Asymptotics of Permutations with Nearly Periodic Patterns of Rises and Falls

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Abstract

Ehrenborg obtained asymptotic results for nearly alternating permutations and conjectured an asymptotic formula for the number of permutations that have a nearly periodic run pattern. We prove a generalization of this conjecture, rederive the fact that the asymptotic number of permutations with a periodic run pattern has the form $Cr^{-n}\,n!$, and show how to compute the various constants. A reformulation in terms of iid random variables leads to an eigenvalue problem for a Fredholm integral equation. Tools from functional analysis establish the necessary properties.

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1 Introduction

Definition 1 (words) *A* word *is a sequence of symbols. If v and w are words, then vw is the concatenation and w^k is the concatenation of k copies of w. The length $|w|$ of w is the number of symbols in the sequence.*

The descent word *of a sequence $\sigma_1, \ldots, \sigma_n$ of numbers is $\alpha = a_1 \cdots a_{n-1} \in \{d, u\}^{n-1}$ where $a_i = d$ if $\sigma_i > \sigma_{i+1}$ and $a_i = u$ otherwise.*

If a permutation has descent word α, then its run word *is a sequence L of positive integers where L_i is the length of the ith run in α. The size $\|L\|$ of a run word L is the sum of its parts plus 1. Thus, its size is one more than the length of the corresponding descent word. In other words, it is the size of the set being permuted.*

Let $\mathrm{Run}(N)$ *be the number of permutations that begin with an ascent and have run word N.*

For example, the descent and run words of the permutation $3, 2, 7, 5, 1, 4, 6$ are *dudduu* and 1122, respectively, and $\|1122\| = 7$. Note that each run word corresponds to two descent words: just interchange the roles of d and u. Thus the total number of permutations with run word N is $2\,\mathrm{Run}(N)$.

We prove the following generalization of Ehrenborg's Conjecture 7.1 [3].

Theorem 1 *Let $L_0, \ldots, L_k$ be (possibly empty) run words and let $M_1, \ldots, M_k$ be nonempty run words. There are nonzero constants $B_0, \ldots, B_k$ such that*

$$\frac{\mathrm{Run}(L_0 M_1^{a_1} L_1 M_2^{a_2} \cdots M_k^{a_k} L_k)}{\|L_0 M_1^{a_1} L_1 M_2^{a_2} \cdots M_k^{a_k} L_k\|!} \quad\sim\quad B_0 \cdots B_k\, \frac{\mathrm{Run}(M_1^{a_1} \ldots M_k^{a_k})}{\|M_1^{a_1} \ldots M_k^{a_k}\|!}$$

as $\min(a_1, \ldots a_k) \to \infty$. *The B_i are given by*

$$B_i \;=\; \begin{cases} \displaystyle\lim_{n\to\infty} \frac{\mathrm{Run}(L_0 M_1^n)\ \|M_1^n\|!}{\mathrm{Run}(M_1^n)\ \|L_0 M_1^n\|!}, & \text{if } \ i = 0, \\[2ex] \displaystyle\lim_{n\to\infty} \frac{\mathrm{Run}(M_i^n L_i M_{i+1}^n)\ \|M_i^n M_{i+1}^n\|!}{\mathrm{Run}(M_i^n M_{i+1}^n)\ \|M_i^n L_i M_{i+1}^n\|!}, & \text{if } \ 0 < i < k, \\[2ex] \displaystyle\lim_{n\to\infty} \frac{\mathrm{Run}(M_k^n L_k)\ \|M_k^n\|!}{\mathrm{Run}(M_k^n)\ \|M_k^n L_k\|!}, & \text{if } \ i = k. \end{cases}$$

We also rederive

Theorem 2 *[6] For a run pattern L there are constants $C(L)$ and $\lambda(L)$ such that the fraction of permutations with run pattern L^n is asymptotic to $C(L)\,\lambda(L)^n$.*

Since $\|L^n\| - 1 \;=\; n(\|L\| - 1)$, the theorem can be rewritten

$$\mathrm{Run}(L^n) \;\sim\; C^*(L)(\lambda^*(L))^{\|L^n\|}\ \|L^n\|!, \tag{1.1}$$

where $\lambda^* = \lambda^{1/(\|L\|-1)}$ and $C^* = C/\lambda^*$.

When $L = 1$, $\mathrm{Run}(L^n)$ counts alternating permutations of size $n+1$ and so we obtain the Euler numbers:[1] $\mathrm{Run}(1^n) = E_{n+1} \sim 2(2/\pi)^{n+2}(n+1)!$. Thus

$$C(1) = 8/\pi^2 \quad \text{and} \quad \lambda(1) = 2/\pi \tag{1.2}$$

in Theorem 2. Permutations with $L = t1$ (in other notation, $u^t d$) were studied by Leeming and MacLeod [5]. They proved that $(\mathrm{Run}(L^{n-1}t))^{1/n(t+1)} \sim \frac{n(t+1)}{e\,|p_{t+1}|}$, where p_ℓ is the zero of $\sum_{n=0}^{\infty} z^{\ell n}/(\ell n)!$ of smallest modulus. It follows that

$$\lambda(L) \sim (\mathrm{Run}(L^n))^{1/n} \sim ((n(t+1)+1)!)^{1/n} \left(\frac{n(t+1)}{e\,|p_{t+1}|}\right)^{t+1}$$

and so

$$\lambda(t1) = |p_{t+1}|^{-(t+1)}. \tag{1.3}$$

Lemma 2 and Theorem 4 in the next section provide the tools for calculating all the constants in Theorems 1 and 2. In Section 3, we illustrate by rederiving (1.3) and computing the associated $C(t1)$ in (3.6).

In giving proofs, we find it more convenient to work with descent words and then translate those results into run-word terms. Our proofs are based on the probabilistic approach of Ehrenborg, Levin and Readdy [4]. This leads us to the study of a Fredholm integral equation. In the next section, we introduce the probabilistic approach and state the relevant probabilistic theorems. In Sections 4–9, we prove the various theorems.

2 A Probabilistic Formulation

Definition 2 (ends of descent words) *The lengths of the longest initial and final constant strings in a descent word α are denoted by $A(\alpha)$ and $Z(\alpha)$, respectively. These are the initial and final integers in the run word corresponding to α.*

We now define the probability distributions and a measure of deviation from independence that play a central role in our approach.

Definition 3 (some probability) *If $\alpha \in \{d,u\}^{n-1}$, then $f(x,y,\alpha)$ is the probability density function for the event that the sequence $X_1,\ldots,X_n$ of iid random variables with the uniform distribution on $[0,1]$ has $X_1 = x$, $X_n = y$ and descent word α. Also, $f(x,y \mid \alpha)$ is the conditional density function. We replace x and/or y with $*$ to indicate marginal distributions. For example, $f(x,*,\alpha) = \int_0^1 f(x,y,\alpha)\,dy$.*

Let $\alpha_1,\alpha_2,\ldots$ be a sequence of descent words with $|\alpha_n| \to \infty$. We call the sequence asymptotically independent *if either*

[1] This works for both odd and even n since 1^{2k} corresponds to $(ud)^k$ and 1^{2k+1} corresponds to $(ud)^k u$.

(a) $\lim_{n\to\infty} A(\alpha_n) = \infty$,

(b) $\lim_{n\to\infty} Z(\alpha_n) = \infty$, *or*

(c) $A(\alpha_n)$ *and* $Z(\alpha_n)$ *are bounded and*

$$\lim_{n\to\infty}\Bigl(\sup_{(x,y)}\Bigl|f(x,y\mid\alpha_n) - f(x,*\mid\alpha_n)f(*,y\mid\alpha_n)\Bigr|\Bigr) \;=\; 0. \tag{2.1}$$

We call the sequence stable *if* $\lim_{n\to\infty} f(x,*\mid\alpha_n)$ *and* $\lim_{n\to\infty} f(*,y\mid\alpha_n)$ *exist or are delta functions.*

Clearly any infinite subsequence of an asymptotically independent or stable sequence also has that property.

The following lemma, noted in [4], connects the probability distributions with permutations.

Lemma 1 *If $X_1,\dots,X_n$ are independent identically distributed (iid) random variables with a continuous density function, then the probability that the sequence $X_1,\dots,X_n$ has descent word α equals the probability that a random permutation of $\{1,\dots,n\}$ has descent word α. In other words, the number of permutations with descent word α is $(1+|\alpha|)!\, f(*,*,\alpha)$.*

Due to the lemma, we may study permutations via the probability distributions. Stability and asymptotic independence imply a result needed to prove Theorem 1:

Theorem 3 *Fix $k>0$. Suppose that, for each $1\le i\le k$, the sequence $\alpha_{i,1},\alpha_{i,2},\dots$ is stable and asymptotically independent. Suppose that β_i are possibly empty descent words for $0\le i\le k$. Let*

$$\delta_n \;=\; \beta_0\alpha_{1,n}\beta_1\cdots\alpha_{k,n}\beta_k.$$

Let $a(\beta)$ and $z(\beta)$ be the first and last letters in β, respectively. If β_i is not empty, assume both

- *that $Z(\alpha_{i,n}a(\beta_i))$ is bounded for all n when $0<i\le k$ and*
- *that $A(z(\beta_i)\alpha_{i+1,n})$ is bounded for all n when $0\le i<k$.*

If β_i is empty and $0<i<k$, assume either

- *that $Z(\alpha_{i,n})$ and $A(\alpha_{i+1,n})$ are bounded for all n or*
- *that $z(\alpha_{i,n})\ne a(\alpha_{i+1,n})$ for all n.*

Then

$$\frac{f(*,*,\delta_n)}{\prod_{i=1}^{k} f(*,*,\alpha_{i,n})} \sim \prod_{i=0}^{k} C_i, \tag{2.2}$$

where the C_i are nonzero and given by

$$C_i = \begin{cases} \lim_{n\to\infty} \dfrac{f(*,*,\beta_0\alpha_{1,n})}{f(*,*,\alpha_{1,n})}, & \text{if } i=0, \\[2ex] \lim_{n\to\infty} \dfrac{f(*,*,\alpha_{i,n}\beta_i\alpha_{i+1,n})}{f(*,*,\alpha_{i,n})\, f(*,*,\alpha_{i+1,n})}, & \text{if } 0<i<k, \\[2ex] \lim_{n\to\infty} \dfrac{f(*,*,\alpha_{k,n}\beta_k)}{f(*,*,\alpha_{k,n})}, & \text{if } i=k. \end{cases}$$

Theorem 4 below proves stability and asymptotic independence for repeated descent patterns.

Conjecture 1 *While stability clearly depends on the form of the words $\alpha_{i,1}, \alpha_{i,2}, \ldots$, we conjecture that $|\alpha_{i,n}| \to \infty$ implies asymptotic independence.*

We now provide the tools for calculating the constants in Theorems 1 and 2.

Definition 4 (reversal of descent words) *For any descent word α, define α^R to be α read in reverse order and $\overline{\alpha}$ to be α with the roles of d and u reversed.*

Lemma 2 *Let α and β be arbitrary descent words, We have*

$$f(x,y,u) = \begin{cases} 0, & \text{if } x>y, \\ 1, & \text{otherwise,} \end{cases} \tag{2.3}$$

$$f(x,y,\overline{\alpha}) = f(y,x,\alpha^R) = f(1-x,1-y,\alpha) \tag{2.4}$$

$$f(x,y,\alpha\beta) = \int_0^1 f(x,t,\alpha) f(t,y,\beta)\, dt \tag{2.5}$$

$$f(*,*,\alpha) \geq f(*,*,\alpha\beta) \tag{2.6}$$

We omit the proof of the lemma since it is simple and is essentially contained in Section 2 of [4].

Theorem 4 *Let $\mu = m_1 \ldots m_{|\mu|}$ be a descent word containing both d and u. The sequence $\mu, \mu^2, \mu^3, \ldots$ is asymptotically independent and stable.*

Let $\omega = e^{2\pi i/|\mu|}$. Define the $|\mu| \times |\mu|$ matrix M for $0 \leq k, \ell < |\mu|$ by

$$M_{k,\ell} = \begin{cases} \omega^{k\ell}, & \text{if } m_{k+1} = d\,, \\ \omega^{k\ell}\exp(r\omega^{\ell}), & \text{if } m_{k+1} = u. \end{cases}$$

Let r be the smallest magnitude number for which the matrix M is not invertible. Let $U(\mu)$ be the number of u's in μ. Then, uniformly for $(x,y) \in [0,1]^2$,

$$f(x,y,\mu^n) \;=\; C(\mu)\Big(\phi(x,\mu)\phi(y,\overline{\mu}^R) + o(1)\Big)\lambda(\mu)^n, \tag{2.7}$$

where

$$\lambda(\mu) \;=\; \frac{(-1)^{U(\mu)}}{r^{|\mu|}}, \tag{2.8}$$

$$\phi(x,\mu) \;=\; \sum_{t=0}^{|\mu|-1} D_t \exp(r\omega^t x), \tag{2.9}$$

$$C(\mu) \;=\; \frac{1}{\int_0^1 \phi(x,\mu)\,\phi(x,\overline{\mu}^R)\;dx}, \tag{2.10}$$

and $\mathbf{D} = (D_0,\ldots,D_{|\mu|-1})^t$ is the solution of $M\mathbf{D} = \mathbf{0}$ such that $\int_0^1 \phi(x,\mu)\,dx = 1$. The value of $\phi(y,\overline{\mu}^R)$ is found by replacing x with y and μ with $\overline{\mu}^R$. The values of λ and $|r|$ are the same for μ and $\overline{\mu}^R$. We may assume $\arg r = 0$ if $U(\mu)$ is even, and $\arg r = \pi/|\mu|$ otherwise.

In particular, $f(*,*,\mu^n) \sim C(\mu)\,\lambda(\mu)^n$.

Remark If (2.7) is integrated over x or y, we obtain Theorem 2 of Shapiro, Shapiro and Vainshtein [6], including the same formulas for calculating C, λ and ϕ. Their method of proof differs from ours. If our Conjecture 1 were proved, then our Theorem 4 would follow from Theorem 2 [6].

Remark The second smallest magnitude r, say r_2, for which M is singular gives the "second largest eigenvalue" $\lambda_2 = 1/|r_2|^{|\mu|}$, which is discussed in later sections. This can be used to obtain information about rate of convergence because of (6.1). See also Section 8.

Using the lemma, one can compute $f(x,y,\alpha)$ for any particular descent word α. We use (2.4) to convert results for d into results for u and results for the left end of α into results for the right, generally without comment. To study the asymptotics of something like $f(*,*,\alpha^k\beta\mu^\ell)$ as $k,\ell \to \infty$, one combines the lemma and theorem:

$$\begin{aligned} f(*,*,\alpha^k\beta\mu^\ell) \;&=\; \int_0^1\int_0^1 f(*,x,\alpha^k)\,f(x,y,\beta)\,f(y,*,\mu^\ell)\;dx\,dy \\ &\sim\; C(\alpha)\,C(\mu)\,\lambda(\alpha)^k\,\lambda(\mu)^\ell \int_0^1\int_0^1 \phi(x,\overline{\alpha}^R)\,f(x,y,\beta)\,\phi(y,\mu)\;dx\,dy. \end{aligned}$$

3 An Illustration: $\mu = ud^{\ell-1}$

We now obtain equations for C, ϕ and λ when $\mu = ud^{\ell-1}$. The value of λ is given implicitly and, since ϕ and C depend on λ, they are given implicitly as well. Of course, our equation for λ will be the same as Leeming and MacLeod's result. Note that $|\mu| = \ell$.

The matrix equation $M\mathbf{D} = \mathbf{0}$ in Theorem 4 is written as $|\mu|$ separate equations in (8.11). With $\omega = e^{2\pi i/\ell}$, these are

$$\sum_{t=0}^{\ell-1} \omega^{kt} D_t \;=\; 0 \quad \text{for } 1 \le k \le \ell - 1.$$

It is easily seen that these equations have the one parameter solution given by $D_0 = D_1 = \cdots = D_{\ell-1}$ The condition for $k = 0$ is

$$0 \;=\; \sum_{t=0}^{\ell-1} D_t \exp(r\omega^t) \;=\; D_0 \sum_{t=0}^{\ell-1} \exp(r\omega^t). \tag{3.1}$$

Since we do not want the identically zero solution, (3.1) gives us the complex transcendental equation $\sum_{t=0}^{\ell-1} \exp(r\omega^t) \;=\; 0$ for r. This can be simplified by using the Taylor series for e^z to expand $\exp(r\omega^t)$ and then collecting terms according to powers of r:

$$\ell \sum_{k=0}^{\infty} \frac{r^{k\ell}}{(k\ell)!} \;=\; 0, \tag{3.2}$$

since the sum of ω^{tn} over t vanishes when n is not a multiple of ℓ. This is the result of Leeming and MacLeod [5] mentioned after Theorem 2. In their notation, $r = p_\ell$, the smallest magnitude zero of (3.2). By (2.8), we can rewrite (3.2) as

$$0 \;=\; \sum_{k=0}^{\infty} \frac{(-1/\lambda)^k}{(k\ell)!}, \tag{3.3}$$

which can be solved numerically for the largest $\lambda > 0$. By (2.9) and Taylor series expansion of the exponentials,

$$\phi(x,\mu) \;=\; D_0 \sum_{t=0}^{\ell-1} \exp(r\omega^t x) \;=\; \ell D_0 \sum_{k=0}^{\infty} \frac{(rx)^{k\ell}}{(k\ell)!} \;=\; \ell D_0 \sum_{k=0}^{\infty} \frac{(-1/\lambda)^k x^{k\ell}}{(k\ell)!}.$$

Integrating over $[0,1]$ gives $1 = \ell D_0 \sum_{k=0}^{\infty} \frac{(-1/\lambda)^k}{(k\ell+1)!}$ and so

$$\phi(x, ud^{\ell-1}) \;=\; \sum_{k=0}^{\infty} \frac{(-1/\lambda)^k x^{k\ell}}{(k\ell)!} \Bigg/ \sum_{k=0}^{\infty} \frac{(-1/\lambda)^k}{(k\ell+1)!}. \tag{3.4}$$

For $\overline{\mu}^R \;=\; u^{\ell-1}d$, the conditions (8.11) for $0 \le k \le \ell - 2$ become

$$0 \;=\; \sum_{t=0}^{\ell-1} \omega^{kt} D_t \exp(r\omega^t) \;=\; \sum_{t=0}^{\ell-1} \omega^{(k+1)t} \Big(\omega^{-t} D_t \exp(r\omega^t)\Big).$$

With $E_t = \omega^{-t}\exp(r\omega^t)D_t$, these become $\sum_{t=0}^{\ell-1}\omega^{jt}E_t = 0$ for $1 \le j \le \ell-1$ and so, as before, $E_0 = E_1 = \cdots = E_{\ell-1}$. For $k = \ell-1$ we have

$$0 = \sum_{t=0}^{\ell-1}\omega^{t(\ell-1)}D_t = \sum_{t=0}^{\ell-1}\omega^{-t}(\omega^t\exp(-r\omega^t)E_t) = E_0\sum_{t=0}^{\ell-1}\exp(-r\omega^t).$$

This is the same as (3.1) with $-r$ replacing r. Thus $r = -p_\ell$ and

$$\begin{aligned}\phi(x, u^{\ell-1}d) &= \sum_{t=0}^{\ell-1}D_t\exp(-p_\ell x\omega^t) = E_0\sum_{t=0}^{\ell-1}\omega^t\exp(p_\ell\omega^t)\exp(-p_\ell x\omega^t)\\ &= E_0\sum_{t=0}^{\ell-1}\omega^t\exp(p_\ell(1-x)\omega^t) = \frac{\ell E_0}{p_\ell}\sum_{k=1}^{\infty}\frac{(-1/\lambda)^k(1-x)^{k\ell-1}}{(k\ell-1)!},\end{aligned}$$

by expanding the exponentials in Taylor series as before. Integrating over $[0,1]$ gives

$$1 = \frac{\ell E_0}{p_\ell}\sum_{k=1}^{\infty}\frac{(-1/\lambda)^k}{(k\ell)!} = -\frac{\ell E_0}{p_\ell}$$

by (3.3). Thus

$$\phi(x, u^{\ell-1}d) = -\sum_{k=1}^{\infty}\frac{(-1/\lambda)^k(1-x)^{k\ell-1}}{(k\ell-1)!}. \tag{3.5}$$

Combining (3.4) and (3.5) with the (2.10), we have

$$\begin{aligned}C(ud^{\ell-1}) &= \frac{-\sum_{k=0}^{\infty}\frac{(-1/\lambda)^k}{(k\ell+1)!}}{\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\int_0^1\frac{(-1/\lambda)^s x^{s\ell}}{(s\ell)!}\frac{(-1/\lambda)^t(1-x)^{t\ell-1}}{(t\ell-1)!}\,dx}\\ &= \frac{-\sum_{k=0}^{\infty}\frac{(-1/\lambda)^k}{(k\ell+1)!}}{\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1/\lambda)^{s+t}}{((s+t)\ell)!}} = \frac{-\sum_{k=0}^{\infty}\frac{(-1/\lambda)^k}{(k\ell+1)!}}{\sum_{k=1}^{\infty}\frac{k(-1/\lambda)^k}{(k\ell)!}}.\end{aligned} \tag{3.6}$$

The following table contains some values of $\lambda(ud^{\ell-1})$ and $C(ud^{\ell-1})$ and well as the denominator of (3.4) and $\lambda^{1/\ell}$. The denominator is needed in computing ϕ and $\lambda^{1/\ell}$ is used in (1.1).

ℓ	$\lambda(ud^{\ell-1})$	$C(ud^{\ell-1})$	(3.4) denom.	$\lambda^{1/\ell}$
2	0.405285	0.810569	0.636620	0.636620
3	0.157985	0.786835	0.744142	0.540595
4	4.1064e−2	0.810569	0.798696	0.450158
5	8.3001e−3	0.836374	0.833030	0.383546
6	1.3874e−3	0.858002	0.857071	0.333964
7	1.9835e−4	0.875238	0.874983	0.295844
8	2.4800e−5	0.888954	0.888885	0.265647
9	2.7557e−6	0.900018	0.899999	0.241128

To further illustrate the calculation procedure, we compute asymptotics in Theorem 1 when the permutation alternates up/down, except for some internal cases of uu. To do this, we take all a_i to be even except possibly a_k, $M_i = 1$ for all i, $L_i = 1$ for $1 \le i \le k-1$, and L_0 and L_k empty. We need to compute B_i.

Ehrenborg's Theorem 4.1 [3] gives the value. When a is even and b is odd, what he calls $\beta(1^a, 2, 1^b)$ is the number of permutations with pattern $(ud)^{a/2}u(ud)^{(b+1)/2}$. He compares this with E_n, the number of alternating permutations of the same length $n = \|1^a 2 1^b\|$. On the other hand, B_i compares it with E_{n-1} Since the fraction of n-long permutations that alternate is asymptotic to $C(1)\lambda(1)^n$, we obtain an extra factor of $\lambda(1) = 2/\pi$:

$$B_i \;\sim\; \lambda(1)\frac{\beta(1^a, 2, 1^b)}{E_n} \;\sim\; \frac{4}{\pi^2}, \quad \text{where} \quad \min(a,b) \to \infty.$$

Thus $B_i = 4/\pi^2$. Ehrenborg also discusses computing $\beta(1^a, L, 1^b)$.

To illustrate the use of our formulas, we now compute B_i without using Ehrenborg's result. Note that $\mathrm{Run}(M_i^n M_{i+1}^n)$ is just counting alternating permutations. To evaluate $\mathrm{Run}(M_i^n L_i M_{i+1}^n)$, we apply (2.5) twice to compute $f(x, y, (ud)^m u (ud)^m)$ and integrate this over x and y. Since

$$f(x, y, (ud)^m) \;=\; C(1)\Big(\phi(x,u)\phi(y,u) + o(1)\Big)\lambda(1)^{2m},$$

where C and λ are given by (1.2), we need to know $\phi(x,u)$. One can use (3.5) with $\ell = 2$ or [4] to conclude that

$$\phi(x,u) \;=\; (\pi/2)\sin(\pi x/2).$$

In computing B_i in Theorem 1, the formulas we are using are probabilities and so we will be estimating $\mathrm{Run}(P)/\|P\|!$ for patterns P. Remembering that $\int_0^1 \phi(x,\alpha)\,dx \;=\; 1$,

$$B_i \;=\; \frac{C(1)^2\lambda(1)^{2n}\int_0^1\int_0^1 \sin(\pi s/2)\sin(\pi t/2) f(s,t,u)\,ds\,dt}{C(1)^2\lambda(1)^{2n}\int_0^1 \sin(\pi s/2)^2\,ds},$$

where $f(s,t,u)$ is given by (2.3). The integral in the denominator is 1/2 and the integral in the numerator is

$$\int_{0<s<t<1} \sin(\pi s/2)\sin(\pi t/2)\, ds\, dt \;=\; \int_0^1 \sin(\pi s/2)(2/\pi)\cos(\pi s/2)\, ds$$
$$= \; (1/\pi)\int_0^1 \sin(\pi s)\, ds \;=\; 2/\pi^2.$$

Hence $B_i \;=\; 4/\pi^2$.

4 Proof of Theorem 3

Lemma 3 *Let α and β be descent words.*

(a) If $|\alpha| > 1$, then $f(x,y,\alpha)$ is a monotonic uniformly continuous function of x and y on the unit square. In fact, it is increasing in x if and only if α begins with d and is increasing in y if and only if α ends with u.

(b) $f(x,y,\alpha du\beta) \geq f(x,*,\alpha d)\, f(*,y,u\beta)$.

(c) If α contains both u and d, there is are functions $U(k)$ and $L(x,y,k)$ such that, for each k, $L(x,y,k)$ is strictly positive for (x,y) in the interior of $[0,1]^2$ and such that

$$U(k) \;\geq\; f(x,y \mid \alpha) \;\geq\; L(x,y,k) \quad \text{where} \quad k \;=\; \max\{A(\alpha), Z(\alpha)\}.$$

Similarly,

$$U_1(A(\alpha)) \;\geq\; f(x,* \mid \alpha) \;\geq\; L_1(x, A(\alpha))$$

and

$$U_1(Z(\alpha)) \;\geq\; f(*,y \mid \alpha) \;\geq\; L_1(y, Z(\alpha))$$

for functions U_1 and L_1 where $L_1(x,k)$ is strictly positive for $0 < x < 1$.

Proof It is easily seen that (2.3) is monotonic. It follows by induction that $f(x,y,\alpha)$ is continuous if $|\alpha| > 1$. Suppose $\alpha = u\beta$ where β is not the empty word. By (2.5)

$$f(x,y,\alpha) \;=\; \int_x^1 f(t,y,\beta)\, dt,$$

which is clearly a decreasing function of x.

We now prove (b). By (2.5),

$$f(x,y,\alpha du\beta) \;=\; \int_0^1 f(x,t,\alpha d)\, f(t,y,u\beta)\, dt.$$

By (a), both $f(x,t,\alpha d)$ and $f(t,y,u\beta)$ are monotonic decreasing functions of t. By the integral form of Chebyshev's integral inequality [7],

$$\int_0^1 f(x,t,\alpha d)\, f(t,y,u\beta)\, dt \;\geq\; \int_0^1 f(x,t,\alpha d)\, dt \;\int_0^1 f(t,y,u\beta)\, dt \;=\; f(x,*,\alpha d)\, f(*,y,u\beta).$$

This completes the proof of (b).

We now prove (c). Let $B(m)$ be an upper bound for $f(x,y,u^m)$. Suppose $\alpha = a^m\beta b^n$ where either

- β is empty and $b = \overline{a}$ or
- $\beta = \overline{a}\delta\overline{b}$.

By part (b) of the lemma,

$$f(x,y,\alpha) \;\geq\; f(x,*,a^m)\, f(*,*,\beta)\, f(*,y,b^n).$$

On the other hand,

$$f(x,y,\alpha) \;=\; \int_0^1\int_0^1 f(x,s,a^m)\, f(s,t,\beta)\, f(t,y,b^n)\, ds\, dt \;\leq\; B(m)B(n)f(*,*,\beta).$$

Thus we have

$$B(m)B(n)f(*,*,\beta) \;\geq\; f(x,y,\alpha) \;\geq\; f(x,*,a^m)\, f(*,*,\beta)\, f(*,y,b^n)$$

and so

$$f(*,*,a^m)\, f(*,*,\beta)\, f(*,*,b^n) \;\leq\; f(*,*,\alpha) \;\leq\; B(m)B(n)f(*,*,\beta).$$

Dividing gives

$$\frac{B(m)B(n)}{f(*,*,a^m)\, f(*,*,b^n)} \;\geq\; f(x,y\mid\alpha) \;\geq\; \frac{f(x,*,a^m)\, f(*,y,b^n)}{B(m)B(n)}.$$

Let $U(k)$ be the maximum of the left side over $m,n\leq k$ and let $L(x,y,k)$ be the minimum of the right side over $m,n\leq k$ and $a,b\in\{d,u\}$. The last statement for (c) is proved in a similar manner. $\square$

Proof (of Theorem 3) We assume all β_i are nonempty. The modifications for an empty β_i are straightforward.

Let $V_m(x)$ be the m-dimensional unit cube $[0,1]^m$ in coordinates $x_0,\ldots,x_{m-1}$. Using (2.5) we have

$$\frac{f(*,*,\delta\,)}{\prod_{i=1}^k f(*,*,\alpha_{i,n})} \;=\; \int_{V_{k+1}(s)}\int_{V_{k+1}(t)} f(t_0,s_0,\beta_0)\times \Big(\prod_{i=1}^{k}(f(s_{i-1},t_i\mid\alpha_{i,n})\, f(t_i,s_i,\beta_i))\, dt_i\, ds_i\Big)\, dt_0\, ds_0. \tag{4.1}$$

Our goal is to show that, asymptotically, we can replace

$$\int_0^1 f(t_{i-1},s_{i-1},\beta_{i-1})\, f(s_{i-1},t_i\mid\alpha_{i,n})\, ds_{i-1} \tag{4.2}$$

with

$$\int_0^1 f(t_{i-1}, s_{i-1}, \beta_{i-1})\, f(s_{i-1}, * \mid \alpha_{i,n})\, f(*, t_i \mid \alpha_{in})\, ds_{i-1}. \tag{4.3}$$

Since the $f(t_i, s_i, \beta_i)$ are either uniformly continuous by Lemma 3(a) or a step function as in (2.3), we can rearrange limits and integrals to obtain (2.2), except for showing that the C_i exist and are nonzero. Note that this gives

$$C_i \;=\; \lim_{n\to\infty} \int_0^1 \int_0^1 f(*, s \mid \alpha_{i,n})\, f(s, t, \beta_i)\, f(t, * \mid \alpha_{i+1,n})\, ds\, dt \tag{4.4}$$

for $0 < i < k$ and similar results for $i = 0$ and k. These C_i are easily seen to be equivalent to those in the theorem.

We distinguish cases according to whether or not $A(\alpha_{i,n})$ and/or $Z(\alpha_{i,n})$ are bounded.

First suppose both $A(\alpha_{i,n})$ and $Z(\alpha_{i,n})$ are bounded. In this case, the definition of asymptotic independence gives us

$$f(s_{i-1}, t_i \mid \alpha_{i,n}) \;=\; f(s_{i-1}, * \mid \alpha_{i,n})\, f(*, t_i \mid \alpha_{i,n}) + o(1)$$

uniformly over the range of integration. Thus we can replace (4.2) with (4.3) plus $\int f(t_{i-1}, s_{i-1}, \beta_{i-1})\, o(1)$. The effect of this latter is to add a term of products of C_j's with $C_{i-1}C_i$ replaced by $o(1)$. Since the C_i will be shown to be nonzero, the asymptotics are unchanged.

Now suppose $A(\alpha_{i,n}) \to \infty$ and $a(\alpha_{i,n}) = u$ the cases of Z and d are handled by (2.4). For simplicity, we drop the i subscripts. Write $\alpha_n = u^m\gamma$ where $m \to \infty$ and $a(\gamma) = d$. By assumption, $z(\beta) = d$. Note that $f(s, t, \beta)$, $f(t, x, u^m)$ and $f(x, y, \gamma)$ are decreasing functions of t and x. Also, for each fixed $x > 0$, $f(t, x \mid u^m)$ approaches a delta function as $m \to \infty$ and so, for $0 < s, x < 1$,

$$\int_0^1 f(s, t, \beta)\, f(t, x, u^m)\, ds \;=\; (f(s, 0, \beta) + o(1))\, f(*, x, u^m).$$

By the uniform continuity of $f(s, t, \beta)$ when $|\beta| > 1$ or (2.3) when $\beta = d$, this is also true for $s = 0$. Multiplying by $f(x, y, \gamma)$, integrating on x, using the monotonicity in x and dividing by $f(*, *, \alpha_n)$,

$$\int_0^1 f(s, t, \beta)\, f(t, y \mid \alpha_n)\, ds \;=\; (f(s, 0, \beta) + o(1))\, f(*, y \mid \alpha_n).$$

Since $f(t, y \mid \alpha_n)$ approaches a delta function for each $y > 0$, we finally have

$$\int_0^1 f(s, t, \beta)\, f(t, y \mid \alpha_n)\, ds \;=\; \left(\int_0^1 (f(s, t, \beta) + o(1))\, f(t, * \mid \alpha_n)\, dt\right) f(*, y \mid \alpha_n). \tag{4.5}$$

We consider $0 < i < k$ and write

$$C_i \;=\; \lim_{n\to\infty} \int_0^1 \int_0^1 f(*, s \mid \alpha_{i,n})\, f(s, t, \beta_i)\, f(t, * \mid \alpha_{i+1,n})\, ds\, dt.$$

The case of unbounded runs at the end of $\alpha_{i,n}$ can be handled as in the derivation of (4.5). Otherwise, stability guarantees that $f(*, s \mid \alpha_{i,n})$ and $f(t, * \mid \alpha_{i+1,n})$ approach a limit and Lemma 3(c) guarantees that the limits are bounded. Since $f(s, t, \beta_i)$ is well behaved, C_i exists. Furthermore, it is positive because of the lower bound in Lemma 3(c). $\square$

5 A Functional Analysis Formulation of Theorem 4

Suppose μ is a descent word containing both d and u. Without loss of generality, we suppose that μ begins with d. Define $K(x,y) \;=\; f(x,y,\mu)$ and

$$K_m(x,y) \;=\; \begin{cases} K(x,y) \;=\; f(x,y,\mu), & \text{if } \; m = 1, \\ \int_0^1 K(x,t)K_{m-1}(t,y)\,dt, & \text{if } \; m > 1. \end{cases}$$

Since $f(x,y,\mu^n) \;=\; K_n(x,y)$, studying $f(x,y,\mu^n)$ as $n \to \infty$ is equivalent to studying large powers of an integral operator T whose kernel is K. If we were dealing with matrices, we would simply be taking powers of a matrix K with strictly positive entries and so K would have a unique eigenvalue λ_1 of maximum modulus. It would be positive real and have left and right eigenspaces of dimension 1. Thus we would have $K^n \;\sim\; C\lambda_1^n \mathbf{u}\mathbf{v}'$ for left and right eigenvectors $\mathbf{u}$ and $\mathbf{v}$. This is the discrete form of Theorem 4. We prove analogous results for the function $K(x,y)$ using functional analysis.

We begin with some relevant properties of the kernel.

Lemma 4 *Let $K(x,y) \;=\; f(x,y,\mu)$, where μ is descent word beginning with d and containing u. Then we have the following.*

(a) $K(x,y)$ is uniformly continuous on the unit square $[0,1]^2$ and strictly positive on $(0,1] \times (0,1)$.

There is a continuous strictly increasing function $\tilde{e}(x)$ on $[0,1]$ with $\tilde{e}(0) = 0$ such that

(b) for every positive Borel measure ν on $[0,1]$ with $\nu(\,(0,1)\,) > 0$, there is a number $\tau_\nu > 0$ such that

$$\tau_\nu \, \tilde{e}(x) \;\le\; \int K(x,y) d\nu_y \quad \textit{for all} \;\; x \in [0,1]; \tag{5.1}$$

(c) there is a constant M_K such that, for every Borel measure ν_y on $[0,1]$,

$$\left| \int K(x,y) d\nu_y \right| \;\le\; \left(M_K \int d|\nu_y| \right) \tilde{e}(x) \quad \textit{for all} \;\; x \in [0,1]; \tag{5.2}$$

(d) The function

$$q(x,y) \;=\; \begin{cases} \tilde{e}(x)^{-1} K(x,y), & \textit{if} \;\; x > 0, \\ \lim_{x \to 0^+} \tilde{e}(x)^{-1} K(x,y), & \textit{if} \;\; x = 0, \end{cases}$$

is continuous on $[0,1]^2$ and is strictly positive on $[0,1] \times (0,1)$.

Proof Lemma 3 implies (a).

We now prove (b). Without loss of generality, $\mu = d^k u\beta$ for some $k > 0$ and so, by Lemma 3(b), $K(x,y) \geq f(x,*,d^k)f(*,y,u\beta)$. Set

$$\tilde{e}(x) = f(x,*,d^k) \quad \text{and} \quad \tau_\nu = \frac{1}{2}\int f(*,y,u\beta)d\nu_y. \tag{5.3}$$

Note that $f(x,*,\delta) > 0$ on $(0,1)$ for any δ, so $\tilde{e}(x)$ is also positive there. Also note that $f(*,y,\delta)$ is strictly positive and continuous on $(0,1)$, so $\tau_\nu > 0$.

We now prove (c). Since $f(x,y,\mu)$ is nonnegative and (uniformly) continuous in the unit square, there is a constant M_K such that $f(x,y,\mu) \leq M_K f(x,*,\mu)$. Combine this with the fact that

$$f(x,*,\delta\gamma) \leq f(x,*,\delta) \quad \text{for all} \quad \delta,\gamma$$

to get

$$K(x,y) \leq M_K f(x,*,\mu) \leq M_K f(x,*,d^k) = M_K \tilde{e}(x).$$

Thus

$$\left|\int K(x,y)\, d\nu_y\right| \leq \int K(x,y)\, d|\nu_y| \leq M_K \tilde{e}(x) \int d|\nu_y|.$$

We now prove (d). Since $e(x)$ is continuous and strictly positive on $(0,1]$ and $K(x,y)$ is continuous on $[0,1]^2$, the claim holds on $(0,1] \times [0,1]$. Since $K(x,y)$ is monotonic in y, so is $e(x)^{-1}K(x,y)$. It suffices to study the limit of this ratio as $x \to 0$. We claim that

$$f(x,y,d^k) = \begin{cases} 0, & \text{if } y \geq x, \\ (x-y)^{k-1}/(k-1)!, & \text{otherwise.} \end{cases} \tag{5.4}$$

To see this, consider the sequence of independent, identically distributed, random variables $X_1,\ldots,X_{k+1}$ conditioned on $X_1 = x > y = X_{k+1}$. The probability that $X_2,\ldots,X_k$ all lie in $[y,x]$ is $(x-y)^{k-1}$ and the probability that they are in increasing order is $1/(k-1)!$ since there $(k-1)!$ ways to arrange them. Since these two events are independent, (5.4) follows.

By repeated application of l'Hospital's Rule

$$\begin{aligned} \lim_{x\to 0}\frac{K(x,y)}{\tilde{e}(x)} &= \lim_{x\to 0}\frac{\int_0^x (x-t)^{k-1} f(t,y,u\beta)\, dt}{\int_0^x (x-t)^{k-1}\, dt} \\ &= \lim_{x\to 0}\frac{\int_0^x (x-t)^{k-2} f(t,y,u\beta)\, dt}{\int_0^x (x-t)^{k-2}\, dt} = \cdots = \lim_{x\to 0}\frac{\int_0^x f(t,y,u\beta)\, dt}{\int_0^x dt} \\ &= \lim_{x\to 0} f(x,y,u\beta) = f(0,y,u\beta). \end{aligned}$$

This completes the proof since $f(0,y,u\beta) > 0$ for $0 < y < 1$. ∎

6 Operators Which Preserve Cones

Before considering integral operators like $K(x,y)$, we develop some general properties of linear operators that are needed for our proof. We follow the terminology in [2] and try to keep the expositions reasonably self-contained.

Definition 5 (cones, a partial order, semi-monotonic norms) *Suppose* $(B, \| \|)$ *is a real Banach space. A* cone $\mathcal{P}$ *is a closed convex set with*

- $\mathcal{P} \neq \{0\}$,
- $\lambda\mathcal{P} \subset \mathcal{P}$ *for any number* $\lambda \geq 0$ *and*
- $\mathcal{P} \cap -\mathcal{P} \ = \ 0$.

Given a cone $\mathcal{P}$*, define a partial order by* $x \geq y$ *if and only if* $x - y \in \mathcal{P}$ *and let* $[a,c] \ = \ \{b : a \leq b \leq c\}$.
A norm $\| \|$ *is called* semi-monotonic *with respect to* $\mathcal{P}$ *if there is a* $\gamma \in \mathbb{R}^+$ *such that* $0 \leq x \leq y$ *implies* $\|x\| \ \leq \ \gamma\|y\|$.

A real Banach space can be complexified to a (unique) complex Banach space B^c and an operator T on B extends uniquely to B^c. (See [2], Chapter 9.8.)

Here is the Krein-Rutman Theorem as stated in Theorem 19.3 of [2]. It plays a central role in our analysis of $K(x,y)$.

Theorem 5 (Krein-Rutman) *Suppose* T *is a compact linear operator* $T : B \to B$ *which maps* $\mathcal{P}$*, except for 0, into its interior, denoted* $\mathcal{P}^0$*, then the maximum magnitude eigenvalue* λ_1 *of* T *extended to the complexification* B^c *is real and positive. The eigenvector* ϕ *corresponding to* λ_1 *is unique (up to a scalar multiple) and lies in* $\mathcal{P}^0$*. Any other eigenvector of* T *does not lie in* $\mathcal{P}$.

As is often the case with Krein-Rutman applications we shall find that our map T maps a cone $\mathcal{P}$ into itself, but not into its interior. We now describe a standard patch which allows one to still use the theorem.

Definition 6 (the norm $\| \|_e$) *Given a cone* $\mathcal{P}$ *in the Banach space* $(B, \| \|)$*, recall the partial order of Definition 5. Pick* $e \in \mathcal{P}$*, set*

$$B_e \ = \ \bigcup_{t>0} t[-e,e]$$

and define a norm on B_e *by*

$$\|b\|_e \ = \ \inf\{t > 0 : b \in t[-e,e]\} \quad \text{for } \ b \in B_e.$$

Define a cone $\mathcal{P}_e$ *by*

$$\mathcal{P}_e \ = \ B_e \cap \mathcal{P} \ = \ \{b \in \mathcal{P} : te - b \in \mathcal{P} \text{ for some } t > 0\}.$$

Beware: B_e is *not* complete in this norm. Note that $[-e,e]$ is the unit ball in B_e. The key facts about $\| \, \|_e$ are as follows.

Lemma 5 *If $(B, \| \, \|)$ is a real Banach space with cone $\mathcal{P}$ and $e \in B$, the following are true.*

(a) The norm $\| \, \|_e$ is semi-monotonic on B_e with respect to the cone $\mathcal{P}_e$.

(b) If $\| \, \|$ is semi-monotonic on B with respect to the cone $\mathcal{P}$, then $(B_e, \| \, \|_e)$ is complete and hence a real Banach space. Also there is a number γ such that $\gamma \|b\|_e \geq \|b\|$ for all b in B_e.

(c) If $T : B \to B$ is a operator such that

(i) T maps the cone $\mathcal{P}$ into $\mathcal{P}$,

(ii) $\| \, \|$ is semi-monotonic on B with respect to the cone $\mathcal{P}$,

(iii) for each b in B there is a number τ_b such that $-\tau_b \, e \;\leq\; T(b) \;\leq\; \tau_b \, e$,

(iv) for each $b \in \mathcal{P}$ there is a number $M_b > 0$ such that $e \;\leq\; M_b \, T(b)$,

then T maps $\mathcal{P}_e$ into its interior.
If in addition T is a compact operator on B_e, then Theorem 5 applies on B_e to give $\lambda_1 \in \mathbb{R}^+$ and $\phi \in \mathcal{P}_e$.

Proof Parts (a) and (b) are Proposition 19.9 of [2].

We prove (c). We claim that the interior of $\mathcal{P}_e$ is $\{b \in B_e : b \geq te$ for some $t > 0\}$. To prove this, first note that $b \in B_e$ is in the interior of $\mathcal{P}_e$ if and only if

$$b + t[-e, e] \;\subset\; \mathcal{P}_e \quad \text{ for some } \; t > 0,$$

which is true if and only if

$$b \pm te \;\in\; \mathcal{P}_e \quad \text{ for some } \; t > 0,$$

which is true if and only if

$$t'e \;\geq\; b \pm te \;\geq\; 0 \quad \text{ for some } \; t, t' > 0.$$

This gives four inequalities that must hold. All follow automatically from $b \in B_e$ except the inequality $b \geq te$. This proves the claim. By (iii), $T : B \to B_e$, and so, by the claim and (iv), we are done. $\square$

We now turn our attention to powers of operators.

Definition 7 (operator norms) *The norm $\| \, \|_{\mathcal{L}(B)}$ on operators on B that is induced by $\| \, \|$ is defined by*

$$\|T\|_{\mathcal{L}(B)} \;=\; \sup_{b \neq 0} \left(\frac{\|Tb\|}{\|b\|} \right).$$

Lemma 6 *Suppose T is a linear operator $T : B \to B$. Suppose $\lambda_1 > 0$ is the only maximum magnitude eigenvalue λ_1 of T on B^c and has multiplicity one. Also suppose λ_1 is isolated from the rest of the spectrum of T on B^c; that is, there is some λ_2 such that*

$$\text{if } \lambda \in (\text{spectrum}(T) \ominus \{\lambda_1\}), \quad \text{then } |\lambda| \ \leq \ |\lambda_2| \ < \ |\lambda_1|.$$

Then there is a rank one operator $Q : B \to B$ such that

$$\|T^m - \lambda_1^{m-1} Q\|_{\mathcal{L}(B)} \ \leq \ (|\lambda_2| + \varepsilon)^m \tag{6.1}$$

for all $\varepsilon > 0$ and large enough $m \ = \ m(\varepsilon)$.

Proof First use the Riesz functional calculus (see [1] Proposition 4.11) with one contour around λ_1 and another around the remaining spectrum of T on B^c to write T on B^c as $T = Q + E$ where Q and E act on B^c, $QE = 0 = EQ$, spectrum$(Q) \ = \ \{\lambda_1, 0\}$, and spectrum$(E) \ = \ $spectrum$(T) \ominus \{\lambda_1\}$. Now $T^m \ = \ Q^m + E^m$, so

$$\|T^m - Q^m\|_{\mathcal{L}(B^c)} \ = \ \|E^m\|_{\mathcal{L}(B^c)}.$$

Since the spectral radius $\rho(A)$ of any continuous operator satisfies

$$\rho(A) \ = \ \lim_{m \to \infty} \left(\|A^m\|_{\mathcal{L}(B^c)}\right)^{1/m}$$

we have for any $\varepsilon > 0$ and large enough $m \ = \ m(\varepsilon)$

$$\|T^m - Q^m\|_{\mathcal{L}(B^c)} \ \leq \ (\rho(E) + \varepsilon)^m. \tag{6.2}$$

Let ϕ be an eigenvector of Q associated with λ_1. Since λ_1 is an eigenvalue of T of multiplicity 1, ϕ spans the range of Q and

$$Qb \ = \ \phi \ell(b) \tag{6.3}$$

where $\ell : B^c \to \mathbb{C}$ is a linear functional satisfying $\ell(\phi) = \lambda_1$. Thus

$$Q^m(b) \ = \ \ell(\phi)^{m-1} \phi \ell(b) \ = \ \lambda_1^{m-1} Q(b). \tag{6.4}$$

Since $T^m : B \to B$ and, by (6.4) and (6.2),

$$\|T^m / \lambda_1^{m-1} - Q\|_{\mathcal{L}(B^c)} \ = \ \frac{\|T^m - Q^m\|_{\mathcal{L}(B^c)}}{\lambda_1^{m-1}} \ \to \ 0,$$

it follows that $Q : B \to B$. Thus we may take $\| \ \|_{\mathcal{L}(B)}$ rather than $\| \ \|_{\mathcal{L}(B^c)}$ in (6.2). ☐

Next we look at the *adjoint* T' of T. The dual Banach space of B will be denoted B'. Define the dual cone $\mathcal{P}'$ to $\mathcal{P}$ by

$$\mathcal{P}' \ = \ \{\ell \in B' : \ell(b) \geq 0 \text{ for all } b \in \mathcal{P}\}.$$

We note some properties of the adjoint.

- If $T(\mathcal{P}) \subset \mathcal{P}$ then $T'(\mathcal{P}') \subset \mathcal{P}'$, since if $\ell \in \mathcal{P}'$, then

$$[T'\ell](b) \;=\; \ell(Tb) \;\geq\; 0.$$

- The adjoint of a compact operator is compact and, if the sequence of operators $T^m/\lambda_1^{m-1} : B \to B$ converges in $\| \; \|_{\mathcal{L}(B)}$ to an operator Q, then the sequence $(T')^m/\lambda_1^{m-1} : B' \to B'$ converges in $\| \; \|_{\mathcal{L}(B')}$ to Q'.

- spectrum(T' on $B^{c'}$) equals spectrum(T on B^c).

These facts allow us to apply Theorem 5 and Lemma 6 to T' and obtain, like before,

$$\|(T')^m - \lambda_1^{m-1} Q'\|_{\mathcal{L}(B')} \;\leq\; (|\lambda_2| + \varepsilon)^n. \tag{6.5}$$

As before Q' has the form

$$Q'(\eta) \;=\; L\alpha(\eta) \quad \text{for all } \eta \in B'.$$

Here $L \in \mathcal{P}$ and $\alpha \in (B')'$. The definition of adjoint and (6.3) imply

$$L(b)\alpha(\eta) \;=\; Q'(\eta)(b) \;=\; \eta(Q(b)) \;=\; \eta(\phi)\ell(b) \tag{6.6}$$

for all η and b. Thus $L = \ell$ and $\alpha(\eta) = \eta(\phi)$ for all $\eta \in B'$. Thus we have proved

Lemma 7 *In the notation of Lemma 6, there exist $\phi \in \mathcal{P}$ and $L \in \mathcal{P}'$ such that*

$$Q(b) \;=\; \phi L(b), \quad T\phi \;=\; \lambda_1 \phi \quad \text{and} \quad T'(L) \;=\; \lambda_1 L.$$

7 Asymptotics for Integral Operators

We now take $K(x,y)$ of Section 5 to be the kernel of an integral operator acting on a space of measures as follows. The space $C[0,1]$ of continuous functions on $[0,1]$ with norm given by $\|g\|_\infty = \sup\limits_{0 \leq x \leq 1} |g(x)|$ is a real Banach space whose dual $\mathcal{M}$ is the Banach space of finite total mass Borel measures with $\|\nu\|_{\mathcal{M}} = \int_0^1 d|\nu|$, the total mass of ν. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T(\nu_y) \;=\; \left(\int K(x,y) d\nu_y \right) dx \tag{7.1}$$

on any ν in $\mathcal{M}$. If K is continuous on the unit square, $T(\nu)$ is a continuous function times dx. Moreover,

$$\{T(\nu) : \|\nu\|_{\mathcal{M}} \leq 1\}$$

is an equicontinuous family of continuous functions times dx.[2] By the Arzela-Ascoli Theorem every sequence has a $\| \|_\infty$ convergent subsequence, which implies it is convergent in $\| \|_{\mathcal{M}}$. Thus T is a compact operator on $\mathcal{M}$ and it maps into $C[0,1]\ dx$.

Let $\mathcal{P}$ denote the cone of positive measures in $\mathcal{M}$. Let $\tilde{e} \in C[0,1]$ be given by Lemma 4 and denote the measure $\tilde{e}(x)dx$ by $e \in \mathcal{P}$. Construct $\mathcal{P}_e$ and $\| \|_e$ as in Definition 6. Note that measures in $\mathcal{P}_e$ all have the form of continuous functions times Lesbeque measure. Semi-monotonicity of $\| \|_{\mathcal{M}}$ follows because $0 \le \nu^1 \le \nu^2$ implies $\|\nu^1\|_{\mathcal{M}} \le \|\nu^2\|_{\mathcal{M}}$. Thus the conclusions of Lemma 5(a,b) are true. Moreover, observe that the estimates (5.1) and (5.2) imply the hypothesis of Lemma 5(c), provided that we can show that $T : \mathcal{M}_e \to \mathcal{M}_e$ is a compact operator. We do that next.

Lemma 8 *View T of (7.1) as an operator on $\mathcal{M}_e$. Then $T : \mathcal{M}_e \to \mathcal{M}_e$ is a compact operator.*

Proof First we show

$$\|g\ dx\|_{\mathcal{M}_e} \;=\; \sup_x |g(x)/\tilde{e}(x)| \;=\; \|g/\tilde{e}\|_\infty . \tag{7.2}$$

In proving it we must verify this for all g in $C[0,1]$ which vanish faster than $\tilde{e}(x)$ at $x=0$, since we can complete this to obtain $\mathcal{M}_e$. Such $g\,dx$ is in the unit ball of $\mathcal{M}_e$ if and only if $|g(x)| \le \tilde{e}(x)$, which holds if and only if $\frac{|g(x)|}{|\tilde{e}(x)|} \le 1$. Since (7.2) is linear in g, we may rescale to prove the formula.

Define

$$\check{K}(x,y) \;=\; \tilde{e}(x)^{-1}K(x,y),$$

which by Lemma 4(d) is continuous on the closed square. The integral operator

$$\check{T}(\nu) \;=\; \int \check{K}(x,y)d\nu_y$$

maps the unit ball of $\mathcal{M}$ to a precompact set of $C[0,1]$ in $\| \|_\infty$, using the same type of estimate as in the previous footnote. If ν_n is a sequence in the unit ball of $\mathcal{M}$, then set $h_n\,dx \;=\; T(\nu_n)$ and observe

$$\|h_n\ dx - h_k\ dx\|_{\mathcal{M}_e} \;=\; \left\|\frac{h_n}{e} - \frac{h_k}{e}\right\|_\infty \;=\; \left\|\check{T}(\nu_n) - \check{T}(\nu_k)\right\|_\infty . \tag{7.3}$$

[2]Equicontinuity follows from

$$\begin{aligned} \left|T(\nu)(x_1) - T(\nu)(x_2)\right| &\le \int \left|K(x_1,y) - K(x_2,y)\right|\ d|\nu_y| \\ &\le \sup_y \left|K(x_1,y) - K(x_2,y)\right| \int d|\nu_y| \;\le\; \sup_y \left|K(x_1,y) - K(x_2,y)\right|, \end{aligned}$$

which for any $\varepsilon > 0$ is less than ε, provided $|x_1 - x_2| < \delta_\varepsilon$, by Lemma 4(a).

Precompactness of the range of $\check{T}$ forces a subsequence of $\check{T}(\nu_n)$ to be a Cauchy sequence, and the estimate (7.3) forces $T(\nu_n)$ to be Cauchy in $\| \, \|_{\mathcal{M}_e}$.

To this point we have $T : \mathcal{M} \to \mathcal{M}_e$ is a compact operator. Now use $\|b\|_{\mathcal{M}_e} \geq \|b\|_{\mathcal{M}}$ to see that the unit ball of $\mathcal{M}$ is contained in the unit ball of $C[0,1]$. We are done. $\square$

We conclude from all of this the first part of

Lemma 9 *The point in* spectrum(T *on* $\mathcal{M}_e$) *having largest absolute value is* λ_1 *and the remaining points in the spectrum of* T *on the complexification of* $\mathcal{M}_e$ *have absolute value at most* $|\lambda_2|$. *There is a unique (up to scalar multiple) eigenfunction,* ϕdx *in* $\mathcal{P}_e$, *of* T. *Moreover,*

$$\text{spectrum}(T \text{ on } \mathcal{M}^c) \;=\; \text{spectrum}(T \text{ on } \mathcal{M}_e^c).$$

Proof It remains to prove the last assertion of the lemma. An eigenvector ν of T on $\mathcal{M}$ has the form $g(x)\ dx$ where $g \in C[0,1]$ and $|g(x)| \leq \tau\tilde{e}(x)$ for some τ. This is true because T maps $\mathcal{M}^c$ to $\mathcal{M}_e^c$. Compactness of T implies that its spectrum, except possibly 0, consists solely of eigenvalues. $\square$

This lemma permits us to apply Lemma 6 to powers of T on $\mathcal{M}$, in order to obtain the operator $Q : \mathcal{M} \to \mathcal{M}$.

To characterize Q more precisely we consider the adjoint T' of T which is defined on $\mathcal{M}'$, a rather unpleasant space. Fortunately, $C[0,1] \subset \mathcal{M}'$ through the isometry which takes $g \in C[0,1]$ to the functional on measures $\int g\, d\nu$. This reflects the duality $C[0,1]' \;=\; \mathcal{M}$. For $g \in C[0,1]$,

$$T'(g) \;=\; \int g(x)K(x,y)dx \;\in\; C[0,1].$$

The estimate (6.5) implies convergence

$$\left(\frac{T'(g)}{\lambda_1}\right)^m \;\to\; \frac{Q'(g)}{\lambda_1}$$

of continuous functions in $\| \, \|_\infty$ to $Q'(g)$; since $\| \, \|_{\mathcal{M}'}$ on $C[0,1]$ equals $\| \, \|_\infty$. Thus $Q'(g) \in C[0,1]$. Moreover, the integral operator form of T' implies

$$Q'(g) \;=\; \psi(x)\tilde{\ell}(g) \;\text{ for some }\; \psi \in C[0,1] \text{ and linear functional } \tilde{\ell} : C[0,1] \to R.$$

Now use Lemma 7 to obtain the precise structure of Q:

Lemma 10 *We have*

$$Q(\nu) \;=\; \int \varphi(x)\psi(y)d\nu_y. \tag{7.4}$$

where

- λ_1 *is the unique eigenvalue of* T *(resp.* T'*) of maximum modulus and has multiplicity 1,*

- *φ is an eigenfunction of T corresponding to λ_1 and satisfying*

$$0 \ < \ \tau \ \tilde{e}(x) \ \leq \ \varphi(x) \ \leq \ M \ \tilde{e}(x)$$

- *ψ is an eigenfunction of T' corresponding to λ_1 and satisfying*

$$0 \ < \ \tau' \ \tilde{e}'(x) \ \leq \ \psi(x) \ \leq \ M' \ \tilde{e}'(x)$$

where $\tilde{e}'(x) \ = f(x,*,d^k)$ is positive except at $x=0$.

Note $\int \varphi(x)\psi(x)dx = \lambda_1$.

The proof of the last estimate on ψ follows exactly the same track as the one already completed in detail for φ.

8 Proof of Theorem 4

Note that powers of the operator T on $\mathcal{M}$ are the integral operators

$$T^{m+1}(\nu) \ = \ \left(\int\int K(x,s)K_m(x,y)ds \ d\nu_y\right)dx \ = \ \left(\int K_{m+1}(x,y)d\nu_y\right)dx.$$

We now go from earlier conclusions about iterates of integral operators to strong conclusions about the kernels K_m. This can be done because we went to the trouble of having our integral operators act on the (very big) space of measures. Our first goal is to show that (2.1) holds for $\alpha_n = \mu^n$. That is, we want

$$\lim_{m\to\infty}\left(\sup_{(x,y)}\left|\frac{K_m(x,y)}{\int\int K_m(x,y)dx\ dy} \ - \ \frac{\int K_m(x,y)dy}{\int\int K_m(x,y)dx\ dy}\frac{\int K_m(x,y)dx}{\int\int K_m(x,y)dx\ dy}\right|\right) \ = \ 0,$$

where all integrals are over $[0,1]$.

The delta function $\nu = \delta_{y_0}$ is a measure in $\mathcal{M}$ and the bound on $T^m(\delta_{y_0}) - \lambda_1^{m-1}Q(\delta_{y_0})$ in (6.1) is

$$\left|K_m(x,y_0) - \lambda_1^{m-1}\int \varphi(x)\psi(y)\delta_{y_0}(y)dy\right| \leq (|\lambda_2|+\varepsilon)^m \int \delta_{y_0}dy$$

for all x, y_0 in $[0,1]$, that is

$$\left|K_m(x,y_0) - \lambda_1^{m-1}\varphi(x)\psi(y_0)\right| \leq (|\lambda_2|+\varepsilon)^m. \tag{8.1}$$

Similarly, take $\nu = dy$

$$\left|\int K_m(x,y)dy - \lambda_1^{m-1}\varphi(x)\int \psi(y)dy\right| \leq (|\lambda_2|+\varepsilon)^m \tag{8.2}$$

and apply $(T')^m$ to 1 to get

$$\left|\int K_m(x,y)dx - \lambda_1^{m-1}\psi(y)\int \varphi(x)dx\right| \le (|\lambda_2|+\varepsilon)^m. \tag{8.3}$$

Integrate (8.1) on y to obtain

$$\left|\int\int K_m(x,y)dx\ dy - \lambda_1^{m-1}\int \psi(y)dy\ \int \varphi(x)dx\right| \ = \ (|\lambda_2|+\varepsilon)^m.$$

Thus

$$\int\int K_m(x,y)dx\ dy \ \sim \ \lambda_1^{m-1}\int \psi(y)dy\ \int \varphi(x)dx.$$

Use this and the estimates (8.1), (8.2) and (8.3), respectively, to get

$$\begin{aligned}
\frac{K_m(x,y)}{\int\int K_m(x,y)\,dx\ dy} \ &= \ \frac{\varphi(x)\psi(y)}{\int\psi(y)dy\ \int\varphi(x)dx} + o(1),\\
\frac{\int K_m(x,y)dy}{\int\int K_m(x,y)\,dx\ dy} \ &= \ \frac{\varphi(x)}{\int\varphi(x)dx} + o(1), \qquad (8.4)\\
\frac{\int K_m(x,y)dx}{\int\int K_m(x,y)\,dx\ dy} \ &= \ \frac{\psi(y)}{\int\psi(y)dy} + o(1). \qquad (8.5)
\end{aligned}$$

Thus

$$\frac{K_m(x,y)}{\int\int K_m(x,y)dx\ dy} \ = \ \frac{\int K_m(x,y)dy}{\int\int K_m(x,y)dx\ dy}\frac{\int K_m(x,y)dx}{\int\int K_m(x,y)dx\ dy} + o(1),$$

which proves asymptotic independence. Stability is (8.4) and (8.5). This completes the proof of the first part of Theorem 4.

Equation (2.7) and the remark after the theorem follow from (8.1), provided we obtain formulas for C, λ and ϕ. (There is no C in (8.1), but it is needed now because we normalize ϕ to have $\int_0^1 \phi = 1$ and we incorporate a factor of λ in C.)

One can interpret finding the eigenfunction ϕ in terms of the inverse of the integral operator. Equivalently, one can use basic calculus and arrive at the same destination: a simple differential equation with boundary conditions. We begin by obtaining the formulas for λ and ϕ and then for C.

The integral equation for $\phi(x)$ is

$$\lambda\,\phi(x) \ = \ \int_0^1 f(x,y,\mu)\,\phi(y)\,dy, \tag{8.6}$$

where $\lambda > 0$ is as large as possible.

For a functional analysis approach, observe that the two operators

$$N_d g(x) \ = \ \int_0^x g(y)\,dy \qquad \text{and} \qquad N_u g(x) \ = \ \int_x^1 g(y)\,dy$$

correspond to prepending d and u to α when $g(y) = f(y, *, \alpha)$. To study the behavior of $f(x, *, ab\dots z)$ one studies the eigenfunctions of $T = N_a N_b \cdots N_z$. Equivalently, we can work with T^{-1} and thus deal with eigenfunctions of the differential operators N_d^{-1} and N_u^{-1}, which are d/dx and $-d/dx$, respectively, with boundary conditions at 0 and 1, respectively. Using this approach, it can be shown that (8.6) becomes

$$\lambda \frac{d^{|\mu|}\phi}{dx^{|\mu|}}(x) \; = \; (-1)^{U(\mu)}\phi(x) \tag{8.7}$$

with boundary conditions

$$\frac{d^k\phi}{dx^k} \; = \; 0 \quad \text{at} \quad \begin{cases} x = 0, & \text{if } \; m_{k+1} = d, \\ x = 1, & \text{if } \; m_{k+1} = u, \end{cases} \tag{8.8}$$

for $0 \le k < |\mu|$; however, we will derive it using elementary calculus.

Observe that

$$\frac{\partial f(x, y, d\alpha)}{\partial x} \; = \; f(x, y, \alpha) \qquad \text{and} \qquad \frac{\partial f(x, y, u\alpha)}{\partial x} \; = \; -f(x, y, \alpha). \tag{8.9}$$

Using this and differentiating (8.6), we have

$$\lambda \frac{d\phi(x)}{dx} \; = \; (-1)^{U(m_1)} \int_0^1 f(x, y, m_2 \dots m_{|\mu|})\, \phi(y)\, dy,$$

where $\mu = m_1 \dots m_{|\mu|}$ and $U(\alpha)$ is the number of u's in α. Differentiating (8.6) $k < |\mu|$ times, we obtain

$$\lambda \frac{d^k\phi(x)}{dx^k} \; = \; (-1)^{U(m_1 \dots m_k)} \int_0^1 f(x, y, m_{k+1} \dots m_{|\mu|})\, \phi(y)\, dy.$$

Since $f(0, y, d\alpha) = f(1, y, u\alpha) = 0$, we have the boundary conditions (8.8). A final differentiation to obtain $d^{|\mu|}\phi/dx^{|\mu|}$ gives us (8.7).

We now solve the differential equation. The general solution to (8.7) is

$$\phi(x) \; = \; \sum_{t=0}^{|\mu|-1} D_t \exp(r\omega^t x), \quad \text{where} \quad \omega \; = \; e^{2\pi i/|\mu|} \quad \text{and} \quad r^{|\mu|} \; = \; \frac{(-1)^{U(\mu)}}{\lambda}. \tag{8.10}$$

Since $\lambda \in \mathbb{R}^+$, we may assume $\arg r = 0$ if $U(\mu)$ is even and $\arg r = \pi/|\mu|$ if $U(\mu)$ is odd. Since λ is to be as large as possible, $|r|$ is to be as small as possible. Substituting (8.10) into (8.8) and dividing out by r^k gives us

$$0 \; = \; \begin{cases} \displaystyle\sum_{t=0}^{|\mu|-1} \omega^{tk} D_t, & \text{if } \; m_{k+1} \; = \; d, \\ \displaystyle\sum_{t=0}^{|\mu|-1} \omega^{tk} D_t \exp(r\omega^t), & \text{if } \; m_{k+1} \; = \; u, \end{cases} \tag{8.11}$$

for $0 \leq k < |\mu|$. Since ϕ is an eigenvector, the value of r must be such that these $|\mu|$ linear equations in $D_1, \ldots, D_{|\mu|}$ are singular. The requirement that the determinant of the coefficients vanish gives a transcendental equation in r. We want the smallest magnitude r and may restrict $\arg r$ as described earlier. Given r, one has linear equations in the D_t, which can be solved up to a scalar multiple. The multiple is determined by the requirement that $\int \phi = 1$.

Now that we can calculate λ and ϕ, there is a simple way to calculate C. By (2.4), $f(*, y, \alpha) = f(y, *, \overline{\alpha}^R)$. Since $f(*, *, \alpha) = f(*, *, \overline{\alpha}^R)$, it follows that $C(\overline{\mu}^R) = C(\mu)$ and $\lambda(\overline{\mu}^R) = \lambda(\mu)$. Thus

$$f(*, *, \mu^{2n}) \;=\; \int_0^1 f(*, x, \mu^n)\, f(x, *, \mu^n)\, dx \;\sim\; C^2 \lambda^{2n} \int_0^1 \phi(x, \overline{\mu}^R)\, \phi(x, \mu)\, dx.$$

Since $f(*, *, \mu^{2n}) \sim C\lambda^{2n}$, we have (2.10).

Finally (2.7) implies the last claim in the theorem.

9 Proofs of Theorems 1 and 2

All mentions of $\mathrm{Run}(\cdots)$ in Theorem 1 are divided by factorials and so can be thought of as functions of the form $f(*, *, \gamma)$ according to Lemma 1. Thus one could apply Theorems 3 and 4 and deduce Theorems 1 and 2 except for two minor complications which we now discuss.

The first complication is the fact that the forms are not quite the same: A direct application would give

$$\prod_{i=1}^{k} \frac{\mathrm{Run}(M_i^{a_i})}{\|M_i^{a_i}\|!} \quad \text{instead of} \quad \frac{\mathrm{Run}(M_1^{a_1} \ldots M_k^{a_k})}{\|M_1^{a_1} \ldots M_k^{a_k}\|!}$$

and

$$C_i \;=\; \lim_{n\to\infty} \frac{\mathrm{Run}(M_i^n L_i M_{i+1}^n)\; \|M_i^n\|!\; \|M_{i+1}^n\|!}{\mathrm{Run}(M_i^n)\; \mathrm{Run}(M_{i+1}^n)\; \|M_i^n L_i M_{i+1}^n\|!}$$

in place of B_i for $0 < i < k$. (For the "end" values, $B_0 = C_0$ and $B_k = C_k$.) This complication is taken care of by writing down the same formula for all the L_i empty and dividing one formula by the other, obtaining for $0 < i < k$,

$$\begin{aligned} B_i \;&=\; \lim_{n\to\infty} \frac{\mathrm{Run}(M_i^n L_i M_{i+1}^n)\; \|M_i^n\|!\; \|M_{i+1}^n\|!}{\mathrm{Run}(M_i^n)\; \mathrm{Run}(M_{i+1}^n)\; \|M_i^n L_i M_{i+1}^n\|!} \times \\ &\quad\; \lim_{n\to\infty} \frac{\mathrm{Run}(M_i^n)\; \mathrm{Run}(M_{i+1}^n)\; \|M_i^n M_{i+1}^n\|!}{\mathrm{Run}(M_i^n M_{i+1}^n)\; \|M_i^n\|!\; \|M_{i+1}^n\|!} \\ &=\; \lim_{n\to\infty} \frac{\mathrm{Run}(M_i^n L_i M_{i+1}^n)\; \|M_i^n M_{i+1}^n\|!}{\mathrm{Run}(M_i^n M_{i+1}^n)\; \|M_i^n L_i M_{i+1}^n\|!}, \end{aligned}$$

as stated in Theorem 1.

The second complication is more tedious to deal with. Let M be a run word with corresponding descent word μ. If $|M|$ is even, then M^k corresponds to μ^k; however, if $|M|$ is odd, then M^{2k} corresponds to $(\mu\overline{\mu})^k$ and M^{2k+1} corresponds to $(\mu\overline{\mu})^k\mu$. As a result, for each M_i of odd length we must consider two cases depending on whether a_i is odd or even. It suffices to consider the case in which M_1 and M_k have odd length and all other M_i have even length since it illustrates all the ideas involved. We must consider four types of descent words:

$$
\begin{array}{lll}
a_1 = 2b_1 & a_k = 2b_k & \delta_1 \;=\; \beta_0(\alpha_1\overline{\alpha_1})^{b_1}\beta_1\alpha_2^{a_2}\ldots\beta_{k-1}(\alpha_k\overline{\alpha_k})^{b_k}\beta_k \\
a_1 = 2b_1 & a_k = 2b_k+1 & \delta_2 \;=\; \beta_0(\alpha_1\overline{\alpha_1})^{b_1}\beta_1\alpha_2^{a_2}\ldots\beta_{k-1}(\alpha_k\overline{\alpha_k})^{b_k}\overline{\overline{\alpha_k}\beta_k} \\
a_1 = 2b_1+1 & a_k = 2b_k & \delta_3 \;=\; \beta_0(\alpha_1\overline{\alpha_1})^{b_1}\overline{\overline{\alpha_1}\beta_1\alpha_2^{a_2}\ldots\beta_{k-1}(\alpha_k\overline{\alpha_k})^{b_k}\beta_k} \\
a_1 = 2b_1+1 & a_k = 2b_k+1 & \delta_4 \;=\; \beta_0(\alpha_1\overline{\alpha_1})^{b_1}\overline{\overline{\alpha_1}\beta_1\alpha_2^{a_2}\ldots\beta_{k-1}(\alpha_k\overline{\alpha_k})^{b_k}}\overline{\alpha_k}\beta_k
\end{array}
$$

To deal with this, we consider each limit separately, replacing β_i with $\overline{\alpha_i}\beta_i$ when a_i and $|M_i|$ are odd. Because of (2.4), the long overlines can be removed from the formulas in Theorem 3. We now show how the resulting four formulas can be reduced to a single formula. It suffices to consider δ_1 and δ_2 since the others are similar. For δ_2, the denominator on the right of (2.2) contains the factor $f(*,*,(\alpha_1\overline{\alpha_1})^{b_1})$ while we want $f(*,*,(\alpha_1\overline{\alpha_1})^{b_1}\alpha_1)$. Also,

$$C_1 \;=\; \lim_{b_1,a_2\to\infty}\frac{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1}\alpha_1\overline{\beta_1\alpha_2^{a_2}})}{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1})\,f(*,*,\overline{\alpha_2^{a_2}})}$$

while we want

$$C_1(\delta_2) \;=\; \lim_{b_1,a_2\to\infty}\frac{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1}\alpha_1\overline{\beta_1\alpha_2^{a_2}})}{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1}\alpha_1)\,f(*,*,\overline{\alpha_2^{a_2}})}$$

and want to know that this is the same value of C_1 as is obtained for δ_1, namely

$$C_1(\delta_1) \;=\; \lim_{b_1,a_2\to\infty}\frac{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1}\beta_1\alpha_2^{a_2})}{f(*,*,(\alpha_1\overline{\alpha_1})^{b_1})\,f(*,*,\alpha_2^{a_2})}.$$

The denominator differences between the left sides of the two versions of (2.2) and between C_1 and $C_1(\delta_2)$ can be adjusted by moving denominator factors from one side to another through the limit because $C_1 \neq 0$ and, as we shall see $C_1(\delta_2) = C_1(\delta_1) \neq 0$. (The nonzero results are due to Theorem 3.) It remains to show that $C_1(\delta_2) = C_1(\delta_1)$. With $\gamma = (\alpha_1\overline{\alpha_1})^{b_1}$ and noting that $\overline{\gamma\alpha_1} = \overline{\alpha_1}\gamma$, we have

$$
\begin{aligned}
C_1(\delta_1) \;&\sim\; \frac{f(*,*,\gamma\beta_1\alpha_2^{a_2})}{f(*,*,\gamma)\,f(*,*,\alpha_2^{a_2})} \\
&=\; \frac{f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\gamma\beta_1\alpha_2^{a_2})}{f(*,*,\gamma)\,f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\alpha_2^{a_2})} \\
&=\; \frac{f(*,*,\gamma)\int_0^1\int_0^1 f(*,s,\overline{\alpha_1})\,f(s,*\mid\gamma)\,f(*,t\mid\gamma)\,f(t,*,\beta_1\alpha_2^{a_2})\,ds\,dt}{f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\alpha_2^{a_2})}
\end{aligned}
$$

$$\begin{aligned}
&\sim \frac{f(*,*,\gamma)\int_0^1\int_0^1 f(*,s,\overline{\alpha_1})\,f(s,t\mid\gamma)\,f(t,*,\beta_1\alpha_2^{a_2})\,ds\,dt}{f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\alpha_2^{a_2})}\\
&= \frac{\int_0^1\int_0^1 f(*,s,\overline{\alpha_1})\,f(s,t,\gamma)\,f(t,*,\beta_1\alpha_2^{a_2})\,ds\,dt}{f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\alpha_2^{a_2})}\\
&= \frac{f(*,*,\overline{\gamma\alpha_1}\beta_1\alpha_2^{a_2})}{f(*,*,\overline{\alpha_1}\gamma)\,f(*,*,\alpha_2^{a_2})} = \frac{f(*,*,\gamma\alpha_1\overline{\beta_1\alpha_2^{a_2}})}{f(*,*,\gamma\alpha_1)\,f(*,*,\overline{\alpha_2^{a_2}})} \sim C_1(\delta_2).
\end{aligned}$$

This completes the proof of Theorem 1.

We now turn our attention to Theorem 2. If $|L|$ is even and corresponds to the descent word μ, the fraction of permutations with run pattern L^n is $f(*,*,\mu^n)$ and so we are done by Theorem 4.

Suppose $|L|$ is odd and L corresponds to descent word α. Let $M = LL$. Then M corresponds to $\mu = \alpha\overline{\alpha}$. Us the previous paragraph and define $C(L) = C(M)$ and $\lambda(L) = \sqrt{\lambda(M)}$. This completes the proof for L^{2n}. We now consider L raised to an odd power. We have

$$\begin{aligned}
f(*,*,(\alpha\overline{\alpha})^{2k+1}) &= \int_0^1 f(*,s,(\alpha\overline{\alpha})^k\alpha)\,f(s,*,\overline{\alpha}(\alpha\overline{\alpha})^k)\,ds\\
&= \int_0^1 f(*,s,\alpha(\overline{\alpha}\alpha)^k)^2\,ds\\
&= \int_0^1 \Big(\int_0^1 f(*,t,\alpha)\,f(t,s,(\overline{\alpha}\alpha)^k)\,dt\Big)^2\,ds\\
&\sim \frac{\int_0^1 \Big(\int_0^1 f(*,t,\alpha)\,f(t,*,(\overline{\alpha}\alpha)^k)\,f(*,s,(\overline{\alpha}\alpha)^k)\,dt\Big)^2\,ds}{f(*,*,(\overline{\alpha}\alpha)^k)^2}\\
&= \frac{f(*,*,\alpha(\overline{\alpha}\alpha)^k)^2\int_0^1 f(*,s,(\overline{\alpha}\alpha)^k)^2\,ds}{f(*,*,(\overline{\alpha}\alpha)^k)^2}\\
&= \frac{f(*,*,\alpha(\overline{\alpha}\alpha)^k)^2 f(*,*,(\overline{\alpha}\alpha)^{2k})}{f(*,*,(\overline{\alpha}\alpha)^k)^2}.
\end{aligned}$$

Using the even-length case, this becomes

$$C(M)\lambda(M)^{2k+1} \sim \frac{f(*,*,\alpha(\overline{\alpha}\alpha)^k)^2\,C(M)\lambda(M)^{2k})}{(C(M)\lambda(M)^k)^2}.$$

Hence $f(*,*,\alpha(\overline{\alpha}\alpha)^k)^2 \sim C(M)^2\lambda(M)^{2k+1}$. Take the square root of both sides.

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