Global defensive alliances in graphs

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Abstract

A *defensive alliance* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that for every vertex $v \in S$, the number of neighbors v has in S plus one (counting v) is at least as large as the number of neighbors it has in $V - S$. Because of such an alliance, the vertices in S, agreeing to mutually support each other, have the strength of numbers to be able to defend themselves from the vertices in $V - S$. A defensive alliance S is called *global* if it effects every vertex in $V - S$, that is, every vertex in $V - S$ is adjacent to at least one member of the alliance S. Note that a global defensive alliance is a dominating set. We study global defensive alliances in graphs.

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1 Introduction

Alliances in graphs were first defined and studied by Hedetniemi, Hedetniemi, and Kristiansen in [4]. In this paper we initiate the study of global defensive alliances (listed as an open problem in [4]), but first we give some terminology and definitions. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. An *endvertex* is a vertex which is only adjacent to one vertex. An endvertex in a tree T is also called a *leaf*, while a *support vertex* of T is a vertex adjacent to a leaf. For a nonempty subset $S \subseteq V$, we denote the subgraph of G induced by S by $\langle S \rangle$. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u\colon uv \in E\}$, while the *closed neighborhood of* v is the set $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* $N[S] = N(S) \cup S$. A set S is a *dominating set* if $N[S] = V$, and is a *total dominating set* or an *open dominating set* if $N(S) = V$. The minimum cardinality of a dominating set (respectively, total dominating set) of G is the *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$). The concept of domination in graphs, with its many variations, is now well studied in graph theory (see [2, 3]). For other graph theory terminology and notation, we follow [1] and [2].

In [4] Hedetniemi, Hedetniemi, and Kristiansen introduced several types of alliances, including defensive alliances that we consider here. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this case, by strength of numbers, we say that every vertex in S is *defended* from possible attack by vertices in $V - S$. A defensive alliance S is called *strong* if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap V - S|$. In this case we say that every vertex in S is *strongly defended*.

In this paper, any reference to an alliance will mean a defensive alliance. Any two vertices u, v in an (strong) alliance S are called *allies* (*with respect to* S); we also say that u and v are *allied*. An (strong) alliance S is called *critical* if no proper subset of S is an (strong) alliance. The *alliance number* $a(G)$ is the minimum cardinality of any critical alliance in G, and the *strong alliance number* $\hat{a}(G)$ is the minimum cardinality of any critical strong alliance in G.

An alliance S is called *global* if it effects every vertex in $V - S$, that is, every vertex in $V - S$ is adjacent to at least one member of the alliance S. In this case, S is a dominating set. The *global alliance number* $\gamma_a(G)$ (respectively, *global strong alliance number* $\gamma_{\hat{a}}(G)$) is the minimum cardinality of an alliance (respectively, strong alliance) of G that is also a dominating set of G. The entire vertex set is a global (strong) alliance for any graph G, so every graph G has a global (strong) alliance number. Note that a global alliance of minimum cardinality is not necessarily a critical alliance, and a critical alliance is not necessarily a dominating set. It is observed in [4] that any critical (strong) alliance S in a graph G must induce a connected subgraph of G. This is obvious, since any component of the induced subgraph $\langle S \rangle$ is a strictly smaller alliance (of the same type). However, for a global alliance this is not necessarily true. For example, the two endvertices of the path P_4 form a global alliance. We refer to a minimum dominating set of G as a $\gamma(G)$-set. Similarly, we call a minimum global alliance (respectively, a minimum global strong alliance) of G a $\gamma_a(G)$-set (respectively, $\gamma_{\hat{a}}(G)$-set).

Many applications of alliances, including national defense, are listed in [4]. Global alliances have similar applications in cases where all the vertices of the graph are involved. In the context of computing networks, a dominating set S represents a set of nodes, each of which has a desired resource, or service capacity, such as a large database, and each node which does not have this resource, or desires this service, can gain access to it by accessing a node at distance at most one from it. However, if all of the nodes in $V - S$ which are adjacent to a particular node $v \in S$ desire simultaneous access to the resource at v, then node v alone may not be able to provide such access. But if S is a global alliance, then the neighbors of v in S would be sufficient in number to satisfy (within distance two) the simultaneous demands of the neighbors of v in $V - S$.

Since every global strong alliance is a global alliance, and every global alliance is both an alliance and dominating, our first observation is immediate.

Observation 1 *For any graph G,*
(i) $1 \leq \gamma(G) \leq \gamma_a(G) \leq \gamma_{\hat{a}}(G) \leq n$,
(ii) $1 \leq a(G) \leq \gamma_a(G) \leq n$, *and*
(iii) $1 \leq a(G) \leq \hat{a}(G) \leq \gamma_{\hat{a}}(G) \leq n$.

2 Examples

We first give the global alliance and global strong alliance numbers for complete graphs and complete bipartite graphs.

Proposition 2 *For the complete graph K_n,*
(i) $\gamma_a(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$, *and*
(ii) $\gamma_{\hat{a}}(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let S be a $\gamma_a(K_n)$-set and let $v \in S$. Then S contains at least $\lfloor (\deg v)/2 \rfloor = \lfloor (n-1)/2 \rfloor$ neighbors of v, and so $\gamma_a(K_n) \geq \lfloor (n+1)/2 \rfloor$. The set consisting of v and $\lfloor (n-1)/2 \rfloor$ of its neighbors is a global alliance, and so $\gamma_a(K_n) \leq \lfloor (n+1)/2 \rfloor$. This establishes (i).

Let D be a $\gamma_{\hat{a}}(K_n)$-set and let $v \in D$. Then D contains at least $\lceil (\deg v)/2 \rceil = \lceil (n-1)/2 \rceil$ neighbors of v, and so $\gamma_{\hat{a}}(K_n) \geq \lceil (n+1)/2 \rceil$. The set consisting of v and $\lceil (n-1)/2 \rceil$ of its neighbors is a global strong alliance, and so $\gamma_{\hat{a}}(K_n) \leq \lceil (n+1)/2 \rceil$. This establishes (ii). $\Box$

Proposition 3 *For the complete bipartite graph $K_{r,s}$,*
(i) $\gamma_a(K_{1,s}) = \left\lfloor \frac{s}{2} \right\rfloor + 1$,
(ii) $\gamma_a(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor$ *if $r, s \geq 2$, and*
(iii) $\gamma_{\hat{a}}(K_{r,s}) = \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil$.

Proof. We first establish (i). The result is immediate when $s = 1$. Suppose $s \geq 2$ and S is a $\gamma_a(K_{1,s})$-set. Since S is a dominating set, the central vertex, v say, belongs to S and therefore, S contains at least $\lfloor (\deg v)/2 \rfloor = \lfloor s/2 \rfloor$ neighbors of v. Hence, $\gamma_a(K_{1,s}) \geq \lfloor s/2 \rfloor + 1$. The set consisting of v and $\lfloor s/2 \rfloor$ of its neighbors is a global alliance, and so $\gamma_a(K_{1,s}) \leq \lfloor s/2 \rfloor + 1$. This establishes (i).

It is given in [4] that $a(K_{r,s}) = \lfloor r/2 \rfloor + \lfloor s/2 \rfloor$ and $\hat{a}(K_{r,s}) = \lceil r/2 \rceil + \lceil s/2 \rceil$. Thus by Observation 1, we have $\gamma_a(K_{r,s}) \geq \lfloor r/2 \rfloor + \lfloor s/2 \rfloor$ and $\gamma_{\hat{a}}(K_{r,s}) \geq \lceil r/2 \rceil + \lceil s/2 \rceil$.

The set consisting of $\lfloor r/2 \rfloor$ vertices in the one partite set and $\lfloor s/2 \rfloor$ vertices in the other partite set is a global alliance, and so $\gamma_a(K_{r,s}) \leq \lfloor r/2 \rfloor + \lfloor s/2 \rfloor$. This establishes (ii). Similarly, the set consisting of $\lceil r/2 \rceil$ vertices in the one partite set and $\lceil s/2 \rceil$ vertices in the other partite set is a global strong alliance establishing (iii). $\Box$

We show that the global alliance and total domination numbers are the same for graphs with minimum degree at least two and maximum degree at most three.

Lemma 4 *For any graph G with $\delta(G) \geq 2$, $\gamma_t(G) \leq \gamma_a(G)$. Furthermore, if $\Delta(G) \leq 3$, then $\gamma_t(G) = \gamma_a(G)$.*

Proof. For any $\gamma_a(G)$-set S and vertex $v \in S$, S contains at least $\lfloor (\deg v)/2 \rfloor \geq 1$ neighbors of v, and so S is a total dominating set. Thus, $\gamma_t(G) \leq \gamma_a(G)$. Furthermore, if $\Delta(G) \leq 3$, then for any $\gamma_t(G)$-set D and vertex $u \in D$, $|N[u] \cap D| \geq 2 \geq |N(u) \cap (V - D)|$. Hence, D is a global alliance, and so $\gamma_a(G) \leq \gamma_t(G)$. $\Box$

As a special case of Lemma 4, if G is a cubic graph, then $\gamma_t(G) = \gamma_a(G)$. Since every total dominating set of a cycle is also a global strong alliance, we also have the following immediate consequence of Lemma 4.

Proposition 5 *For cycles C_n, $n \geq 3$, $\gamma_a(C_n) = \gamma_{\hat{a}}(C_n) = \gamma_t(C_n)$.*

The minimum degree condition is necessary for Lemma 4 to hold. In fact, there exist connected graphs G for which the difference $\gamma_t(G) - \gamma_a(G)$ can be arbitrarily large. For $2 \leq s \leq k - 1$ and $k \geq 3$, consider the graph G obtained by attaching (with an edge) s disjoint copies of P_3 to each vertex of a complete graph K_k. For $k = 3$ and $s = 2$, the graph G is shown in Figure 2. Since a support vertex must be in every $\gamma_t(G)$-set, it follows that at least two vertices from each attached copy of P_3 must be in every $\gamma_t(G)$-set. Moreover, the set of support vertices of G along with their neighbors of degree two totally dominate G. Hence, $\gamma_t(G) = 2sk$. But since $s \leq k - 1$, the set of endvertices together with the vertices of K_k form a global alliance of G of minimum cardinality, and so $\gamma_a(G) = (s + 1)k$.

We show next that for any graph without isolated vertices, the total domination number is bounded above by the global strong alliance number.

Lemma 6 *For any graph G with no isolated vertices, $\gamma_t(G) \leq \gamma_{\hat{a}}(G)$.*

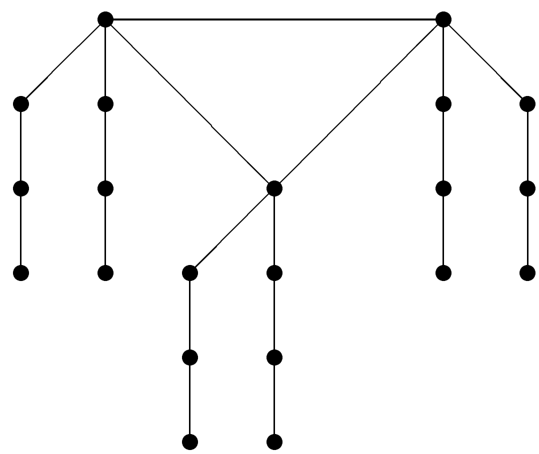

Figure 1: A graph G with $\gamma(G) = 7$, $\gamma_a(G) = 9$, and $\gamma_{\hat{a}} = \gamma_t(G) = 12$.

Proof. For any $\gamma_{\hat{a}}(G)$-set S and vertex $v \in S$, S contains at least $\lceil (\deg v)/2 \rceil \geq 1$ neighbors of v, and so S is a total dominating set. Thus, $\gamma_t(G) \leq \gamma_{\hat{a}}(G)$. □

The total domination number of paths P_n and cycles C_n is well known: For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. For paths, we show that the global strong alliance number equals the total domination number. However, the global alliance number of a path is not necessarily equal to its total domination number.

Proposition 7 *For $n \geq 3$, $\gamma_{\hat{a}}(P_n) = \gamma_t(P_n)$.*

Proof. By Lemma 6, $\gamma_t(P_n) \leq \gamma_{\hat{a}}(P_n)$. For any $\gamma_t(P_n)$-set D and vertex $u \in D$, $|N[u] \cap D| \geq 2 > |N(u) \cap (V - D)|$. Hence, D is a global strong alliance, and so $\gamma_{\hat{a}}(P_n) \leq \gamma_t(P_n)$. □

Proposition 8 *For $n \geq 2$, $\gamma_a(P_n) = \gamma_t(P_n)$ unless $n \equiv 2 \pmod 4$, in which case $\gamma_a(P_n) = \gamma_t(P_n) - 1$.*

Proof. Let $T = P_n$. Since $\Delta(T) \leq 2$, every total dominating set of T is also a global alliance of T, and so $\gamma_a(T) \leq \gamma_t(T)$. Suppose $n \equiv 2 \pmod 4$. If v denotes an endvertex of T, then either $n = 2$, in which case $\gamma_a(T) = 1 = \gamma_t(T) - 1$, or $n \geq 6$, in which case $\gamma_a(T) \leq |\{v\}| + \gamma_t(T - N[v]) = 1 + \gamma_t(P_{n-2}) = n/2 = \gamma_t(T) - 1$. Hence, $\gamma_a(T) \leq \gamma_t(T)$ and if $n \equiv 2 \pmod 4$, then $\gamma_a(T) \leq \gamma_t(T) - 1$.

On the other hand, let A be a $\gamma_a(T)$-set. Then A is a dominating set of T. If the subgraph $\langle A \rangle$ induced by A contains an isolated vertex, then this vertex must be an endvertex of T. Hence, $\langle A \rangle$ contains a most two isolated vertices. If $\langle A \rangle$ contains no isolated vertex, then A is a total dominating set, and so $\gamma_t(T) \leq |A|$. If $\langle A \rangle$ contains one isolated vertex v, then $A - \{v\}$ is a total dominating set of $T - N[v] = P_{n-2}$, and so $\gamma_t(P_{n-2}) \leq |A| - 1$. If now $n \not\equiv 2 \pmod 4$, then $\gamma_t(T) = \gamma_t(P_{n-2}) + 1 \leq |A|$, while if $n \equiv 2 \pmod 4$, then $\gamma_t(T) = \gamma_t(P_{n-2}) + 2 \leq |A| + 1$. If $\langle A \rangle$ contains two isolated vertices u and v, then either $T = P_4$, in which case $\gamma_t(T) = 2 = |A|$, or $|A| \geq 4$, in which case $A - \{u, v\}$ is a total dominating set of $T - N[u] - N[v] = P_{n-4}$. Therefore, $\gamma_t(P_{n-4}) \le |A| - 2$, and so $\gamma_t(T) = \gamma_t(P_{n-4}) + 2 \le |A|$. Since $|A| = \gamma_a(T)$, we have shown that $\gamma_a(T) \ge \gamma_t(T)$ unless $n \equiv 2 \pmod 4$, in which case $\gamma_a(T) \ge \gamma_t(T) - 1$. The desired result follows. □

A *double star* is a tree that contains exactly two vertices that are not endvertices. If one of these vertices is adjacent to r leaves and the other to s leaves, then we denote this double star by $S(r, s)$.

Proposition 9 *For $r, s \ge 1$, $\gamma_a(S_{r,s}) = \lfloor (r-1)/2 \rfloor + \lfloor (s-1)/2 \rfloor + 2$.*

Proof. Let u and v be the two central vertices of $S_{r,s}$, where u is adjacent to r leaves. Let S be a $\gamma_a(S_{r,s})$-set. Since S is a dominating set, if u (respectively, v) is not in S, then all the leaves adjacent to u (respectively, v) are in S. Hence we may assume $\{u, v\} \subseteq S$. Then S contains at least $\lfloor (r-1)/2 \rfloor$ leaves adjacent to u, and at least $\lfloor (s-1)/2 \rfloor$ leaves adjacent to v. Hence, $\gamma_a(S_{r,s}) \ge \lfloor (r-1)/2 \rfloor + \lfloor (s-1)/2 \rfloor + 2$. The set consisting of u, v, $\lfloor (r-1)/2 \rfloor$ leaves adjacent to u, and $\lfloor (s-1)/2 \rfloor$ leaves adjacent to v is a global alliance, and so $\gamma_a(S_{r,s}) \le \lfloor (r-1)/2 \rfloor + \lfloor (s-1)/2 \rfloor + 2$. The desired result follows. □

Using a similar proof to the one for Proposition 9, we obtain the global strong alliance number of a double star.

Proposition 10 *For $r, s \ge 1$, $\gamma_{\hat{a}}(S_{r,s}) = \lfloor r/2 \rfloor + \lfloor s/2 \rfloor + 2$.*

3 Lower Bounds

Our aim in this section is to give lower bounds on the global alliance and global strong alliance numbers of a graph in terms of its order.

3.1 General Graphs

Theorem 11 *If G is a graph of order n, then*

$$\gamma_a(G) \ge (\sqrt{4n+1} - 1)/2,$$

and this bound is sharp.

Proof. Let $\gamma_a(G) = k$. For any $\gamma_a(G)$-set S and vertex $v \in S$, S contains at least $\lfloor (\deg v)/2 \rfloor$ neighbors of v. Hence, $k = |S| \ge |\{v\}| + \lfloor (\deg v)/2 \rfloor \ge (\deg v + 1)/2$. Thus, $V - S$ contains at most $\lceil (\deg v)/2 \rceil \le (\deg v + 1)/2 \le k$ neighbors of v. Therefore, each vertex in S has at most k neighbors in $V - S$, and so $n - k = |V - S| \le k^2$, or, equivalently, $k^2 + k - n \ge 0$. Hence, $k \ge (\sqrt{4n+1} - 1)/2$.

That this bound is sharp may be seen as follows. Let $F_1 = K_2$ and for $k \ge 2$, let F_k be the graph obtained from the disjoint union of k stars $K_{1,k}$ by adding all edges between the central vertices of the k stars. Then, $G = F_k$ for some $k \ge 1$ has order $n = k(k+1)$, and so $k = (\sqrt{4n+1} - 1)/2$. If $k = 1$, then $\gamma_a(G) = 1 = (\sqrt{4n+1} - 1)/2$. If $k \ge 2$, then the k central vertices of the stars form a global alliance, and so $\gamma_a(G) \leq (\sqrt{4n+1}-1)/2$. Consequently, $\gamma_a(G) = (\sqrt{4n+1}-1)/2$. $\square$

Using an argument similar to that used in the proof of Theorem 11 one can also obtain the following result and corollary.

Proposition 12 *If G is a graph of order n, then*

$$\gamma_a(G) \geq \frac{n}{\lceil \frac{r}{2} \rceil + 1}.$$

Corollary 13 *If G is a cubic graph or a 4-regular graph of order n, then $\gamma_a(G) \geq \frac{n}{3}$.*

Theorem 14 *If G is a graph of order n, then*

$$\gamma_{\hat{a}}(G) \geq \sqrt{n},$$

and this bound is sharp.

Proof. Let $\gamma_{\hat{a}}(G) = k$. For any $\gamma_{\hat{a}}(G)$-set S and vertex $v \in S$, S contains at least $\lceil (\deg v)/2 \rceil$ neighbors of v. Hence, $k = |S| \geq |\{v\}| + \lceil (\deg v)/2 \rceil \geq (\deg v + 2)/2$. Thus $V - S$ contains at most $\lfloor (\deg v)/2 \rfloor \leq (\deg v)/2 \leq k-1$ neighbors of v. Therefore, each vertex in S has at most $k-1$ neighbors in $V-S$, and so $n - k = |V - S| \leq k(k-1)$, or, equivalently, $k \geq \sqrt{n}$.

That this bound is sharp, may be seen as follows. Let $G_1 = K_1$, $G_2 = P_4$, and for $k \geq 3$, let G_k be the graph obtained from the disjoint union of k stars $K_{1,k-1}$ by adding all edges between the central vertices of the k stars. Then, $G = G_k$ for some $k \geq 1$ has order $n = k^2$, and so $k = \sqrt{n}$. If $k = 1$, then $\gamma_{\hat{a}}(G) = 1 = \sqrt{n}$, while if $k = 2$, then $\gamma_{\hat{a}}(G) = 2 = \sqrt{n}$. If $k \geq 3$, then the k central vertices of the stars form a global strong alliance, and so $\gamma_{\hat{a}}(G) \leq \sqrt{n}$. Thus, $\gamma_{\hat{a}}(G) = \sqrt{n}$. $\square$

3.2 Bipartite Graphs

Theorem 15 *If G is a bipartite graph of order n and maximum degree Δ, then*

$$\gamma_a(G) \geq \frac{2n}{\Delta + 3},$$

and this bound is sharp.

Proof. Let $\gamma_a(G) = k$. Let S be a $\gamma_a(G)$-set. Since G is a bipartite graph, so too is the induced subgraph $\langle S \rangle$. Let $\mathcal{L}$ and $\mathcal{R}$ denote the bipartite sets of $\langle S \rangle$. Let Δ_L denote the maximum degree in G of a vertex in $\mathcal{L}$, and let Δ_R denote the maximum degree in G of a vertex in $\mathcal{R}$. We may assume (renaming if necessary) that $\Delta_L \geq \Delta_R$.

Let $u \in \mathcal{L}$ and $v \in \mathcal{R}$. Since S is a global alliance, S contains at least $\lfloor (\deg u)/2 \rfloor$ neighbors of u and at least $\lfloor (\deg v)/2 \rfloor$ neighbors of v. Hence, $V - S$ contains at most $\lceil (\deg u)/2 \rceil \leq \lceil \Delta_L/2 \rceil \leq (\Delta_L + 1)/2$ neighbors of u and at most $\lceil (\deg v)/2 \rceil \leq \lceil \Delta_R/2 \rceil \leq$

$(\Delta_R+1)/2$ neighbors of v. Therefore, each vertex in $\mathcal{L}$ has at most $(\Delta_L+1)/2$ neighbors in $V-S$, while each vertex in $\mathcal{R}$ has at most $(\Delta_R+1)/2$ neighbors in $V-S$. Hence, since $n-k=|V-S|$ and $k=|\mathcal{L}|+|\mathcal{R}|$,

$$\begin{aligned} n-k &\le |\mathcal{L}|\cdot\left(\frac{\Delta_L+1}{2}\right)+|\mathcal{R}|\cdot\left(\frac{\Delta_R+1}{2}\right) \\ &\le \left(\frac{\Delta_L+1}{2}\right)(|\mathcal{L}|+|\mathcal{R}|) \\ &\le \left(\frac{\Delta+1}{2}\right)k, \end{aligned}$$

and so $k \ge 2n/(\Delta+3)$.

That this bound is sharp may be seen as follows. For $k\ge 1$, let H_k be the bipartite graph obtained from the disjoint union of $2k$ stars $K_{1,k+1}$ with centers $\{x_1,x_2,\ldots,x_k,\ y_1,\ y_2,\ \ldots,y_k\}$ by adding all edges of the type x_iy_j, $1\le i\le j\le k$. Then, $G=H_k$ for some $k\ge 1$ has maximum degree $\Delta=2k+1$ and order $n=2k(k+2)$. The $2k$ central vertices of the stars form a global alliance, and so $\gamma_a(G)\le 2k=n/(k+2)=2n/(\Delta+3)$. Consequently, $\gamma_a(G)=2n/(\Delta+3)$. $\square$

Theorem 16 *If G is a bipartite graph of order n and maximum degree Δ, then*

$$\gamma_{\hat{a}}(G)\ge\frac{2n}{\Delta+2},$$

and this bound is sharp.

Proof. Let $\gamma_{\hat{a}}(G)=k$. Let S be a $\gamma_{\hat{a}}(G)$-set. Using the notation employed in the proof of Theorem 15, let $u\in\mathcal{L}$ and $v\in\mathcal{R}$. Since S is a global strong alliance, S contains at least $\lceil(\deg u)/2\rceil$ neighbors of u and at least $\lceil(\deg v)/2\rceil$ neighbors of v. Hence, $V-S$ contains at most $\lfloor(\deg u)/2\rfloor\le\lfloor\Delta_L/2\rfloor\le\Delta_L/2$ neighbors of u and at most $\lfloor(\deg v)/2\rfloor\le\lfloor\Delta_R/2\rfloor\le\Delta_R/2$ neighbors of v. Therefore, each vertex in $\mathcal{L}$ has at most $\Delta_L/2$ neighbors in $V-S$, while each vertex in $\mathcal{R}$ has at most $\Delta_R/2$ neighbors in $V-S$. Hence,

$$n-k\le|\mathcal{L}|\cdot\left(\frac{\Delta_L}{2}\right)+|\mathcal{R}|\cdot\left(\frac{\Delta_R}{2}\right)\le\left(\frac{\Delta_L}{2}\right)(|\mathcal{L}|+|\mathcal{R}|)\le\left(\frac{\Delta}{2}\right)k$$

and so $k\ge 2n/(\Delta+2)$.

That this bound is sharp, may be seen as follows. For $k\ge 1$, let M_k be the bipartite graph obtained from the disjoint union of $2k$ stars $K_{1,k}$ with centers $\{x_1,x_2,\ldots,x_k,\ y_1,\ y_2,\ \ldots,y_k\}$ by adding all edges of the type x_iy_j, $1\le i\le j\le k$. Then, $G=M_k$ for some $k\ge 1$ has maximum degree $\Delta=2k$ and order $n=2k(k+1)$. The $2k$ central vertices of the stars form a global strong alliance, and so $\gamma_{\hat{a}}(G)\le 2k=n/(k+1)=2n/(\Delta+2)$. Hence, $\gamma_{\hat{a}}(G)=2n/(\Delta+2)$. $\square$

3.3 Trees

Theorem 17 *If T is a tree of order n, then*

$$\gamma_a(T) \geq \frac{n+2}{4},$$

and this bound is sharp.

Proof. Let $\gamma_a(G) = k$ and let S be a $\gamma_a(T)$-set. Let $F = \langle S \rangle$. Since F is a forest, $\sum_{v \in S} \deg_F v = 2|E(F)| \leq 2(|V(F)| - 1) = 2(k-1)$. For each $v \in S$, $V - S$ contains at most $\deg_F v + 1$ neighbors of v. Therefore, $n - k = |V - S| \leq \sum_{v \in S}(\deg_F v + 1) \leq 2(k-1) + k = 3k - 2$, and so $k \geq (n+2)/4$.

That this bound is sharp, may be seen as follows. Let T be the tree obtained from a tree F of order k by adding $\deg_F v + 1$ new vertices for each vertex v of F and joining them to v. Then, T has order $n = |V(F)| + \sum_{v \in V(F)}(\deg_F v + 1) = 2k + \sum_{v \in V(F)} \deg_F v = 2k + 2(k-1) = 4k - 2$. Since $V(F)$ is a global alliance of T, $\gamma_a(T) \leq k = (n+2)/4$. Consequently, $\gamma_a(T) = (n+2)/4$. $\square$

Theorem 18 *If T is a tree of order n, then*

$$\gamma_{\hat{a}}(T) \geq \frac{n+2}{3},$$

and this bound is sharp.

Proof. Let $\gamma_{\hat{a}}(G) = k$ and let S be a $\gamma_{\hat{a}}(T)$-set. Let $F = \langle S \rangle$. Then, $\sum_{v \in S} \deg_F v \leq 2(k-1)$. For each $v \in S$, $V - S$ contains at most $\deg_F v$ neighbors of v. Therefore, $n - k = |V - S| \leq \sum_{v \in S} \deg_F v \leq 2(k-1)$, and so $k \geq (n+2)/3$.

That this bound is sharp, may be seen as follows. Let T be the tree obtained from a tree F of order k by adding $\deg_F v$ new vertices for each vertex v of F and joining them to v. Then, T has order $n = |V(F)| + \sum_{v \in V(F)} \deg_F v = k + 2(k-1) = 3k - 2$. Since $V(F)$ is a global strong alliance of T, $\gamma_{\hat{a}}(T) \leq k = (n+2)/3$. Thus, $\gamma_{\hat{a}}(T) = (n+2)/3$. $\square$

4 Upper Bounds

Our aim in this section is to give upper bounds on the global alliance and global strong alliance numbers of a graph in terms of its order.

4.1 General Graphs

Proposition 19 *For any graph G with no isolated vertices and minimum degree δ,*

(i) $\gamma_a(G) \leq n - \lceil \delta/2 \rceil$, *and*

(ii) $\gamma_{\hat{a}}(G) \leq n - \lfloor \delta/2 \rfloor$,

and these bounds are sharp.

Proof. Let v be a vertex of minimum degree, and let S be the set of vertices formed by removing $\lceil \delta/2 \rceil$ neighbors of v from V. Then, S dominates G. For each $u \in S$, $|N(u) \cap (V - S)| \leq \lceil \delta/2 \rceil \leq \lceil (\deg u)/2 \rceil$, and so $|N[u] \cap S| \geq \lfloor (\deg u)/2 \rfloor + 1 \geq |N(u) \cap (V - S)|$. Thus, S is a global alliance, and so $\gamma_a(G) \leq |S|$. This establishes (i). That this bound is sharp follows from Proposition 2 (take $G = K_n$ with n odd).

Let D be the set of vertices formed by removing $\lfloor \delta/2 \rfloor$ neighbors of v from V. Then, D dominates G. For each $u \in D$, $|N(u) \cap (V - D)| \leq \lfloor \delta/2 \rfloor \leq \lfloor (\deg u)/2 \rfloor$, and so $|N[u] \cap D| \geq \lceil (\deg u)/2 \rceil + 1 > |N(u) \cap (V - D)|$. Thus, D is a global strong alliance, and so $\gamma_{\hat{a}}(G) \leq |D|$. This establishes (ii). That this bound is sharp follows from Proposition 2 (take $G = K_n$). □

Corollary 20 *For any graph G, $\gamma_a(G) = n$ if and only if $G = \overline{K}_n$.*

4.2 Trees

In order to establish a sharp upper bound on the global alliance number of a tree and to characterize the trees achieving this bound, we introduce some more notation. For a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. We also introduce a family $\mathcal{T}_1$ of trees as follows: Let $T = P_5$ or $T = K_{1,4}$ or let T be the tree obtained from $tK_{1,4}$ (the disjoint union of t copies of $K_{1,4}$) by adding $t - 1$ edges between leaves of these copies of $K_{1,4}$ in such a way that the center of each $K_{1,4}$ is adjacent to exactly three leaves in T. Let $\mathcal{T}_1$ be the family of all such trees T.

Theorem 21 *If T is a tree of order $n \geq 4$, then*

$$\gamma_a(T) \leq \frac{3n}{5},$$

with equality if and only if $T \in \mathcal{T}_1$.

Proof. We proceed by induction on $n \geq 4$. If $n = 4$, then either $T = P_4$ or $T = K_{1,3}$, and so $\gamma_a(T) = 2 < 3n/5$. Suppose, then, that for all trees T' of order n', where $4 \leq n' < n$, $\gamma_a(T') \leq 3n'/5$. Let T be a tree of order n. If T is a star, then, by Proposition 3, $\gamma_a(K_{1,n-1}) = \lfloor (n-1)/2 \rfloor + 1 \leq 3n/5$ with equality if and only if $n = 5$, i.e., if and only if $T = K_{1,4} \in \mathcal{T}_1$. If T is a double star, then it follows from Proposition 9 that $\gamma_a(T) < 3n/5$. If $T = P_5$, then, by Proposition 8, $\gamma_a(T) = 3 = 3n/5$. Hence we may assume that $\operatorname{diam}(T) \geq 4$ and that $T \neq P_5$.

Among all support vertices of T of eccentricity $\operatorname{diam}(T) - 1$, let v be one of minimum degree. Let r be a vertex at distance $\operatorname{diam}(T) - 1$ from v and root T at r. Let u denote the parent of v, and x the parent of u.

Let T' be the tree obtained from T by deleting v and its children, i.e., $T' = T - D[v]$. Let T' have order n'. Since $\operatorname{diam}(T) \geq 4$ and $T \neq P_5$, it follows from our choice of v that $n' \geq 4$. Applying the inductive hypothesis to T', $\gamma_a(T') \leq 3n'/5$. Let S' be a $\gamma_a(T')$-set. Let $|C(v)| = \ell$, and so $n = n' + \ell + 1$.

If $u \in S'$, then adding v and $\lfloor(\ell-1)/2\rfloor$ children of v to S' produces a global alliance of T, and so $\gamma_a(T) \le |S'| + (\ell+1)/2 \le 3(n-\ell-1)/5 + (\ell+1)/2 < 3n/5$. Hence we may assume that $u \notin S'$ (for otherwise $\gamma_a(T) < 3n/5$).

If $\deg v = 2$, then adding the child of v to S' produces a global alliance of T, and so $\gamma_a(T) \le |S'| + 1 \le 3(n-2)/5 + 1 < 3n/5$. Hence we may assume that $|C(v)| \ge 2$ (for otherwise $\gamma_a(T) < 3n/5$).

Suppose $\deg u \ge 3$. If u has a child v' different from v that is a support vertex, then, by our choice of v, $|C(v')| \ge 2$. But then we can always choose S' to contain u and v', contradicting our assumption that $u \notin S'$. Hence every child of u different from v must be a leaf. If u is adjacent to more than one leaf, then we can choose $u \in S'$, a contradiction. Hence, $\deg u = 3$ and the child v' (say) of u different from v is a leaf. Thus, $v' \in S'$. Deleting v' from S', and adding u and v and $\lfloor(\ell-1)/2\rfloor$ children of v to S' produces a global alliance of T, and so $\gamma_a(T) \le |S'| + (\ell+1)/2 < 3n/5$. Hence we may assume that $\deg u = 2$ (for otherwise $\gamma_a(T) < 3n/5$).

If $\mathrm{diam}(T) = 4$, then x is a support vertex (of degree at least $\deg v \ge 3$), and so $T - D[u]$ is a star $K_{1,k}$ with x as its center where $k \ge \ell \ge 2$. The set consisting of $\{x, u, v\}$, together with $\lfloor(\ell-1)/2\rfloor$ leaves adjacent to v and $\lfloor(k-1)/2\rfloor$ leaves adjacent to x is a global alliance of T, and so $\gamma_a(T) \le (k+\ell+4)/2 = (n+1)/2$. Since $n \ge 7$, $\gamma_a(T) < 3n/5$. Hence we may assume that $\mathrm{diam}(T) \ge 5$.

Let $T^* = T - N[v]$, i.e., $T^* = T - D[u]$. Let T^* have order n^*. Since $\mathrm{diam}(T) \ge 5$, it follows from our choice of v that $n^* \ge 4$. Applying the inductive hypothesis to T^*, $\gamma_a(T^*) \le 3n^*/5$ with equality if and only if $T^* \in \mathcal{T}_1$.

Let S^* be a $\gamma_a(T^*)$-set. Adding u and v and $\lfloor(\ell-1)/2\rfloor$ children of v to S^* produces a global alliance of T. Hence if $\ell = 2$, then $\gamma_a(T) \le |S^*| + 2 = 3(n-4)/5 + 2 < 3n/5$, while if $\ell \ge 3$, then $\gamma_a(T) \le |S^*| + (\ell+3)/2 \le 3(n-\ell-2)/5 + (\ell+3)/2 \le 3n/5$. Furthermore, suppose $\gamma_a(T) = 3n/5$. Then $\ell = 3$ and $\gamma_a(T^*) = |S^*| = 3n^*/5$. By the inductive hypothesis, $T^* \in \mathcal{T}_1$. If $T^* = P_5$, then T is a tree of order $n = 10$ obtained from $K_{1,4}$ by subdividing one edge five times. But then $\gamma_a(T) = 5 < 3n/5$. Hence, $T^* \ne P_5$. If $T^* = K_{1,4}$, then $T \in \mathcal{T}_1$. So we may assume $T^* \ne K_{1,4}$. Thus, T^* is obtained from $t \ge 2$ copies of $K_{1,4}$ with $t-1$ edges added as in the description of $\mathcal{T}_1$.

Suppose x is a central vertex of one of the copies of $K_{1,4}$ in T^*. Now S^* contains at least one child of x that is a leaf in T^*. Deleting this child of x from S^*, and adding u, v and one child of v, produces a global alliance of T of cardinality $|S^*| + 2 = 3(n-5)/5 + 2 < 3n/5$, a contradiction. Hence, x must be a leaf of one of the copies of $K_{1,4}$ in T^*.

Let z be the center of the $K_{1,4}$ in T^* that contains x. Let $N(z) = \{z_1, z_2, z_3, x\}$.

Suppose first that x is a leaf in T^*. Then in T, z is adjacent to exactly two leaves, z_1 and z_2 say. Now let D^* be a $\gamma_a(T^*)$-set that contains all the central vertices of the $K_{1,4}$s in T^*, exactly one leaf adjacent to each central vertex and all the leaves of $K_{1,4}$ that are incident to added edges when constructing T^*. In particular, $z, z_3 \in D^*$. We may assume $x \in D^*$. Let $D = (D^* - \{x, z, z_3\}) \cup \{z_1, z_2, u, v, w\}$, where w is any child of v. Then, D is a global alliance of T of cardinality $\gamma_a(T^*) + 2 < 3n/5$, a contradiction. Hence, x cannot be a leaf in T^*.

Suppose next that x is not a leaf in T^*. Then x is adjacent to a vertex (a leaf) in a copy of $K_{1,4}$ in T^*. Thus, z_1, z_2, and z_3 are leaves in T^* and it follows that $T \in \mathcal{T}_1$. $\Box$

Next we establish a sharp upper bound on the global strong alliance number of a tree and characterize the trees achieving this bound. For this purpose, we introduce a family $\mathcal{T}_2$ of trees as follows: Let T be the tree obtained from the disjoint union $tK_{1,3}$ of $t \geq 1$ copies of $K_{1,3}$ by adding $t-1$ edges between leaves of these copies of $K_{1,3}$ in such a way that the center of each $K_{1,3}$ is adjacent to at least one leaf in T. Let $\mathcal{T}_2$ be the family of all such trees T.

Theorem 22 *If T is a tree of order $n \geq 3$, then*

$$\gamma_{\hat{a}}(T) \leq \frac{3n}{4},$$

with equality if and only if $T \in \mathcal{T}_2$.

Proof. We proceed by induction on $n \geq 3$. If $n = 3$, then $T = P_3$ and $\gamma_{\hat{a}}(T) = 2 < 3n/4$. Suppose, then, that for all trees T' of order n', where $3 \leq n' < n$, $\gamma_{\hat{a}}(T') \leq 3n'/4$. Let T be a tree of order n.

If T is a star, then, by Proposition 3, $\gamma_{\hat{a}}(K_{1,n-1}) = \lceil (n-1)/2 \rceil + 1$. If $n = 3$, then $\gamma_{\hat{a}}(K_{1,n-1}) = 2 < 3n/4$. If $n \geq 4$, then $\gamma_{\hat{a}}(K_{1,n-1}) \leq (n+2)/2 \leq 3n/4$ with equality if and only if $n = 4$, i.e., if and only if $T = K_{1,3} \in \mathcal{T}_2$. If T is a double star, then it follows from Proposition 10 that $\gamma_{\hat{a}}(T) < 3n/4$. Hence we may assume that $\mathrm{diam}(T) \geq 4$.

Among all support vertices of T of eccentricity $\mathrm{diam}(T) - 1$, let v be one of minimum degree and let r be a vertex at distance $\mathrm{diam}(T) - 1$ from v. Let T be rooted at r. Let u denote the parent of v, and x the parent of u.

Let T' be the tree obtained from T by deleting v and its children, i.e., $T' = T - D[v]$. Let T' have order n'. Since $\mathrm{diam}(T) \geq 4$, $n' \geq 3$. Applying the inductive hypothesis to T', $\gamma_{\hat{a}}(T') \leq 3n'/4$. Let S' be a $\gamma_{\hat{a}}(T')$-set. Let $|C(v)| = \ell$, and so $n = n' + \ell + 1$.

Suppose $\deg u \geq 3$. Then in T', u is a support vertex or is adjacent to a support vertex. Since S' is a global strong alliance, every support vertex is in S' and at least one neighbor of every support vertex is in S'. In particular, we can choose S' to contain u. Hence, adding v and $\lfloor \ell/2 \rfloor$ children of v to S' produces a global alliance of T. Thus if $\ell = 1$, then $\gamma_{\hat{a}}(T) \leq |S'| + 1 \leq 3(n-2)/4 + 1 < 3n/4$, while if $\ell \geq 2$, then $\gamma_{\hat{a}}(T) \leq |S'| + (\ell+2)/2 \leq 3(n-\ell-1)/4 + (\ell+2)/2 < 3n/4$. Hence we may assume that $\deg u = 2$ (for otherwise $\gamma_{\hat{a}}(T) < 3n/4$).

Let $T^* = T - N[v]$, i.e., $T^* = T - D[u]$. Let T^* have order n^*. If $T = P_5$, then, by Proposition 7, $\gamma_{\hat{a}}(T) = 3 < 3n/4$. Hence we may assume $T \neq P_5$. Thus it follows from our choice of v that $n^* \geq 3$. Applying the inductive hypothesis to T^*, $\gamma_{\hat{a}}(T^*) \leq 3n^*/4$ with equality if and only if $T^* \in \mathcal{T}_2$. Let S^* be a $\gamma_{\hat{a}}(T^*)$-set.

If $\ell = 1$, then adding u and v to S^* produces a global strong alliance of T, and so $\gamma_{\hat{a}}(T) \leq |S^*| + 2 \leq 3(n-3)/4 + 2 < 3n/4$. Hence we may assume $\ell \geq 2$. Adding u and v and $\lfloor \ell/2 \rfloor$ children of v to S^* produces a global strong alliance of T, and so $\gamma_{\hat{a}}(T) \leq |S^*| + 2 + \ell/2 \leq 3(n-\ell-2)/4 + (\ell+4)/2 \leq 3n/4$. Furthermore, suppose

$\gamma_{\hat{a}}(T) = 3n/4$. Then $\ell = 2$ and $\gamma_{\hat{a}}(T^*) = |S^*| = 3n^*/4$. By the inductive hypothesis, $T^* \in \mathcal{T}_2$. We show that $T \in \mathcal{T}_2$.

Suppose x is a central vertex of one of the copies of $K_{1,3}$ in T^*. It follows that x is adjacent to at least one leaf in T^*, and hence, $x \in S^*$. Now S^* contains at least one child of x, say x'. Note that x' is either a leaf or all its neighbors in T^* are dominated by support vertices in S^*. Adding u, v and one child of v to $S^* - \{v'\}$, produces a global strong alliance of T of cardinality $|S^*| + 2 = 3(n-4)/4 + 2 < 3n/4$, a contradiction. Hence, x must be a leaf of one of the copies of $K_{1,3}$ in T^*.

Let z be the center of the $K_{1,3}$ in T^* that contains x. Let $N(z) = \{x, z_1, z_2\}$. Since $T^* \in \mathcal{T}_2$, each central vertex of the $K_{1,3}$s in T^* are support vertices in T^* and therefore belong to S^*. Thus, S^* contains all the central vertices of the $K_{1,3}$s in T^* and two (of the three) neighbors of each of these central vertices. We may assume without loss of generality that S^* contains the nonleaf neighbors of these central vertices. Suppose neither z_1 nor z_2 are leaves in T^*. Then $\{z_1, z_2\} \subseteq S^*$. Moreover, $S^* - \{z, z_1, z_2\}$ dominates $\{z_1, z_2\}$. But then $(S^* - \{z, z_1, z_2\}) \cup \{x, u, v, w\}$, where w is any child of v is a global strong alliance of T of cardinality $\gamma_{\hat{a}}(T^*) + 1 < 3n/4$, a contradiction. Hence either z_1 or z_2 must be a leaf. Thus, z is a support vertex in T. It follows that $T \in \mathcal{T}_2$. $\square$

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