Colored Prüfer codes for k-edge colored trees

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Submitted: Dec 31, 2002; Accepted: Oct 15, 2003; Published: Jul 19, 2004
MR Subject Classifications: 05C05, 05C30

Abstract

A combinatorial bijection between k-edge colored trees and colored Prüfer codes for labelled trees is established. This bijection gives a simple combinatorial proof for the number $k(n-2)!\binom{nk-n}{n-2}$ of k-edge colored trees with n vertices.

1 Introduction

A k-edge colored tree is a labelled tree whose edges are colored from a set of k colors such that any two edges with a common vertex have different colors [2, p81, 5.28]. For a pair (n, k) of positive integers, let $\mathcal{C}_{n,k}$ denote the set of all k-edge colored trees on vertex set $[n] = \{1, 2, \ldots, n\}$, with color set $[k]$. The number of k-edge colored trees in $\mathcal{C}_{n,k}$ is already known:

Theorem 1. *The number of k-edge colored trees on vertex set $[n]$, $n \geq 2$, is*

$$k(nk-n)(nk-n-1)\cdots(nk-2n+3) = k(n-2)!\binom{nk-n}{n-2}.$$

Stanley in [2, p124] introduces a proof of the above formula and asks whether there is a simple bijective proof. In this paper we provide a combinatorial bijection between k-edge colored trees and 'colored Prüfer codes', thus establishing a simple bijective proof of the above formula.

The Prüfer code $\varphi(T) = (a_1, \ldots, a_{n-2}, 1)$ of a labelled tree T with vertex set $[n]$ is obtained from the tree by successively pruning the leaf with the largest label. To obtain the code from T, we remove the largest leaf in each step, recording its neighbor a_i, from the tree, until the single vertex 1 is left. The inverse of φ can be described easily. Let $\sigma = (a_1, \ldots, a_{n-2}, 1)$ be a sequence of positive integers with $a_i \in [n]$ for all i. We can find the tree T whose code is σ as follows:

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†Partially supported by the Korea Research Foundation Grant(KRF-2001-015-DP0055).

- Let $V = \{1\}$ and $E = \emptyset$.
- For each i from $n-2$ to 1,
  - if $a_i \notin V$, then set $b_{i+1} = a_i$,
  - otherwise set $b_{i+1} = \min\{x : x \in [n] \setminus V\}$;
  - set $V := V \cup \{b_{i+1}\}$ and $E := E \cup \{\{a_{i+1}, b_{i+1}\}\}$.
- Let b_1 be the unique element in $[n] \setminus V$.
- Finally, set $V := V \cup \{b_1\}$ and $E := E \cup \{\{a_1, b_1\}\}$.
- Let T be the tree with vertex set V and edge set E.

Example. Let T be the tree in Figure 1. The Prüfer code of T is $(1, 6, 1, 3, 3, 1)$. We

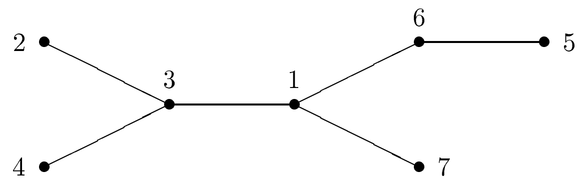

Figure 1: The tree T corresponding to $(1, 6, 1, 3, 3, 1)$

can recover T from its Prüfer code by the above algorithm.

Clearly, Prüfer codes are in one-to-one correspondence with labelled trees. The following is a well known result. See [1, 2].

Theorem 2. *The number of the tree on $[n]$ vertices is n^{n-2}.*

Proof. Any sequence $(a_1, a_2, \ldots, a_{n-2}) \in [n]^{n-2}$ of integers corresponds to a Prüfer code $(a_1, a_2, \ldots, a_{n-2}, 1)$ which in turn determines a unique labelled tree with vertex set $[n]$. $\square$

2 Colored Prüfer code

Let $\mathcal{P}_{n,k}$ denote the set of all arrays of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix},$$

such that $(a_1, c_1), (a_2, c_2), \ldots, (a_{n-2}, c_{n-2}) \in [n] \times [k-1]$ are distinct and $c_{n-1} \in [k]$. An array like the above is called a *colored Prüfer code*, since its first row is a Prüfer code and its second row can be interpreted as an edge-coloring.

Lemma 3. *The cardinality of* $\mathcal{P}_{n,k}$ *is*

$$k(n-2)!\binom{nk-n}{n-2}.$$

Proof. Consider an element $\sigma \in \mathcal{P}_{n,k}$:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}.$$

The conditions for σ are: $(a_i, c_i) \in [n] \times [k-1]$ for $1 \le i \le n-2$, $c_{n-1} \in [k]$ and the first $n-2$ columns of σ are distinct. So the number of possible σ is

$$k(nk-n)(nk-n-1)(nk-n-2)\cdots(nk-2n+3) = k(n-2)!\binom{nk-n}{n-2}.$$

$\square$

Recall that $\mathcal{C}_{n,k}$ is the set of all k-edge colored trees on vertex set $[n]$ with color set $[k]$. Let T be a k-edge colored tree in $\mathcal{C}_{n,k}$ with vertex set $V(T)$ and edge set $E(T)$. Let $C_T : E(T) \to [k]$ denote the edge-coloring of T, i.e. $C_T(e)$ is the color of edge e in T.

For each pair of distinct edges e and e' in T, define the *distance* between e and e', denoted by $d(e, e')$, to be $l-1$ when l is the shortest length of paths containing e and e'. Note that the distance between edges sharing a vertex is one.

When x is the smallest neighbor of 1 in T, we call the edge $\alpha = \{1, x\}$ the *root edge* of T. For any two edges e, e' in T with a common vertex, we call e the *parent edge* of e' and e' the *child edge* of e, if $d(e,\alpha) + 1 = d(e', \alpha)$.

Let $\widetilde{\mathcal{C}}_{n,k}$ denote the set of labelled trees with vertex set $[n]$ whose edges are colored from a set of k colors, say $[k]$, in such a way that

1. the root edge is colored from $[k]$,
2. any pair of edges sharing a vertex with a common parent edge have distinct colors, and
3. edges which are not the root edge are colored from $[k-1]$.

For a tree T in $\widetilde{\mathcal{C}}_{n,k}$, let $\widetilde{C}_T$ denote the edge-coloring of T, i.e. $\widetilde{C}_T(e)$ is the color of edge e in T.

Bijection ϕ

We define a mapping $\phi : \widetilde{\mathcal{C}}_{n,k} \to \mathcal{P}_{n,k}$ through the following steps:

- Set $T_0 := T$.

- For any i, $1 \le i \le n-1$, assuming that T_{i-1} is defined already, define a_i, b_i, c_i and T_i: b_i is the largest leaf in T_{i-1}, a_i is the vertex adjacent to b_i, T_i is the tree obtained by removing the vertex b_i and the edge $\{a_i, b_i\}$ from T_{i-1}, and $c_i = \widetilde{C}_T(\{a_i, b_i\})$.

- Define $\phi(T)$ by

$$\phi(T) = \left(\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{array} \right)$$

Note that the first row of $\phi(T)$ is the Prüfer code of T, so ϕ is one-to-one.

Clearly, the first $n-2$ columns of $\phi(T)$ are distinct, and $c_i \in [k-1]$ for $1 \le i \le n-2$, $c_{n-1} \in [k]$. So $\phi(T)$ is an element in $\mathcal{P}_{n,k}$.

Bijection ψ

We now define a mapping $\psi : \mathcal{P}_{n,k} \to \widetilde{\mathcal{C}}_{n,k}$, which is the inverse of ϕ. Let σ be an element in $\mathcal{P}_{n,k}$:

$$\sigma = \left(\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{array} \right).$$

We construct, by the following algorithm, a labelled tree whose Prüfer code is the first row of σ, with an edge-coloring $\widetilde{C}_T$:

- Let $V = \{1\}$ and $E = \emptyset$.

- For each i from $n-2$ to 1,

  - if $a_i \notin V$, then set $b_{i+1} = a_i$,
  - otherwise set $b_{i+1} = \min\{x : x \in [n] \setminus V\}$;
  - set $V := V \cup \{b_{i+1}\}$ and $E := E \cup \{\{a_{i+1}, b_{i+1}\}\}$.

- Let b_1 be the unique element in $[n] \setminus V$.

- Finally, set $V := V \cup \{b_1\}$ and $E := E \cup \{\{a_1, b_1\}\}$.

- Let T be the tree with vertex set V and edge set E.

- Set $\widetilde{C}_T(\{a_i, b_i\}) = c_i$ for $i \in [n-2]$ and $\widetilde{C}_T(\{1, b_{n-1}\}) = c_{n-1}$.

Let $\psi(\sigma)$ be the resulting tree with edge-coloring $\widetilde{C}_T$. Clearly $\psi(\sigma)$ is in $\widetilde{\mathcal{C}}_{n,k}$ and ψ is the inverse of ϕ. So we have the following.

Lemma 4. *The mapping $\phi : \widetilde{\mathcal{C}}_{n,k} \to \mathcal{P}_{n,k}$ is a bijection and thus the cardinality of $\widetilde{\mathcal{C}}_{n,k}$ is*

$$k(n-2)! \left(\begin{array}{c} nk-n \\ n-2 \end{array} \right).$$

Main result

We now define a mapping Δ from $\mathcal{C}_{n,k}$ to $\widetilde{\mathcal{C}}_{n,k}$. For any $T \in \mathcal{C}_{n,k}$, define $\widetilde{C}_T : E(T) \to [k]$ as follows:

- Let x be the smallest neighbor of 1 and α denote edge $\{1, x\}$. Set $\widetilde{C}_T(\alpha) = C_T(\alpha)$.
- Assume that $\widetilde{C}_T(f)$ is defined for all edges f such that $d(\alpha, f) < i$. For an edge g with $d(\alpha, g) = i$, let h be the unique edge such that $d(\alpha, h) = i-1$ and $d(h, g) = 1$. Define $\widetilde{C}_T(g)$ by

$$\widetilde{C}_T(g) = \begin{cases} C_T(g), & \text{if } C_T(g) \leq \widetilde{C}_T(h), \\ C_T(g) - 1, & \text{otherwise.} \end{cases}$$

Note that $\widetilde{C}_T(f) \leq k-1$ for all $f \neq \alpha$. Let $\Delta(T)$ be the tree T with its edge-coloring C_T replaced by $\widetilde{C}_T$. Clearly $\Delta(T)$ is an element in $\widetilde{\mathcal{C}}_{n,k}$.

We next define a mapping Λ from $\widetilde{\mathcal{C}}_{n,k}$ to $\mathcal{C}_{n,k}$. For any $T \in \widetilde{\mathcal{C}}_{n,k}$, define $C_T : E(T) \to [k]$ as follows:

- Let x be the smallest neighbor of 1 and α denote the edge $\{1, x\}$. Set $C_T(\alpha) = \widetilde{C}_T(\alpha)$.
- Assume that $C_T(f)$ is defined for all edges f such that $d(\alpha, f) < i$. For an edge g with $d(\alpha, g) = i$, let h be the unique edge such that $d(\alpha, h) = i-1$ and $d(h, g) = 1$. Define $C_T(g)$ by

$$C_T(g) = \begin{cases} \widetilde{C}_T(g), & \text{if } \widetilde{C}_T(g) < C_T(h), \\ \widetilde{C}_T(g) + 1, & \text{otherwise.} \end{cases}$$

Note that $C_T(f) \leq k$ for all f and no pair of two edges with a common vertex have the same color. Let $\Lambda(T)$ be the tree T with its edge-coloring $\widetilde{C}_T$ replaced by C_T. Clearly $\Lambda(T)$ is an element in $\mathcal{C}_{n,k}$.

Clearly, Λ is the inverse of Δ. Hence we have the following crucial lemma:

Lemma 5. *The mapping $\Delta : \mathcal{C}_{n,k} \to \widetilde{\mathcal{C}}_{n,k}$ is a bijection.*

Example. A k-edge colored tree T in $\mathcal{C}_{10,5}$ and its $\Delta(T)$ are in Figures 2 and 3. The edge $\{1, 3\}$ is the root edge.

We can now count the number of the k-edge colored trees with n vertices. The following is the restatement of Theorem 1.

Theorem 6 (Main theorem). *The number of k-edge colored trees on $[n]$ is*

$$k(n-2)! \binom{nk-n}{n-2}.$$

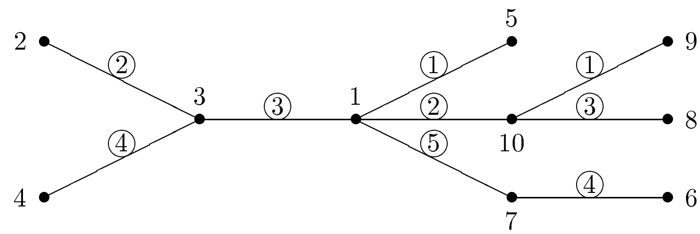

Figure 2: A k-edge colored tree T in $\mathcal{C}_{10,5}$

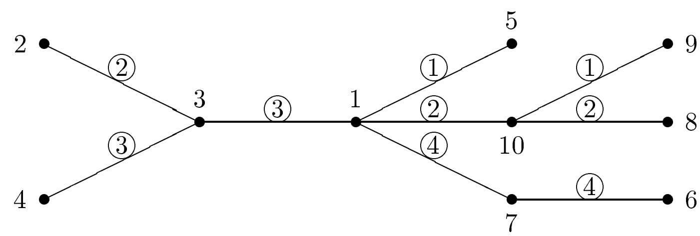

Figure 3: $\Delta(T)$ in $\widetilde{\mathcal{C}}_{10,5}$, i.e. T with $\widetilde{C}_T$

Proof. Since $\Delta : \mathcal{C}_{n,k} \to \widetilde{\mathcal{C}}_{n,k}$ and $\phi : \widetilde{\mathcal{C}}_{n,k} \to \mathcal{P}_{n,k}$ are bijections, it follows from Lemma 3 or 4. $\square$

The colored Prüfer codes can be used to count certain sets of labelled trees with edge-coloring. Recall that a k-edge colored tree is a labelled tree whose edges are colored from a set of k colors such that any two edges with a common vertex have different colors. We now consider slightly different edge-colorings of labelled trees.

Theorem 7. *The number of different labelled trees with vertex set $[n]$ whose edges are colored from a set of k colors in such a way that the color of each edge is different from that of its parent edge is*

$$k(nk-n)^{n-2}.$$

Proof. Let T be a tree with the property in the statement. Following the steps for the definition of ϕ, we can obtain an array σ corresponding to T:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}.$$

There are k possible ways to choose the c_{n-1}. Next, the number of possible ways to choose the $(n-2)$-th column of σ is $n(k-1)$, since the color of an edge is different from that of its parent edge. The i-th column of σ has always $n(k-1)$ choices. Hence the number of such trees is $k(nk-n)^{n-2}$. $\square$

Note that the above theorem can be proved by using a generalization of Δ. The mapping Δ can be defined as long as the colors of children edges are different from that of their parent edge. Then the image of Δ of a tree considered in the theorem just satisfies that non-root edges are colored with $[k-1]$, so that each of the first $n-2$ columns of its colored Prüfer code is an arbitrary element in $[n] \times [k-1]$.

Theorem 8. *The number of different labelled trees with vertex set $[n]$ whose edges are colored from a set of k colors in such a way that any pair of edges sharing a vertex with a common parent edge have distinct colors is*

$$k(n-2)!\binom{nk}{n-2}.$$

Proof. Let T be a tree with the property in the statement. Following the steps for the definition of ϕ, we can obtain an array σ corresponding to T:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & 1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}.$$

There are k possible ways to choose c_{n-1}. Since the c_{n-2} may be identical with c_{n-1}, the number of possible ways to choose the $(n-2)$-th column of σ is nk. Since the i-th column of T is different from the columns from the $(i+1)$-th to the $(n-2)$-th for $1 \leq i \leq n-3$, the number of possible ways to choose the i-th column decreases by 1 when i changes from $n-2$ to 1. So the number of such trees is

$$k(nk)(nk-1)(nk-2)\cdots(nk-n+3) = k(n-2)!\binom{nk}{n-2}.$$

□

References

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