# Bounds on the Turán density of $\mathrm{PG}(3,2)$

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**Abstract**

We prove that the Turán density of $\mathrm{PG}(3,2)$ is at least $\frac{27}{32}=0.84375$ and at most $\frac{27}{28}=0.96428\ldots$.

## 1 Introduction

For $n \geq 2$, let $\mathrm{PG}(n,2)$ be the finite projective geometry of dimension $n$ over $\mathbb{F}_2$, the field of order 2. The elements or points of $\mathrm{PG}(n,2)$ are the one-dimensional vector subspaces of $\mathbb{F}_2^{n+1}$; the lines of $\mathrm{PG}(n,2)$ are the two-dimensional vector subspaces of $\mathbb{F}_2^{n+1}$. Each such one-dimensional subspace $\{0,x\}$ is represented by the non-zero vector $x$ contained in it. For ease of notation, if $\{e_0,e_1,\ldots,e_n\}$ is a basis of $\mathbb{F}_2^{n+1}$ and $x$ is an element of $\mathrm{PG}(n,2)$, then we denote $x$ by $a_1\ldots a_s$, where $x=e_{a_1}+\cdots+e_{a_s}$ is the unique expansion of $x$ in the given basis. For example, the element $x=e_0+e_2+e_3$ is denoted 023. For an $r$-uniform hypergraph $\mathcal{F}$, the Turán number $\mathrm{ex}(n,\mathcal{F})$ is the maximum number of edges in an $r$-uniform hypergraph with $n$ vertices not containing a copy of $\mathcal{F}$. The Turán density of an $r$-uniform hypergraph $\mathcal{F}$ is $\pi(\mathcal{F})=\lim_{n\to\infty}\frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{r}}$. A 3-uniform hypergraph is also called a triple system. The points and the lines of $\mathrm{PG}(n,2)$ form a triple system $\mathcal{H}_n$ with vertex set $V(\mathcal{H}_n)=\mathbb{F}_2^{n+1}\setminus\{0\}$ and edge set $E(\mathcal{H}_n)=\{xyz : x,y,z\in V(\mathcal{H}_n), x+y+z=0\}$. The Turán number(density) of $\mathrm{PG}(n,2)$ is the Turán number(density) of $\mathcal{H}_n$. It was proved in [1] that the Turán density of $\mathrm{PG}(2,2)$, also known as the Fano plane, is $\frac{3}{4}$. The exact Turán number of the Fano plane was later determined for $n$ sufficiently large: it is $\mathrm{ex}(n,\mathrm{PG}(2,2))=\binom{n}{3}-\binom{\lfloor\frac{n}{2}\rfloor}{3}-\binom{\lceil\frac{n}{2}\rceil}{3}$. This result was proved simultaneously and independently in [2] and [4]. In the following sections, we present bounds on the Turán density of $\mathrm{PG}(3,2)$.

## 2 A lower bound

Let $\mathcal{G}$ be the triple system on $n \geq 1$ vertices with vertex set $A \cup B \cup C$, where $A, B$ and $C$ are disjoint, $|A| = \lfloor \frac{\lceil \frac{3n}{4} \rceil}{2} \rfloor \sim \frac{3n}{8}, |B| = \lceil \frac{\lceil \frac{3n}{4} \rceil}{2} \rceil \sim \frac{3n}{8}$ and $|C| = \lfloor \frac{n}{4} \rfloor \sim \frac{n}{4}$. Also let $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where $C_1, C_2, C_3$ and $C_4$ are disjoint and $|C_i| = \lfloor \frac{\lfloor \frac{n}{4} \rfloor + i - 1}{4} \rfloor \sim \frac{n}{16}$ for $1 \leq i \leq 4$. The edge set of $\mathcal{G}$ is obtained by removing from the set of all 3-subsets of $V = A \cup B \cup C$ the following triples

$$\{xyz : x, y, z \in A\} \cup \{xyz : x, y, z \in B\} \cup \{xyz : x, y, z \in C\}$$
$$\cup \{xyz : x \in A \cup B, y, z \in C_i, 1 \leq i \leq 4\} \tag{1}$$

The number of edges of $\mathcal{G}$ is $\frac{27}{32}\binom{n}{3} + O(n^2)$.

**Theorem 2.1.** *$\mathcal{G}$ does not contain $\mathcal{H}_3$.*

*Proof.* It was proved in [5] that the chromatic number of $\mathcal{H}_3$ is 3 and for any 3-coloring of $\mathcal{H}_3$, all three color classes have cardinality 5.

Suppose $\mathcal{H}_3$ is contained in $\mathcal{G}$. Color the vertices in $A$ with color 1, the vertices in $B$ with color 2 and the vertices in $C$ with color 3. From the definition of the edge set of $\mathcal{G}$, it follows that no edge of $\mathcal{G}$ is monochromatic. Since $\mathcal{H}_3$ is contained in $\mathcal{G}$, it follows that $\mathcal{H}$ must admit a 3-coloring such that one color class is included in $A$, another in $B$ and the other in $C$. Thus, we have a color class $D$ of $\mathcal{H}_3$ in $C = C_1 \cup C_2 \cup C_3 \cup C_4$. Since this color class has 5 vertices, from the pigeonhole principle we get that there exists $1 \leq i \leq 4$ such that at least 2 of the vertices of $D$ are in $C_i$. Without loss of any generality, we can assume $i = 1$; let $x$ and $y$ be two of the vertices of $D$ which are contained in $C_1$. From the definition of $\mathcal{H}_3$, it follows that there exists a unique vertex $z$ in $V(\mathcal{H}_3)$ such that $xyz \in E(\mathcal{H}_3)$. But $z$ cannot be contained in $C$, therefore $z \in A \cup B$.

Thus, we have found that $\mathcal{G}$ contains an edge with one endpoint in $A \cup B$ and two endpoints in $C_1$; this is impossible by (1). Hence, $\mathcal{H}_3$ is not contained in $\mathcal{G}$. □

This implies

$$\pi(\mathrm{PG}(3, 2)) \geq \frac{27}{32} = 0.84375.$$

## 3 An upper bound

It follows from [6] that $\pi(\mathrm{PG}(3, 2)) \leq 1 - \frac{1}{|E(\mathcal{H}_3)|} = \frac{34}{35} = 0.971\ldots$. In this section, we provide a slight improvement of this bound and show that $\pi(\mathrm{PG}(3, 2)) \leq \frac{27}{38} = 0.964\ldots$.

Let $m(n, k, r)$ denote the maximum number of edges in a graph on $n$ vertices with the property that any $k$ vertices span at most $r$ edges. It was proved in [3] that the asymptotic density $\mathrm{ex}(k, r) = \lim_{n\to\infty} \frac{m(n,k,r)}{\binom{n}{2}}$ exists for all $k$ and $r \geq 0$ and that $m(n, k, r) = \mathrm{ex}(k, r)\binom{n}{2} + O(n)$.

Let $\mathcal{G}$ be a triple system with $n$ vertices such that $\mathcal{G}$ doesn't contain $\mathcal{H}_3$. In obtaining an upper bound on $\pi(\mathcal{H}_3)$, we may assume that $\mathcal{G}$ contains a copy $\mathcal{F}$ of the Fano plane, otherwise $\pi(\mathcal{H}_3) \leq \pi(\mathcal{F}) = \frac{3}{4} = 0.75$ which contradicts $\pi(\mathcal{H}_3) \geq 0.84375$. Given any vertex $a \in V(\mathcal{G})$, the link $L_S(a)$ of $a$ restricted to a subset $S$ of $V(\mathcal{G})$ is $\{\{b,c\} : \{a,b,c\} \in E(\mathcal{G}), b,c \in S\}$. The proof of the next result is technical and it is presented in the next section.

**Theorem 3.1.** *Let $\mathcal{G}$ be a triple system that contains a Fano plane $\mathcal{F}$. Suppose there is a subset $S$ of 8 elements of $V(\mathcal{G}) \setminus V(\mathcal{F})$ so that the link multigraph of $\mathcal{F}$ restricted to $S$ has 192 edges. Then $\mathcal{G}$ contains $\mathcal{H}_3$.*

Thus, for any set $S$ of 8 vertices included in $V(\mathcal{G}) \setminus V(\mathcal{F})$, the union $\cup_{x\in\mathcal{F}} L_S(x)$ contains at most 191 edges. It follows that the number of edges in $\cup_{x\in\mathcal{F}} L_S(x)$ is at most $m(n,8,191) + O(n)$. This implies that there exists a vertex $x$ in $\mathcal{F}$ that is contained in at most $\frac{m(n,8,191)}{7} + O(n)$ edges of $\mathcal{G}$. From Theorem 9(page 24) in [3] it follows that $\mathrm{ex}(8,191) = 6 + \mathrm{ex}(8,23) = 6 + \frac{3}{4} = \frac{27}{4}$. Thus, $x$ will be contained in at most $\frac{27}{28}\binom{n}{2} + O(n)$ edges of $\mathcal{G}$. Deleting $x$ and applying the same argument as before to $\mathcal{G} \setminus \{x\}$, we get that the number of edges in $\mathcal{G}$ is at most $\frac{27}{28}\binom{n}{3} + O(n^2)$ which implies

$$\pi(\mathrm{PG}(3,2)) \leq \frac{27}{28} = 0.96428\ldots.$$

Hence,

$$0.84375 = \frac{27}{32} \leq \pi(\mathrm{PG}(3,2)) \leq \frac{27}{28} = 0.96428\ldots.$$

# 4 Proof of theorem 3.1.

As usual, $C_4$ will denote the cycle on 4 vertices, $K_4$ will be the complete graph on 4 vertices and $Q_3$ will be the cube on 8 vertices.

*Proof.* Let $\mathcal{F} = \{0,1,2,01,02,12,012\}$ be the Fano plane included in $\mathcal{G}$. For $a \in \mathcal{F}$, we will denote by $L(a)$ the link of $a$ restricted to $S$. Let $x_1, x_2, \ldots, x_7$ denote the sizes of the links of the vertices of $\mathcal{F}$ restricted to $S$ with $x_1 \leq x_2 \leq \cdots \leq x_7 \leq 28$.

The solutions $(y_1, y_2, \ldots, y_7)$ of the equation $y_1 + y_2 + \cdots + y_7 = 192, y_1 \leq y_2 \leq \cdots \leq y_7$ and $y_i \in \mathbb{N}$ for all $1 \leq i \leq 7$ are the following:

1. (24,28,28,28,28,28,28)
2. (25,27,28,28,28,28,28)
3. (26,26,28,28,28,28,28)
4. (26,27,27,28,28,28,28)
5. (27,27,27,27,28,28,28)

Then $(x_1, x_2, \ldots, x_7)$ is one of the 7-tuples above. The following result is folklore and it will be used in the proof of our theorem.

**Lemma 4.1.** *If $G$ is a graph on $2n$ vertices and $\binom{2n-1}{2}+1$ edges, then $G$ contains a perfect matching.*

The automorphism group of PG$(2,2)$ acts transitively on the lines of PG$(2,2)$ and also, acts transitively on the 3-subsets of PG$(2,2)$ that are not lines. This fact is used in analyzing **Case 4** and **Case 5**.

- **Case 1** $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)=(24,28,28,28,28,28,28)$

  We can assume that $|L(0)|=24$. It follows that there exists a perfect matching $M(0)$ of $S$ that is included in $L(0)$. Label this matching as
  $M(0)=\{\{3,03\},\{13,013\},\{23,023\},\{123,0123\}\}$. The choices of perfect matchings for the remaining vertices of $\mathcal{F}$ are obvious since $x_i=28$ for all $i, 2\le i\le 7$. We choose
  $M(01)=\{\{3,013\},\{03,13\},\{23,0123\},\{123,023\}\}$,
  $M(1)=\{\{3,13\},\{03,013\},\{23,123\},\{023,0123\}\}$,
  $M(2)=\{\{3,23\},\{13,123\},\{03,023\},\{013,0123\}\}$,
  $M(02)=\{\{3,023\},\{03,23\},\{13,0123\},\{013,123\}\}$,
  $M(12)=\{\{3,123\},\{03,0123\},\{13,23\},\{013,023\}\}$ and
  $M(012)=\{\{3,0123\},\{03,123\},\{13,023\},\{23,013\}\}$.
  Then $\mathcal{F}$ with the edges containing all these perfect matchings will form $\mathcal{H}_3$.

- **Case 2** $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)=(25,27,28,28,28,28,28)$

  We can assume that $|L(0)|=25$ and $|L(1)|=27$. There exists a perfect matching $M(0)$ of $S$ that is included in $L(0)$. It can be easily checked that there are exactly 12 perfect matchings $Q$ of $S$ such that $M(0)\cup Q=2C_4$. Also, for every pair $\{u,v\}\notin M(0)$ with $u,v\in S$, there exist precisely 2 perfect matchings $R$ of $S$ such that $M(0)\cup R=2C_4$ and $\{u,v\}\in R$. Thus, for every pair $\{u,v\}\notin M(0)$ with $u,v\in S$, there exist exactly 10 perfect matchings $Q$ of $S$ such that $M(0)\cup Q=2C_4$ and $\{u,v\}\notin Q$. Since $|L(1)|=27$, it follows that there exist at least 10 perfect matchings $Q$ of $S$ such that $Q\subset L(1)$ and $M(0)\cup Q=2C_4$. We choose one of these $Q$'s to be $M(1)$. Thus, we have $M(0)\subset L(0)$, $M(1)\subset L(1)$ and $M(0)\cup M(1)=2C_4$. We label these two matchings as follows:
  $M(0)=\{\{3,03\},\{13,013\},\{23,023\},\{123,0123\}\}$ and
  $M(1)=\{\{3,13\},\{03,013\},\{23,123\},\{023,0123\}\}$.
  We can continue the labelling as in **Case 1**.

- **Case 3** $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)=(26,26,28,28,28,28,28)$

  We can assume that $|L(0)|=26$ and $|L(1)|=26$. There exists a perfect matching $M(0)$ of $S$ that is included in $L(0)$. Again, there are exactly 12 perfect matchings $Q$ of $S$ such that $M(0)\cup Q=2C_4$. A pair $\{u,v\}\notin M(0)$ with $u,v\in S$ belongs to exactly 2 perfect matchings $Q$ of $S$ such that $M(0)\cup Q=2C_4$. It follows that for any two pairs $\{u,v\},\{u',v'\}\notin M(0)$ with $u,v,u',v'\in S$, there exist at most 4 perfect matchings $R$ of $S$ such that $M(0)\cup R=2C_4$ and $\{\{u,v\},\{u',v'\}\}\cap R\neq\emptyset$. Since $|L(1)|=26$, it follows that there are at least 8 perfect matchings $Q$ of $S$ such that $Q \subset L(1)$ and $M(0) \cup Q = 2C_4$. We choose one of these $Q$'s to be $M(1)$. Thus, we have $M(0) \subset L(0), M(1) \subset L(1)$ and $M(0) \cup M(1) = 2C_4$. We label these two matchings as follows:

$M(0) = \{\{3, 03\}, \{13, 013\}, \{23, 023\}, \{123, 0123\}\}$ and
$M(1) = \{\{3, 13\}, \{03, 013\}, \{23, 123\}, \{023, 0123\}\}$.

We can continue the labelling as in **Case 1**.

- **Case 4** $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (26, 27, 27, 28, 28, 28, 28)$

Without loss of generality we can assume that $|L(0)| = 26$ and $|L(1)| = |L(01)| = 27$ or $|L(0)| = 26$ and $|L(1)| = |L(2)| = 27$. There exists a perfect matching $M(0)$ of $S$ that is included in $L(0)$.

Suppose that $|L(1)| = |L(01)| = 27$. There exist 24 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $M(0) \cup Q \cup R = 2K_4$. For a pair $\{u, v\} \notin M(0)$ with $u, v \in S$, there are 4 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $M(0) \cup Q \cup R = 2K_4$ and $\{u, v\} \in Q \cup R$. Thus, for two pairs $\{u, v\}, \{u', v'\} \notin M(0)$ with $u, v, u', v' \in S$, there are at most 16 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $M(0) \cup Q \cup R = 2K_4$ and $\{\{u, v\}, \{u', v'\}\} \cap (Q \cup R) \neq \emptyset$. Since $|L(1)| = |L(01)| = 27$, it follows that there exist at least 8 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $Q \subset L(1)$, $R \subset L(01)$ and $M(0) \cup Q \cup R = 2K_4$. Choose one of these pairs and let $M(1) = Q$ and $M(01) = R$. We label these matchings as follows:

$M(0) = \{\{3, 03\}, \{13, 013\}, \{23, 023\}, \{123, 0123\}\}$,
$M(1) = \{\{3, 13\}, \{03, 013\}, \{23, 123\}, \{023, 0123\}\}$ and
$M(01) = \{\{3, 013\}, \{03, 13\}, \{23, 0123\}, \{123, 023\}\}$.

We then continue as in **Case 1**.

Suppose that $|L(1)| = |L(2)| = 27$. Since $|L(1)| = 27$, it is obvious from the previous cases that we can find a perfect matching $M(1) \subset L(1)$ of $S$ such that $M(0) \cup M(1) = 2C_4$. Now, because $|L(2)| = 27$, it is easy to see that there are at least 6 perfect matchings $R$ of $S$ such that $R \subset L(2)$ and $M(0) \cup M(1) \cup R = Q_3$. Choose one of them and let $M(2) = R$. We now label these matchings as follows:

$M(0) = \{\{3, 03\}, \{13, 013\}, \{23, 023\}, \{123, 0123\}\}$,
$M(1) = \{\{3, 13\}, \{03, 013\}, \{23, 123\}, \{023, 0123\}\}$ and
$M(2) = \{\{3, 23\}, \{13, 123\}, \{03, 023\}, \{013, 0123\}\}$.

We then continue as in **Case 1**.

- **Case 5** $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (27, 27, 27, 27, 28, 28, 28)$

Without loss of generality we can assume that $|L(0)| = |L(1)| = |L(01)| = |L(2)| = 27$ or $|L(0)| = |L(1)| = |L(2)| = |L(012)| = 27$.

Suppose that $|L(0)| = |L(1)| = |L(01)| = |L(2)| = 27$. From **Case 4**, it follows that there exist perfect matchings $M(0), M(1)$ and $M(01)$ of $S$ such that $M(0) \subset L(0), M(1) \subset L(1), M(01) \subset L(01)$ and $M(0) \cup M(1) \cup M(01) = 2K_4$. Since $|L(2)| = 27$, it is easy to observe that we can find a perfect matching $M(2) \subset L(2)$

of $S$ such that $M(2) \cup M(x) \cup M(y) = Q_3$ for any $\{x, y\} \subset \{0, 1, 01\}$. We label these matchings as follows:

$M(0) = \{\{3, 03\}, \{13, 013\}, \{23, 023\}, \{123, 0123\}\}$
$M(01) = \{\{3, 013\}, \{03, 13\}, \{23, 0123\}, \{123, 023\}\}$,
$M(1) = \{\{3, 13\}, \{03, 013\}, \{23, 123\}, \{023, 0123\}\}$ and
$M(2) = \{\{3, 23\}, \{13, 123\}, \{03, 023\}, \{013, 0123\}\}$.

The rest of the matchings are labelled as in **Case 1**.

Suppose now that $|L(0)| = |L(1)| = |L(2)| = |L(012)| = 27$. From **Case 4**, we can find perfect matchings $M(0), M(1)$ of $S$ such that $M(0) \subset L(0)$, $M(1) \subset L(1)$, and $M(0) \cup M(1) = 2C_4$. There exist 16 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $X \cup Y \cup Z = Q_3$ for any $\{X, Y, Z\} \subset \{M(0), M(1), Q, R\}$. For $\{u, v\} \notin M(0) \cup M(1)$ with $u, v \in S$, there are at most 2 perfect matchings $Q$ of $S$ such that $M(0) \cup M(1) \cup Q = Q_3$ and $\{u, v\} \in Q$. It follows that for $\{u, v\}, \{u', v'\} \notin M(0) \cup M(1)$ with $u, v, u', v' \in S$, there are at most 8 ordered pairs $(Q, R)$ of perfect matchings of $S$ such that $\{\{u, v\}, \{u', v'\}\} \cap (Q \cup R) \neq \emptyset$ and $X \cup Y \cup Z = Q_3$ for any $\{X, Y, Z\} \subset \{M(0), M(1), Q, R\}$. This implies that we can find perfect matchings $M(2) \subset L(2)$ and $M(012) \subset L(012)$ of $S$ such that $M(x) \cup M(y) \cup M(z) = Q_3$ for any $\{x, y, z\} \subset \{0, 1, 2, 012\}$. We label these matchings as follows:

$M(0) = \{\{3, 03\}, \{13, 013\}, \{23, 023\}, \{123, 0123\}\}$,
$M(1) = \{\{3, 13\}, \{03, 013\}, \{23, 123\}, \{023, 0123\}\}$,
$M(2) = \{\{3, 23\}, \{13, 123\}, \{03, 023\}, \{013, 0123\}\}$ and
$M(012) = \{\{3, 0123\}, \{03, 123\}, \{13, 023\}, \{23, 013\}\}$.

We then continue as in **Case 1**.

□

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