On Feasible Sets of Mixed Hypergraphs

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Abstract

A mixed hypergraph H is a triple $(V, \mathcal{C}, \mathcal{D})$ where V is the vertex set and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V, called $\mathcal{C}$-edges and $\mathcal{D}$-edges. A vertex coloring of H is proper if each $\mathcal{C}$-edge contains two vertices with the same color and each $\mathcal{D}$-edge contains two vertices with different colors. The spectrum of H is a vector $(r_1, \ldots, r_m)$ such that there exist exactly r_i different colorings using exactly i colors, $r_m \geq 1$ and there is no coloring using more than m colors. The feasible set of H is the set of all i's such that $r_i \neq 0$.

We construct a mixed hypergraph with $O(\sum_i \log r_i)$ vertices whose spectrum is equal to $(r_1, \ldots, r_m)$ for each vector of non-negative integers with $r_1 = 0$. We further prove that for any fixed finite sets of positive integers $A_1 \subset A_2$ ($1 \notin A_2$), it is NP-hard to decide whether the feasible set of a given mixed hypergraph is equal to A_2 even if it is promised that it is either A_1 or A_2. This fact has several interesting corollaries, e.g., that deciding whether a feasible set of a mixed hypergraph is gap-free is both NP-hard and coNP-hard.

1 Introduction

Graph coloring problems are intensively studied both from the theoretical point view and the algorithmic point of view. A *hypergraph* is a pair $(V, \mathcal{E})$ where $\mathcal{E}$ is a family of subsets

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of V of size at least 2. The elements of V are called *vertices* and the elements of $\mathcal{E}$ are called *edges*. A *mixed hypergraph* H is a triple $(V, \mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V of size at least 2. The elements of $\mathcal{C}$ are called $\mathcal{C}$*-edges* and the elements of $\mathcal{D}$ are called $\mathcal{D}$*-edges*. A *proper ℓ-coloring* c of H is a mapping $c : V \to \{1, \ldots, \ell\}$ such that there are two vertices with $\mathcal{D}$ifferent colors in each $\mathcal{D}$-edge and there are two vertices with a $\mathcal{C}$ommon color in each $\mathcal{C}$-edge. A proper coloring c is a *strict ℓ-coloring* if it uses all ℓ colors. A mixed hypergraph is *colorable* if it has a proper coloring. Mixed hypergraphs were introduced in [23]. The concept of mixed hypergraphs can find its applications in different areas, e.g. list-coloring of graphs [14], graph homomorphisms [9], coloring block designs [2, 3, 16, 17, 18], etc. The importance and interest of the concept is witnessed by a recent monograph on the subject by Voloshin [21]. As an example, we present here the following construction described in [14]: Let G be a graph and let L be a function which assigns each vertex a set of colors. A coloring c of G is a *proper list-coloring with respect to the lists L* if $c(v) \in L(v)$ for each vertex v of G and $c(u) \neq c(v)$ for each edge uv of G. Let $\mathcal{L}$ be the union of the lists of all the vertices of G. Consider a mixed hypergraph H with the vertex set $V(G) \cup \mathcal{L}$ and the following edges: a $\mathcal{D}$-edge $\{u, v\}$ for each $uv \in E(G)$, a $\mathcal{D}$-edge $\{x, y\}$ for any $x, y \in \mathcal{L}$ $(x \neq y)$ and a $\mathcal{C}$-edge $\{v\} \cup L(v)$ for each vertex of G. H has a proper coloring iff G has a proper list-coloring. Similar constructions have been found by the author [9] for graph homomorphisms, the channel assignment problem, $L(p, q)$-labelings of graphs and some other graph coloring problems.

The *feasible set* $\mathcal{F}(H)$ of a mixed hypergraph H is the set of all ℓ's such that there exists a strict ℓ-coloring of H. The *(lower) chromatic number* $\chi(H)$ of H is the minimum number contained in $\mathcal{F}(H)$ and the *upper chromatic number* of $\bar{\chi}(H)$ of H is the maximum number. The feasible set of H is *gap-free (unbroken)* if $\mathcal{F}(H) = [\chi(H), \bar{\chi}(H)]$ where $[a, b]$ is the set of all the integers between a and b (inclusively). If the feasible set of H contains a gap, we say it is *broken*. The *spectrum* of a mixed hypergraph H is the vector $(r_1, \ldots, r_{\bar{\chi}(H)})$ where r_ℓ is the number of different strict ℓ-colorings of H. Two colorings c_1 and c_2 are considered to be different if there is no permutation of colors changing one of them to the other, i.e., it is not true that $c_1(u) = c_1(v)$ iff $c_2(u) = c_2(v)$ for each two vertices u and v. We remark that the spectrum is usually defined to be a vector $(r_1, \ldots, r_n)$ where n is the number of vertices of the mixed hypergraph, but we prefer using the definition without trailing zeroes in the vector. If $\mathcal{F}$ is a set of positive integers, we say that a mixed hypergraph H is a *realization* of $\mathcal{F}$ if $\mathcal{F}(H) = \mathcal{F}$. A mixed hypergraph H is a *one-realization* of $\mathcal{F}$ if it is a realization of $\mathcal{F}$ and all the entries of the spectrum of H are either 0 or 1.

A necessary and sufficient condition on a set of positive integers to be the feasible set of a mixed hypergraph was proved in [6]:

Theorem 1 *A set $\mathcal{F}$ of positive integers is a feasible set of a mixed hypergraph iff $1 \notin \mathcal{F}$ or $\mathcal{F}$ is an interval. If $1 \in \mathcal{F}$, then all the mixed hypergraphs with this feasible set contain only $\mathcal{C}$-edges.*

In particular, there exists a mixed hypergraph such that its feasible set contains a gap. On the other hand, it was proved that feasible sets of mixed hypertrees [10], mixed strong hypercacti [13] and of mixed hypergraphs with maximum degree two [11, 12] are gap-free. Feasible sets of mixed hypergraphs with maximum degree three need not to be gap-free. The feasible sets of planar mixed hypergraphs, i.e., hypergraphs whose bipartite incidence graphs of their vertices and edges are planar [4, 15], are exactly intervals $[k_1, k_2], 1 \leq k_1 \leq 4, k_1 \leq k_2$ and sets $\{2\} \cup [4, k], k \geq 4$ as proved in [7].

Necessary or sufficient conditions for a vector to be a spectrum of a mixed hypergraph were not addressed in detail so far. However, Voloshin [22] conjectured a sufficient condition for a vector to be the spectrum of a mixed hypergraph (Conjecture 2 in [22]): *If $n_0, \dots, n_t$ is a sequence of positive integers such that $n_i \geq (n_{i-1}+n_{i+1})/2$ for $1 \leq i \leq t-1$ and $\max\{n_{\lfloor t/2 \rfloor}, n_{\lceil t/2 \rceil}\} = \max_{0 \leq i \leq t}\{n_i\}$, then there exists a mixed hypergraph H such that $\chi(H)+t = \bar{\chi}(H)$ and H allows exactly n_i different strict $(\chi(H)+i)$-colorings ($0 \leq i \leq t$).* We prove this conjecture. In fact, Theorem 3 implies that the only hypothesis needed is that $n_0, \dots, n_t$ is a sequence of non-negative integers. Let us remark at this point that Conjecture 1 from [22] on co-perfect mixed hypergraphs was disproved in [8].

We study several problems posed in [22] (Problem 10, 11, Conjecture 2) and in [6]. In particular, we are interested in the size of the smallest (one-)realization of a given feasible set. In [6], two constructions of a mixed hypergraph with a given feasible set $\mathcal{F}$ are presented, but both of them can have exponentially many vertices in terms of $\max \mathcal{F}$ and $|\mathcal{F}|$. The second construction from [6] does not even give one-realization of $\mathcal{F}$. We present an algorithmic construction (Theorem 2) which gives a small one-realization for a given feasible set $\mathcal{F}$. The number of vertices of this realization is at most $|\mathcal{F}|+2\max\mathcal{F}-1$ and the number of edges is cubic in the number of vertices.

Theorem 2 from Section 2 can be restated as follows: Let $(r_1, \dots, r_m)$ be a vector such that $r_1 = 0$ and $r_i \in \{0, 1\}$ for $2 \leq i \leq m$. Then, there exists a mixed hypergraph H such that the spectrum of H is $(r_1, \dots, r_m)$. Note that the condition $r_1 = 0$ is the condition $1 \notin \mathcal{F}$ mentioned earlier. We generalize this theorem in Section 3. We prove that for each vector $(r_1, \dots, r_m)$ of non-negative integers such that $r_1 = 0$ there exists a mixed hypergraph such that its spectrum is equal to $(r_1, \dots, r_m)$ (Theorem 3). The number of vertices of the mixed hypergraph from Theorem 3 is $2m + 2\sum_{i=1, r_i \neq 0}^{m}(1 + \lfloor \log_2 r_i \rfloor)$ and the number of its edges is cubic in the number of its vertices. Theorem 3 provides an affirmative answer to Conjecture 2 from [22] which was mentioned above.

We deal with complexity questions related to feasible sets of mixed hypergraphs in Section 4. We prove that for any fixed finite sets of positive integers $A_1 \subset A_2$, it is NP-hard to decide whether the feasible set of a given mixed hypergraph H is equal to A_2 even if it is promised that $\mathcal{F}(H)$ is either A_1 or A_2. This theorem has several interesting corollaries: It is NP-complete to decide whether a given mixed hypergraph is colorable, it is both NP-hard and coNP-hard for a fixed non-empty finite set of positive integers A to decide whether the feasible set of a mixed hypergraph is equal to A, it is both NP-hard and coNP-hard to decide whether the feasible set of a given mixed hypergraph is gap-free. This particular result was previously obtained in [11]. It was also known before that it is NP-hard to decide whether a given mixed hypergraph is uniquely colorable [20] and that it is NP-hard to compute the upper chromatic number even when restricted to several special classes of mixed hypergraphs [1, 10, 12, 19].

There is also no polynomial-time $o(n)$-approximation algorithm for the lower or the upper chromatic number unless $P = NP$ where n is the number of vertices of an input mixed hypergraph. We remark there is an $O(n\frac{(\log\log n)^2}{\log^3 n})$-approximation algorithm for the chromatic number of ordinary graphs [5]. Recall that an algorithm for a maximization (minimization) problem is said to be K-*approximation* algorithm if it always finds a solution whose value is at least OPT/K (at most $K \cdot \mathrm{OPT}$) where OPT is the value of the optimum solution.

2 Small realizations of feasible sets

If $\mathcal{F} = \{m\}$, then the complete graph of order m is the one-realization of $\mathcal{F}$ with the fewest number of vertices. In this section, we present a one-realization with few vertices for the case $|\mathcal{F}| \geq 2$:

Theorem 2 *Let $\mathcal{F}$ be a finite non-empty set of positive integers with $1 \notin \mathcal{F}$. There exists a mixed hypergraph with at most $|\mathcal{F}|+2\max\mathcal{F}-\min\mathcal{F}$ vertices whose feasible set is $\mathcal{F}$ and every entry of its spectrum is 0 or 1. The number of the edges of this mixed hypergraph is cubic in the number of its vertices.*

Proof: The proof proceeds by induction on $\max\mathcal{F}$. If $2 \notin \mathcal{F}$, then let H' be a one-realization of $\mathcal{F}' = \{i-1 | i \in \mathcal{F}\}$. Let H be the mixed hypergraph obtained from H' by adding a vertex x and $\mathcal{D}$-edges $\{x, v\}$ for all $v \in V(H')$. This operation was also used in [6] under the name "elementary shift". It is clear that proper ℓ-colorings of H are in one-to-one correspondence with proper $(\ell+1)$-colorings of H', since the color of the vertex x has to be different from the color of any other vertex and it does not affect coloring of any edge except for the added $\mathcal{D}$-edges of size two. Hence, H is one-realization of $\mathcal{F}$. The number of vertices of H is at most $1+|\mathcal{F}'|+2\max\mathcal{F}'-\min\mathcal{F}' \leq |\mathcal{F}|+2\max\mathcal{F}-\min\mathcal{F}$.

It remains to consider the case when $\min\mathcal{F} = 2$. The case of $\mathcal{F} = \{2\}$ is described before Theorem 2. In the rest, we assume that $\max\mathcal{F} > 2$. For this purpose, we define a mixed hypergraph H with the vertex set $\{v_2^+, \ldots, v_m^+, v_1^-, \ldots, v_m^-, v_1^\oplus\} \cup \{v_i^\oplus | i \in \mathcal{F} \setminus \{2, m\}\}$ where $m = \max\mathcal{F}$. Let $\mathcal{F}(H) = \{c_1, \ldots, c_k\}$, $c_1 < \ldots < c_k$, and set $c_1' = 1$ and $c_i' = c_i$ for $2 \leq i \leq k$. Next, we describe the edges of H. The mixed hypergraph H contains the following edges for each l, $2 \leq l \leq k$:

$$\{v_i^-, v_j^+\} \text{ is a } \mathcal{D}\text{-edge for } c_{l-1}' \leq i \leq c_l' \text{ and } c_{l-1}' < j \leq c_l' \text{ with } i \neq j \quad (1)$$

$$\{v_i^-, v_{c_{l-1}'}^\oplus\} \text{ is a } \mathcal{D}\text{-edge for } c_{l-1}' < i \leq c_l' \quad (2)$$

$$\{v_i^+, v_i^-, v_j^+\} \text{ is a } \mathcal{C}\text{-edge for } c_{l-1}' < i, j \leq c_l' \text{ and } i \neq j \quad (3)$$

$$\{v_i^+, v_i^-, v_{c_{l-1}'}^\oplus\} \text{ is a } \mathcal{C}\text{-edge for } c_{l-1}' < i \leq c_l' \quad (4)$$

$$\{v_i^+, v_i^-, v_j^-\} \text{ is a } \mathcal{C}\text{-edge for } c_{l-1}' < i, j \leq c_l' \text{ and } i \neq j \quad (5)$$

$$\{v_{c_{l-1}'}^\oplus, v_{c_{l-1}'}^-, v_j^+\} \text{ is a } \mathcal{C}\text{-edge for } c_{l-1}' < j \leq c_l' \quad (6)$$

$$\{v_{c_{l-1}'}^\oplus, v_{c_{l-1}'}^-, v_j^-\} \text{ is a } \mathcal{C}\text{-edge for } c_{l-1}' < j \leq c_l' \quad (7)$$

$$\{v^+_{c'_l}, v^-_{c'_l}, v^\oplus_{c'_l}\} \text{ is a } \mathcal{C}\text{-edge if } l < k \tag{8}$$

$$\{v^\oplus_{c'_{l-1}}, v^+_{c'_l}, v^\oplus_{c'_l}\} \text{ is a } \mathcal{C}\text{-edge if } l < k \tag{9}$$

$$\{v^-_{c'_{l-1}}, v^-_{c'_l}, v^\oplus_{c'_l}\} \text{ is a } \mathcal{D}\text{-edge if } l < k \tag{10}$$

$$\{v^-_i, v^+_j, v^-_j\} \text{ is a } \mathcal{D}\text{-edge for } 1 \le i \le c'_l\text{ , } c'_{l-1} < j \le c'_l \text{ and } i \ne j \tag{11}$$

We prove that the mixed hypergraph H has the properties claimed in the statement of the theorem. Note that H is exactly the mixed hypergraph H^k defined in the next paragraph. In the rest, we slightly abuse the notation and we call the $\mathcal{D}$-edges described in (1) just $\mathcal{D}$-edges (1) and we call other kinds of edges in a similar way.

Let H^l be the mixed hypergraph obtained from by H (for $l \ge 2$) restricting to the vertices $\{v^+_2, \ldots, v^+_{c'_l}, v^-_1, \ldots, v^-_{c'_l}, v^\oplus_1\} \cup \{v^\oplus_i | i \in \mathcal{F} \wedge 3 \le i < c'_l\}$. The edges of H^l are those edges of H which are fully contained in the vertex set of H^l. We prove that the following statements hold for all l, $2 \le l \le k$:

1. $\mathcal{F}(H^l) = \{c_1, \ldots, c_l\}$

2. Any proper coloring c of H^l with $c(v^+_{c'_l}) \ne c(v^-_{c'_l})$ uses less than c_l colors. In addition, c satisfies that $c(v^\oplus_{c'_{l-1}}) = c(v^+_{c'_l})$, $c(v^-_{c'_{l-1}}) = c(v^-_{c'_l}) \ne c(v^+_{c'_l})$.

3. Any proper coloring c of H^l with $c(v^+_{c'_l}) = c(v^-_{c'_l})$ uses exactly c_l colors. In addition, the coloring c colors the vertices $v^-_1, \ldots, v^-_{c_l}$ with mutually different colors and the colors of v^-_i and v^+_i (and $v^\oplus_i$ if it exists) are the same.

4. There is exactly one proper coloring using exactly λ colors for each $\lambda \in \mathcal{F}(H^l)$.

These four claims are proved simultaneously by induction on l.

We first deal with the case that $l = 2$. Let c be a proper coloring of H^2. If $c(v^\oplus_1) \ne c(v^-_1)$, then this coloring uses exactly two colors on the vertices $v^-_1, \ldots, v^-_{c'_2}, v^\oplus_1, v^+_2, \ldots, v^+_{c'_2}$ due to the presence of $\mathcal{C}$-edges (6) and (7). Furthermore, the $\mathcal{D}$-edges (1) and (2) force that $c(v^-_1) = c(v^-_2) = \ldots = c(v^-_{c'_2})$ and $c(v^+_1) = c(v^+_2) = \ldots = c(v^+_{c'_2}) = c(v^\oplus_1)$ Thus, the vertices are colored as described in the second claim.

Let us now suppose that $c(v^\oplus_1) = c(v^-_1)$. If $c(v^+_i) \ne c(v^-_i)$ for some $2 \le i \le c'_2$, then $c(v^\oplus_1) \ne c(v^-_1)$ due to the presence of $\mathcal{C}$-edges (4) and (5) and $\mathcal{D}$-edges (1) and (2). Thus $c(v^+_i) = c(v^-_i)$ for all $2 \le i \le c'_2$. The colors of $c(v^-_i)$ for $1 \le i \le c'_2$ are mutually distinct due to the presence of $\mathcal{D}$-edges (1) and (2). We may infer that any such coloring c assigns c'_2 colors to the vertices $v^-_1, \ldots, v^-_{c'_2}, v^\oplus_1, v^+_2, \ldots, v^+_{c'_2}$. Hence, the coloring c uses exactly $c_2 = c'_2$ colors. This finishes the proof of all the four claims for H^2. It is straightforward to check that the two given colorings of H^2 are proper.

Let us prove the claims 1, 2, 3 and 4 for H^l ($l \ge 3$) assuming them proved for H^{l-1}. Consider a proper coloring c of a mixed hypergraph H^l. If $c(v^-_{c'_{l-1}}) \ne c(v^+_{c'_{l-1}})$, then the $\mathcal{C}$-edge (8) and the $\mathcal{D}$-edge (10) together with the second claim assure that $c(v^+_{c'_{l-1}}) = c(v^\oplus_{c'_{l-1}})$. Note that in this case c uses less than c_{l-1} colors to color vertices of

H^{l-1}. If $c(v^-_{c'_{l-1}}) = c(v^+_{c'_{l-1}})$, then $c(v^-_{c'_{l-2}}) = c(v^\oplus_{c'_{l-2}}) \neq c(v^-_{c'_{l-1}})$. Thus, the color $c(v^\oplus_{c'_{l-1}})$ is either $c(v^-_{c'_{l-1}}) = c(v^+_{c'_{l-1}})$ or $c(v^-_{c'_{l-2}}) = c(v^\oplus_{c'_{l-2}})$ due to the presence of the $\mathcal{C}$-edge (9) (and both is possible). In both cases, the coloring c must use exactly c_{l-1} colors to color vertices of H^{l-1}.

We distinguish two cases (similar to those above): $c(v^-_{c'_{l-1}}) \neq c(v^\oplus_{c'_{l-1}})$ and $c(v^-_{c'_{l-1}}) = c(v^\oplus_{c'_{l-1}})$. If $c(v^-_{c'_{l-1}}) \neq c(v^\oplus_{c'_{l-1}})$, then the same argumentation as used before yields that $c(v^-_{c'_{l-1}}) = \ldots = c(v^-_{c'_l})$ and $c(v^\oplus_{c'_{l-1}}) = c(v^+_{c'_{l-1}+1}) = \ldots = c(v^+_{c'_l})$. On the other hand, if $c(v^-_{c'_{l-1}}) = c(v^\oplus_{c'_{l-1}})$, then we can again infer that $c(v^\oplus_{c'_{l-1}}) = c(v^-_{c'_{l-1}})$, $c(v^+_{c'_{l-1}+1}) = c(v^-_{c'_{l-1}+1}) \neq \ldots \neq c(v^+_{c'_l}) = c(v^-_{c'_l})$. The colors $c(v^-_{c'_{l-1}}), \ldots, c(v^-_{c'_l})$ are also mutually distinct because of the $\mathcal{D}$-edges (1) and (2) and they are different from colors $c(v^-_1), \ldots, c(v^-_{c'_{l-1}-1})$ due to the presence of $\mathcal{D}$-edges (11) and the third claim used for H^{l-1}. This proves the first, the second and the third claim for H^l. We again leave a straightforward check that all the described colorings are proper. As to the fourth claim: If $c(v^-_{c'_{l-1}}) = c(v^\oplus_{c'_{l-1}})$, then exactly c_{l-1} colors are used to color the vertices of H^{l-1} and new $c_l - c_{l-1}$ colors are used to color the vertices of $v^+_{c'_{l-1}+1}, \ldots, v^+_{c'_l}$ and $v^-_{c'_{l-1}+1}, \ldots, v^-_{c'_l}$ due to the presence of $\mathcal{D}$-edges 11. On the other hand, if $c(v^-_{c'_{l-1}}) \neq c(v^\oplus_{c'_{l-1}})$, there exists unique extension of any proper coloring of H^{l-1} to H^l. This finishes the proof of all the four claims on H^l.

We conclude that $H = H^k$ has the desired properties. The bound on the number of edges follows from the fact that each edge has size at most three.

■

We immediately have the following corollary of Theorems 1 and 2:

Corollary 1 *There exists a polynomial-time algorithm which for a given set $\mathcal{F}$ decides whether it is a feasible set of some mixed hypergraph and if so it outputs a mixed hypergraph H such that $\mathcal{F}(H) = \mathcal{F}$.*

Proof: If $1 \notin \mathcal{F}$, then the algorithm returns the construction from Theorem 2. If $1 \in \mathcal{F}$ and $\mathcal{F}$ is not interval, then the algorithm returns that no such mixed hypergraph exists (Theorem 1). If $1 \in \mathcal{F}$ and $\mathcal{F}$ is an interval, then the algorithm outputs a mixed hypergraph consisting of $\max \mathcal{F}$ vertices and no edges.

■

3 Realizations of spectra

We first slightly alter the construction from Theorem 2:

Lemma 1 *Let $\mathcal{F} = \{c_1, \ldots, c_k\}$ be a set of positive integers with $1 \notin \mathcal{F}$. There exists a mixed hypergraph H^* with at most $2(|\mathcal{F}| + \max \mathcal{F})$ vertices which is a one-realization of $\mathcal{F}$. Moreover, H^* contains $3l$ vertices $w^+_i, w^\oplus_i, w^\ominus_i$ ($1 \le i \le k$) with the following property: Let c be any proper coloring of H^*, then*

- *The vertices $w_i^+, w_i^\oplus, w_i^\ominus$ are colored by c with exactly two colors for each i.*
- $c(w_i^\ominus) = c(w_i^+) \neq c(w_i^\oplus)$ *iff c uses exactly c_i colors.*
- $c(w_i^\ominus) \neq c(w_i^+) = c(w_i^\oplus)$ *iff c does not use exactly c_i colors.*

Proof: If $\mathcal{F} = \{2\}$, then consider the following mixed hypergraph H^*: $V(H^*) = \{w_1^+, w_1^\oplus, w_1^\ominus\}$ where $\{w_1^\ominus, w_1^+\}$ is the only $\mathcal{C}$-edge of H^* and $\{w_1^\ominus, w_1^\oplus\}$ is the only $\mathcal{D}$-edge of H^*.

Assume now that $2 \in \mathcal{F}$ and $\mathcal{F} \neq \{2\}$. The case $2 \notin \mathcal{F}$ is considered later. We extend the construction from the proof of Theorem 2. Let H^k be the mixed hypergraph obtained in the construction and let us continue using notation from the proof of Theorem 2. We add a vertex v_1^+ together with a $\mathcal{C}$-edge $\{v_1^+, v_1^-\}$ and a vertex $v_{c'_k}^\oplus$ together with a $\mathcal{C}$-edge $\{v_{c'_{k-1}}^\oplus, v_{c'_k}^\oplus\}$. It is routine to check that the following two claims hold:

- $c(v_{c'_i}^-) = c(v_{c'_i}^+) \neq c(v_{c'_i}^\oplus)$ iff c uses exactly c_i colors.
- $c(v_{c'_i}^+) = c(v_{c'_i}^\oplus)$ iff c does not use exactly c_i colors.

Let $w_i^+ = v_{c'_i}^+$, $w_i^- = v_{c'_i}^-$ and $w_i^\oplus = v_{c'_i}^\oplus$. We add new vertices $w_i^\ominus$ for all $1 \le i \le k$ to the mixed hypergraph together with $\mathcal{C}$-edges $\{w_i^\ominus, w_i^+, w_i^\oplus\}$ and $\{w_i^\ominus, w_i^+, w_i^-\}$ for all $1 \le i \le k$, $\mathcal{C}$-edges $\{w_i^\ominus, w_i^-, v_1^-\}$ for all $2 \le i \le k$ and $\mathcal{D}$-edges $\{w_i^\ominus, w_i^\oplus\}$ for all $1 \le i \le k$. We further add a $\mathcal{C}$-edge $\{w_1^\ominus, w_1^-, v_2^-\}$. The resulting mixed hypergraph is H^*.

Let c be a proper coloring of H^*. If $c(w_i^+) \neq c(w_i^\oplus)$ (and thus $c(w_i^+) = c(w_i^-)$), then the $\mathcal{C}$-edge $\{w_i^\ominus, w_i^+, w_i^\oplus\}$ and the $\mathcal{D}$-edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(w_i^+) = c(w_i^-)$ (and this extension is possible) — this describes the case when the coloring c uses exactly c_i colors. Let us assume further $c(w_i^+) = c(w_i^\oplus)$. If $c(w_i^+) \neq c(w_i^-)$, then the $\mathcal{C}$-edge $\{w_i^\ominus, w_i^+, w_i^-\}$ and the $\mathcal{D}$-edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(w_i^-)$ (and this extension is possible). If $c(w_i^+) = c(w_i^\oplus) = c(w_i^-)$, then the $\mathcal{C}$-edge $\{w_i^\ominus, w_i^-, v_1^-\}$ (the $\mathcal{C}$-edge $\{w_1^\ominus, w_1^-, v_2^-\}$ in case $i = 1$) and the $\mathcal{D}$-edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(v_1^-)$ (the color $c(v_2^-)$). This requires that $c(v_1^-) \neq c(w_i^+) = c(w_i^\oplus) = c(w_i^-)$ ($c(v_2^-) \neq c(w_i^\oplus)$ where $w_i^\oplus = v_1^\oplus$, since i is 1 in this case). The last non-equality is assured by the presence of the $\mathcal{D}$-edge (11) ($\mathcal{D}$-edge (2)) in the construction of Theorem 2. This implies that each coloring c of H^k can be uniquely extended to H^*.

It is straightforward to check that all the three properties stated by the lemma hold. The second and the third one are established due to the presence of a $\mathcal{C}$-edge $\{w_i^\ominus, w_i^+, w_i^-\}$ and a $\mathcal{D}$-edge $\{w_i^\ominus, w_i^\oplus\}$ ($1 \le i \le k$) and due to the analogous claims stated in the previous paragraph for $v_{c'_i}^+$ and $v_{c'_i}^\oplus$. The first one is established by the presence of the $\mathcal{C}$-edges $\{w_i^\ominus, w_i^+, w_i^-\}$ and $\mathcal{D}$-edges $\{w_i^\ominus, w_i^\oplus\}$.

The final case to consider is that $2 \notin \mathcal{F}$. In this case, we first construct a mixed hypergraph for $\mathcal{F}' = \{i - 1 | i \in \mathcal{F}\}$ and then add a new vertex x together with $\mathcal{D}$-edges $\{x, v\}$ for all vertices v as in the beginning of the proof of Theorem 2.

∎

Lemma 2 *Let l be a given positive integer. There is a mixed hypergraph H_l with three special vertices w^+, $w^\ominus$, $w^\oplus$ which satisfies: Let c be any precoloring of w^+, $w^\ominus$ and $w^\oplus$ using two colors such that $c(w^\ominus) \neq c(w^\oplus)$, then:*

- *Any extension of c to a proper coloring of H_l uses no additional colors.*
- *If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can be uniquely extended to a proper coloring of H_l.*
- *If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to exactly l different proper colorings of H_l.*

The number of vertices of H_l does not exceed $3 + 2\lfloor \log_2 l \rfloor$.

Proof: The proof proceeds by induction on l. The statement is trivial for $l = 1$. We distinguish two cases:

- **The number l is even.**
  Let H_l be the mixed hypergraph obtained from $H_{l/2}$ by adding a new vertex x, a $\mathcal{C}$-edge $\{w^\oplus, w^\ominus, x\}$ and a $\mathcal{D}$-edge $\{w^\oplus, w^+, x\}$. If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can be extended uniquely to $H_{l/2}$ and also to x, since the added edges force that $c(x) = c(w^\ominus)$. If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to $l/2$ different proper colorings to $H_{l/2}$ and it can be extended by setting $c(x)$ to either $c(w^\oplus)$ or $c(w^\ominus)$. Altogether, we obtain l different extensions.
- **The number l is odd.**
  Let $l = 2t + 1$ and consider the mixed hypergraph H_t with the properties described in the statement of the lemma. Let $w'^+, w'^\oplus, w'^\ominus$ be the special vertices of H_t. The mixed hypergraph H_l is constructed as follows: We set the vertex $w^\oplus$ to be $w'^\ominus$ and $w^\ominus$ to $w'^\oplus$. In addition, new vertices w^+ and x are introduced. Now add $\mathcal{C}$-edges $\{w^\oplus, w^\ominus, w'^+\}$ and $\{w^\oplus, w^\ominus, x\}$, and $\mathcal{D}$-edges $\{w^\oplus, w^+, w'^+\}$ and $\{w^\ominus, w'^+, x\}$. If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then the added $\mathcal{C}$-edges and $\mathcal{D}$-edges force that $c(w'^+) = c(w^\ominus)$ and $c(x) = c(w^\oplus)$. The coloring c can be uniquely extended to the remaining vertices of H_t. If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then $c(w'^+)$ can be either $c(w^\oplus)$ or $c(w^\ominus)$. We consider these two possibilities. If $c(w'^+)$ is $c(w^\ominus)$, then $c(x)$ has to be $c(w^\oplus)$ and c can be uniquely extended to the remaining vertices of H_t. If $c(w'^+)$ is $c(w^\oplus)$, then $c(x)$ can be either $c(w^\oplus)$ or $c(w^\ominus)$ and c can be extended in t different ways to the remaining vertices of H_t. Thus, c can be extended altogether in $2t + 1 = l$ different ways.

The bound on the number of vertices of H_l is obviously fulfilled in both the cases.

■

We combine Lemmas 1 and 2 to get the main result of this section:

Theorem 3 *Let $(r_1, \ldots, r_m)$ be any vector of non-negative integers such that $r_1 = 0$. Then there exists a mixed hypergraph with at most $2m + 2\sum_{i=1, r_i \neq 0}^{m}(1 + \lfloor \log_2 r_i \rfloor)$ vertices such that its spectrum is equal to $(r_1, \ldots, r_m)$. Moreover, the number of edges of this mixed hypergraph is cubic in the number of its vertices.*

Proof: Let $\mathcal{F} = \{j | r_j \neq 0\}$ and let H^* be the mixed hypergraph from Lemma 1. We keep the notation of Lemma 1. We apply the following procedure for each $c_i \in \mathcal{F}(H^*)$: We add a copy of $H_{r_{c_i}}$ from Lemma 2 to H^* and we identify vertices w_i^+ and w^+, $w_i^\oplus$ and $w^\oplus$ and $w_i^\ominus$ and $w^\ominus$. Lemmas 1 and 2 now yield that the spectrum of the just constructed mixed hypergraph is $(r_1, \ldots, r_m)$. The bound on the number of vertices easily follows from counting the number of the vertices of H^* and the vertices of $H_{r_{c_i}}$ (and realizing that some of the vertices have been identified). The bound on the number of edges follows from the fact that each edge has size at most three.

■

4 Complexity results

The main theorem of this section is proved in a similar way as Theorem 3, except that instead of Lemma 2, we use the following lemma:

Lemma 3 *Let Φ be a given formula with clauses of size three and let n be the number of variables and m the number of clauses of Φ. There exists a mixed hypergraph H_Φ with three special vertices w^+, $w^\ominus$, $w^\oplus$ which satisfies: Let c be any precoloring of w^+, $w^\ominus$ and $w^\oplus$ using two colors such that $c(w^\ominus) \neq c(w^\oplus)$, then:*

- *Any extension of c to a proper coloring of H_Φ uses no additional colors.*
- *If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can always be extended to a proper coloring of H_Φ.*
- *If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to a proper colorings of H_Φ iff Φ is satisfiable.*

H_Φ has $2n + 3$ vertices and $3n + m$ edges.

Proof: Let $x_1, \ldots, x_m$ be the variables of the given formula. Let H_Φ be a mixed hypergraph with vertices $w^+, w^\ominus, w^\oplus, v_1^{\mathrm{T}}, v_1^{\mathrm{F}}, \ldots, v_n^{\mathrm{T}}, v_n^{\mathrm{F}}$ and the following edges:

- $\mathcal{C}$-edges $\{w^\oplus, w^\ominus, v_i^{\mathrm{T}}\}$ and $\{w^\oplus, w^\ominus, v_i^{\mathrm{F}}\}$ for $1 \leq i \leq n$,
- $\mathcal{D}$-edges $\{v_i^{\mathrm{T}}, v_i^{\mathrm{F}}\}$ for $1 \leq i \leq n$ and
- $\mathcal{D}$-edges $\{w^\ominus, w^+, w_i^X, w_j^Y, w_k^Z\}$ for each clause of the formula containing the variables x_i, x_j and x_k where $X = \mathrm{T}$ if the occurrence of x_i in the clause is positive and $X = \mathrm{F}$ otherwise; Y and Z are set in the same manner.

The bounds on the size of H_Φ are clearly fulfilled.

Any extension of a precoloring c of the vertices $w^+, w^\ominus, w^\oplus$ with $c(w^\ominus) \neq c(w^\oplus)$ to the vertices $v_1^{\mathrm{T}}, v_1^{\mathrm{F}}, \dots, v_n^{\mathrm{T}}, n_n^{\mathrm{F}}$ uses only the colors $c(w^\ominus)$ and $c(w^\oplus)$ due to the presence of $\mathcal{C}$-edges $\{w^\oplus, w^\ominus, v_i^{\mathrm{T}}\}$ and $\{w^\oplus, w^\ominus, v_i^{\mathrm{F}}\}$ for all $1 \leq i \leq n$. If $c(w^\ominus) \neq c(w^+)$, then all the $\mathcal{D}$-edges corresponding to the clauses of Φ are properly colored already by the precoloring and thus assigning all the vertices v_i^{T} the color $c(w^\oplus)$ and all the vertices v_i^{F} the color $c(w^\ominus)$ yields a proper extension of c.

Let us assume in the rest of the proof that $c(w^\ominus) = c(w^+)$. The color $c(w^\oplus)$ represents true and the color $c(w^\ominus)$ represents false in our construction. The presence of $\mathcal{D}$-edges $\{v_i^{\mathrm{T}}, v_i^{\mathrm{F}}\}$ assures that each variable and its negation have opposite values (recall that the value of x_i is represented by the color of v_i^{T}). The $\mathcal{D}$-edges $\{w^\ominus, w^+, v_i^X, v_j^Y, v_k^Z\}$ force that each clause contains at least one true literal (a vertex colored by the color $c(w^\oplus)$. Hence, c can be extended to H_Φ iff there is a satisfying assignment of Φ.

■

We now combine Lemmas 1 and 3 to get the following theorem:

Theorem 4 *Let A_2 be a finite non-empty subset of $\{2, 3, \dots\}$ and A_1 a proper (possibly empty) subset of A_2. It is NP-hard to decide whether the feasible set of a given mixed hypergraph H is equal to A_2 even if it is promised that $\mathcal{F}(H)$ is either A_1 or A_2.*

Proof: We present a reduction from the well-known NP-complete problem 3SAT. Let Φ be a given formula with n variables and H_Φ the mixed hypergraph from Lemma 3. Consider the mixed hypergraph H^* from Lemma 1 for the set $\mathcal{F} = A_2 = \{c_1, \dots, c_k\}$. Let $A_2 \setminus A_1 = \{c_{i_1}, \dots, c_{i_{k'}}\}$. We create $|A_2| - |A_1| = k' \geq 1$ copies of H_Φ and we identify the vertices $w^\ominus, w^+, w^\oplus$ of the j-th copy with the vertices $w^\ominus_{c_{i_j}}, w^+_{c_{i_j}}, w^\oplus_{c_{i_j}}$ of H^*. Let H be the obtained mixed hypergraph.

It is easy to verify that H can be constructed in time polynomial in the number of variables and clauses of the formula Φ. In particular, the number of vertices of H is at most $3 \cdot \max A_2 + 2 \cdot |A_2 \setminus A_1| \cdot n$ and the number of its edges is cubic in the number of its vertices.

The mixed hypergraph H has a strict ℓ-coloring for $\ell \in A_1$ since any strict ℓ-coloring of H^* can be extended to the copies of H_Φ due to Lemma 3. Recall that $c(w^\ominus_{c_{i_j}}) \neq c(w^+_{c_{i_j}}) = c(w^\oplus_{c_{i_j}})$ for $1 \leq j \leq k'$ for every strict ℓ-coloring of H^* with $\ell \in A_1$. On the other hand, H has a strict ℓ-coloring for $\ell \in A_2 \setminus A_1$ iff Φ is satisfiable: Since it holds that $c(w^\ominus_{c_{i_j}}) = c(w^+_{c_{i_j}}) \neq c(w^\oplus_{c_{i_j}})$ for every strict c_{i_j}-coloring c of H^*, the coloring c can be extended to the j-th copy of H_Φ iff Φ is satisfiable by Lemma 3.

■

Several interesting computational complexity corollaries follow almost immediately. These results are new except for Corollary 4 which was proved in a weaker form in [11]:

Corollary 2 *It is NP-complete to decide whether a given mixed hypergraph H is colorable.*

Proof: This problem clearly belongs to the class NP. It is enough to set $A_1 = \emptyset$ and $A_2 = \{2\}$ in Theorem 4 to get the result.

■

Corollary 3 *Let A be a fixed finite subset of $\{2, 3, \ldots\}$. It is coNP-hard to decide whether the feasible set of a given mixed hypergraph H is equal to A. If $A \neq \emptyset$, then this problem is NP-hard, too.*

Proof: The coNP-hardness is established by setting $A_1 = A$ and A_2 to a proper finite superset of A omitting 1 in Theorem 4. The NP-hardness is established by setting A_2 to A and A_1 to a proper subset of A.

■

Corollary 4 *It is both NP-hard and coNP-hard to decide whether the feasible set of a given mixed hypergraph H is gap-free even for a mixed hypergraph H with $\bar{\chi}(H) = 4$.*

Proof: The NP-hardness is established by setting $A_1 = \{2, 4\}$ and $A_2 = \{2, 3, 4\}$ in Theorem 4. The coNP-hardness is established by setting $A_1 = \{4\}$ and $A_2 = \{2, 4\}$.

■

Corollary 5 *There is no polynomial-time $o(n)$-approximation algorithm for the lower or the upper chromatic number of a mixed hypergraph where n is the number of its vertices unless $P = NP$.*

Proof: Suppose that there is a polynomial-time $f(n)$-approximation algorithm for the lower chromatic number where $f(n) \in o(n)$ and n is the number of vertices of a given mixed hypergraph. Let Φ be a given formula with clauses of size three with N variables. Choose m such that $m > 2 \cdot f(3m + 2N)$. It is not hard to see that there is such an integer $m \in O(N)$ since $f(n) \in o(n)$. Let H be the mixed hypergraph from the construction of Theorem 4 for $A_1 = \{m\}$ and $A_2 = \{2, m\}$. Note that the number of vertices of H is at most $3k + 2N$. The approximation algorithm for the lower chromatic number outputs a number which is less then m iff the feasible set of the input mixed hypergraph is A_2. Recall that $\mathcal{F}(H) = A_2$ iff Φ is satisfiable. Hence, the existence of the polynomial-time $o(n)$-approximation algorithm implies that 3SAT$\in P$ and consequently $P = NP$. The non-existence (unless P=NP) of a polynomial-time $o(n)$-approximation algorithm for the upper chromatic number can be proved similarly.

■

5 Conclusion

There exists a mixed hypergraph whose feasible set is $\mathcal{F}$ for any set $\mathcal{F}$ of positive integers with $1 \notin \mathcal{F}$. We proved that there exists a mixed hypergraph whose spectrum is $(r_1, \ldots, r_m)$ for any vector $(r_1, \ldots, r_m)$ of non-negative integers such that $r_1 = 0$. The number of vertices of the smallest mixed hypergraph which is a realization of a given set $\mathcal{F}$ has been substantially decreased from exponential to linear in $\max \mathcal{F}$. But the following question has not been answered: What is the number of vertices of the smallest mixed hypergraph whose feasible set is equal to a given set $\mathcal{F}$? Or even, what is the number of vertices of the smallest mixed hypergraph whose spectrum is equal to a given spectrum $(r_1, \ldots, r_m)$? The answer to any of these questions probably requires some very fine analysis.

We have not dealt with mixed hypergraphs containing only $\mathcal{C}$-edges in this paper. It is clear that if $r_1 \neq 0$ (this is equivalent to the fact that a mixed hypergraphs contains only $\mathcal{C}$-edges), then $r_1 = 1$. Furthermore, $r_2 = (2^n - 2)/2$ for some n since $\mathcal{C}$-edges of size two can be contracted without affecting the spectrum and any two-coloring of a mixed hypergraph on n-vertices with no $\mathcal{D}$-edges and with no $\mathcal{C}$-edges of size two is proper. This leads to the following problem: What are necessary and sufficient conditions for a vector $(r_1, \ldots, r_m)$ with $r_1 = 1$ to be the spectrum of a mixed hypergraph?

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