Chromatically Unique Multibridge Graphs

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Abstract

Let $\theta(a_1, a_2, \cdots, a_k)$ denote the graph obtained by connecting two distinct vertices with k independent paths of lengths $a_1, a_2, \cdots, a_k$ respectively. Assume that $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k$. We prove that the graph $\theta(a_1, a_2, \cdots, a_k)$ is chromatically unique if $a_k < a_1 + a_2$, and find examples showing that $\theta(a_1, a_2, \cdots, a_k)$ may not be chromatically unique if $a_k = a_1 + a_2$.

Keywords: Chromatic polynomials, χ-unique, χ-closed, polygon-tree

1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G), E(G), v(G), e(G)$, $g(G), P(G,\lambda)$ respectively be the vertex set, edge set, order, size, girth and chromatic polynomial of G. Two graphs G and H are *chromatically equivalent* (or simply χ-*equivalent*),

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symbolically denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. Note that if $H \sim G$, then $v(H) = v(G)$ and $e(H) = e(G)$. The *chromatic equivalence class* of G, denoted by $[G]$, is the set of graphs H such that $H \sim G$. A graph G is *chromatically unique* (or simply *χ-unique*) if $[G] = \{G\}$. Whenever we talk about the chromaticity of a graph G, we are referring to questions about the chromatic equivalence class of G.

Let k be an integer with $k \geq 2$ and let $a_1, a_2, \cdots, a_k$ be positive integers with $a_i + a_j \geq 3$ for all i, j with $1 \leq i < j \leq k$. Let $\theta(a_1, a_2, \cdots, a_k)$ denote the graph obtained by connecting two distinct vertices with k independent (internally disjoint) paths of lengths $a_1, a_2, \cdots, a_k$ respectively. The graph $\theta(a_1, a_2, \cdots, a_k)$ is called a *multibridge* (more specifically *k-bridge*) graph (see Figure 1).

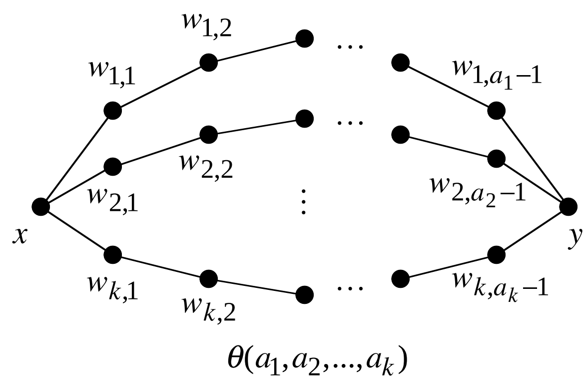

Figure 1

Given positive integers $a_1, a_2, \cdots, a_k$, where $k \geq 2$, what is a necessary and sufficient condition on $a_1, a_2, \ldots, a_k$ for $\theta(a_1, a_2, \ldots, a_k)$ to be chromatically unique? Many papers [2, 4, 10, 6, 11, 12, 13, 14] have been published on this problem, but it is still far from being completely solved [8, 9]. In this paper, we shall solve this problem under the condition that $\max\limits_{1\leq i\leq k} a_i \leq \min\limits_{1\leq i<j\leq k}(a_i + a_j)$.

2 Known results

For two non-empty graphs G and H, an *edge-gluing* of G and H is a graph obtained from G and H by identifying one edge of G with one edge of H. For example, the graph $K_4 - e$ (obtained from K_4 by deleting one edge) is an edge-gluing of K_3 and K_3. There are many edge-gluings of G and H. Let $\mathcal{G}_2(G, H)$ denote the family of all edge-gluings of G and H. Zykov [15] showed that any member of $\mathcal{G}_2(G, H)$ has chromatic polynomial

$$P(G,\lambda)P(H,\lambda)/(\lambda(\lambda-1)). \tag{1}$$

Thus any two members in $\mathcal{G}_2(G, H)$ are χ-equivalent.

For any integer $k \geq 2$ and non-empty graphs $G_0, G_1, \cdots, G_k$, we can recursively define

$$\mathcal{G}_2(G_0, G_1, \cdots, G_k) = \bigcup_{\substack{0\leq i\leq k \\ G'\in\mathcal{G}_2(G_0,\cdots,G_{i-1},G_{i+1},\cdots,G_k)}} \mathcal{G}_2(G_i, G'). \tag{2}$$

Each graph in $\mathcal{G}_2(G_0, G_1, \cdots, G_k)$ is also called an *edge-gluing* of $G_0, G_1, \cdots, G_k$. By (1), any two graphs in $\mathcal{G}_2(G_0, G_1, \cdots, G_k)$ are χ-equivalent.

Let C_p denote the cycle of order p. It was shown independently in [12] and [13] that if G is χ-equivalent to a graph in $\mathcal{G}_2(C_{i_0}, C_{i_1}, \cdots, C_{i_k})$, then $G \in \mathcal{G}_2(C_{i_0}, C_{i_1}, \cdots, C_{i_k})$. In other words, this family is a χ-equivalence class.

For $k = 2, 3$, the graph $\theta(a_1, a_2, \cdots, a_k)$ is a cycle or a generalized θ-graph respectively, and it is χ-unique in both cases (see [10]). Assume therefore that $k \geq 4$. It is clear that if $a_i = 1$ for some i, say $i = 1$, then $\theta(a_1, a_2, \cdots, a_k)$ is a member of $\mathcal{G}_2(C_{a_2+1}, C_{a_3+1}, \cdots, C_{a_k+1})$ and thus $\theta(a_1, a_2, \cdots, a_k)$ is not χ-unique. Assume therefore that $a_i \geq 2$ for all i. For $k = 4$, Chen, Bao and Ouyang [2] found that $\theta(a_1, a_2, a_3, a_4)$ may not be χ-unique.

Theorem 2.1 ([2]) *(a) Let a_1, a_2, a_3, a_4 be integers with $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. Then $\theta(a_1, a_2, a_3, a_4)$ is χ-unique if and only if $(a_1, a_2, a_3, a_4) \neq (2, b, b+1, b+2)$ for any integer $b \geq 2$.*

(b) The χ-equivalence class of $\theta(2, b, b+1, b+2)$ is

$$\{\theta(2, b, b+1, b+2)\} \cup \mathcal{G}_2(\theta(3, b, b+1), C_{b+2}).$$

□

Thus the problem of the chromaticity of $\theta(a_1, a_2, \cdots, a_k)$ has been completely settled for $k \leq 4$. For $k \geq 5$, we have

Theorem 2.2 ([14]) *For $k \geq 5$, $\theta(a_1, a_2, \cdots, a_k)$ is χ-unique if $a_i \geq k-1$ for $i = 1, 2, \cdots, k$.* □

Theorem 2.3 ([11]) *Let $h \geq s+1 \geq 2$ or $s = h+1$. Then for $k \geq 5$, $\theta(a_1, a_2, \cdots, a_k)$ is χ-unique if $a_2 - 1 = a_1 = h$, $a_j = h+s$ $(j = 3, \cdots, k-1)$, $a_k \geq h+s$ and $a_k \notin \{2h, 2h+s, 2h+s-1\}$.* □

Theorems 2.2 and 2.3 do not include the case where $a_1 = a_2 = \cdots = a_k < k-1$.

Theorem 2.4 ([4], [6] and [13]) *$\theta(a_1, a_2, \cdots, a_k)$ is χ-unique if $k \geq 2$ and $a_1 = a_2 = \cdots = a_k \geq 2$.* □

3 χ-closed families of g.p. trees

A *k-polygon tree* is a graph obtained by edge-gluing a collection of k cycles successively, i.e., a graph in $\mathcal{G}_2(C_{i_1}, C_{i_2}, \cdots, C_{i_k})$ for some integers $i_1, i_2, \cdots, i_k$ with $i_j \geq 3$ for all $j = 1, 2, \cdots, k$. A *polygon-tree* is a k-polygon tree for some integer k with $k \geq 1$. A graph is called a *generalized polygon tree* (g.p. tree) if it is a subdivision of some polygon tree.

Let $\mathcal{GP}$ denote the set of all g.p. trees. Dirac [3] and Duffin [5] proved independently that a 2-connected graph is a g.p. tree if and only if it contains no subdivision of K_4.

A family $\mathcal{S}$ of graphs is said to be *chromatically closed* (or simply χ-*closed*) if $\bigcup_{G\in\mathcal{S}}[G] = \mathcal{S}$. By using Dirac's and Duffin's result, Chao and Zhao [1] obtained the following result.

Theorem 3.1 ([1]) *The set $\mathcal{GP}$ is χ-closed.* □

The family $\mathcal{GP}$ can be partitioned further into χ-closed subfamilies. Let $G \in \mathcal{GP}$. A pair $\{x, y\}$ of non-adjacent vertices of G is called a *communication pair* if there are at least three independent $x - y$ paths in G. Let $c(G)$ denote the number of communication pairs in G. For any integer $r \geq 1$, let $\mathcal{GP}_r$ be the family of all g.p. trees G with $c(G) = r$.

Theorem 3.2 ([13]) *The family $\mathcal{GP}_r$ is χ-closed for every integer $r \geq 1$.* □

Let G be a g.p. tree. We call a pair $\{x, y\}$ of vertices in G a *pre-communication pair* of G if there are at least three independent x-y paths in G. If x and y are non-adjacent, then $\{x, y\}$ is a communication pair. Assume that $c(G) = 1$. Then G is a subdivision of a k-polygon tree H for some $k \geq 2$. It is clear that G and H have the same pre-communication pairs. But not every pre-communication pair in H is a communication pair. Since $c(G) = 1$, only one pre-communication pair in H is transformed into a communication pair in G. If G has only one pre-communication pair, then G is a multibridge graph. Otherwise, G is an edge-gluing of a multibridge graph and some cycles. Therefore

$$\mathcal{GP}_1 = \bigcup_{k\geq 3} \bigcup_{\substack{3\leq t\leq k \\ b_1,b_2,\cdots,b_k\geq 2}} \mathcal{G}_2(\theta(b_1, b_2, \cdots, b_t), C_{b_{t+1}+1}, \cdots, C_{b_k+1}). \tag{3}$$

Hence we have

Lemma 3.1 *Let $a_i \geq 2$ for $i = 1, 2, \cdots, k$, where $k \geq 3$. If $H \sim \theta(a_1, a_2, \cdots, a_k)$, then H is either a k-bridge graph $\theta(b_1, \cdots, b_k)$ with $b_i \geq 2$ for all i or an edge-gluing of a t-bridge graph $\theta(b_1, \cdots, b_t)$ with $b_i \geq 2$ for all i and $k - t$ cycles for some integer t with $3 \leq t \leq k - 1$.* □

Note that for $G \in \mathcal{G}_2(\theta(b_1, b_2, \cdots, b_t), C_{b_{t+1}+1}, \cdots, C_{b_k+1})$,

$$e(G) = v(G) + k - 2. \tag{4}$$

4 A graph function

For any graph G and real number τ, write

$$\Psi(G, \tau) = (-1)^{1+e(G)}(1 - \tau)^{e(G)-v(G)+1}P(G, 1 - \tau). \tag{5}$$

Observe that $\Psi(G, \tau) = \Psi(H, \tau)$ if $G \sim H$. However, the converse is not true. For example, $\Psi(G, \tau) = \Psi(G \cup mK_1, \tau)$ but $G \not\sim G \cup mK_1$ for any $m \geq 1$, where $G \cup mK_1$ is the graph obtained from G by adding m isolated vertices. However, we have

Lemma 4.1 *For graphs G and H, if $G \sim H$, then $\Psi(G,\tau) = \Psi(H,\tau)$; if $v(G) = v(H)$ and $\Psi(G,\tau) = \Psi(H,\tau)$, then $G \sim H$.*

Proof. We need to prove only the second assertion. Observe from (5) that $\Psi(G,\tau)$ is a polynomial in τ with degree $e(G)+1$. Thus $e(G) = e(H)$. Since $v(G) = v(H)$ and $\Psi(G,\tau) = \Psi(H,\tau)$, we have $P(G, 1-\tau) = P(H, 1-\tau)$. Therefore $G \sim H$. $\square$

Thus, by Lemma 4.1, for any graph G, $[G]$ is the set of graphs H such that $v(H) = v(G)$ and $\Psi(H,\tau) = \Psi(G,\tau)$. In this paper, we shall use this property to study the chromaticity of $\theta(a_1, a_2, \cdots, a_k)$. We first derive an expression for $\Psi(\theta(a_1, a_2, \cdots, a_k), \tau)$.

The following lemma is true even if $k = 1$ or $a_i = 1$ for some i.

Lemma 4.2 *For positive integers $k, a_1, a_2, \cdots, a_k$,*

$$\Psi(\theta(a_1, a_2, \cdots, a_k), \tau) = \tau \prod_{i=1}^{k} (\tau^{a_i} - 1) - \prod_{i=1}^{k} (\tau^{a_i} - \tau). \tag{6}$$

Proof. By the deletion-contraction formula for chromatic polynomials, it can be shown that

$$\begin{aligned} & P(\theta(a_1, a_2, \cdots, a_k), \lambda) \\ = \;& \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}} \prod_{i=1}^{k} \left((\lambda-1)^{a_i+1} + (-1)^{a_i+1}(\lambda-1)\right) \\ & + \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k} \left((\lambda-1)^{a_i} + (-1)^{a_i}(\lambda-1)\right). \end{aligned}$$

Let $\tau = 1 - \lambda$. Then

$$\begin{aligned} & (-1)^{1+a_1+a_2+\cdots+a_k}(1-\tau)^{k-1} P(\theta(a_1, a_2, \cdots, a_k), 1-\tau) \\ = \;& \tau \prod_{i=1}^{k} (\tau^{a_i} - 1) - \prod_{i=1}^{k} (\tau^{a_i} - \tau). \end{aligned}$$

Since $v(G) = 2 - k + \sum_{i=1}^{k} a_i$ and $e(G) = \sum_{i=1}^{k} a_i$, by definition of $\Psi(G,\tau)$, (6) is obtained. $\square$

Corollary 4.1 *For positive integers $k, a_1, a_2, \cdots, a_k$,*

$$\begin{aligned} \Psi(\theta(a_1, a_2, \cdots, a_k), \tau) = & (-1)^k(\tau - \tau^k) \\ & + \sum_{\substack{1 \le r \le k \\ 1 \le i_1 < i_2 < \cdots < i_r \le k}} (-1)^{k-r} \left(\tau - \tau^{k-r}\right) \tau^{a_{i_1} + a_{i_2} + \cdots + a_{i_r}}. \end{aligned} \tag{7}$$

$\square$

We are now going to find an expression for $\Psi(H,\tau)$ for any H in

$$\mathcal{G}_2(\theta(b_1,b_2,\cdots,b_t),C_{b_{t+1}+1},\cdots,C_{b_k+1}).$$

Lemma 4.3 *Let G and H be non-empty graphs, and $M\in\mathcal{G}_2(G,H)$. Then*

$$\Psi(M,\tau)=\Psi(G,\tau)\Psi(H,\tau)/((-\tau)(1-\tau)). \tag{8}$$

Proof. Since $v(M)=v(G)+v(H)-2$, $e(M)=e(G)+e(H)-1$ and

$$P(M,\lambda)=P(G,\lambda)P(H,\lambda)/(\lambda(\lambda-1)), \tag{9}$$

by (5), (8) is obtained. □

Lemma 4.4 *Let $k,t,b_1,b_2,\cdots,b_k$ be integers with $3\le t<k$ and $b_i\ge 1$ for $i=1,2,\cdots,k$. If $H\in\mathcal{G}_2(\theta(b_1,b_2,\cdots,b_t),C_{b_{t+1}+1},\cdots,C_{b_k+1})$, then*

$$\Psi(H,\tau)=\tau\prod_{i=1}^{k}(\tau^{b_i}-1)-\prod_{i=1}^{t}(\tau^{b_i}-\tau)\prod_{i=t+1}^{k}(\tau^{b_i}-1). \tag{10}$$

Proof. By (5), we have $\Psi(C_{b_i+1},\tau)=(-\tau)(1-\tau)(\tau^{b_i}-1)$. Thus by (6) and (8), (10) is obtained. □

5 χ-unique multibridge graphs

By Lemma 4.2, we can prove that $\theta(a_1,a_2,\cdots,a_k)\cong\theta(b_1,b_2,\cdots,b_k)$ if $\theta(a_1,a_2,\cdots,a_k)\sim\theta(b_1,b_2,\cdots,b_k)$.

Lemma 5.1 *Let a_i and b_i be integers with $1\le a_1\le a_2\le\cdots\le a_k$ and $1\le b_1\le b_2\le\cdots\le b_k$, where $k\ge 3$. If*

$$\theta(a_1,a_2,\cdots,a_k)\sim\theta(b_1,b_2,\cdots,b_k), \tag{11}$$

then $b_i=a_i$ for $i=1,2,\cdots,k$.

Proof. By Lemma 4.1 and Corollary 4.1, we have

$$\begin{aligned}&\sum_{\substack{1\le r\le k\\ 1\le i_1<i_2<\cdots<i_r\le k}}(-1)^{k-r}\left(\tau-\tau^{k-r}\right)\tau^{a_{i_1}+a_{i_2}+\cdots+a_{i_r}}\\ =&\sum_{\substack{1\le r\le k\\ 1\le i_1<i_2<\cdots<i_r\le k}}(-1)^{k-r}\left(\tau-\tau^{k-r}\right)\tau^{b_{i_1}+b_{i_2}+\cdots+b_{i_r}},\end{aligned} \tag{12}$$

after we cancel the terms $(-1)^k(\tau-\tau^k)$ from both sides. The terms with lowest power in both sides have powers $1+a_1$ and $1+b_1$ respectively. Hence $a_1=b_1$.

Suppose that $a_i = b_i$ for $i = 1, \cdots, m$ but $a_{m+1} \neq b_{m+1}$ for some integer m with $1 \leq m \leq k-1$. Since $a_i = b_i$ for $i = 1, 2, \cdots, m$, by (12), we have

$$\begin{aligned} & \sum_{\substack{1\leq r\leq k \\ 1\leq i_1<i_2<\cdots<i_r\leq k \\ i_r>m}} (-1)^{k-r}\left(\tau-\tau^{k-r}\right)\tau^{a_{i_1}+a_{i_2}+\cdots+a_{i_r}} \\ = & \sum_{\substack{1\leq r\leq k \\ 1\leq i_1<i_2<\cdots<i_r\leq k \\ i_r>m}} (-1)^{k-r}\left(\tau-\tau^{k-r}\right)\tau^{b_{i_1}+b_{i_2}+\cdots+b_{i_r}}. \end{aligned} \tag{13}$$

The terms with lowest power in both sides of (13) have powers $1 + a_{m+1}$ and $1 + b_{m+1}$ respectively. Hence $a_{m+1} = b_{m+1}$, a contradiction. Therefore $b_i = a_i$ for $i = 1, 2, \cdots, k$. □

Let a_i be an integer with $a_i \geq 2$ for $i = 1, 2, \cdots, k$ and suppose that $a_1 \leq a_2 \leq \cdots \leq a_k$. We shall show that $\theta(a_1, a_2, \cdots, a_k)$ is χ-unique if $a_k < a_1 + a_2$. It is well known (see [8]) that

Lemma 5.2 *If $G \sim H$, then $g(G) = g(H)$.* □

Theorem 5.1 *If $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k < a_1 + a_2$, where $k \geq 3$, then $\theta(a_1, a_2, \cdots, a_k)$ is chromatically unique.*

Proof. By Theorem 2.2, we may assume that $a_1 \leq k-2$.

By Lemmas 3.1 and 5.1, it suffices to show that $\theta(a_1, a_2, \cdots, a_k) \not\sim H$ for any graph $H \in \mathcal{G}_2(\theta(b_1, b_2, \cdots, b_t), C_{b_{t+1}+1}, \cdots, C_{b_k+1})$, where t and b_i are integers with $3 \leq t < k$ and $b_i \geq 2$ for $i = 1, 2, \cdots, k$. We may assume that $b_1 \leq b_2 \leq \cdots \leq b_t$ and $b_{t+1} \leq \cdots \leq b_k$.

Suppose that $H \sim \theta(a_1, a_2, \cdots, a_k)$. The girth of $\theta(a_1, a_2, \cdots, a_k)$ is $a_1 + a_2$. Since

$$g(H) = \min\left\{\min_{1\leq i<j\leq t}(b_i + b_j), \min_{t+1\leq i\leq k}(b_i + 1)\right\}, \tag{14}$$

by Lemma 5.2, we have $g(H) = a_1 + a_2$ and

$$\begin{cases} b_i + b_j \geq a_1 + a_2, & 1 \leq i < j \leq t, \\ b_i \geq a_1 + a_2 - 1, & t+1 \leq i \leq k. \end{cases} \tag{15}$$

As $e(H) = e(\theta(a_1, a_2, \cdots, a_k))$, we have

$$a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k. \tag{16}$$

By Lemma 4.1, (6) and (10), we have

$$\tau\prod_{i=1}^{k}(\tau^{a_i} - 1) - \prod_{i=1}^{k}(\tau^{a_i} - \tau) = \tau\prod_{i=1}^{k}(\tau^{b_i} - 1) - \prod_{i=1}^{t}(\tau^{b_i} - \tau)\prod_{i=t+1}^{k}(\tau^{b_i} - 1). \tag{17}$$

We expand both sides of (17), delete $(-1)^k\tau$ from them and keep only the terms with powers at most a_1+a_2. Since $a_i+a_j \geq a_1+a_2$ and $b_i+b_j \geq a_1+a_2$ for all i,j with $1 \leq i < j \leq k$, we have

$$\begin{aligned}
& (-1)^{k-1}\sum_{i=1}^{k}\tau^{a_i+1}+(-1)^{k-1}\tau^k+(-1)^k\sum_{i=1}^{k}\tau^{k-1+a_i} \\
\equiv\ & (-1)^{k-1}\sum_{i=1}^{k}\tau^{b_i+1}+(-1)^{k-1}\tau^t+(-1)^k\sum_{i=1}^{t}\tau^{b_i+t-1} \\
& +(-1)^k\sum_{i=t+1}^{k}\tau^{b_i+t} \qquad (\mathrm{mod}\ \tau^{a_1+a_2+1}). \qquad (18)
\end{aligned}$$

Observe that $b_i+t > a_1+a_2$ for $t+1 \leq i \leq k$ and $k-1+a_i > a_1+a_2$ for $2 \leq i \leq k$. Thus

$$\begin{aligned}
& (-1)^{k-1}\sum_{i=1}^{k}\tau^{a_i+1}+(-1)^{k-1}\tau^k+(-1)^k\tau^{k-1+a_1} \\
\equiv\ & (-1)^{k-1}\sum_{i=1}^{k}\tau^{b_i+1}+(-1)^{k-1}\tau^t+(-1)^k\sum_{i=1}^{t}\tau^{b_i+t-1} \quad (\mathrm{mod}\ \tau^{a_1+a_2+1}).
\end{aligned}$$

Hence

$$\sum_{i=1}^{k}\tau^{a_i+1}+\tau^k+\sum_{i=1}^{t}\tau^{b_i+t-1} \equiv \sum_{i=1}^{k}\tau^{b_i+1}+\tau^t+\tau^{k-1+a_1} \quad (\mathrm{mod}\ \tau^{a_1+a_2+1}). \qquad (19)$$

Since $t \geq 3$, we have $b_i+t-1 > a_1+a_2$ for $i \geq t+1$. If $b_2+t-1 \leq a_1+a_2$, then since $k-1+a_1 > k$, the left side of (19) contains more terms with powers at most a_1+a_2 than does the right side, a contradiction. Hence $b_i+t-1 > a_1+a_2$ for $2 \leq i \leq t$. Therefore

$$\sum_{i=1}^{k}\tau^{a_i+1}+\tau^k+\tau^{b_1+t-1} \equiv \sum_{i=1}^{k}\tau^{b_i+1}+\tau^t+\tau^{k-1+a_1} \quad (\mathrm{mod}\ \tau^{a_1+a_2+1}). \qquad (20)$$

Note that $t \leq a_1+a_2$; otherwise, since $k > t$ and $a_1, b_1 \geq 2$, (20) becomes

$$\sum_{i=1}^{k}\tau^{a_i+1} = \sum_{i=1}^{k}\tau^{b_i+1},$$

which implies the equality of the multisets $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ in contradiction to (17).

Claim 1: There are no i, j such that

$$\{b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_k\} = \{a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_k\}$$

as multisets.

Otherwise, by (16), $\{b_1, \cdots, b_k\} = \{a_1, \cdots, a_k\}$ as multisets, which leads to a contradiction by (17).

Claim 2: $a_2 \geq k-1$.

If $a_2 < k-1$, then $a_1+k-1 > a_1+a_2$. But $a_i+1 \leq a_1+a_2$ for $1 \leq i \leq k$. So, by (20), the multiset $\{a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_k\}$ is a subset of the multiset $\{b_1, \cdots, b_k\}$ for some j with $1 \leq j \leq k$, which contradicts Claim 1.

Claim 3: $a_1 = t-1$.

Since τ^t is a term of the right side of (20), the left side also contains τ^t. But $k > t$, $b_1+t-1 > t$ and, by Claim 2, $a_i+1 \geq k > t$ for $i \geq 2$. Therefore $a_1+1 = t$.

By Claim 3, (20) is simplified to

$$\sum_{i=2}^{k} \tau^{a_i+1} + \tau^k + \tau^{b_1+t-1} \equiv \sum_{i=1}^{k} \tau^{b_i+1} + \tau^{k-1+a_1} \pmod{\tau^{a_1+a_2+1}}. \tag{21}$$

Claim 4: $b_1 = k-1$.

Note that $k < a_1+a_2$, by Claim 2. As τ^k is a term of the left side of (21), the right side also contains this term. Thus $b_i+1 = k$ for some i. If $i > t$, then by (15) and Claims 2 and 3, we

$$b_i \geq a_1+a_2-1 \geq k+t-3 \geq k,$$

a contradiction. Thus $i \leq t$ and $b_1 \leq b_i = k-1$. If $b_1 \leq k-2$, then the right side of (21) has a term with power at most $k-1$. But the left side has no such term, a contradiction. Hence $b_1 = k-1$.

By Claims 3 and 4, we have $\tau^{b_1+t-1} = \tau^{k-1+a_1}$. Thus (21) is further simplified to

$$\sum_{i=2}^{k} \tau^{a_i+1} + \tau^k \equiv \sum_{i=1}^{k} \tau^{b_i+1} \pmod{\tau^{a_1+a_2+1}}. \tag{22}$$

Therefore the multiset $\{a_2, a_3, \cdots, a_k\}$ is a subset of the multiset $\{b_1, b_2, \cdots, b_k\}$, in contradiction to Claim 1.

Therefore $H \not\sim \theta(a_1, a_2, \cdots, a_k)$ and we conclude that $\theta(a_1, a_2, \cdots, a_k)$ is χ-unique. □

6 χ-equivalent graphs

In Section 5, we proved that $\theta(a_1, a_2, \cdots, a_k)$ is χ-unique if

$$\max_{1\leq i\leq k} a_i < \min_{1\leq i<j\leq k}(a_i+a_j). \tag{23}$$

Lemma 6.1 shows that, for any non-negative integer n, there exist examples where the graph $\theta(a_1, a_2, \ldots, a_k)$ is not χ-unique and

$$\max_{1\leq i\leq k} a_i - \min_{1\leq i<j\leq k}(a_i+a_j) = n. \tag{24}$$

Lemma 6.1 *(i) $\theta(2,2,2,3,4) \sim H$ for every $H \in \mathcal{G}_2(\theta(2,2,3), C_4, C_4)$.*

(ii) For $k \geq 4$ and $a \geq 2$, $\theta(k-2, a, a+1, \cdots, a+k-2) \sim H$ for every $H \in \mathcal{G}_2(\theta(k-1, a, a+1, \cdots, a+k-3), C_{a+k-2})$.

(iii) For $k \geq 5$, $\theta(2,3,\cdots,k-1,k,k-3) \sim H$ for every graph H in $\mathcal{G}_2(\theta(2,3,\cdots,k-1),C_{k-1},C_k)$. □

It is straightforward to verify Lemma 6.1 by using Lemmas 4.1, 4.2 and 4.4.

It is natural to ask the following question: for which choices of $(a_1,a_2,\ldots,a_k)$ satisfying $k \geq 5$ and

$$\max_{1\leq i\leq k} a_i = \min_{1\leq i<j\leq k}(a_i+a_j)$$

is the graph $\theta(a_1,a_2,\ldots,a_k)$ chromatically unique? If $\theta(a_1,a_2,\cdots,a_k)$ is not χ-unique, what is its χ-equivalence class? The solution to this question will be given in another paper.

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