The q-Binomial Theorem and two Symmetric q-Identities

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Abstract

We notice two symmetric q-identities, which are special cases of the transformations of ${}_2\phi_1$ series in Gasper and Rahman's book (Basic Hypergeometric Series, Cambridge University Press, 1990, p. 241). In this paper, we give combinatorial proofs of these two identities and the q-binomial theorem by using conjugation of 2-modular diagrams.

1 Introduction

We follow the notation and terminology in [7], and we always assume that $0 \leq |q| < 1$. The *q-shifted factorial* is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n \in \mathbb{N}, \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The following theorem is usually called the *q-binomial theorem*. It was found by Rothe, and was rediscovered by Cauchy (see [1, p. 5]).

Theorem 1.1 *If $|z| < 1$, then*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.1}$$

Various proofs (1.1) are known. For simple proofs of (1.1), see Andrews [3, Section 2.2] and Gasper [6], and for combinatorial proofs, see Alladi [2] and Pak [8].

The following two theorems are special cases of the transformations of ${}_2\phi_1$ series in Gasper and Rahman [7, p. 241].

Theorem 1.2 *For $|a| < 1$ and $|b| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(az;q)_n}{(a;q)_{n+1}} b^n = \sum_{n=0}^{\infty} \frac{(bz;q)_n}{(b;q)_{n+1}} a^n. \tag{1.2}$$

Theorem 1.3 *We have*

$$\sum_{k=0}^{n} \frac{(q/z;q)_k (z;q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{mk} z^k = \sum_{k=0}^{m} \frac{(q/z;q)_k (z;q)_{m-k}}{(q;q)_k (q;q)_{m-k}} q^{nk} z^k. \tag{1.3}$$

Clearly, the left-hand side of (1.2) may be written as

$$\frac{1}{(1-a)}\, {}_2\phi_1(az, q; qa; q, b). \tag{1.4}$$

By the Heine's transformation (III.1) in Gasper and Rahman [7, p. 241], (1.4) is equal to

$$\frac{1}{(1-a)} \frac{(q, abz;q)_\infty}{(qa, b;q)_\infty}\, {}_2\phi_1(a, b; abz; q, q),$$

which is symmetric in a and b. Note that the special case $z = 0$ of (1.2) has also appeared in the literature (see Stockhofe [9] and Pak [8, 2.2.4]).

Rewrite the left-hand side of (1.3) as

$$\frac{(z;q)_n}{(q;q)_n}\, {}_2\phi_1(q^{-n}, q/z; q^{1-n}/z; q, q^{m+1}).$$

Applying the transformation (III.6) in [7, p. 241], we get

$$q^{mn}\, {}_3\phi_2(q^{-n}, q^{-m}, z; q, 0; q, q),$$

which is symmetric in m and n.

The purpose of this paper is to give combinatorial proofs of (1.1), (1.2), and (1.3) by using conjugation of 2-modular diagrams.

As usual, a *partition* λ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \dots, \lambda_m)$ in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. A nonzero λ_i is called a part of λ. The numbers of odd parts and even parts of λ are denoted by $\mathrm{odd}(\lambda)$ and $\mathrm{even}(\lambda)$, respectively. Define $\ell(\lambda) = \mathrm{odd}(\lambda) + \mathrm{even}(\lambda)$, called the *length* of λ. Write $|\lambda| = \sum_{i=1}^{m} \lambda_i$, called the *weight* of λ.

The set of all partitions into even parts is denoted by $\mathcal{P}_{\text{even}}$. The set of all partitions into distinct odd parts is denoted by $\mathcal{D}_{\text{odd}}$. Let $\mathcal{P}_1$ (respectively, $\mathcal{P}_2$) denote the set of partitions with no repeated odd (respectively, even) parts.

For partitions λ and μ, we define $\lambda \cup \mu$ to be the partition obtained by putting all parts of λ and μ together in decreasing order.

2 A Theorem on Partitions

The following theorem is crucial to prove Theorems 1.1–1.3 combinatorially.

Theorem 2.1 *Given $m \geq 1$, the number of partitions of n into at most m parts with no repeated odd parts is equal to the number of partitions of n with the largest part at most $2m$ and with no repeated odd parts.*

Theorem 2.1 was established by Chapman [5] in his proof of the q-identity

$$\sum_{n=1}^{\infty} n \frac{-q^{2n-1}+q^{2n}}{1-q^{2n}} \prod_{j=1}^{n-1} \frac{1-q^{2j-1}}{1-q^{2j}} = \prod_{j=1}^{\infty} \frac{1-q^{2j-1}}{1-q^{2j}} \sum_{d=1}^{\infty} (-1)^d \frac{q^d}{1-q^d},$$

which is due to Andrews, Jiménez-Urroz, and Ono [4]. Here we describe Chapman's proof.

Proof of Theorem 2.1. We shall construct an involution σ on $\mathcal{P}_1$ such that σ preserves $|\lambda|$ while interchanging $\ell(\lambda)$ and $\lceil \lambda_1/2 \rceil$.

We construct a diagram for each $\lambda \in \mathcal{P}_1$. Each part λ_i will yield a row of length $\lceil \lambda_i/2 \rceil$. An even part $2k$ will give a row of k 2's, while an odd part $2k+1$ will give a row of k 2's followed by a 1. Such a diagram is called a 2-*modular diagram*. As an example, let $\lambda = (10, 9, 7, 4, 4, 4, 3, 2, 2, 1)$. Then, λ gives the 2-modular diagram

$$\begin{array}{lllll}
2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 & \\
2 & 2 & & & \\
2 & 2 & & & \\
2 & 2 & & & \\
2 & 1 & & & \\
2 & & & & \\
2 & & & & \\
1 & & & &
\end{array}$$

Since no odd part of λ is repeated, the 1's can only occur at the bottom of columns. We identify elements of $\mathcal{P}_1$ with their diagrams, and then define σ to be conjugation of diagrams. For the above λ, $\sigma(\lambda)$ gives the 2-modular diagram

$$\begin{array}{llllllllll}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & & & \\
2 & 2 & 2 & & & & & & & \\
2 & 2 & 1 & & & & & & & \\
2 & 1 & & & & & & & &
\end{array}$$

Namely, $\sigma(\lambda) = (19, 13, 6, 5, 3)$. Clearly, the number of rows in the diagram of λ is $\ell(\lambda)$, while the number of columns is $\lceil \lambda_1/2 \rceil$. Thus, σ has the required properties and Theorem 2.1 is proved. ∎

Note that the above involution σ on $\mathcal{P}_1$ also preserves $\mathrm{odd}(\lambda)$.

3 Combinatorial Proofs of Theorems 1.1, 1.2, and 1.3

In this section, we give combinatorial proofs of the q-binomial theorem and Theorems 1.2 and 1.3. Our combinatorial proof of the q-binomial theorem is based on Theorem 2.1, and is essentially the same as that of Alladi [2] or Pak [8].

Proof of Theorem 1.1. Replacing q and a by q^2 and $-aq$, respectively, (1.3) becomes

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n}{(q^2;q^2)_n} z^n = \frac{(-aqz;q^2)_\infty}{(z;q^2)_\infty}. \tag{3.1}$$

It is easy to see that the coefficient of z^n on the left-hand side of (3.1) is equal to

$$\sum_{\substack{\mu \in \mathcal{P}_1 \\ \mu_1 \le 2n}} q^{|\mu|} a^{\mathrm{odd}(\mu)},$$

while the coefficient of z^n on the right-hand side is equal to

$$\sum_{\substack{\mu \in \mathcal{P}_1 \\ \ell(\mu) \le n}} q^{|\mu|} a^{\mathrm{odd}(\mu)}.$$

The proof then follows from the involution σ in the proof of Theorem 2.1. ∎

Proof of Theorem 1.2. Replacing q and z by q^2 and $-zq$, respectively, (1.2) becomes

$$\sum_{n=0}^{\infty} \frac{(-azq;q^2)_n}{(a;q^2)_{n+1}} b^n = \sum_{n=0}^{\infty} \frac{(-bzq;q^2)_n}{(b;q^2)_{n+1}} a^n. \tag{3.2}$$

It is easy to see that the coefficient of $a^m b^n$ on the left-hand side of (3.2) is equal to

$$\sum_{\substack{\mu \in \mathcal{P}_1 \\ \ell(\mu) \le m \\ \mu_1 \le 2n}} q^{|\mu|} z^{\mathrm{odd}(\mu)},$$

while the coefficient of $a^m b^n$ on the right-hand side is equal to

$$\sum_{\substack{\mu \in \mathcal{P}_1 \\ \ell(\mu) \le n \\ \mu_1 \le 2m}} q^{|\mu|} z^{\mathrm{odd}(\mu)}.$$

By the involution σ in the proof of Theorem 2.1, we have

$$\sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\leq n\\ \mu_1\leq 2m}} q^{|\mu|}z^{\mathrm{odd}(\mu)} = \sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\leq m\\ \mu_1\leq 2n}} q^{|\mu|}z^{\mathrm{odd}(\mu)}. \tag{3.3}$$

This completes the proof. ∎

Replacing q and z by q^2 and $-zq$, respectively, (1.3) may be written as

$$\sum_{k=0}^{n}(-1)^k\frac{(-q/z;q^2)_k(-zq;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}q^{(2m+1)k}z^k$$
$$=\sum_{k=0}^{m}(-1)^k\frac{(-q/z;q^2)_k(-zq;q^2)_{m-k}}{(q^2;q^2)_k(q^2;q^2)_{m-k}}q^{(2n+1)k}z^k. \tag{3.4}$$

We will prove (3.4) combinatorially by first establishing the following two lemmas.

Lemma 3.1 *For $m\geq 0$ and $n\geq 1$, we have*

$$\sum_{k=0}^{n}(-1)^k\frac{(-q/z;q^2)_k(-zq;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}q^{(2m+1)k}z^k$$
$$=\sum_{\substack{(\lambda,\mu)\in\mathcal{P}_2\times\mathcal{P}_1\\ \ell(\lambda)+\ell(\mu)\leq n\\ \lambda_{\ell(\lambda)}\geq 2m+1}}(-1)^{\ell(\lambda)}q^{|\lambda|+|\mu|}z^{\mathrm{odd}(\lambda)+\mathrm{odd}(\mu)}. \tag{3.5}$$

Proof. It is easy to see that

$$\frac{(-q/z;q^2)_k}{(q^2;q^2)_k}z^k = \sum_{\substack{\lambda\in\mathcal{D}_{\mathrm{odd}}\\ \lambda_1\leq 2k-1}} q^{|\lambda|}z^{k-\ell(\lambda)}\sum_{\substack{\mu\in\mathcal{P}_{\mathrm{even}}\\ \mu_1\leq 2k}} q^{|\mu|}$$
$$=\sum_{\substack{\tau\in\mathcal{P}_1\\ \tau_1\leq 2k}} q^{|\tau|}z^{k-\mathrm{odd}(\tau)},$$

where $\tau=\lambda\cup\mu$.

By the involution σ in the proof of Theorem 2.1, we have

$$\sum_{\substack{\tau\in\mathcal{P}_1\\ \tau_1\leq 2k}} q^{|\tau|}z^{k-\mathrm{odd}(\tau)} = \sum_{\substack{\tau\in\mathcal{P}_1\\ \ell(\tau)\leq k}} q^{|\tau|}z^{k-\mathrm{odd}(\tau)},$$

Hence,

$$\begin{aligned}
\frac{(-q/z;q^2)_k}{(q^2;q^2)_k} q^{(2m+1)k} z^k &= \sum_{\substack{\tau\in\mathcal{P}_1\\ \ell(\tau)\le k}} q^{|\tau|+(2m+1)k} z^{k-\mathrm{odd}(\tau)}\\
&= \sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathcal{P}_2\\ \lambda_k\ge 2m+1}} q^{|\lambda|} z^{k-\mathrm{even}(\lambda)}\\
&= \sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathcal{P}_2\\ \lambda_k\ge 2m+1}} q^{|\lambda|} z^{\mathrm{odd}(\lambda)},
\end{aligned}$$

where $\lambda_i = \tau_i + 2m + 1$ $(1 \le i \le k)$.

Similarly, we have

$$\frac{(-zq;q^2)_{n-k}}{(q^2;q^2)_{n-k}} = \sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\le n-k}} q^{|\mu|} z^{\mathrm{odd}(\mu)}.$$

Therefore, the left-hand side of (3.5) is equal to

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k \sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathcal{P}_2\\ \lambda_k\ge 2m+1}} q^{|\lambda|} z^{\mathrm{odd}(\lambda)} \sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\le n-k}} q^{|\mu|} z^{\mathrm{odd}(\mu)}\\
&= \sum_{\substack{(\lambda,\mu)\in\mathcal{P}_2\times\mathcal{P}_1\\ \ell(\lambda)+\ell(\mu)\le n\\ \lambda_{\ell(\lambda)}\ge 2m+1}} (-1)^{\ell(\lambda)} q^{|\lambda|+|\mu|} z^{\mathrm{odd}(\lambda)+\mathrm{odd}(\mu)}, \qquad (3.6)
\end{aligned}$$

as desired. ∎

Lemma 3.2 *For $m \ge 0$ and $n \ge 1$, we have*

$$\sum_{\substack{(\lambda,\mu)\in\mathcal{P}_2\times\mathcal{P}_1\\ \ell(\lambda)+\ell(\mu)\le n\\ \lambda_{\ell(\lambda)}\ge 2m+1}} (-1)^{\ell(\lambda)} q^{|\lambda|+|\mu|} z^{\mathrm{odd}(\lambda)+\mathrm{odd}(\mu)} = \sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\le n\\ \mu_1\le 2m}} q^{|\mu|} z^{\mathrm{odd}(\mu)}. \qquad (3.7)$$

Proof. Let

$$\mathcal{B} := \{(\lambda,\mu)\in\mathcal{P}_2\times\mathcal{P}_1 \colon \ell(\lambda)+\ell(\mu)\le n \text{ and } \lambda_{\ell(\lambda)}\ge 2m+1\}.$$

We will construct an involution ϕ on the subset

$$\mathcal{B}_m := \{(\lambda,\mu)\in\mathcal{B}\colon \lambda\ne 0 \text{ or } \mu_1\ge 2m+1\}$$

of $\mathcal{B}$, with the properties that ϕ preserves $|\lambda|+|\mu|$ and $\text{odd}(\lambda)+\text{odd}(\mu)$ while sign-reversing $(-1)^{\ell(\lambda)}$.

For any $(\lambda,\ \mu) \in \mathcal{B}_m$, note that no even part of λ is repeated while no odd part of μ is repeated. Define

$$\phi((\lambda,\mu)) = \begin{cases} ((\mu_1,\lambda_1,\lambda_2,\ldots),(\mu_2,\mu_3,\ldots)), & \text{if } \lambda_1 < \mu_1 \text{ or } \lambda_1 = \mu_1 = 2s+1, \\ ((\lambda_2,\lambda_3,\ldots),(\lambda_1,\mu_1,\mu_2,\ldots)), & \text{if } \lambda_1 > \mu_1 \text{ or } \lambda_1 = \mu_1 = 2s. \end{cases}$$

It is straightforward to verify that ϕ is an involution on $\mathcal{B}_m$ with the required properties. This proves that

$$\sum_{(\lambda,\ \mu)\in\mathcal{B}_m} (-1)^{\ell(\lambda)} q^{|\lambda|+|\mu|} z^{\text{odd}(\lambda)+\text{odd}(\mu)} = 0,$$

which implies (3.7). ∎

Proof of Theorem 1.3. Combining Lemmas 3.1 and 3.2, we obtain

$$\sum_{k=0}^{n} (-1)^k \frac{(-q/z;q^2)_k(-zq;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}} q^{(2m+1)k} z^k = \sum_{\substack{\mu\in\mathcal{P}_1 \\ \ell(\mu)\le n \\ \mu_1\le 2m}} q^{|\mu|} z^{\text{odd}(\mu)}.$$

By symmetry, we have

$$\sum_{k=0}^{m} (-1)^k \frac{(-q/z;q^2)_k(-zq;q^2)_{m-k}}{(q^2;q^2)_k(q^2;q^2)_{m-k}} q^{(2n+1)k} z^k = \sum_{\substack{\mu\in\mathcal{P}_1 \\ \ell(\mu)\le m \\ \mu_1\le 2n}} q^{|\mu|} z^{\text{odd}(\mu)}.$$

The proof then follows from (3.3). ∎

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