Skolem-type Difference Sets for Cycle Systems

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Abstract

Cyclic m-cycle systems of order v are constructed for all $m \geq 3$, and all $v \equiv 1 (\bmod 2m)$. This result has been settled previously by several authors. In this paper, we provide a different solution, as a consequence of a more general result, which handles all cases using similar methods and which also allows us to prove necessary and sufficient conditions for the existence of a cyclic m-cycle system of $K_v - F$ for all $m \geq 3$, and all $v \equiv 2 (\bmod 2m)$.

1 Introduction

Throughout this paper, K_v will denote the complete graph on v vertices and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is a set $\mathcal{C}$ of m-cycles in G whose edges partition the edge set of G. A survey on cycle systems is given in [12] and necessary and sufficient conditions for the existence of an m-cycle system of G in the cases $G = K_v$ and $G = K_v - F$ (the complete graph of order v with a 1-factor removed) were given in [1, 15]. Such m-cycle systems exist if and only if $v \geq m$, every vertex of G has even degree, and m divides the number of edges in G.

Let ρ denote the permutation $(0, 1, \ldots, v-1)$. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_v$ is *cyclic* if for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. If X is a set of m-cycles in a graph G with vertex set $\mathbb{Z}_v$ such that $\mathcal{C} = \{\rho^\alpha(C) \mid C \in X, \alpha = 0, 1, \ldots, v-1\}$ is an m-cycle system of G, then X is called a *starter set* for $\mathcal{C}$, the m-cycles in X are called *starter cycles*, and $\mathcal{C}$ is said to be *cyclically generated*, or just *generated*, by the m-cycles in X.

The existence question for cyclic m-cycle systems of complete graphs has attracted much interest, and a complete answer for $m = 3$ [11], 5 and 7 [13] has been found. For m even and $v \equiv 1 (\mathrm{mod}\, 2m)$, cyclic m-cycle systems of K_v are constructed for $m \equiv 0 (\mathrm{mod}\, 4)$ in [10] and for $m \equiv 2 (\mathrm{mod}\, 4)$ in [13]. Both of these cases are also handled in [7]. For m odd and $v \equiv 1 (\mathrm{mod}\, 2m)$, cyclic m-cycle systems of K_v are found using different methods in [4, 3, 8], and, for $v \equiv m (\mathrm{mod}\, 2m)$ cyclic m-cycle systems of K_v are given [5] for $m \notin M$, where $M = \{p^e \mid p \text{ is prime}, e > 1\} \cup \{15\}$, and in [18] for $m \in M$. In this paper, as a consequence of a more general result, we find cyclic m-cycle systems of K_v for all positive integers m and $v \equiv 1 (\mathrm{mod}\, 2m)$ with $v \geq m \geq 4$ using similar methods. We also settle the existence question for cyclic m-cycle systems of $K_v - F$ for $v \equiv 2 (\mathrm{mod}\, 2m)$.

For $x \not\equiv 0 (\mathrm{mod}\, v)$, the *modulo v length* of an integer x, denoted $|x|_v$, is defined to be the smallest positive integer y such that $x \equiv y (\mathrm{mod}\, v)$ or $x \equiv -y (\mathrm{mod}\, v)$. Note that for any integer $x \not\equiv 0 (\mathrm{mod}\, v)$, it follows that $|x|_v \in \{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\}$. If L is a set of modulo v lengths, we define $\langle L \rangle_v$ to be the graph with vertex set $\mathbb{Z}_v$ and edge set $\{\{i, j\} \mid |i - j|_v \in L\}$. Observe that $K_v \cong \langle \{1, 2, \ldots, \lfloor v/2 \rfloor\} \rangle_v$. An edge $\{i, j\}$ in a graph with vertex set $\mathbb{Z}_v$ is called an *edge of length* $|i - j|_v$.

Let $v > 0$ be an integer and suppose there exists an ordered m-tuple $(d_1, d_2, \ldots, d_m)$ satisfying each of the following:

(i) d_i is an integer for $i = 1, 2, \ldots, m$;

(ii) $|d_i|_v \neq |d_j|_v$ for $1 \leq i < j \leq m$;

(iii) $d_1 + d_2 + \ldots + d_m \equiv 0 (\mathrm{mod}\, v)$; and

(iv) $d_1 + d_2 + \ldots + d_r \not\equiv d_1 + d_2 + \ldots + d_s (\mathrm{mod}\, v)$ for $1 \leq r < s \leq m$.

Then $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \ldots + d_{m-1})$ generates a cyclic m-cycle system of the graph $\langle \{|d_1|_v, |d_2|_v, \ldots, |d_m|_v\} \rangle_v$. An m-tuple satisfying (i)-(iv) is called a *modulo v difference m-tuple*, it *corresponds* to the starter m-cycle $\{(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \ldots + d_{m-1})\}$, and it *uses* edges of lengths $|d_1|_v, |d_2|_v, \ldots, |d_m|_v$. A *modulo v m-cycle difference set of size t*, or an *m-cycle difference set of size t* when the value of v is understood, is a set consisting of t modulo v difference m-tuples that use edges of distinct lengths $l_1, l_2, \ldots, l_{tm}$; the m-cycles corresponding to the difference m-tuples generate a cyclic m-cycle system $\mathcal{C}$ of $\langle\{l_1, l_2, \ldots, l_{tm}\}\rangle_v$. Thus the modulo v m-cycle difference set *generates* $\mathcal{C}$.

A *Skolem sequence* of order t is a sequence $S = (s_1, s_2, \ldots, s_{2t})$ of $2t$ integers satisfying the conditions

(S1) for every $k \in \{1, 2, \ldots, t\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;

(S2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

It is well-known that a Skolem sequence of order t exists if and only if $t \equiv 0, 1 (\mathrm{mod}\, 4)$ [17]. For $t \equiv 2, 3 (\mathrm{mod}\, 4)$, the natural alternative is a hooked Skolem sequence. A *hooked Skolem sequence of order t* is a sequence $HS = (s_1, s_2, \ldots, s_{2t+1})$ of $2t+1$ integers satisfying conditions (S1) and (S2) above and

(S3) $s_{2t} = 0$.

It is well-known that a hooked Skolem sequence of order t exists if and only if $t \equiv 2, 3 (\mathrm{mod}\, 4)$ [9].

Skolem sequences and their generalisations have been used widely in the construction of combinatorial designs, a survey on Skolem sequences can be found in [6], and perhaps the most well-known use of Skolem sequences is in the construction of cyclic Steiner triple systems. A *Steiner triple system of order v* is a pair (V, B) where V is a v-set and B is a set of 3-subsets, called triples, of V such that every 2-subset of V occurs in exactly one triple of B. A Steiner triple system of order v is equivalent to a 3-cycle system of K_v, and a Skolem sequence $S = (s_1, s_2, \ldots, s_{2t})$ or a hooked Skolem sequence $HS = (s_1, s_2, \ldots, s_{2t+1})$ of order t can be used to construct the 3-cycle difference set

$$\{(k, t+i, -(t+j)) \mid k = 1, 2, \ldots, t,\ s_i = s_j = k, i < j\}$$

of size t which generates a cyclic 3-cycle system of K_{6t+1} (the m-tuple $(k, 3t+1-k, -(3t+1))$ obtained from a hooked Skolem sequence of order t uses edges of lengths $k, 3t+1-k$ and $3t$).

Notice that if $(d_1, d_2, \ldots, d_m)$ is a modulo v difference m-tuple with $d_1+d_2+\ldots+d_m = 0$, not just $d_1 + d_2 + \ldots + d_m \equiv 0 (\mathrm{mod}\, v)$, then $(d_1, d_2, \ldots, d_m)$ is a modulo w difference m-tuple for all $w \geq M/2 + 1$ where $M = |d_1| + |d_2| + \cdots + |d_m|$. All the difference triples obtained from Skolem sequences and hooked Skolem sequences are of the form (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = 0$. In the literature, difference triples obtained from Skolem sequences are usually written (a, b, c) with $a + b = c$. However, the equivalent representation we are using here, with c replaced by $-c$ so that $a + b + c = 0$, is more convenient for the purpose of extending these ideas to m-cycle systems with $m > 3$. We make the following definition.

Definition 1.1 A difference m-tuple $(d_1, d_2, \ldots, d_m)$ is of *Skolem-type* if $d_1 + d_2 + \ldots + d_m = 0$. An m-cycle difference set using edges of lengths $1, 2, \ldots, mt$, and in which all of the m-tuples are of Skolem type, is called a *Skolem-type m-cycle difference set of size t*. An m-cycle difference set using edges of lengths $1, 2, \ldots, mt-1, mt+1$, and in which all of the m-tuples are of Skolem type, is called a *hooked Skolem-type m-cycle difference set of size t*.

Clearly, (hooked) Skolem sequences of order t yield (hooked) Skolem-type 3-cycle difference sets of size t. In this paper, we prove necessary and sufficient conditions for the existence of Skolem-type and hooked Skolem-type m-cycle difference sets of size t for all $m \geq 3$ and all $t \geq 1$ (see Theorem 2.3). As a corollary, we obtain several existence results on cyclic m-cycle systems. These include necessary and sufficient conditions for the existence of cyclic m-cycle systems of K_v for all $v \equiv 1 (\bmod 2m)$ and $K_v - F$ for all $v \equiv 2 (\bmod 2m)$.

As remarked earlier, several cases of these results have been settled previously. However, in this paper, we provide a complete solution in which all of the cases are dealt with using similar methods. Moreover, since the difference sets are of Skolem-type, we also obtain cyclic m-cycle systems of $\langle\{1, 2, \ldots, \lfloor\frac{v}{2}\rfloor\}\rangle_w$ or $\langle\{1, 2, \ldots, \frac{v}{2}-1, \lfloor\frac{v}{2}\rfloor+1\}\rangle_w$ for infinitely many values of w, which have not been previously found. All of our Skolem-type m-cycle difference sets will have the additional property that the number of positive integers in each m-tuple differs from the number of negative integers by at most one. In other words, when m is even the number of positive integers equals the number of negative integers, and when m is odd the number of positive integers and the number of negative integers differ by one.

To construct our sets of Skolem-type difference tuples we will use *Langford sequences*. A *Langford sequence of order t and defect d* is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t})$ of $2t$ integers satisfying the conditions

(L1) for every $k \in \{d, d+1, \ldots, d+t-1\}$ there exists exactly two elements $\ell_i, \ell_j \in L$ such that $\ell_i = \ell_j = k$, and

(L2) if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

A *hooked Langford sequence of order t and defect d* is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t+1})$ of $2t+1$ integers satisfying conditions (L1) and (L2) above and

(L3) $\ell_{2t} = 0$.

Clearly, a (hooked) Langford sequence with defect 1 is a (hooked) Skolem sequence. The following theorem gives necessary and sufficient conditions for the existence of Langford sequences.

Theorem 1.2 [16] *There exists a Langford sequence of order t and defect d if and only if*

(1) $t \geq 2d - 1$, and

(2) $t \equiv 0, 1 \pmod 4$ and d is odd, or $t \equiv 0, 3 \pmod 4$ and d is even.

There exists a hooked Langford sequence of order t and defect d if and only if

(1) $t(t - 2d + 1) + 2 \geq 0$, and

(2) $t \equiv 2, 3 \pmod 4$ and d is odd, or $t \equiv 1, 2 \pmod 4$ and d is even.

In a similar manner to which 3-cycle difference sets are constructed from Skolem and hooked Skolem sequences, a Langford sequence or hooked Langford sequence of order t can be used to construct a 3-cycle difference set of size t that uses edges of lengths $d, d+1, d+2, \ldots, d+3t-1$ or $d, d+1, d+2, \ldots, d+3t-2, d+3t$ respectively.

2 Construction of Difference Sets for Cycle Systems

Before proving the main theorem, we need the following two lemmas which are used in extending m-cycle difference sets of size t to $(m+4)$-cycle difference sets of size t. Lemma 2.1 is for ordinary Skolem-type m-cycle difference sets and Lemma 2.2 is for hooked Skolem-type m-cycle difference sets.

Lemma 2.1 *Let n, r and t be positive integers. There exists a $t \times 4r$ matrix $Y(r, n, t) = [y_{i,j}]$ such that $\{|y_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq 4r\} = \{n+1, n+2, \ldots, n+4rt\}$, the sum of the entries in each row of $Y(r, n, t)$ is zero, and $|y_{i,1}| < |y_{i,2}| < \ldots < |y_{i,4r}|$ for $i = 1, 2, \ldots, t$.*

Proof. Let $Y'(r, n, t)$ be the matrix

$$\begin{bmatrix} 2t-1 & 2t & 4t-1 & 4t & & 4rt-1 & 4rt \\ 2t-3 & 2t-2 & 4t-3 & 4t-2 & & 4rt-3 & 4rt-2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 3 & 4 & 2t+3 & 2t+4 & & (4r-2)t+3 & (4r-2)t+4 \\ 1 & 2 & 2t+1 & 2t+2 & & (4r-2)t+1 & (4r-2)t+2 \end{bmatrix} + \begin{bmatrix} n & \cdots & n \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ n & \cdots & n \end{bmatrix}$$

and let Y be the matrix obtained from Y' by multiplying by -1 each entry in column j for all $j \equiv 2, 3 \pmod 4$. It is straightforward to verify that Y has the required properties. ■

Lemma 2.2 *Let n, r and t be positive integers. There exists a $t \times 4r$ matrix $Y(r, n, t) = [y_{i,j}]$ such that $\{|y_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq 4r\} = \{n, n+2, n+3, \ldots, n+4rt-1, n+4rt+1\}$, the sum of the entries in each row is zero, and $|y_{i,1}| < |y_{i,2}| < \ldots < |y_{i,4r}|$ for $i = 1, 2, \ldots, t$.*

Proof. Let $Y'(r, n, t)$ be the matrix

$$\begin{bmatrix} 0 & 2 & 4t-1 & 4t & & 4rt-1 & 4rt+1 \\ 2t-1 & 2t & 4t-3 & 4t-2 & & 4rt-3 & 4rt-2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 5 & 6 & 2t+3 & 2t+4 & & (4r-2)t+3 & (4r-2)t+4 \\ 3 & 4 & 2t+1 & 2t+2 & & (4r-2)t+1 & (4r-2)t+2 \end{bmatrix} + \begin{bmatrix} n & \cdots & n \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ n & \cdots & n \end{bmatrix}$$

and let Y be the matrix obtained from Y' by multiplying by -1 each entry in column j for all $j \equiv 2, 3 \pmod 4$. It is straightforward to verify that Y has the required properties. ■

We are now ready to prove necessary and sufficient conditions for the existence of Skolem-type and hooked Skolem-type m-cycle difference sets of size t.

Theorem 2.3 *Let m and t be integers with $m \geq 3$ and $t \geq 1$. There exists a Skolem-type m-cycle difference set of size t if and only if $mt \equiv 0, 3 \pmod 4$. There exists a hooked Skolem-type m-cycle difference set of size t if and only if $mt \equiv 1, 2 \pmod 4$.*

Proof. If $mt \equiv 1, 2 \pmod 4$ and $\{|x_1|, |x_2|, \ldots, |x_{mt}|\} = \{1, 2, \ldots, mt\}$ then $x_1 + x_2 + \ldots + x_{mt}$ is odd, and it follows that there is no Skolem-type m-cycle difference set of size t. Similarly, if $mt \equiv 0, 3 \pmod 4$ and $\{|x_1|, |x_2|, \ldots, |x_{mt}|\} = \{1, 2, \ldots, mt-1, mt+1\}$ then $x_1 + x_2 + \ldots + x_{mt}$ is odd, and it follows that there is no hooked Skolem-type m-cycle difference set of size t. Hence it remains to construct a Skolem-type m-cycle difference set of size t whenever $mt \equiv 0, 3 \pmod 4$ and a hooked Skolem-type m-cycle difference set of size t whenever $mt \equiv 1, 2 \pmod 4$.

The proof splits into four cases depending on the congruence class of m modulo 4. For each case we construct a $t \times m$ matrix $X = [x_{i,j}]$ with entries $1, 2, \ldots, mt$ when $mt \equiv 0, 3 \pmod 4$ or with entries $1, 2, \ldots, mt-1, mt+1$ when $mt \equiv 1, 2 \pmod 4$ such that for each $i = 1, 2, \ldots, t$, we have

$$\sum_{j=1}^{m} x_{i,j} = 0.$$

The entries in each row of our matrices will also satisfy various inequalities which will allow us to arrange them so that for $1 \leq r < s \leq m$ and $v \geq 2mt + 1$, we have $d_1 + d_2 + \ldots, d_r \not\equiv d_1 + d_2 + \ldots, d_s \pmod v$, so that a Skolem-type m-cycle difference set of size t can be obtained.

Case 1. *Suppose that $m \equiv 0 \pmod 4$.* In this case, $mt \equiv 0 \pmod 4$ for all t and let $X = [x_{i,j}]$ be the $t \times m$ matrix $Y(\frac{m}{4}, 0, t)$ given by Lemma 2.1. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < \cdots < |x_{i,m}|$ and $x_{i,j} < 0$ precisely when $j \equiv 2, 3 \pmod 4$. Hence the required set of m-tuples can be constructed directly from the rows of X by including the m-tuple

$$(x_{i,1}, x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,2}, x_{i,m})$$

for $i = 1, 2, \ldots, t$.

Case 2. *Suppose that* $m \equiv 2 (\mathrm{mod}\, 4)$. In this case, $mt \equiv 0 (\mathrm{mod}\, 4)$ when t is even and $mt \equiv 2 (\mathrm{mod}\, 4)$ when t is odd. If t is even, let

$$X = \left[\begin{array}{llllllc} 1 & -2 & 3 & -4 & -5 & 7 & \\ 6 & -8 & 10 & -9 & -11 & 12 & \\ 13 & -14 & 15 & -16 & -17 & 19 & \\ 18 & -20 & 22 & -21 & -23 & 24 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & Y(\frac{m-6}{4}, 6t, t) \\ 6t-12 & -(6t-10) & 6t-8 & -(6t-9) & -(6t-7) & 6t-6 & \\ 6t-5 & -(6t-4) & 6t-3 & -(6t-2) & -(6t-1) & 6t+1 & \end{array}\right]$$

where $Y(\frac{m-6}{4}, 6t, t)$ is the $t \times \frac{m-6}{4}$ matrix given by Lemma 2.1, and if t is odd, let

$$X = \left[\begin{array}{llllllc} 1 & -2 & 3 & -4 & -5 & 7 & \\ 6 & -8 & 10 & -9 & -11 & 12 & \\ 13 & -14 & 15 & -16 & -17 & 19 & \\ 18 & -20 & 22 & -21 & -23 & 24 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & Y(\frac{m-6}{4}, 6t, t) \\ 6t-11 & -(6t-10) & 6t-9 & -(6t-8) & -(6t-7) & 6t-5 & \\ 6t-6 & -(6t-4) & 6t-2 & -(6t-3) & -(6t-1) & 6t & \end{array}\right]$$

where $Y(\frac{m-6}{4}, 6t, t)$ is the $t \times \frac{m-6}{4}$ matrix given by Lemma 2.2. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,2}| < |x_{i,3}| < |x_{i,5}|$, and $x_{i,j} < 0$ precisely when $j = 2$ and when $j \equiv 0, 1 (\mathrm{mod}\, 4)$ with $j \geq 4$. Hence, the required set of m-tuples can be constructed directly from the rows of X by including the m-tuple

$$(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,5}, x_{i,7} \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,m}).$$

for $i = 1, 2, \ldots, t$.

Case 3. *Suppose that* $m \equiv 3 (\mathrm{mod}\, 4)$. In this case, $mt \equiv 0, 3 (\mathrm{mod}\, 4)$ when $t \equiv 0, 1 (\mathrm{mod}\, 4)$ and $mt \equiv 1, 2 (\mathrm{mod}\, 4)$ when $t \equiv 2, 3 (\mathrm{mod}\, 4)$. If $t \equiv 0, 1 (\mathrm{mod}\, 4)$, there exists a Skolem sequence of order t, and let $\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $\{1, 2, \ldots, 3t\}$ constructed from such a sequence. If $t \equiv 2, 3 (\mathrm{mod}\, 4)$, there exists a hooked Skolem sequence of order t, and let $\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $\{1, 2, \ldots, 3t-1, 3t+1\}$ constructed from such a sequence. Furthermore, when $t \equiv 2, 3 (\mathrm{mod}\, 4)$, we ensure that $3t+1 \notin \{a_1, b_1, c_1\}$. Let

$$X = \left[\begin{array}{cccc} a_1 & c_1 & b_1 & \\ a_2 & c_2 & b_2 & \\ \vdots & \vdots & \vdots & Y(\frac{m-3}{4}, 3t, t) \\ a_t & c_t & b_t & \end{array}\right]$$

where $Y(\frac{m-3}{4}, 3t, t)$ is the $t \times \frac{m-3}{4}$ matrix given by Lemma 2.1 or 2.2 if $t \equiv 0, 1 (\mathrm{mod}\, 4)$ or $t \equiv 2, 3 (\mathrm{mod}\, 4)$ respectively. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| <$

$|x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,3}| < |x_{i,5}|$, and $x_{i,j} < 0$ precisely when $j \geq 2$ and $j \equiv 1, 2 (\mathrm{mod}\, 4)$. Hence, the required set of m-tuples can be constructed directly from the rows of X by including the m-tuple

$$(x_{i,1}, x_{i,2}, x_{i,4}, x_{i,6}, x_{i,8}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,5}, x_{i,3}, x_{i,m})$$

for $i = 1, 2, \ldots, t$.

Case 4. *Suppose that* $m \equiv 1 (\mathrm{mod}\, 4)$. In this case, $mt \equiv 0, 3 (\mathrm{mod}\, 4)$ when $t \equiv 0, 3 (\mathrm{mod}\, 4)$ and $mt \equiv 1, 2 (\mathrm{mod}\, 4)$ when $t \equiv 1, 2 (\mathrm{mod}\, 4)$. The matrix X is slightly different for each of the four congruence classes of t modulo 4.

When $t \equiv 0 (\mathrm{mod}\, 4)$, there exists a Langford sequence of order $t-1$ and defect 2, and let $\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq t-1\}$ be a set of $t-1$ difference triples using edges of lengths $2, 3, \ldots, 3t-2$ constructed from such a sequence. Let

$$X = \left[\begin{array}{llllll}
1 & -2 & 3 & 5t-2 & -(5t) & \\
a_1+2 & c_1-2 & b_1+2 & 5t-6 & -(5t-4) & \\
a_2+2 & c_2-2 & b_2+2 & 5t-10 & -(5t-8) & \\
 & & & \vdots & \vdots & \\
 & & & 3t+2 & -(3t+4) & \\
\vdots & \vdots & \vdots & 5t-3 & -(5t-1) & Y(\frac{m-5}{4}, 5t, t) \\
 & & & 5t-7 & -(5t-5) & \\
 & & & \vdots & \vdots & \\
a_{t-1}+2 & c_{t-1}-2 & b_{t-1}+2 & 3t+1 & -(3t+3) &
\end{array}\right]$$

where $Y(\frac{m-5}{4}, 5t, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1.

When $t \equiv 3 (\mathrm{mod}\, 4)$, there exists a hooked Langford sequence of order $t-1$ and defect 2 and let $\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq t-1\}$ be a set of $t-1$ difference triples using edges of lengths $2, 3, \ldots, 3t-3, 3t-1$ constructed from such a sequence. Let

$$X = \left[\begin{array}{llllll}
a_1+2 & c_1-2 & b_1+2 & 5t-3 & -(5t-1) & \\
a_2+2 & c_2-2 & b_2+2 & 5t-7 & -(5t-5) & \\
 & & & \vdots & \vdots & \\
 & & & 3t+3 & -(3t+5) & \\
\vdots & \vdots & \vdots & 5t-2 & -(5t) & Y(\frac{m-5}{4}, 5t, t) \\
 & & & 5t-6 & -(5t-4) & \\
 & & & \vdots & \vdots & \\
a_{t-1}+2 & c_{t-1}-2 & b_{t-1}+2 & 3t+4 & -(3t+6) & \\
1 & -2 & 3 & 3t & -(3t+2) &
\end{array}\right]$$

where $Y(\frac{m-5}{4}, 5t, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1.

When $t = 1$, let $X = [1 \ \ -2 \ 3 \ 4 \ \ -6 \ Y(\frac{m-5}{4}, 5, 1)]$ where $Y((m-5)/4, 5, 1)$ is the $1 \times \frac{m-5}{4}$ matrix given by Lemma 2.2. For $t \equiv 1 (\mathrm{mod}\, 4)$, $t \geq 5$, there exists a Langford sequence of order $t-1$ and defect 2, and let $\{\{a_i, b_i, c_i\} \mid 1 \le i \le t-1\}$ be a set of $t-1$ difference triples using edges of lengths $2, 3, \ldots, 3t-2$ constructed from such a sequence. Let

$$X = \left[\begin{array}{llllll}
1 & -2 & 3 & 5t-1 & -(5t+1) & \\
a_1+2 & c_1-2 & b_1+2 & 5t-4 & -(5t-2) & \\
a_2+2 & c_2-2 & b_2+2 & 5t-8 & -(5t-6) & \\
 & & & \vdots & \vdots & \\
 & & & 3t+2 & -(3t+4) & \\
\vdots & \vdots & \vdots & 5t-5 & -(5t-3) & Y(\frac{m-5}{4}, 5t, t \\
 & & & 5t-9 & -(5t-7) & \\
 & & & \vdots & \vdots & \\
a_{t-1}+2 & c_{t-1}-2 & b_{t-1}+2 & 3t+1 & -(3t+3) &
\end{array}\right]$$

where $Y(\frac{m-5}{4}, 5t, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.2.

When $t = 2$, let

$$X = \left[\begin{array}{cccccc}
1 & -5 & 6 & 7 & -9 & Y(\frac{m-5}{4}, 10, 2) \\
2 & -3 & 4 & 8 & -11 &
\end{array}\right]$$

where $Y(\frac{m-5}{4}, 10, 2)$ is the $2 \times \frac{m-5}{4}$ matrix given by Lemma 2.2. For $t \equiv 2 \pmod 4$, $t \ge 6$, there exists a hooked Langford sequence of order $t-1$ and defect 2, and let $\{\{a_i, b_i, c_i\} \mid 1 \le i \le t-1\}$ be a set of $t-1$ difference triples using edges of lengths $2, 3, \ldots, 3t-3, 3t-1$ constructed from such a sequence. Let

$$X = \left[\begin{array}{llllll}
a_1+2 & c_1-2 & b_1+2 & 5t-1 & -(5t+1) & \\
a_2+2 & c_2-2 & b_2+2 & 5t-5 & -(5t-3) & \\
 & & & 5t-9 & -(5t-7) & \\
 & & & \vdots & \vdots & \\
 & & & 3t+3 & -(3t+5) & \\
\vdots & \vdots & \vdots & 5t-4 & -(5t-2) & Y(\frac{m-5}{4}, 5t, t \\
 & & & 5t-8 & -(5t-6) & \\
 & & & \vdots & \vdots & \\
a_{t-1}+2 & c_{t-1}-2 & b_{t-1}+2 & 3t+4 & -(3t+6) & \\
1 & -2 & 3 & 3t & -(3t+2) &
\end{array}\right]$$

where $Y(\frac{m-5}{4}, 5t, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.2.

For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,3}| < |x_{i,5}|$, and $x_{i,j} < 0$ precisely when $j = 2$, $j = 5$ and when $j \equiv 0, 3 \pmod 4$ with $j > 5$. Hence, the required set of m-tuples can be constructed directly from the rows of X by including the m-tuple

$$(x_{i,1}, x_{i,2}, x_{i,4}, x_{i,6}, x_{i,8}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,5}, x_{i,3}, x_{i,m})$$

for $i = 1, 2, \ldots, t$. ■

3 Cyclic Cycle Systems

Theorem 2.3 has the following three theorems on cyclic m-cycle systems as immediate corollaries.

Theorem 3.1 *Let $t \geq 1$ and $m \geq 3$. Then*

(1) for $mt \equiv 0, 3 \pmod 4$ and all $v \geq 2mt+1$, there exists a cyclic m-cycle system of $\langle\{1, 2, \ldots, mt\}\rangle_v$; and

(2) for $mt \equiv 1, 2 \pmod 4$ and all $v \geq 2mt+3$, there exists a cyclic m-cycle system of $\langle\{1, 2, \ldots, mt-1, mt+1\}\rangle_v$.

Proof. When $mt \equiv 0, 3 \pmod 4$, the required cyclic m-cycle system is generated from a Skolem-type m-cycle difference set of order t. When $mt \equiv 1, 2 \pmod 4$, the required cyclic m-cycle system is generated from a hooked Skolem-type m-cycle difference set of order t. ■

Theorem 3.2 *For all integers $m \geq 3$ and $t \geq 1$, there exists a cyclic m-cycle system of K_{2mt+1}.*

Proof. If $mt \equiv 0, 3 \pmod 4$, then the result follows immediately from Theorem 3.1 since $\langle\{1, 2, \ldots, mt\}\rangle_v \cong K_v$ when $v = 2mt+1$. If $mt \equiv 1, 2 \pmod 4$ then since $|mt+1|_{2mt+1} = mt$, the difference m-tuples obtained from a hooked Skolem-type m-cycle difference set of order t form a modulo v difference set that uses edges of lengths $1, 2, \ldots, mt$. ■

Theorem 3.3 *For all integers $m \geq 3$ and $t \geq 1$, there exists a cyclic m-cycle system of $K_{2mt+2} - F$ if and only if $mt \equiv 0, 3 \pmod 4$.*

Proof. The required cyclic m-cycle systems exist by Theorem 3.1, since $\langle\{1, 2, \ldots, mt\}\rangle_v \cong K_v - F$ when $v = 2mt+2$. Hence it remains to prove that there is no cyclic m-cycle system of $K_{2mt+2} - F$ when $mt \equiv 1, 2 \pmod 4$. Suppose $\mathcal{C}$ is a cyclic m-cycle system of $K_v - F$ with $mt \equiv 1, 2 \pmod 4$, suppose $C \in \mathcal{C}$ has an orbit of length r, and let $s = \frac{v}{r}$. Let P be a path in C such that the only two vertices a and b on P for which $|a-b|_v \equiv 0 \pmod r$ are the endvertices of P. It follows that P has $\frac{m}{s}$ edges. Hence s divides m and s divides $2mt+2$, and so $s = 1$ or $s = 2$. That is, $r = v$ or $r = \frac{v}{2}$.

We will now show that C does not contain an edge of length $\frac{v}{2}$. Since there are only $\frac{v}{2}$ edges of length $\frac{v}{2}$, we cannot have $r = v$. If $r = \frac{v}{2}$ then consideration of the path P consisting of a single edge of length $\frac{v}{2}$ tells us that $\frac{m}{2} = 1$, which is impossible. Hence the 1-factor F consists of the edges of length $\frac{v}{2}$.

Now, if $r = v$, then C contains edges of distinct lengths $l_1, l_2, \ldots, l_m$ such that $l_1 + l_2 + \ldots + l_m$ is even, and if $r = \frac{v}{2}$ then C contains edges of distinct lengths $l_1, l_2, \ldots, l_{\frac{m}{2}}$ such that $l_1 + l_2 + \ldots + l_{\frac{m}{2}} \equiv \frac{v}{2} \pmod 2$. However, the sum of all the orbit lengths is vt and so the number of orbits of length $= \frac{v}{2}$ is even. It follows that there are an even number of odd edge lengths, which is a contradiction when $mt \equiv 1, 2 \pmod 4$. ■

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