setminus \langle H \cup \{b\} \rangle$. Hence, the elements b, c, bc belong to $G \setminus H$. Since $ba^n b^{-1} = a^{-n}$ and $ca^n c^{-1} = a^{-n}$, $(bc)a^n(bc)^{-1} = bca^n c^{-1} b^{-1} = ba^{-n} b^{-1} = a^n$, a contradiction. $\square$

Proof of theorem 3. If G' is finite, apply Corollary of Lemma 3. So, let G' is infinite. If G' is periodic, then by Theorem 1, G' is locally finite and then G is locally finite. So, let G' is non-periodic. By Theorem 1, every finitely generated subgroup of G' is almost cyclic. Then by Lemma 5, G' is infinite almost cyclic and so by Lemma 4, G is almost cyclic. $\square$

Remark 1. A. Khelif has recently supplemented Theorem 3 proving that if G is a group with infinite G' and non-periodic G/G', then G is not assymetrically resolvable. From Theorem 3 and Khelif's result it follows that, as in the Abelian case, every assymetrically resolvable group is either almost cyclic or countable locally finite.

Remark 2. As distinguished from the Abelian case, among groups with finitely many elements of order 2 which are not assymetrically resolvable, there are both countable locally finite groups and almost cyclic groups [7].

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