A Binomial Coefficient Identity Associated with Beukers' Conjecture on Apéry numbers

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Abstract

By means of partial fraction decomposition, an algebraic identity on rational function is established. Its limiting case leads us to a harmonic number identity, which in turn has been shown to imply Beukers' conjecture on the congruence of Apéry numbers.

Throughout this work, we shall use the following standard notation:

$$\left.\begin{array}{llll}\text{Harmonic numbers} & H_0 = 0 & \text{and} & H_n = \sum_{k=1}^{n} 1/k \\ \text{Shifted factorials} & (x)_0 = 1 & \text{and} & (x)_n = \prod_{k=0}^{n-1}(x+k)\end{array}\right\} \text{ for } n = 1, 2, \cdots .$$

For a natural number n, let $A(n)$ be Apéry number defined by binomial sum

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and $\alpha(n)$ determined by the formal power series expansion

$$\sum_{m=1}^{\infty} \alpha(m) q^m := q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 + \cdots .$$

Beukers' conjecture [3] asserts that if p is an odd prime, then there holds the following congruence (cf. [1, Theorem 7])

$$A\left(\frac{p-1}{2}\right) \equiv \alpha(p) \pmod{p^2}.$$

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Recently, Ahlgren and Ono [1] have shown that this conjecture is implied by the following beautiful binomial identity

$$\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\Big\{1+2kH_{n+k}+2kH_{n-k}-4kH_k\Big\} \;=\; 0 \tag{1}$$

which has been confirmed successfully by the WZ method in [2].

The purpose of this note is to present a new and classical proof of this binomial-harmonic number identity, which will be accomplished by the following general algebraic identity.

Theorem. *Let x be an indeterminate and n a natural number. There holds*

$$\frac{x(1-x)_n^2}{(x)_{n+1}^2}=\frac{1}{x}+\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\Big\{\tfrac{-k}{(x+k)^2}+\tfrac{1+2kH_{n+k}+2kH_{n-k}-4kH_k}{x+k}\Big\}. \tag{2}$$

The binomial-harmonic number identity (1) is the limiting case of this theorem. In fact, multiplying by x across equation (2) and then letting $x\to+\infty$, we recover immediately identity (1).

Proof of the **Theorem**. By means of the standard partial fraction decomposition, we can formally write

$$f(x):=\frac{x(1-x)_n^2}{(x)_{n+1}^2}=\frac{A}{x}+\sum_{k=1}^{n}\Big\{\frac{B_k}{(x+k)^2}+\frac{C_k}{x+k}\Big\}$$

where the coefficients A and $\{B_k,C_k\}$ remain to be determined.

First, the coefficients A and $\{B_k\}$ are easily computed:

$$\begin{aligned}
A &= \lim_{x\to0} xf(x) \;=\; \lim_{x\to0}\frac{(1-x)_n^2}{(1+x)_n^2} \;=\; 1;\\
B_k &= \lim_{x\to-k}(x+k)^2f(x) \;=\; \lim_{x\to-k}\frac{x(1-x)_n^2}{(x)_k^2(1+x+k)_{n-k}^2}\\
&= \frac{-k(1+k)_n^2}{(-k)_k^2(1)_{n-k}^2} \;=\; -k\binom{n}{k}^2\binom{n+k}{k}^2.
\end{aligned}$$

Applying the L'Hôspital rule, we determine further the coefficients $\{C_k\}$ as follows:

$$\begin{aligned}
C_k &= \lim_{x\to-k}(x+k)\Big\{f(x)-\frac{B_k}{(x+k)^2}\Big\} \;=\; \lim_{x\to-k}\frac{(x+k)^2f(x)-B_k}{x+k}\\
&= \lim_{x\to-k}\frac{d}{dx}\big\{(x+k)^2f(x)-B_k\big\} \;=\; \lim_{x\to-k}\frac{d}{dx}\frac{x(1-x)_n^2}{(x)_k^2(1+x+k)_{n-k}^2}\\
&= \lim_{x\to-k}\frac{(1-x)_n^2}{(x)_k^2(1+x+k)_{n-k}^2}\Big\{1-\sum_{i=1}^{n}\frac{2x}{i-x}-\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{2x}{x+j}\Big\}\\
&= \binom{n}{k}^2\binom{n+k}{k}^2\Big\{1+2kH_{n+k}+2kH_{n-k}-4kH_k\Big\}.
\end{aligned}$$

This completes the proof of the Theorem. □

References

[1] S. Ahlgren - K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences,* J. Reine Angew. Math. 518 (2000), 187-212.

[2] S. Ahlgren - S. B. Ekhad - K. Ono - D. Zeilberger, *A binomial coefficient identity associated to a conjecture of Beukers,* The Electronic J. Combinatorics 5 (1998), #R10.

[3] F. Beukers, *Another congruence for Apéry numbers,* J. Number Theory 25 (1987), 201-210.

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