Homotopy and homology of finite lattices

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Abstract

We exhibit an explicit homotopy equivalence between the geometric realizations of the order complex of a finite lattice and the simplicial complex of coreless sets of atoms whose join is not $\hat{1}$. This result, which extends a theorem of Segev, leads to a description of the homology of a finite lattice, extending a result of Björner for geometric lattices.

1 Introduction

The purpose of this paper is to unify and extend three directions of work that originated from Rota's broken-circuit formula [4] for the Möbius function of a geometric lattice. In this introduction, we shall present the necessary terminology, state Rota's theorem, outline the three developments that are relevant for our purposes, and then describe our results and how they are related to the previous ones.

Throughout this paper, L is a finite lattice, with lattice operations written $\vee$ and $\wedge$ and with ordering written $\leq$. Its smallest and largest elements are $\hat{0}$ and $\hat{1}$, and the least upper bound of a subset X is written $\bigvee X$. We always assume that L is non-degenerate, i.e., that $\hat{0} \neq \hat{1}$. The set of atoms, i.e., minimal non-$\hat{0}$ elements, is called A.

The *Möbius function* $\mu(x, y)$ is defined for all $x \leq y$ in L and is uniquely characterized by the equations

$$\mu(x,x) = 1 \qquad \text{and} \qquad (\forall x < y) \sum_{x \leq z \leq y} \mu(x,z) = 0.$$

We shall be interested primarily in the special case $\mu(\hat{0}, \hat{1})$. The general values $\mu(x, y)$ can be obtained by applying this special case to the intervals $\{z : x \leq z \leq y\}$ of L.

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In the special case that L is a geometric lattice, i.e., the lattice of flats of a matroid (up to isomorphism), a *circuit* of L is defined to be a subset of A that is minimal with respect to the property that for some $a \in A$, $\bigvee A = \bigvee (A - \{a\})$. This agrees with the matroid notion of circuit. If a linear ordering $\preceq$ of A is specified (entirely unrelated to the lattice ordering $\leq$ which is of course trivial on A), then a *broken circuit* is a set obtained from a circuit by deleting its $\preceq$-first element. An *NBC* (abbreviating "no broken circuit") set is a subset of A that includes no broken circuits. With this terminology, we can state Rota's formula for the Möbius function, a generalization of a result of Whitney [7], who introduced broken circuits and NBC sets in the context of graph theory.

Theorem 1 (Rota [4]) *Let $\preceq$ be a linear ordering of the set A of atoms of a finite geometric lattice L. Then, for any $x \in L$,*

$$\mu(\hat{0}, x) = \sum_{B} (-1)^{|B|}$$

where the sum is over all NBC subsets B of A such that $\bigvee B = x$.

In a geometric lattice, all NBC sets with a specified join x have the same cardinality, namely the rank $\rho(x)$ of x. Thus, the sum in Rota's theorem is simply the number of NBC sets with join x, with a sign $(-1)^{\rho(x)}$. We wrote this as a sum for the sake of the generalizations below, where the relevant B's might not all have the same cardinality.

As indicated above, the essential content of Rota's theorem is captured already by the special case where $x = \hat{1}$, for the general case follows by considering the interval of elements $\leq x$. Thus, we shall sometimes refer to this special case as "Rota's theorem." The sets B occurring in this special case, namely the NBC subsets of A whose joins are $\hat{1}$, are called the *NBC bases* of L.

We shall be interested in three sorts of extensions of Rota's result. The first of these extensions, carried out in [5] and [2], removes the restriction to geometric lattices. There are two versions of the main result in [2]. One uses a notion of NBB set, defined relative to an arbitrary partial ordering of the atoms. Theorem 1.1 of [2] says that Rota's formula holds for all finite lattices if one replaces NBC with NBB. When the ordering of atoms is linear and L is geometric, NBB coincides with NBC, so this theorem from [2] subsumes Rota's theorem. A second, even more general version of the result is given in Section 8 of [2]. Here, NBB is replaced by a notion of "coreless," which we shall develop in detail below because it plays a central role in the present paper. Again, the formula for $\mu(0, x)$ is as in Rota's theorem, with B required to be coreless rather than NBC.

To describe the second and third extensions of Rota's theorem, we write Δ for the *order complex* of the partially ordered set $L - \{\hat{0}, \hat{1}\}$. This is the simplicial complex whose underlying set is $L - \{\hat{0}, \hat{1}\}$ and whose simplices are the subsets that are linearly ordered by the lattice ordering $\leq$. We sometimes abuse notation by calling Δ the order complex of L.

It is well known that, in any lattice (indeed in any poset), $\mu(x, y)$ is the number of chains $x < z_1 < \cdots < z_{l-1} < y$ in L, counted with positive or negative signs according as the lengths l are even or odd. It follows that $\mu(\hat{0}, \hat{1})$ is the reduced Euler characteristic of the order complex Δ. Since Euler characteristics of simplicial complexes can be computed from the homology groups (as the alternating sum of their ranks), Rota's theorem implies a connection between NBC bases of a geometric lattice L and the homology of the order complex of L. Björner [1] gave an elegant explicit form of this connection. If a geometric lattice L has rank r (meaning the maximal length of a chain from $\hat{0}$ to $\hat{1}$) then its reduced homology vanishes in all dimensions except $r-2$. Björner gave an explicit function, assigning to each NBC base (with respect to an arbitrary but fixed ordering $\preceq$ of A) an $(r-2)$-cycle of Δ, and he proved that the images of these cycles in homology constitute a free basis for $H_{r-2}(\Delta)$.

For more general finite lattices, the situation is considerably more complicated than for geometric lattices, because the reduced homology need not be concentrated in a single dimension, nor need it be free. Indeed, it is known that for any finite simplicial complex there is a finite lattice with isomorphic homology groups. Nevertheless, we shall obtain a generalization of Björner's theorem to arbitrary finite lattices, in terms of the coreless sets of atoms.

The third extension of Rota's theorem involves looking not only at the homology groups of Δ but at its homotopy type, by which we mean the homotopy type of its geometric realization. This extension was carried out by Segev [6], in the context of general finite lattices and NBB sets. The NBB sets of atoms that are *not* bases, i.e., that have join strictly smaller than $\hat{1}$, are the simplices of a simplicial complex, and Segev proves that this complex and Δ are homotopy equivalent. His proof is rather abstract; it does not explicitly exhibit the maps (in either direction) that constitute a homotopy equivalence.

Our main result is an extension of Segev's. A minor aspect of our extension is that we replace NBB with the more general concept of "coreless." The more important aspect is that we explicitly exhibit the homotopy equivalence, in both directions. One direction is in Section 3, the other in Section 4.

Once this result is established, it provides explicit formulas for Björner-style isomorphisms of reduced homology groups. The formulas given directly by the homotopy equivalence can be simplified somewhat, clarifying their connection with Björner's formula. This is done in Section 5.

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I thank Bruce Sagan for bringing Björner's work [1] to my attention and suggesting that it might be related to our joint work in [2].

2 Coreless Sets

We devote this preliminary section to the notion of a coreless set of atoms, which will play a central role in all our results. This notion was introduced in [2, Section 8] but considered only briefly, so we provide a more extensive treatment here.

Convention 2 Throughout this paper, L is a finite lattice and M is a function assigning to every $x \in L - \{\hat{0}\}$ a nonempty family $M(x)$ of atoms $\leq x$.

Definition 3 The family $\mathcal{N}$ of *coreless* sets of atoms is defined to be the smallest family of sets of atoms such that

- $\varnothing \in \mathcal{N}$ and
- if $X \in \mathcal{N}$ and $y \geq \bigvee X$ in L and $a \in M(y)$, then $X \cup \{a\} \in \mathcal{N}$.

The following rephrasing of the definition is sometimes useful.

Corollary 4 *A set X of atoms is coreless if and only if there is a sequence $X_0, X_1, \ldots, X_k$ of sets of atoms, starting with $X_0 = \varnothing$, ending with $X_k = X$, and satisfying for all $i < k$*

$$X_{i+1} = X_i \cup \{a\} \quad \textit{for some } a \in M(y) \textit{ with } y \geq \bigvee X_i.$$

The name "coreless" comes from the equivalent characterization given by the following definition and proposition; this characterization was used as the definition in [2].

Definition 5 A set B of atoms is a *core* set if, for all $y \geq \bigvee B$, $B \cap M(y) = \varnothing$.

Every set X of atoms has a largest core subset, obtained by iterating the operation

$$X \mapsto S(X) = X - \bigcup_{y \geq \bigvee X} M(y)$$

until it stabilizes. This subset is called the *core* of X. Notice that $\varnothing$ is a core set and singletons are not core sets.

Proposition 6 *A set X of atoms is coreless if and only if its only core subset is $\varnothing$. (Equivalently, $S^n(X) = \varnothing$ for some n.)*

Proof We show first, by induction on the cardinality $|X|$, that if $\varnothing$ is the only core subset of X then X is coreless. If $|X| = 0$ this is correct, because $\varnothing$ is coreless by definition. So suppose $X \neq \varnothing$ and $\varnothing$ is its only core subset. In particular, X itself is not a core set, so we can find some $y \geq \bigvee X$ and some $a \in X \cap M(y)$. Now $X - \{a\}$ has $\varnothing$ as its only core subset (because X does), so $X - \{a\} \in \mathcal{N}$ by induction hypothesis. But $y \geq \bigvee(X - \{a\})$ and $a \in M(y)$, so the definition of $\mathcal{N}$ gives us $X \in \mathcal{N}$, as required.

For the converse implication, it suffices, thanks to "smallest" in the definition of $\mathcal{N}$, to show that the family

$$\mathcal{N}' = \{X \subseteq A : \varnothing \text{ is the only core subset of } X\}$$

satisfies

- $\varnothing \in \mathcal{N}'$ and

- if $X \in \mathcal{N}'$ and $y \geq \bigvee X$ in L and $a \in M(y)$, then $X \cup \{a\} \in \mathcal{N}'$.

The former is obvious. To prove the latter, suppose X, y, and a were a counterexample, and let B be a nonempty core subset of $X \cup \{a\}$. As X has no such subset, we must have $a \in B$. We also have $y \geq \bigvee X$ and $y \geq a$ (as $a \in M(y)$), and so $y \geq \bigvee(X \cup \{a\}) \geq \bigvee B$. But then y and a witness that B is not a core set. □

Corollary 7 *Any subset of a coreless set is coreless.*

Remark 8 This corollary can also be proved directly from the definition or the characterization in Corollary 4. If $Y \subset X$ and we have a chain leading from $\varnothing$ to X as in Corollary 4, then we can get a chain leading to Y by simply omitting the steps that added elements of $X - Y$.

The reason for introducing the notion of coreless sets in [2] was the following theorem, whose proof we reproduce here.

Theorem 9 *For every $x \in L$,*

$$\mu(\hat{0}, x) = \sum_{X \in \mathcal{N},\, \bigvee X = x} (-1)^{|X|}.$$

Proof Let $\nu(x)$ be the sum on the right side of the equation in the theorem. By the definition of the Möbius function, it suffices to prove that $\nu(\hat{0}) = 1$ and that $\sum_{x \leq y} \nu(x) = 0$ for all $y \neq \hat{0}$ in L. The former is obvious, as $\varnothing \in \mathcal{N}$ and no other set of atoms has join $\hat{0}$. For the latter, we have

$$\sum_{x \leq y} \nu(x) = \sum_{X \in \mathcal{N},\, \bigvee X \leq y} (-1)^{|X|}.$$

So it suffices to find a parity-reversing involution on $\{X \in \mathcal{N} : \bigvee X \leq y\}$ for each fixed $y \neq 0$. Given y, choose some $a \in M(y)$ and let the involution be $X \mapsto X \triangle \{a\}$, where $\triangle$ denotes symmetric difference. That is, remove a from X if it was in X, and adjoin it to X otherwise. The preceding corollary ensures that the result of removing a is still in $\mathcal{N}$; the definition of $\mathcal{N}$ ensures that the result of adjoining a is also still in $\mathcal{N}$. □

Remark 10 If we change M by replacing each $M(x)$ with a (nonempty) subset of its original value, then $\mathcal{N}$ changes to a subfamily of what it was before. So, by taking M as small as possible, i.e., all $M(x)$ are singletons, we get the fewest terms in the sum expressing $\mu(\hat{0}, x)$. Larger M's will usually lead to extra terms, which must cancel.

Remark 11 If $\preceq$ is a partial ordering of the set A of atoms, then there is an associated function M assigning to each $x \in L - \{\hat{0}\}$ the set of $\preceq$-minimal elements of $\{a \in A : a \leq x\}$. For this choice of M, nonempty core sets are exactly the bounded below sets of [2] and therefore the coreless sets are the NBB sets.

Since the equivalence of "core" and "bounded below" was stated without proof in [2], the referee suggested that we provide the proof here. For M defined from $\preceq$ as here, the definition of "core" says that $B \subseteq A$ is a core set if and only if, for all $y \geq \bigvee B$, no element of B is $\preceq$-minimal among the atoms below y (where "below" refers, of course, to the lattice ordering $\leq$). That is, for each $b \in B$ (that is below y), there is some $d \prec b$ that is also below y. Here the parentheses around "that is below y" indicate that, although it is part of what we get when applying the definition, it is redundant because $y \geq \bigvee B$.

Notice that the statement "for each $b \in B$ there is some $d \prec b$ that is below y" will hold for all $y \geq \bigvee B$ if and only if it holds for $y = \bigvee B$. Thus, we find that B is coreless if and only if, for each $b \in B$, there is some $d \prec b$ that is below $\bigvee B$. Comparing this with the definition of "B is bounded below" in [2], we see that there are only two differences. One is that bounded below sets are required to be nonempty. The other is that we have $d \leq \bigvee B$ where the definition in [2] required $d < \bigvee B$. But the latter is no real difference; since d and b are distinct atoms (as $d \prec b$), d cannot equal the join of a set B that contains b.

Therefore, the nonempty coreless sets for the M defined from $\preceq$ are exactly the bounded below sets for $\preceq$.

Remark 12 Specializing further, suppose L is a geometric lattice and $\preceq$ is a linear ordering of A. Then the associated M is related to broken circuits as follows. Any broken circuit (with respect to the ordering $\preceq$) is a core set, and any nonempty core set includes a broken circuit. (See the discussion following Theorem 1.2 in [2].) Therefore, the coreless sets are exactly the NBC sets, and our formula for the Möbius function specializes to Rota's.

3 A Homotopy Equivalence

According to Corollary 7, the family $\mathcal{N}$ of coreless sets is an abstract simplicial complex. So are the subfamilies

$$\mathcal{N}_x = \{X \in \mathcal{N} : \bigvee X \leq x\}$$

for all $x \in L$ and

$$\mathcal{N}^- = \bigcup_{x \neq \hat{1}} \mathcal{N}_x = \{X \in \mathcal{N} : \bigvee X < \hat{1}\}.$$

We shall also use the notation

$$\mathcal{N}^+ = \mathcal{N} - \mathcal{N}^- = \{X \in \mathcal{N} : \bigvee X = \hat{1}\},$$

but of course $\mathcal{N}^+$ is not a simplicial complex, i.e., it is not closed under subsets.

We use the standard convention that, when topological concepts (such as homotopy) are applied to simplicial complexes, they are meant to apply to the geometric realizations.

Lemma 13 *For any $x \in L - \{\hat{0}\}$, the simplicial complex $\mathcal{N}_x$ is a cone and therefore contractible.*

Proof Given x, fix an element a of $M(x)$. Then if $X \in \mathcal{N}_x$, we have $\bigvee X \leq x$ and so our choice of a and the definition of $\mathcal{N}$ ensure that $X \cup \{a\} \in \mathcal{N}$. Since $a \leq x$, we have $X \cup \{a\} \in \mathcal{N}_x$. Therefore, $\mathcal{N}_x$ is a cone with vertex a. □

Theorem 14 *The complex $\mathcal{N}^-$ of coreless non-bases and the order complex Δ are homotopy equivalent.*

Proof For topological purposes, we may replace the simplicial complex $\mathcal{N}^-$ by its barycentric subdivision, because their geometric realizations are homeomorphic. We regard the barycentric subdivision as an abstract simplicial complex in its own right. Its vertices are the sets in $\mathcal{N}^- - \{\varnothing\}$, and its simplices are the chains (with respect to set-inclusion) of such sets. In other words, the barycentric subdivision is the order complex of the poset $(\mathcal{N}^- - \{\varnothing\}, \subseteq)$.

There is an order-preserving map

$$j : (\mathcal{N}^- - \{\varnothing\}, \subseteq) \to L - \{\hat{0}, \hat{1}\} : X \mapsto \bigvee X.$$

Like any order-preserving map between posets, j induces a simplicial map of the order complexes, which in turn induces a continuous map $\tilde{\jmath}$ of the geometric realizations. We intend to show that this $\tilde{\jmath}$ is a homotopy equivalence.

By Quillen's theorem (see [3, page 82] and dualize), it suffices to show that, for each $x \in L - \{\hat{0}, \hat{1}\}$, the subcomplex $j^{-1}(\{y : y \leq x\})$ is contractible. But this subcomplex is the barycentric subdivision of $\mathcal{N}_x$ which we already saw is a cone and therefore contractible. □

When M arises from a partial ordering of A, the complex $\mathcal{N}^-$ is the complex of NBB non-spanning sets, and so Theorem 14 specializes to the main theorem of Segev [6]. Unlike Segev's proof, ours exhibits an explicit homotopy equivalence. In the next section, we shall explicitly exhibit a homotopy inverse for it, and in Section 5 we shall study its action on homology.

4 The Inverse Equivalence

Let γ be a function assigning to each $x \in L - \{\hat{0}, \hat{1}\}$ an element $\gamma(x)$ of the set $M(x)$. Thus, γ maps each vertex of the order complex Δ to a vertex of the complex $\mathcal{N}^-$.

Lemma 15 *This γ is a simplicial map from Δ to $\mathcal{N}^-$.*

Proof We must show that for every simplex of Δ, i.e., for every chain $x_0 < x_1 < \cdots < x_k$ in $L - \{\hat{0}, \hat{1}\}$, the image under γ is a simplex of $\mathcal{N}^-$. That is, we must show that $\{\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_k)\}$ is coreless and its join is $< \hat{1}$. For each i, we have $\gamma(x_i) \leq x_i \leq x_k$, and so $\bigvee_{i=0}^{k} \gamma(x_i) \leq x_k < \hat{1}$ as desired. It remains to show that $\{\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_k)\}$ is coreless.

To this end, consider the sets

$$X_j = \{\gamma(x_i) : 0 \le i < j\} \qquad \text{for } 0 \le j \le k+1.$$

Then $X_0 = \varnothing$ and, for $j < k+1$, X_{j+1} is obtained from X_j by adjoining $\gamma(x_j) \in M(x_j)$. Since

$$x_j \ge \bigvee_{i<j} x_i \ge \bigvee_{i<j} \gamma(x_i) = \bigvee X_j,$$

the definition of "coreless" shows that each X_j is coreless. In particular, it shows that $X_{k+1} = \{\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_k)\}$ is coreless, as required. □

It follows immediately from the lemma that γ induces a continuous function $\tilde{\gamma}$ from the geometric realization of Δ to that of $\mathcal{N}^-$.

Proposition 16 *The $\tilde{\gamma}$ defined here is a homotopy equivalence. In fact it is a homotopy inverse of the $\tilde{\jmath}$ of the preceding section.*

Proof In the statement of this proposition, we have used the common convention from topology that the geometric realizations of a simplicial complex and of its barycentric subdivision are identified. Thus, the domain of $\tilde{\jmath}$, the geometric realization of the barycentric subdivision of $\mathcal{N}^-$, agrees with the codomain of $\tilde{\gamma}$, the geometric realization of $\mathcal{N}^-$.

Consider the composite function $\tilde{\gamma} \circ \tilde{\jmath}$ and how it acts on a simplex of the geometric realization of $\mathcal{N}^-$, say the k-simplex with vertices $a_0, \ldots, a_k$. To apply $\tilde{\jmath}$, we regard this simplex as the union of certain simplices of the barycentric subdivision, namely the $(k+1)!$ simplices corresponding to the chains

$$\{a_{\pi(0)}\} \subseteq \{a_{\pi(0)}, a_{\pi(1)}\} \subseteq \cdots \subseteq \{a_{\pi(0)}, a_{\pi(1)} \ldots, a_{\pi(k)}\}$$

in $(\mathcal{N}^- - \{\varnothing\}, \subseteq)$ for arbitrary permutations π of $\{0, 1, \ldots, k\}$. Now we can apply $\tilde{\jmath}$, which maps each of these simplices (linearly) to the corresponding simplex of the geometric realization of Δ, given by the chain

$$a_{\pi(0)} \le a_{\pi(0)} \vee a_{\pi(1)} \le \cdots \le a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)}$$

in L. Applying $\tilde{\gamma}$ to these simplices, we get the simplices in the geometric realization of $\mathcal{N}^-$ spanned by the corresponding sets

$$\{\gamma(a_{\pi(0)}), \gamma(a_{\pi(0)} \vee a_{\pi(1)}), \ldots, \gamma(a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)})\}.$$

Now each vertex of each of these image simplices has the form

$$\gamma(a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(i)}) \le a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(i)} \le a_0 \vee a_1 \vee \cdots \vee a_k.$$

That is, the image under $\tilde{\gamma} \circ \tilde{\jmath}$ of our original simplex with vertices $a_0, a_1, \ldots, a_k$ lies entirely in the geometric realization $C_{a_0 \vee a_1 \vee \cdots \vee a_k}$ of the complex $\mathcal{N}_{a_0 \vee a_1 \vee \cdots \vee a_k}$. This is a subcomplex of $\mathcal{N}^-$ because, with $\{a_0, a_1, \ldots, a_k\} \in \mathcal{N}^-$, the join $a_0 \vee a_1 \vee \cdots \vee a_k$ is not $\hat{1}$.

And it is contractible by Lemma 13. Of course the original simplex is also included in this same $C_{a_0 \vee a_1 \vee \cdots \vee a_k}$. Adding the trivial observation that $C_{a_0 \vee a_1 \vee \cdots \vee a_k}$ is an order-preserving (with respect to set inclusion) function of $\{a_0, a_1, \ldots, a_k\}$, we see that the hypotheses of the Contractible Carrier Lemma of [3, page 74] are satisfied. That lemma then says that $\tilde{\gamma} \circ \tilde{\jmath}$ and the identity map of the geometric realization of $\mathcal{N}^-$ are homotopic.

This shows that $\tilde{\gamma}$ is a left homotopy inverse of $\tilde{\jmath}$. That it is also a right homotopy inverse follows immediately, since we already know, from the proof of Theorem 14, that $\tilde{\jmath}$ is a homotopy equivalence.

Alternatively, one can verify directly that $\tilde{\gamma}$ is a right homotopy inverse for $\tilde{\jmath}$, thus giving a new proof of Theorem 14. This verification again uses the Contractible Carrier Lemma. The carrier associated with a simplex $\{x_0 < x_1 < \cdots < x_k\}$ of Δ is the geometric realization of the subcomplex of Δ that is the order complex of the poset $\{y \in L : \hat{0} < y \leq x_k\}$. This is contractible, because it is a cone with vertex x_k. We leave to the reader the routine verification that it carries both $\tilde{\jmath} \circ \tilde{\gamma}$ and the identity map. $\square$

5 Homology

The homotopy equivalence $\tilde{\jmath}$ exhibited in the proof of Theorem 14 induces, like any homotopy equivalence, an isomorphism of homology groups. In this section, we look at this isomorphism more closely and use it to get a simple representation, extending that in [1], for the reduced homology of Δ.

We work with oriented simplicial homology groups. For any simplicial complex $\mathcal{X}$, the (oriented simplicial) chain complex $C_*(\mathcal{X})$ has, in any dimension k, the free abelian group $C_k(\mathcal{X})$ generated by oriented simplices $[x_0, x_1, \ldots, x_k]$. Here $\{x_0, x_1, \ldots, x_k\}$ is a k-dimensional (i.e., $(k+1)$-element) simplex of $\mathcal{X}$, and, if the entries of such a simplex are permuted, then $[x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(k)}]$ is identified with $\mathrm{sign}(\pi)[x_0, x_1, \ldots, x_k]$. If two of the x_i are equal, then the notation $[x_0, x_1, \ldots, x_k]$ denotes zero. The boundary operator $\partial : C_k \to C_{k-1}$ is given by

$$\partial[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} (-1)^i [x_0, x_1, \ldots, \widehat{x_i}, \ldots, x_k],$$

where the hat over x_i means that this vertex is to be omitted. We include the empty simplex in our simplicial complexes, so our chain complexes include a group $C_{-1}(\mathcal{X})$ isomorphic to $\mathbb{Z}$. It is well known that the homology $H_*(\mathcal{X})$ of $C_*(\mathcal{X})$ is canonically isomorphic to the reduced homology of the geometric realization of $\mathcal{X}$.

We shall be concerned with four simplicial complexes and their homology:

- the order complex Δ of L (strictly speaking, of $L - \{\hat{0}, \hat{1}\}$),
- the complex $\mathcal{N}$ of all coreless sets of atoms,
- the subcomplex $\mathcal{N}^-$ of coreless sets whose join is not $\hat{1}$, and

- the barycentric subdivision $\mathcal{B}$ of $\mathcal{N}^-$.

Recall from the proof of Theorem 14 that the simplicial map $j : \mathcal{B} \to \Delta$ induces a homotopy equivalence $\tilde{\jmath}$ of geometric realizations and therefore an isomorphism of homology groups

$$j_* : H_*(\mathcal{B}) \to H_*(\Delta).$$

On the chain level, j sends a simplex $[X_0, X_1, \dots, X_k]$ of $\mathcal{B}$, defined by a nest $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ of nonempty, coreless sets with joins $< \hat{1}$, to $[\bigvee X_0, \bigvee X_1, \dots, \bigvee X_k]$, the simplex of Δ defined by the increasing sequence $\bigvee X_0 \leq \bigvee X_1 \leq \cdots \leq \bigvee X_k$ in L. The homology isomorphism j_* is thus induced by this $\bigvee$ operation.

We can express this in terms of the complex $\mathcal{N}^-$ instead of its barycentric subdivision $\mathcal{B}$, if we recall how the identification between their geometric realizations works at the chain level. That identification corresponds to the chain map b that sends any oriented simplex $[a_0, a_1, \dots, a_k]$ of $\mathcal{N}^-$ to the alternating sum over all permutations π of $\{0, 1, \dots, k\}$

$$\sum_\pi \operatorname{sign}(\pi)[\{a_{\pi(0)}\}, \{a_{\pi(0)}, a_{\pi(1)}\}, \dots, \{a_{\pi(0)}, a_{\pi(1)}, \dots, a_{\pi(k)}\}].$$

Composing this with j_*, we find that an isomorphism of homology groups, $j_* \circ b_* : H_*(\mathcal{N}^-) \cong H_*(\Delta)$, is induced by the chain map $j \circ b$ that sends any oriented simplex $[a_0, a_1, \dots, a_k]$ of $\mathcal{N}^-$ to the alternating sum

$$\sum_\pi \operatorname{sign}(\pi)[a_{\pi(0)}, a_{\pi(0)} \vee a_{\pi(1)}, \dots, a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)}]$$

of oriented simplices of Δ.

This can be reformulated as follows in terms of the coreless sets with join $\hat{1}$, i.e., the sets in $\mathcal{N}^+$. Although $\mathcal{N}^+$ is not a simplicial complex, it is the set-theoretic difference between the simplicial complexes $\mathcal{N}$ and $\mathcal{N}^-$. So its simplices freely generate the groups Q_k of the chain complex $Q_* = C_*(\mathcal{N})/C_*(\mathcal{N}^-)$. In more detail, we have the following description of Q_*.

Definition 17 Q_* is the chain complex defined as follows. Its group Q_k in dimension k is freely generated by the oriented simplices $[a_0, a_1, \dots, a_k]$ where $\{a_0, a_1, \dots, a_k\} \in \mathcal{N}^+$. Its boundary operator $\partial : Q_k \to Q_{k-1}$ sends $[a_0, a_1, \dots, a_k]$ to

$$\sum_{i=0}^{k} (-1)^i [a_0, a_1, \dots, \widehat{a_i}, \dots, a_k],$$

subject to the convention that, if $\{a_0, a_1, \dots, \widehat{a_i}, \dots, a_k\} \notin \mathcal{N}^+$ (because its join is $< \hat{1}$) then $[a_0, a_1, \dots, \widehat{a_i}, \dots, a_k] = 0$.

Lemma 18 *There is a short exact sequence of chain complexes*

$$0 \to C_*(\mathcal{N}^-) \to C_*(\mathcal{N}) \to Q_* \to 0.$$

The proof is just a matter of inspecting the definitions.

Notice that, in the notation introduced at the beginning of Section 3, $\mathcal{N} = \mathcal{N}_{\hat{1}}$. So, by Lemma 13, $\mathcal{N}$ is contractible and therefore its reduced homology $\tilde{H}_*(\mathcal{N})$ vanishes. That is, in the long exact homology sequence produced from the short exact sequence in Lemma 18, every third group is zero. By exactness, the remaining groups are isomorphic in pairs. Specifically, the boundary homomorphism ∂ induces isomorphisms

$$\partial_* : H_k(Q_*) \cong \tilde{H}_{k-1}(\mathcal{N}^-).$$

Chasing through the relevant definitions, we find that ∂_* of the homology class represented by a cycle z in Q_k can be obtained by sending each oriented simplex (basis element of Q_k) $[a_0, a_1, \ldots, a_k]$ occurring in z to the sum

$$\sum_{i=0}^{k} (-1)^i [a_0, a_1, \ldots, \widehat{a_i}, \ldots, a_k],$$

obtaining a cycle in $C_{k-1}(\mathcal{N}^-)$. Although the sum exhibited here is in general a chain over $\mathcal{N}$, not $\mathcal{N}^-$, the terms in $\partial_*(z)$ involving simplices from $\mathcal{N}^+$ cancel, because z is a cycle of Q_*. This fact and the fact that $\partial_*(z)$ is a cycle, rather than merely a chain, of $C_*(\mathcal{N}^-)$ are part of the general construction of long exact homology sequences.

Composing ∂_* with $j_* \circ b_*$ we obtain an isomorphism (with a shift of dimension because of ∂_*)

$$j_* \circ b_* \circ \partial_* : H_*(Q_*) \cong \tilde{H}_*(\Delta).$$

Inspection of the definitions shows that this isomorphism is given at the chain level by the transformation τ sending $[a_0, a_1, \ldots, a_k]$ to

$$\sum_{i=0}^{k} (-1)^i \sum_{\pi} \operatorname{sign}(\pi) [a_{\pi(0)}, a_{\pi(0)} \vee a_{\pi(1)}, \ldots, a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)}],$$

where the inner sum is over all permutations π of $\{0, 1, \ldots, \hat{\imath}, \ldots, k\}$. Note carefully that i is in neither the domain nor the range of π and that a_i is therefore not involved in any of the joins appearing in the i^{th} summand in the formula above for $\tau[a_0, a_1, \ldots, a_k]$.

The precise meaning of "given at the chain level" is as follows. The formula for $\tau[a_0, a_1, \ldots, a_k]$ can include terms $[a_{\pi(0)}, \ldots, a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)}]$ that are not oriented simplices of Δ because $a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)} = \hat{1}$ and Δ is the order complex of $L - \{\hat{0}, \hat{1}\}$. But when τ is applied term-by-term to a cycle z of Q_* then these improper terms cancel, and $\tau(z)$ is a cycle of the chain complex of Δ, a cycle representing the homology class $j_* \circ b_* \circ \partial_*(z)$. We can arrange for τ to be defined on all chains of Q_*, not just on cycles, by adopting the convention that when $a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)} = \hat{1}$ then $[a_{\pi(0)}, \ldots, a_{\pi(0)} \vee a_{\pi(1)} \vee \cdots \vee a_{\pi(k)}] = 0$.

A change of notation will simplify the formula above for τ. To each i and each π as in that formula, associate the permutation σ of $\{0, 1, \ldots, k\}$ defined by

$$\begin{aligned} \sigma(0) &= i, \\ \sigma(j) &= \pi(j-1) \quad \text{for } 1 \leq j \leq i, \text{ and} \\ \sigma(j) &= \pi(j) \quad \text{for } j > i. \end{aligned}$$

As i ranges from 0 to k and π ranges over all permutations of $\{0, 1, \ldots, \hat{\imath}, \ldots, k\}$ (as in the formula for τ), σ ranges over all permutations of $\{0, 1, \ldots, k\}$. Furthermore, $(-1)^i \mathrm{sign}(\pi) = \mathrm{sign}(\sigma)$. Therefore, the formula for $\tau[a_0, a_1, \ldots, a_k]$ can be rewritten as follows.

Theorem 19 *The complex Q_* of Definition 17 has homology groups isomorphic, with a shift of dimension, to the reduced homology of Δ. An isomorphism $\tau : H_k(Q_*) \to \tilde{H}_{k-1}(\Delta)$ is given at the chain level by the formula*

$$[a_0, a_1, \ldots, a_k] \mapsto \sum_\sigma \mathrm{sign}(\sigma)[a_{\sigma(1)}, a_{\sigma(1)} \vee a_{\sigma(2)}, \ldots, a_{\sigma(1)} \vee a_{\sigma(2)} \vee \cdots \vee a_{\sigma(k)}]$$

where σ ranges over all permutations of $\{0, 1, \ldots, k\}$ and where we adopt the convention that whenever $[a_{\sigma(1)}, a_{\sigma(1)} \vee a_{\sigma(2)}, \ldots, a_{\sigma(1)} \vee a_{\sigma(2)} \vee \cdots \vee a_{\sigma(k)}]$ is not a $(k-1)$-chain of the order complex Δ of $L - \{\hat{0}, \hat{1}\}$ (either because of repeated elements or because $a_{\sigma(1)} \vee a_{\sigma(2)} \vee \cdots \vee a_{\sigma(k)} = \hat{1}$) then this term is 0.

(The absence of $a_{\sigma(0)}$ from the formula is not a typographical error; it corresponds to the absence of a_i in the earlier formula for τ.)

Remark 20 Let us consider the special case where L is a geometric lattice and M is obtained from a linear ordering of the atoms, so "coreless" reduces to NBC. Any set in $\mathcal{N}$ is independent in the matroid associated to L, since it doesn't even contain a broken circuit, much less a full circuit. Therefore any set in $\mathcal{N}^+$ is a basis for the matroid. By a fundamental result of matroid theory, all such bases have the same cardinality r, the rank of the matroid and of the lattice L. Therefore, in the complex Q_*, only one group is non-zero, namely Q_{r-1}, freely generated by the NBC bases. The boundary operator of Q_* therefore vanishes, and the homology of Q_* is isomorphic to Q_* itself. By Theorem 19, $\tilde{H}_*(\Delta)$ is the same except for a shift in dimension: $\tilde{H}_{r-2}(\Delta)$ is a free abelian group of rank equal to the number of NBC bases of L, and $\tilde{H}_k(\Delta) = 0$ for all $k \neq r-2$. Furthermore, the explicit formula for τ in Theorem 19 becomes, thanks to the vanishing of the boundary operator of Q_*, an explicit formula for converting any NBC base (with orientation) $[a_0, a_1, \ldots, a_k]$ into an explicit cycle of Δ representing the corresponding homology class. In this way, Theorem 19 includes Björner's explicit representation [1] of the reduced homology of geometric lattices in terms of NBC bases.

Remark 21 In [2], a partial order $\preceq$ of the atoms of a lattice L was called *perfect* if, for each $x \in L$, either all NBB sets with join x have an even number of elements or they all have an odd number of elements. In other words, there is no cancellation in the formula for $\mu(\hat{0}, x)$. Such orderings produce the fewest possible NBB sets with any specified join x and thus, in some sense, make the calculation of the Möbius function as simple as possible.

It was mentioned in [2] that some finite lattices admit no perfect partial orderings of their atoms; so for these lattices, some cancellation is unavoidable. Since the notion of coreless sets generalizes the notion of NBB sets, it is reasonable to ask whether the generalization makes it possible to attain perfection for all lattices. That is, does every finite lattice L admit a function M, as in Convention 2, such that the sum in Theorem 9 never involves cancellation — for each $x \in L$ the cardinalities of all its coreless bases have the same parity?

Unfortunately, the answer is negative. The reason is that, as mentioned in the introduction, the homology of any finite simplicial complex C, which may well involve torsion, is isomorphic to the homology of some finite lattice. Indeed, if we take the poset of faces of C (including the empty face $\hat{0}$) and adjoin a top element $\hat{1}$, then we get a lattice whose order complex Δ is (as an abstract simplicial complex) the barycentric subdivision of C and therefore has the same homology (up to isomorphism). If that homology involves torsion, then the complex Q_* of Definition 17 must have, in some dimensions, higher rank than its homology; producing torsion in the quotient groups requires some cancellation.

For a specific example, consider the (real) projective plane, whose reduced homology is $\mathbb{Z}/2$ in dimension 1 and zero in all other dimensions. It has a triangulation consisting of 6 vertices, 15 edges (joining all pairs of vertices), and 10 triangles, namely the result of identifying antipodes in a regular icosahedron. By the preceding paragraph, the lattice consisting of these simplices, ordered by inclusion, plus $\hat{0}$ and $\hat{1}$, has $\mu(\hat{0}, \hat{1}) = 0$ (as can be verified by direct computation). But, no matter how cleverly one defines M, there will be some coreless bases, as $H_2(Q_*) \cong \mathbb{Z}/2$. So there will be cancellation in Theorem 9.

Remark 22 Although $\mu(\hat{0}, \hat{1})$ doesn't make sense for infinite lattices, our results about homotopy types and homology groups not only make sense but remain true with the same proofs when the lattices are infinite. (Simplices remain finite.)

Furthermore, we don't really need lattices; bounded join-semilattices suffice. This is because an inspection of our arguments reveals that the only meet we ever used was the meet $\hat{1}$ of the empty family. In the finite case, this observation would be pointless, since any finite, bounded join-semilattice is automatically a lattice, but in the infinite case it increases the generality of the results.

References

[1] A. Björner, "On the homology of geometric lattices," *Alg. Unversalis* 14 (1982) 107–128.

[2] A. Blass and B. E. Sagan, "Möbius functions of lattices," *Adv. Math.* 127 (1997) 94–123.

[3] L. Budach, B. Graw, C. Meinel, and S. Waack, *Algebraic and Topological Properties of Finite Partially Ordered Sets*, Teubner-Texte zur Mathematik 109, Teubner-Verlag (1988).

[4] G.-C. Rota, "On the foundations of combinatorial theory I: Theory of Möbius functions," *Z. Wahrscheinlichkeitstheorie* 2 (1964) 340–368.

[5] B. E. Sagan, "A generalization of Rota's NBC theorem," *Adv. Math.* 111 (1995) 195–207.

[6] Y. Segev, "On the order complex of a prelattice," *European J. Combin.* 18 (1997) 311–314.

[7] H. Whitney, "A logical expansion in mathematics," *Bull. Amer. Math. Soc.* 38 (1932) 572–579.