n-Color partition theoretic interpretations of some mock theta functions

A.K. Agarwal*
Centre for Advanced Study in Mathematics
Panjab University
Chandigarh-160014, India
E-mail: aka@pu.ac.in

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Abstract

Using n-color partitions we provide new number theoretic interpretations of four mock theta functions of S. Ramanujan.

1 Introduction

In his last letter to G.H. Hardy, S. Ramanujan listed 17 functions which he called mock theta functions. He separated these 17 functions into three classes. First containing 4 functions of order 3, second containing 10 functions of order 5 and the third containg 3 functions of order 7. Watson [8] found three more functions of order 3 and two more of order 5 appear in the *lost notebook* [7]. Mock theta functions of order 6 and 8 have also been studied in [3] and [4], respectively. For the definitions of mock theta functions and their order the reader is referred to [6]. The object of this paper is to provide new number theoretic interpretations of the following mock theta functions:

$$\Psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}, \tag{1.1}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m}, \tag{1.2}$$

$$\Phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m, \tag{1.3}$$

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and

$$\Phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m, \tag{1.4}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a.

We remark that $\Psi(q)$ is of order 3 while the remaining three are of order 5.

Number theoretic interpretations of some of the mock theta functions are found in the literature. For example, $\Psi(q)$ has been interpreted as generating function for partitions into odd parts without gaps [5]. We in this paper use n-color partitions (also called partitions with n copies of n and studied first by Agarwal and Andrews in [2]) to give new number theoretic interpretations of the mock theta functions defined above by (1.1)-(1.4). Before we state our main results we recall some definitions from [2].

Definition 1.1. An n-color partition (also called a partition with 'n copies of n') of a positive integer ν is a partition in which a part of size n can come in n different colors denoted by subscripts:$n_1, n_2, ..., n_n$ and the parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < ....$ Thus, for example, the n-color partitions of 3 are

$$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1.$$

Definition 1.2. The weighted difference of two parts $m_i, n_j, m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.

We shall prove that the mock theta functions defined by (1.1)-(1.4) have, respectively, the following number theoretic interpretations:

Theorem 1. For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \Psi(q). \tag{1.5}$$

Example. $A_1(8) = 3$. The relevant n-color partitions are $8_8, 7_5 + 1_1, 6_2 + 2_2$.

Theorem 2. For $\nu \geq 0$, let $A_2(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd greater than 1, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu} = F_0(q). \tag{1.6}$$

Theorem 3. For $\nu \geq 0$, let $A_3(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k-1)_1$ or $(2k)_2$, the minimum part is 1_1 or 2_2, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=0}^{\infty} A_3(\nu) q^{\nu} = \Phi_0(q). \tag{1.7}$$

Theorem 4. For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is 1_1, and the weighted difference of any two consecutive parts is 0. Then,

$$\sum_{\nu=1}^{\infty} A_4(\nu) = \Phi_1(q). \tag{1.8}$$

Remark. We remark that there are 160 n-color partitions of 8 but only one partition viz., $6_2 + 2_2$ is relevant for Theorem 3 and none is relevant for Theorem 4. Out of 859 n-color partitions of 11, none is relevant for Theorems 3-4. Among 18334 n-color partitions of 17 only two viz., $9_1 + 6_2 + 2_2$ and $8_2 + 5_1 + 3_1 + 1_1$ satisfy the conditions of Theorem 3, whereas the lone partition $8_2+5_1+3_1+1_1$ satisfies the conditions of Theoem 4.

Following the method of [1], we give in our next section the detail proof of Theorem 1 and the shortest possible proofs for the remaining theorems. In the sequel $A_i(m,\nu), (1 \leq i \leq 4)$, will denote the number of partitions of ν enumerated by $A_i(\nu)$ into m parts, and we shall write

$$f_i(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_i(m,\nu) z^m q^{\nu}. \tag{1.9}$$

In our last section we illustrate how our new results can be used to yield new combinatorial identities.

2 Proofs

Proof of Theorem 1. We split the partitions enumerated by $A_1(m,\nu)$ into two classes: (1) those that contain 1_1 as a part, and those that contain $k_k, (k > 1)$ as a part. Following the method of [1] it can be easily proved that the partitions in Class (1) are enumerated by $A_1(m-1, \nu-2m+1)$ and in Class (2) by $A_1(m, \nu-2m+1)$, and so

$$A_1(m,\nu) = A_1(m-1, \nu-2m+1) + A_1(m, \nu-2m+1). \tag{2.1}$$

From (1.9), we have

$$f_1(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_1(m,\nu) z^m q^{\nu}. \tag{2.2}$$

Substituting for $A_1(m,\nu)$ from (2.1) in (2.2) and then simplifying we get

$$f_1(z,q) = zqf_1(zq^2,q) + q^{-1} f_1(zq^2,q). \tag{2.3}$$

Setting $f_1(z,q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n$, and then comparing the cofficients of z^n on each side of (2.3), we see that

$$\alpha_n(q) = \frac{q^{2n-1}}{1-q^{2n-1}} \alpha_{n-1}(q). \tag{2.4}$$

Iterating (2.4) n times and observing that $\alpha_0(q) = 1$, we find that

$$\alpha_n(q) = \frac{q^{n^2}}{(q;q^2)_n}. \tag{2.5}$$

Therefore

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q;q^2)_n}. \tag{2.6}$$

Now

$$\begin{aligned} \sum_{\nu=0}^{\infty} A_1(\nu) q^{\nu} &= \sum_{\nu=0}^{\infty} (\sum_{m=0}^{\infty} A_1(m,\nu)) q^{\nu} \\ &= f_1(1,q) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\ &= \Psi(q). \end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

The proof is similar to that of Theorem 1, hence we omit the details and give only the q-functional equation used in this case.

$$f_2(z,q) = zq^2 f_2(zq^4, q) + q^{-1} f_2(zq, q). \tag{2.7}$$

Proof of Theorem 3.

We split the partitions enumerated by $A_3(m,\nu)$ into two classes:(1) those that contain 1_1 as a part, and (2) those that contain 2_2 as a part. By using the usual technique we see that the partitions in Class (1) are enumerated by $A_3(m-1, \nu-2m+1)$ and in Class (2) by $A_3(m-1, \nu-4m+2)$. This leads to the identity

$$A_3(m,\nu) = A_3(m-1, \nu-2m+1) + A_3(m-1, \nu-4m+2). \tag{2.8}$$

Using (2.8) one can easily obtain the following q-functional equation

$$f_3(z,q) = zq f_3(zq^2, q) + zq^2 f_3(zq^4, q). \tag{2.9}$$

Setting $f_3(z,q) = \sum_{n=0}^{\infty} \beta_n(q) z^n$, and noting that $f_3(0,q) = 1$, we can easily check by coefficient comparison in (2.9) that

$$\beta_n(q) = q^{n^2}(-q;q^2)_n. \tag{2.10}$$

Therefore,

$$f_3(z,q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n z^n. \tag{2.11}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_3(\nu)q^\nu &= \sum_{\nu=0}^{\infty}(\sum_{m=0}^{\infty} A_3(m,\nu))q^\nu \\ &= f_3(1,q) \\ &= \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n \\ &= \Phi_0(q).\end{aligned}$$

This proves Theorem 3.

Proof of Theorem 4.

The partitions enumerated by $A_4(m,\nu)$ are precisely those partitions which belong to Class 1 of the previous case. Therefore,

$$A_4(z,\nu) = A_3(m-1, \nu-2m+1). \tag{2.12}$$

Using Equations (2.8) and (2.12), one can easily obtain the following q-functional equation:

$$f_4(z,q) = f_3(z,q) - zq^2 f_3(zq^4,q). \tag{2.13}$$

Setting $f_4(z,q) = \sum_{n=0}^{\infty} \gamma_n(q) z^n$, and then comparing the coefficients of z^n on each side of (2.13), we see that

$$\begin{aligned}\gamma_n(q) &= \beta_n(q) - \beta_{n-1}(q)q^{4n-2} \\ &= q^{n^2}(-q;q^2)_{n-1}.\end{aligned}$$

This implies that

$$f_4(z,q) = \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1} z^n.$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_4(\nu)q^{\nu} &= \sum_{\nu=0}^{\infty}(\sum_{m=0}^{\infty} A_4(m,\nu))q^{\nu} \\ &= f_4(1,q) \\ &= \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_{n-1} \\ &= \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n \\ &= \Phi_1(q).\end{aligned}$$

This completes the proof of Theorem 4.

3 New combinatorial identities

Our Theorems 1-4 can be combined with the known number theoretic interpretations of (1.1)-(1.4) to yield new combinatorial identities. For example, Theorem 1 in view of the known partition theoretic interpretation of $\Psi(q)$ given above in Section 1 gives the following result:

Theorem 5. For $\nu \geq 1$, the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0 equals the number of ordinary partitions of ν into odd parts without gaps.

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