Colouring the petals of a graph

David Cariolaro [*]
Department of Mathematics
University of Reading
RG6 6AX
U.K.
davidcariolaro@hotmail.com

Gianfranco Cariolaro
Dipartimento di Ingegneria dell' Informazione
Università di Padova
35133 Padova
Italia
cariolar@dei.unipd.it

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Abstract

A petal graph is a connected graph G with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a 2–regular graph and the set of vertices of degree two induces an empty graph. We prove here that, with the single exception of the graph obtained from the Petersen graph by deleting one vertex, all petal graphs are Class 1. This settles a particular case of a conjecture of Hilton and Zhao.

[*]corresponding author.

1 Introduction

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. If G is a graph, we let $V(G)$ and $E(G)$ denote, respectively, the vertex and the edge set of G. If S is a set of vertices or edges of G, we let $G[S]$ denote the graph *induced* by S in G. $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. The *core* of G, denoted by G_Δ, is the subgraph of G induced by the vertices of degree $\Delta(G)$. If H is a subgraph of G, we let $\Gamma(H)$ denote the set of vertices of G which are adjacent in G to at least one vertex of H. For standard graph theoretic terminology, not explicitly defined here, we follow [1].

A *petal graph* is a connected graph G such that:

1. $\Delta(G) = 3,\ \delta(G) = 2$;
2. G_Δ is 2-regular;
3. Every edge of G is incident with at least one vertex in G_Δ.

If G is a petal graph and w is a vertex of G of degree two, having neighbours v_1, v_2, then the path $P_w = v_1wv_2$ is a *petal* of G. We name w the *centre* of the petal and v_1, v_2 the *basepoints*. By property 3, the basepoints of the petal P_w are in G_Δ. If $dist_{G_\Delta}(v_1, v_2) = k$, we say that the *size* of the petal P_w is k, or that P_w is a k-petal (we assume the distance between vertices belonging to distinct connected components of a graph to be infinite). The *petal size* of G, denoted $p(G)$, is the minimum size of the petals of G.

A (proper) *k-edge colouring* of a graph G is a map $\varphi : E(G) \to \mathcal{C}$, where $|\mathcal{C}| = k$ and $\varphi(e_1) \neq \varphi(e_2)$ for each pair (e_1, e_2) of adjacent edges of G. We say that the vertex v is *missing* colour $c \in \mathcal{C}$ (with respect to the colouring φ) if no edge incident with v is assigned colour c by the colouring φ. The *chromatic index* of G, denoted $\chi_1(G)$, is the minimum k for which G has a k-edge colouring. For a general introduction to edge colouring, the interested reader is referred to [5]. As we shall only consider edge colourings in this paper, the terms “colouring” and “edge colouring” will be used as synonyms.

A fundamental theorem due to Vizing [13] states that, for any graph G, we have

$$\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1.$$

Correspondingly we say that G is *Class 1* if $\chi_1(G) = \Delta(G)$ and *Class 2* if $\chi_1(G) = \Delta(G) + 1$. We say that G is *critical* if it is connected, Class 2, and $G - e$ is Class 1 for every edge $e \in E(G)$. G is *overfull* if $|E(G)| > \lfloor |V(G)|/2 \rfloor \cdot \Delta(G)$, and it is easy to see that, if G is overfull, then G is Class 2. For more information about overfull graphs see [7].

Classifying a graph as Class 1 or Class 2 is a difficult problem in general (indeed, NP-hard), even when restricted to the class of graphs with maximum degree three (see [12]). As a consequence, this problem is usually considered on particular classes of graphs. One possibility is to consider graphs whose core has a simple structure (see [2, 3, 4, 6, 8, 9, 10, 11, 14]). Vizing [14] proved that, if G_Δ has at most two vertices, then G is Class 1.

Fournier [6] generalized Vizing's result by proving that, if G_Δ contains no cycles, then G is Class 1. Thus a necessary condition for a graph to be Class 2 is to have a core that contains cycles. Hilton and Zhao [9, 10] considered the problem of classifying graphs whose core is the disjoint union of cycles. Only a few such graphs are known to be Class 2. These include the overfull graphs and the graph P^*, which is obtained from the Petersen graph by removing one vertex (see Fig.1). Notice that P^* is a petal graph and is not overfull.

In [9] Hilton and Zhao posed the following conjecture:

Conjecture 1 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$. Let $G \neq P^*$. Then G is Class 2 if and only if G is overfull.*

In [9] the same authors showed this conjecture to be equivalent to the following:

Conjecture 2 *Let G be a connected graph such that $\Delta(G) < \frac{1}{2}(|V(G)|+3)$ and $\Delta(G_\Delta) \leq 2$. Let $G \neq P^*$ and let G not be an odd cycle* [1]. *Then G is Class 1.*

In this paper we shall prove that Conjecture 2 (and hence Conjecture 1) holds for all graphs G with $\Delta(G) = 3$, by proving the following:

Theorem 1 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$ and $\Delta(G) = 3$. Let $G \neq P^*$. Then G is Class 1.*

The notion of petal graph will be particularly useful because, as we shall see, the proof of Theorem 1 will be reduced to the proof of the following theorem:

Theorem 2 *Let G be a petal graph, and let $G \neq P^*$. Then G is Class 1.*

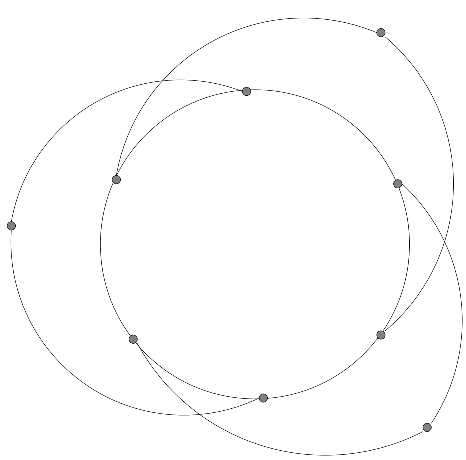

Figure 1: The graph P^*

[1] note that, in the original cited paper of Hilton and Zhao, by an oversight this conjecture was stated without the hypothesis that G is not an odd cycle.

2 Some useful lemmas

The first of the two following lemmas is due to Vizing [14], the second is an important result of Hilton and Zhao [10] which will be essential for us:

Lemma 1 *Let G be a critical graph. Then every vertex of G is adjacent to at least two vertices of G_Δ.*

Lemma 2 *Let G be a connected Class 2 graph with $\Delta(G_\Delta) \leq 2$. Then:*

1. *G is critical;*
2. *$\delta(G_\Delta) = 2$;*
3. *$\delta(G) = \Delta(G) - 1$, unless G is an odd cycle;*
4. *$\Gamma(G_\Delta) = V(G)$.*

The following lemma motivates the introduction of petal graphs:

Lemma 3 *Let G be a connected Class 2 graph with $\Delta(G) = 3$ and $\Delta(G_\Delta) \leq 2$. Then G is a petal graph.*

Proof. Property 1 and 2 of the definition of petal graph follow immediately from Lemma 2. Property 3 follows from Lemma 2 and Lemma 1. $\square$

Notice that Lemma 3 reduces Theorem 1 to Theorem 2. From now on G will denote a petal graph. The colour set will be the set $\{\alpha, \beta, \gamma\}$ and, if $\mathcal{D} \subset \{\alpha, \beta, \gamma\}$, $\overline{\mathcal{D}}$ will denote the set $\{\alpha, \beta, \gamma\} \setminus \mathcal{D}$. We need the following technical lemma:

Lemma 4 *Let L_n denote, for any positive integer n, the graph obtained from a path $v_0v_1v_2 \cdots v_{n-1}v_n$ of length n by inserting at each of the inner vertices $v_1, v_2, \cdots v_{n-1}$ a 2-path $v_iw_iy_i$, as shown in Fig.2 for $n = 10$. Let $f_i = w_iy_i$ and let $\phi : \{v_0v_1, f_1, f_2, \cdots f_{n-1}\} \to \{\alpha, \beta, \gamma\}$ be an arbitrary assignment of colours. Let $\theta \in \overline{\{\phi(f_{n-1})\}}$. Then ϕ can be extended to a proper edge colouring $\hat{\phi} : E(L_n) \to \{\alpha, \beta, \gamma\}$. Moreover such a colouring can be chosen in order to satisfy the additional requirement that $\hat{\phi}(v_{n-1}v_n) \neq \theta$.*

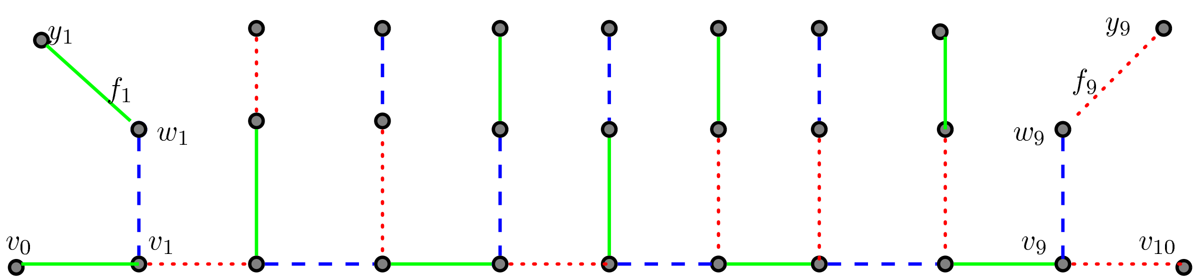

Figure 2: An example of a colouring of L_{10} under constraints

Proof. The proof is by induction on n. Notice that the lemma holds trivially for $n = 1$. Let $n \geq 2$, and assume that the lemma holds for all positive integers less than n. Let $H = L_n \setminus \{v_0, w_1, y_1\}$. Notice that $H \cong L_{n-1}$. Let $c \in \overline{\{\phi(v_0v_1), \phi(f_1)\}}$ and let $c' \in \overline{\{\phi(v_0v_1), c\}}$. Consider the assignment $\psi : \{v_1v_2, f_2, \cdots f_{n-1}\} \rightarrow \{\alpha, \beta, \gamma\}$ given by $\psi(v_1v_2) = c'$ and $\psi(e) = \phi(e)$ for all $e \in \{f_2, f_3, \cdots f_{n-1}\}$. Using the inductive hypothesis and the fact that $H \cong L_{n-1}$, there exists a proper 3-colouring $\hat{\psi} : E(H) \rightarrow \{\alpha, \beta, \gamma\}$ extending ψ, and such that $\hat{\psi}(v_{n-1}v_n) \neq \theta$. Extend $\hat{\psi}$ to a colouring $\hat{\phi}$ of G in the following manner: let $\hat{\phi}\mid_{E(H)} = \hat{\psi}$, $\hat{\phi}(v_0v_1) = \phi(v_0v_1)$, $\hat{\phi}(f_1) = \phi(f_1)$ and $\hat{\phi}(v_1w_1) = c$. Notice that $\hat{\phi}(v_{n-1}v_n) \neq \theta$. Thus $\hat{\phi}$ satisfies all the requirements of the statement of the lemma, so that the lemma holds for the integer n as well. By induction, the proof is completed. □

The next few lemmas concern the colourability of particular classes of petal graphs. The following notation will be useful: if G is a petal graph and φ is a 3-colouring of G, we let $G(\alpha, \beta)$ denote the subgraph of G induced by the set of edges coloured by φ either α or β. Note that this graph has maximum degree two, so its connected components consist of paths and even cycles. If $e \in E(G(\alpha, \beta))$, we let $G(\alpha, \beta; e)$ denote the connected component of $G(\alpha, \beta)$ containing the edge e.

Lemma 5 *Let G be a petal graph such that $p(G) = 1$. Then G is Class 1.*

Proof. Suppose that G is Class 2. By Lemma 2 , G is critical. Let $P_w = v_1wv_2$ be a 1-petal of G. Let $G_1 = G - w - v_1v_2$. Since G is critical, G_1 is 3-colourable. Suppose that the length of the cycle of G_Δ containing v_1, v_2 is at least four. Let u_1v_1, u_2v_2 be the two edges adjacent to the edge v_1v_2 in G_Δ. Let G^* be the graph obtained from G_1 by the identification of v_1 and v_2, and let v^* be the vertex obtained from this identification. Notice that there is a natural one-to-one correspondence between the set of 3-colourings of G^* and the set of those 3-colourings of G_1 which assign different colours to the edges u_1v_1 and u_2v_2. It is immediate to see that the graph G^* is not a petal graph, but G^* is connected, $\Delta(G^*) = 3$ and $\Delta(G^*_\Delta) \leq 2$. Applying Lemma 3, we then have that G^* is Class 1. By the above remark, there exists a 3-colouring of G_1 under which u_1v_1 and u_2v_2 get different colours. However, this colouring can easily be extended to a 3-colouring of G, which gives a contradiction. Therefore we can assume that the cycle K of G_Δ containing v_1, v_2 has length three, say $K = uv_1v_2u$. However, in this case, any 3-colouring of G_1 satisfies the property of assigning different colours to the edges uv_1 and uv_2. Again, any such colouring can easily be extended to a 3-colouring of G, which gives another contradiction. Therefore G cannot be Class 2 and hence is Class 1. □

Lemma 6 *Let G be a petal graph such that $p(G) = 2$. Then G is Class 1.*

Proof. We will argue by contradiction, as before, so suppose that G is Class 2. By Lemma 2, G is critical. Let $P_w = v_1wv_2$ be a 2-petal of G with centre w. Let $u_1v_1xv_2u_2$ be a 4-path (or possibly a 4-cycle) in G_Δ containing v_1v_2 and let $P_t = xty$ be the petal of G containing the vertex x. Since G is critical, $G - w$ is Class 1. Notice that under no 3-colouring of $G - w$ the vertices v_1 and v_2 can miss different colours, otherwise the colouring itself can be immediately extended to a 3-colouring of G, thus contradicting the assumption that G is Class 2. Let then φ_0 be a 3-colouring of $G - w$. By the above remark, we can assume, without loss of generality, that $\varphi_0(u_1v_1) = \alpha,\ \varphi_0(v_1x) = \beta,\ \varphi_0(xv_2) = \alpha,\ \varphi_0(v_2u_2) = \beta,\ \varphi_0(xt) = \gamma$. Assume also, without loss of generality, that $\varphi_0(ty) = \beta$. Exchanging the colours between the edge xv_2 and the edge xt, we obtain a colouring of $G - w$ under which the vertices v_1 and v_2 miss different colours, which contradicts the above remark. Therefore G cannot be Class 2, and hence is Class 1. □

Lemma 7 *Let G be a petal graph such that $p(G) = \infty$. Then G is Class* 1.

Proof. Again we will argue by contradiction, so let us assume that G is Class 2. Let $v_0 \in V(G_\Delta)$ and let $K = v_0v_1 \cdots v_kv_0$ be the cycle of G_Δ containing v_0. For each $i = 0, 1, 2, \cdots k$, let $P_{w_i} = v_iw_iy_i$ be the petal of G containing v_i, and let $f_i = w_iy_i$. Let $G_0 = G - v_0w_0$ and let $G_1 = G \setminus V(K)$. By Lemma 2 , G is critical so that G_1 is Class 1. Suppose that there exists a 3-colouring $\varphi_1 : E(G_1) \to \{\alpha, \beta, \gamma\}$ such that $\varphi_1(f_k) \neq \varphi_1(f_0)$, say $\varphi_1(f_k) = \beta$ and $\varphi_1(f_0) = \alpha$. Consider the graph $H = G[E(K) \cup \bigcup_{i=1}^{k} E(P_{w_i})]$. Let H^* be the graph obtained from H by splitting the vertex v_0 into a pair of vertices z_1, z_k, with z_1 adjacent to v_1 and z_k adjacent to v_k in H^*. Note that $H^* \cong L_{k+1}$, where L_{k+1} is the graph defined in Lemma4. Also note that there is an obvious one-to-one correspondence between the 3-colourings of H and those 3-colourings of H^* in which the edges z_1v_1 and z_kv_k receive different colours. By Lemma 4, there exists a proper colouring $\varphi^* : E(H^*) \to \{\alpha, \beta, \gamma\}$ of H^* satisfying the conditions $\varphi^*(z_1v_1) = \alpha,\ \varphi^*(f_i) = \varphi_1(f_i)$ for each $i = 1, 2, \cdots k$ and $\varphi^*(z_kv_k) \neq \alpha$. By the above observation, this implies the existence of a 3-colouring of H, which we still denote by φ^*, which satisfies $\varphi^*(f_i) = \varphi_1(f_i)$ for each $i = 1, 2, \cdots k$, and $\varphi^*(v_0v_1) = \alpha$. This colouring can be extended to a 3-colouring φ of G in the following way: we let $\varphi\ |_{E(G_1)} = \varphi_1,\ \varphi\ |_{E(H)} = \varphi^*$ and $\varphi(v_0w_0) \in \overline{\{\varphi^*(v_0v_k), \alpha\}}$. However this is in contradiction with the assumption that G is Class 2, so that the condition $\varphi_1(f_k) \neq \varphi_1(f_0)$ cannot hold. Similarly, $\varphi_1(f_1) \neq \varphi_1(f_0)$ cannot hold, so that, for all 3-colourings φ_1 of G_1, we have:

$$\varphi_1(f_1) = \varphi_1(f_0) = \varphi_1(f_k)\,. \tag{1}$$

Let then φ_1 be one such colouring, and assume $\varphi_1(f_0) = \alpha$. Consider the graph $G_1(\alpha, \beta)$. In this graph the vertices w_k, w_0, w_1 all have degree one, so that not all of them belong to the same connected component of $G_1(\alpha, \beta)$. In particular, by exchanging the colours of the edges in $G_1(\alpha, \beta; f_0)$, we obtain a proper colouring of G_1 in which not all the edges f_k, f_0, f_1 receive the same colour, which contradicts (1). This contradiction shows that G cannot be Class 2, and thus G is Class 1. □

3 Proof of the main result

In this section we prove Theorem 1 and Theorem 2. We begin with Theorem 2, which we prove by using all the previous lemmas. We first show that no Class 2 petal graph can have petal size other than three. We then continue the proof by induction on the order of G. In particular, by associating to each petal graph $G \neq P^*$, with $p(G) = 3$, a smaller petal graph G^*, whose colourability implies the colourability of G, we conclude that any petal graph G, other than P^*, must be Class 1.

Proof of Theorem 2. Let G be a petal graph, $G \neq P^*$, and let $p = p(G)$. By Lemma 5, Lemma 6 and Lemma 7, we can assume that $3 \leq p < \infty$. We argue by contradiction, so suppose that G is Class 2. By Lemma 2, G is critical. Let $u_0 v_0 v_1 v_2 \cdots v_p u_p$ be a $(p+2)$-path in G_Δ containing the p-path $Y = v_0 v_1 v_2 \cdots v_p$, where $P_{w_0} = v_0 w_0 v_p$ is a p-petal of G. For each $i = 1, 2, \cdots p-1$, let $P_{w_i} = v_i w_i y_i$ be the petal of G containing v_i, and let $f_i = w_i y_i$. Notice that, by the definition of p, all the w_i's and y_i's are distinct, and none of the vertices y_i lies in Y, for $i = 1, 2, \cdots p-1$. Let $G_0 = G - w_0$. Since G is critical, G_0 is Class 1.

We will repeatedly use the fact that, for each 3-colouring φ_0 of G_0, the vertices v_0 and v_p miss the same colour (otherwise the colouring φ_0 could immediately be extended to a proper 3-colouring of G). Suppose that there exists a 3-colouring φ_0 of G_0 such that $\varphi_0(u_0 v_0) = \varphi_0(f_1)$, or $\varphi_0(u_p v_p) = \varphi_0(f_{p-1})$. Exchanging the colours between the edges $v_0 v_1$ and $v_1 w_1$ (or $v_p v_{p-1}$ and $v_{p-1} w_{p-1}$), we obtain a proper colouring of G_0 under which the vertices v_0 and v_p miss different colours. This is a contradiction, as observed above. Therefore, for any 3-colouring φ_0 of G_0, we have:

$$\varphi_0(f_1) \neq \varphi_0(u_0 v_0) \text{ and } \varphi_0(f_{p-1}) \neq \varphi_0(u_p v_p). \tag{2}$$

Next, consider the graph $G_1 = G_0 - \{v_1, v_2, \cdots, v_{p-1}\}$. Obviously G_1 is Class 1. We consider the interrelation between colourings of G_1 and colourings of G_0. Let φ_1 be a 3-colouring of G_1 satisfying (2) (for example consider the restriction of any colouring of G_0 to G_1). Without loss of generality, assume $\varphi_1(u_0 v_0) = \alpha$. Suppose that $\varphi_1(u_p v_p) \neq \alpha$, say $\varphi_1(u_p v_p) = \beta$. Let $H = G_0[\bigcup_{i=1}^{p-1} E(P_{w_i}) \cup E(Y)]$. Notice that $H \cong L_p$, where L_p is the graph introduced by Lemma 4. Applying Lemma 4 to the graph H, there exists a proper 3-colouring $\hat{\varphi}$ of H such that $\hat{\varphi}(v_0 v_1) = \gamma$, $\hat{\varphi}(f_i) = \varphi_1(f_i)$ for $1 \leq i \leq p-1$, and $\hat{\varphi}(v_p v_{p-1}) \neq \beta$. But now the colouring $\tilde{\varphi}$ of G_0 given by

$$\tilde{\varphi}(e) = \begin{cases} \varphi_1(e) & \text{if } e \in E(G_1) \\ \hat{\varphi}(e) & \text{if } e \in E(H) \end{cases}$$

is a proper 3-colouring of G_0 under which v_0 misses colour β but v_p does not! This is a contradiction, hence we must conclude that

$$\varphi_1(u_p v_p) = \varphi_1(u_0 v_0) = \alpha. \tag{3}$$

Suppose now that $\varphi_1(f_{p-2}) \neq \varphi_1(f_{p-1})$. By (2) and (3), $\varphi_1(f_{p-1}) \neq \alpha$. Without loss of generality, assume that $\varphi_1(f_{p-1}) = \gamma$. Let $H_1 = H - \{v_p, w_{p-1}, y_{p-1}\}$. Clearly $H_1 \cong L_{p-1}$. Notice that, by our assumption, $\varphi_1(f_{p-2}) \neq \gamma$. By Lemma 4 , there exists a proper 3-colouring $\hat{\varphi}$ of H_1 such that $\hat{\varphi}(v_0 v_1) = \beta$, $\hat{\varphi}(f_i) = \varphi_1(f_i)$ for each $i = 1, 2, \cdots p-2$, and $\hat{\varphi}(v_{p-2} v_{p-1}) \neq \gamma$. Extend this to a colouring $\hat{\varphi}$ of H by letting $\hat{\varphi}(v_p v_{p-1}) = \gamma$, $\hat{\varphi}(f_{p-1}) = \gamma$, and $\hat{\varphi}(v_{p-1} w_{p-1}) \in \overline{\{\hat{\varphi}(v_{p-1} v_{p-2}), \gamma\}}$. Now define a colouring $\tilde{\varphi}$ of G_0 as follows:

$$\tilde{\varphi}(e) = \begin{cases} \varphi_1(e) & \text{if } e \in E(G_1) \\ \hat{\varphi}(e) & \text{if } e \in E(H). \end{cases}$$

Notice that this is a proper 3-colouring of G_0 in which vertex v_p misses colour β and vertex v_0 does not! This a contradiction, so that it must be the case that $\varphi_1(f_{p-2}) = \gamma$.

By considering the graph $H_2 = H_1 - \{v_{p-1}, w_{p-2}, y_{p-2}\} \cong L_{p-2}$, we can repeat the same argument to show that $\varphi_1(f_{p-3}) = \gamma$ and, similarly, that

$$\varphi_1(f_{p-i}) = \varphi_1(f_{p-1}) \text{ for all } \ i = 1, 2, \cdots p-1. \tag{4}$$

Suppose now that $p > 3$. Since in $G_1(\beta, \gamma)$ the vertices w_1, w_2, w_{p-1} have degree 1, not all of them belong to the same component of $G_1(\beta, \gamma)$. By interchanging the colours in $G_1(\beta, \gamma; f_2)$ we obtain a colouring $\tilde{\varphi}$ of G_1 which still satisfies (2), but does not satisfy (4). However this is a contradiction, and therefore G cannot be Class 2 and hence is Class 1.

We are left with the case $p = 3$. The proof continues by induction on the order of G. Let $n = |V(G)|$. Since P^* is the only petal graph with $p = 3$ and $n \leq 9$, the statement of the theorem holds trivially for $n \leq 9$. Assume now that $n > 9$, and that the statement of the theorem holds for any petal graph with order less than n. Continuing with the notations introduced earlier, let K be the cycle of G_Δ containing the path $Y = v_0v_1v_2v_3$, and let k be the length of K. Suppose that $k > 6$. Let G^* be the graph obtained from G_1 by deleting v_0 and v_3, joining u_0, u_3 by an edge, and identifying w_1 and w_2. Let w^* denote the vertex of G^* obtained by means of the identification of w_1 and w_2. It is easy to see that G^* is a petal graph of order $n - 6$. By the inductive hypothesis, G^* is Class 1, or $G^* = P^*$. If $G^* = P^*$, then G is necessarily one of the three graphs shown in Fig. 3, all of which are Class 1. Therefore we can assume that G^* is Class 1. Let φ^* be a 3-colouring of G^*. Define a colouring of G_1 in the following way:

$$\varphi_1(e) = \begin{cases} \varphi^*(e) & \text{if } e \in E(G_1) - \{u_0v_0, u_3v_3, f_1, f_2\} \\ \varphi^*(u_0u_3) & \text{if } e = u_0v_0 \text{ or } e = u_3v_3 \\ \varphi^*(w^*y_1) & \text{if } e = f_1 \\ \varphi^*(w^*y_2) & \text{if } e = f_2 \ . \end{cases}$$

Notice that this is a proper 3-colouring of G_1 which either satisfies (2) but not (4), which is a contradiction, or is immediately extendable to a 3-colouring of G_0 in which the vertices v_0 and v_3 miss two different colours, which is also a contradiction.

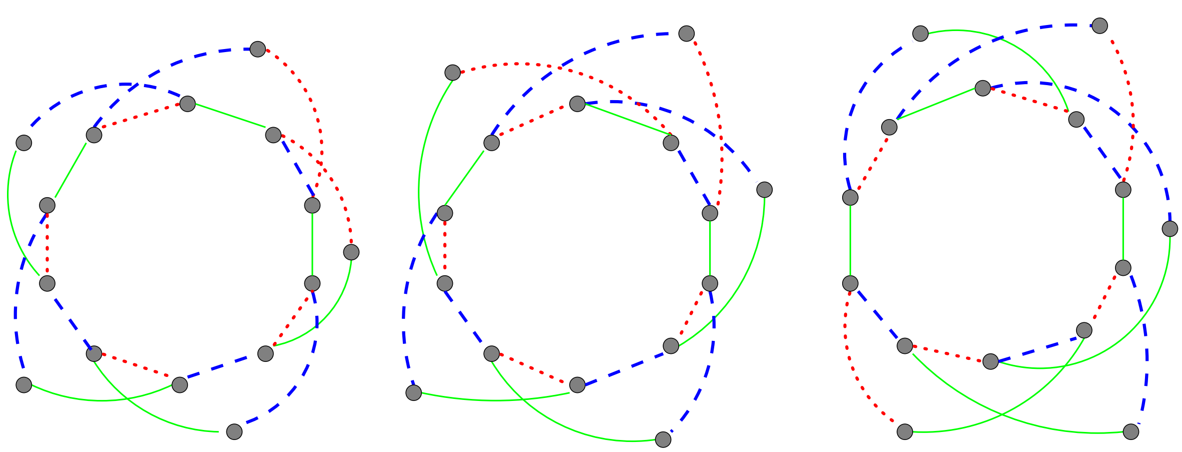

Figure 3: A 3-colouring of the only graphs G for which $G^* = P^*$.

This contradiction shows that we cannot have $k > 6$. Notice that, by the fact that $p = 3$, we cannot have $k < 6$ either, so that we are left with the only possibility that $k = 6$. Thus $K = u_0v_0v_1v_2v_3u_3u_0$. Notice that there are two distinct v_0v_3-paths in K of length three. In particular, we can apply all the previous considerations to both paths. Let f_1', f_2' denote the pair of edges corresponding to f_1, f_2 with respect to the path $v_0u_0u_3v_3$. Let φ_0 be a 3-colouring of $G_0 = G - w_0$. By considering the restriction of φ_0 to $G_0 - \{v_1, v_2\}$ and $G_0 - \{u_0, u_3\}$, and applying (4), we have that $\varphi_0(f_1) = \varphi_0(f_2)$ and $\varphi_0(f_1') = \varphi_0(f_2')$. Let $\overline{\varphi} = \varphi_0 \mid_{E(G_0 \setminus V(K))}$. It is easy to see that, if G has at least one petal of infinite size based on K, then the colouring $\overline{\varphi}$ can be extended immediately to a 3-colouring of G (see Fig. 4). Therefore we can assume that all the petals on K have finite size. Since G is connected, we then have $G_\Delta = K$ and, since $p(G) = 3$, the only possibility left is that $G = P^*$. But we assumed $G \neq P^*$, so that we have again a contradiction. This contradiction concludes the proof of the theorem. □

Proof of Theorem 1. Follows immediately from Lemma 3 and Theorem 2. □

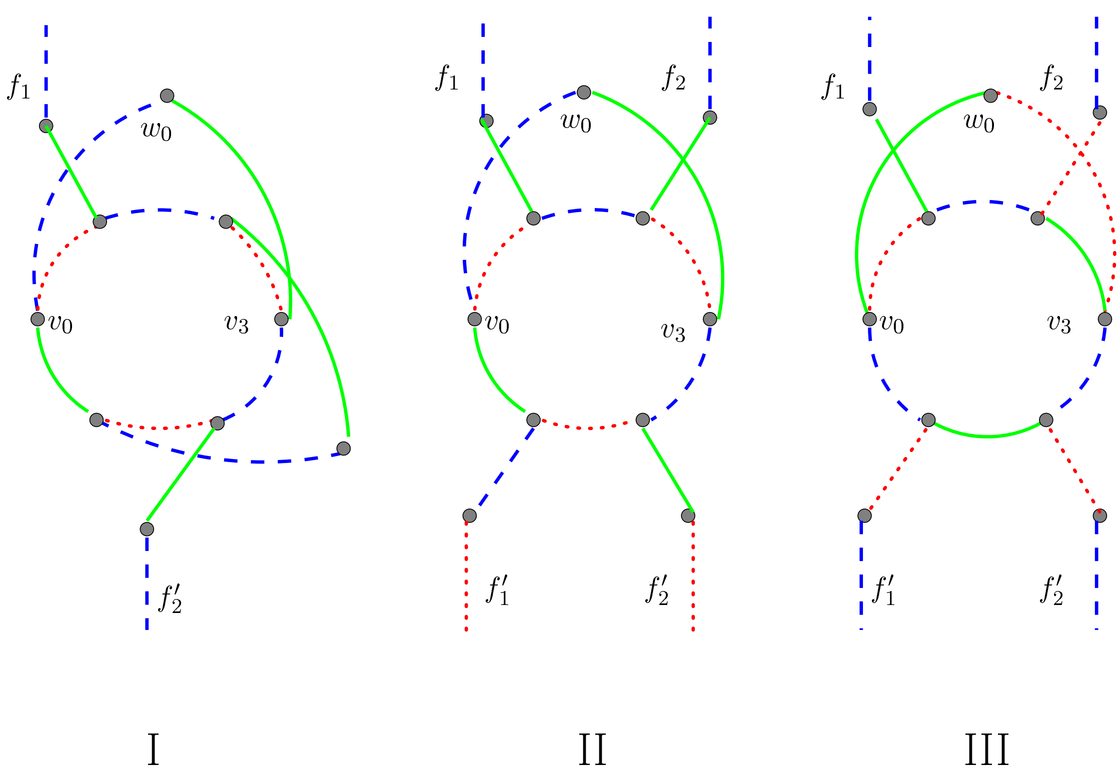

Figure 4: A 3-colouring of the edges of a 6-cycle component of G_Δ containing a petal of infinite size. Notice that the colouring in I is still valid when the colour of the edge f_2' is assumed to be one of the other two colours, and thus it is representative of the correct extension of $\overline{\varphi}$ in the most general situation.

Conclusions. In this paper we made use of a technique which we believe is a novelty in edge colouring. More specifically, we used the concept of critical graph to explicitly construct a 3-colouring of a petal graph G. This technique could be effective in proving more general results, e.g. other cases of Conjecture 2. One possible way to make further progress on Conjecture 2 is to obtain some sort of generalization of Lemma 3.

The proof of Theorem 2 is intrinsically algorithmic and can be used to construct a 3-colouring of any given petal graph, other than P^*. We have indeed written a computer program, using `Mathematica`, which accepts as an input a petal graph $G \neq P^*$ and returns, in linear time, a proper 3-colouring of G. The program is available on request from the second author.

We end with the hope that this paper will help stimulating further research around the mysterious properties of the Petersen graph.

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