Goldberg-Coxeter Construction for 3- and 4-valent Plane Graphs

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Abstract

We consider the Goldberg-Coxeter construction $GC_{k,l}(G_0)$ (a generalization of a simplicial subdivision of the dodecahedron considered in [Gold37] and [Cox71]), which produces a plane graph from any 3- or 4-valent plane graph for integer parameters k, l. A *zigzag* in a plane graph is a circuit of edges, such that any two, but no three, consecutive edges belong to the same face; a *central circuit* in a 4-valent plane graph G is a circuit of edges, such that no two consecutive edges belong to the same face. We study the zigzag (or central circuit) structure of the resulting graph using the algebraic formalism of the *moving group*, the (k, l)*-product* and a finite index subgroup of $SL_2(\mathbb{Z})$, whose elements preserve the above structure. We also study the intersection pattern of zigzags (or central circuits) of $GC_{k,l}(G_0)$ and consider its *projections*, obtained by removing all but one zigzags (or central circuits).

Key words. Plane graphs, polyhedra, zigzags, central circuits.

1 Introduction

As initial graph G_0 for the Goldberg-Coxeter construction, we consider mainly:

(i) 3- and 4-valent 1-skeleton of Platonic and semiregular polyhedra, prisms and antiprisms (see Table 1),

(ii) 3-valent graphs related to *fullerenes* and other chemically-relevant polyhedra,

(iii) 4-valent plane graphs, which are minimal projections for some interesting alternating links; those links are denoted according to Rolfsen's notation [Rol76] (see also, for example, [Kaw96]).

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name	$\|Mov(G_0)\|$	reference
Tetrahedron	4	Theorem 6.5
Cube	12	Theorem 6.5, Theorem 6.7, Proposition 7.4 and Conjecture 7.7
Dodecahedron	60	Theorem 6.5, Proposition 7.4 and Conjecture 7.7
Octahedron	24	Theorem 6.5, Theorem 6.7 and Proposition 7.4
Cuboctahedron	576	$= GC_{1,1}(Octahedron)$
Icosidodecahedron	7200	Conjecture 6.9
trunc. Tetrahedron	12	$= GC_{1,1}(Tetrahedron)$
trunc. Octahedron	576	$= GC_{1,1}(Cube)$
trunc. Cube	20736	Theorem 6.7
trunc. Icosahedron	648000	$= GC_{1,1}(Dodecahedron)$
trunc. Dodecahedron	648000	Theorem 6.7
Rhombicuboctahedron	165888	$= GC_{2,0}(Octahedron)$
Rhombicosidodecahedron	51840000	$= GC_{1,1}(Icosidodecahedron)$
trunc. Cuboctahedron	1327104	Theorem 6.7
trunc. Icosidodecahedron	139968000000	Conjecture 6.9
$Prism_m$	$12(\frac{m}{gcd(m,4)})^3$	Conjecture 6.11
$APrism_m$	$\frac{24}{gcd(m,2)}(\frac{m}{gcd(m,3)})^3$	Conjecture 6.12

Table 1: The Goldberg-Coxeter construction from 3 or 4-valent regular and semiregular polyhedra

The group of all rotations, preserving a plane graph G, will be denoted by $Rot(G)$; it is a subgroup of index 1 or 2 of the full automorphism group $Aut(G)$. For 3-connected plane graphs without 2-gonal faces, the following theorem of Mani ([Mani71], a refinement of Steinitz's theorem [Ste16], see also [Grün67]) is useful: the symmetry group (i.e. automorphism group) of a graph can be realized as the point group of a convex polyhedron, having this graph as the skeleton, and so, it can be identified with this point group. In the presence of 2-gonal faces (i.e. multiple edges), one cannot speak of convex polyhedra; however, for the graphs with 2-gonal faces, considered in this paper, one can still identify the symmetry group of the graph with a point group.

We consider here plane graphs with restrictions on their valency (namely, having valency 3 or 4) and face sizes. It turns out, that some classes of such graphs with maximal symmetry can be described in terms of what we call *the Goldberg-Coxeter construction* $GC_{k,l}(G_0)$ with G_0 being the initial graph (see Section 5). Since the Goldberg-Coxeter construction will concern only 3- and 4-valent plane graphs, there are two cases, whose main features are depicted in Table 2.

A *zigzag* in a 3-valent plane graph is a circuit (possibly, with self-intersections) of edges, such that any two, but no three, consecutive edges belong to the same face. A *central circuit* in a 4-valent plane graph is a circuit of edges, such that no two consecutive edges belong to the same face. Many results for 3- and 4-valent graphs will be similar; in such case we will use general notion of "either zigzag, or central circuit" and call it *ZC-circuit*.

	3-valent graph G_0	4-valent graph G_0
lattice	root lattice A_2	square lattice $\mathbb{Z}^2$
ring	Eisenstein integers $\mathbb{Z}[\omega]$	Gaussian integers $\mathbb{Z}[i]$
$t(k,l)$	k^2+kl+l^2	k^2+l^2
Euler formula	$\sum_i(6-i)p_i=12$	$\sum_i(4-i)p_i=8$
zero-curvature	hexagons	squares
ZC-circuits	zigzags	central circuits
case $k=l=1$	leapfrog graph	medial graph

Table 2: Main features of GC-construction

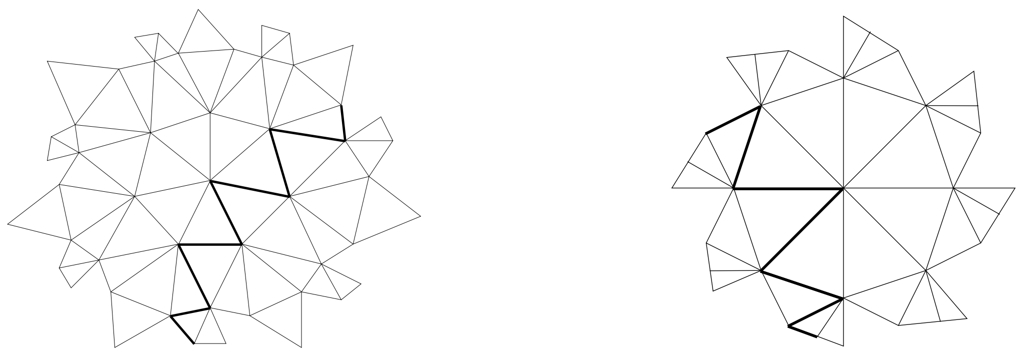


Figure 1: A zigzag in Klein map $\{3^7\}$ and Dyck map $\{3^8\}$

A *road* in a 3- or 4-valent plane graph is a non-extendible sequence (possibly, with self-intersections) of either hexagonal faces or of square faces, such that any non-end face is adjacent to its neighbors on opposite edges. If the sequence stops on a non-hexagon or, respectively, a non-square face, then it is called a *pseudo-road*; otherwise, it is called a *railroad* and it is a circuit by finiteness of the graph. A graph without railroads is called *tight*; in other words, every ZC-circuit of a tight graph is incident on each, the left and right side, to at least one non-hexagonal or, respectively, non-square face (in [DeSt03] and [DDS03] the term "*irreducible*" was used instead of "tight").

Those notions can be also defined for maps on orientable surfaces; see, for example, on Figure 1 a zigzag for the Klein map $\{3^7\}$ and the Dyck map $\{3^8\}$, which are dual triangulations for such 3-valent maps. The notion of zigzag (respectively, central circuit) is used here in 3-valent (respectively, 4-valent) case, but they can be defined on any plane graph (respectively, Eulerian plane graph). Moreover, the notion of zigzag extends naturally to infinite plane graphs and to higher dimension (see [DeDu04]).

For any plane graph G the *dual* graph G^* is the graph with vertex-set being the set of faces of G and two faces being adjacent if they share an edge of G.

Definition 1.1 *A ZC circuit with an orientation will be denoted by* $\overrightarrow{ZC}$.

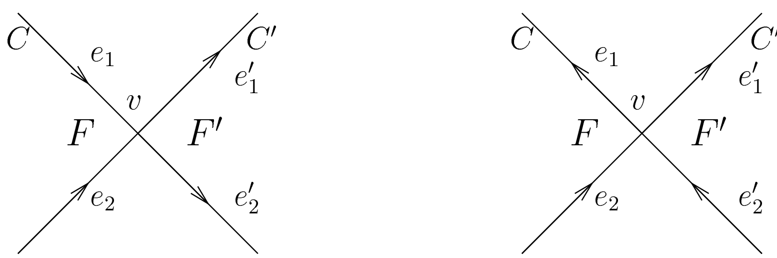
(i) Let Z and Z' be two (possibly, identical) zigzags of a plane graph and let an orientation be selected on them. An edge e of intersection is called of type I *or* type II, *if*

Z and Z' traverse it in opposite or same direction, respectively (see picture below).

Type I Type II

The intersection $I(\overrightarrow{Z}, \overrightarrow{Z}')$ *of two zigzags Z and Z' with an orientation fixed on them, is the pair (α_1, α_2), where α_1, α_2 are, respectively, the numbers of edges of intersection of type I, type II, respectively, between $\overrightarrow{Z}$ and $\overrightarrow{Z}'$. If $Z = Z'$, then the type of intersection is independent of the chosen orientation; hence, the intersection of Z with itself, which we will call its* signature *is well-defined.*

(ii) Let G be a 4-valent plane graph and denote by $\mathcal{C}_1$, $\mathcal{C}_2$ a bipartition of the face-set of G (it exists, since G^* is bipartite). Let C and C' be two (possibly, identical) central circuits of G and let an orientation be selected on them. A vertex v of the intersection between C and C' is contained in two faces, say, F and F', of $\mathcal{C}_1$. The vertex v is incident to two edges of F, say, e_1 and e_2, and to two edges of F', say, e'_1 and e'_2. If e_1 and e_2 have both arrows pointing to the vertex or both arrows pointing out of the vertex, then e'_1 and e'_2 are in the same case. The type *of the vertex v, relatively to the pair $(\mathcal{C}_1, \mathcal{C}_2)$, is said to be I in this case and II, otherwise.*



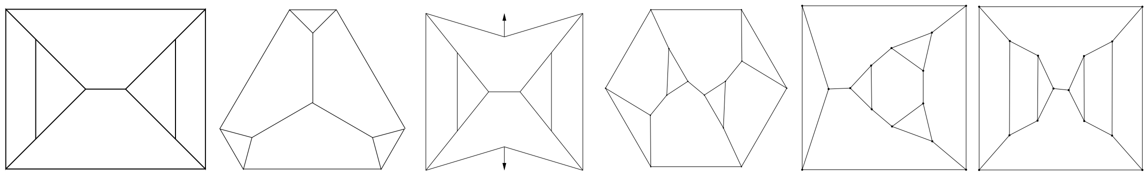
Type I Type II

If one interchanges $\mathcal{C}_1$ and $\mathcal{C}_2$, while keeping the same orientation, then the types of intersection of vertices are interchanged. The intersection $I_{\mathcal{C}_1,\mathcal{C}_2}(\overrightarrow{C}, \overrightarrow{C}')$ *of two central circuits C and C', with an orientation fixed on them, is the pair (α_1, α_2), where α_1, α_2 are, respectively, the numbers of vertices of the intersection between C and C' of type I, II, respectively, relatively to $\mathcal{C}_1$, $\mathcal{C}_2$.*

If $C = C'$, then the type of intersection is independent of the chosen orientation; hence, the intersection of C with itself, which we will call its signature, *relatively to $\mathcal{C}_1$, $\mathcal{C}_2$ is well-defined.*

Since interchanging $\mathcal{C}_1$ and $\mathcal{C}_2$ interchanges α_1 and α_2, there is an ambiguity in the definition of α_1 and α_2, which can be resolved either by specifying $\mathcal{C}_1$ or if not precised by requiring $\alpha_1 \geq \alpha_2$.

For any 3-valent plane graph G, the *leapfrog* of G is defined to be the truncation of G^* (see [FoMa95]).



10-2 (C_{2v}) 10-3 (C_{3v}) 12-6 (C_{2v}) 14-21 (C_2) 14-23 (C_s) 14-24 (C_{2v})

Figure 2: Some z-uniform 3-valent graphs with their symmetry group

The *medial graph* of a plane graph G, denoted by $Med(G)$, is defined by taking, as vertex-set, the set of edges of G with two edges being adjacent if they share a vertex and belong to the same face of G. $Med(G)$ is 4-valent and its central circuits $C_1, \dots, C_p$ correspond to zigzags $Z_1, \dots, Z_p$ of G. Moreover, an orientation of a zigzag Z_i induces an orientation of a central circuit C_i. The set of faces of $Med(G)$ corresponds to the set of vertices and faces of G. If one takes $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$) to be the set of faces of $Med(G)$ corresponding to faces (respectively, vertices) of G, then (if we keep the same orientation) the intersection numbers of C_i and C_j are the same as the intersection numbers of Z_i and Z_j.

The *z-vector* (or *CC-vector*) of a graph G is the vector enumerating *lengths*, i.e. the numbers of edges, of all its zigzags (or, respectively, central circuits) with their signature as subscript. The simple ZC-circuits are put in the beginning, in non-decreasing order of length, without their signature $(0,0)$, and separated by a semicolon from others. The self-intersecting ones are also ordered by non-decreasing lengths. If there are $m > 1$ ZC-circuits of the same length l and the same signature (α_1, α_2), then we write l^m if $\alpha_1 = \alpha_2 = 0$ and $l^m_{\alpha_1,\alpha_2}$, otherwise. For a ZC-circuit ZC, its *intersection vector* $(\alpha_1, \alpha_2); \dots, c_k^{m_k}, \dots$ is such that $\dots, c_k, \dots$ is an increasing sequence of sizes of its intersection with all other ZC-circuits, while m_k denote respective multiplicity. Given a 3-valent plane graph G_0, its z-vector is equal to the CC-vector of $Med(G_0)$.

A 3- or 4-valent graph is called *ZC-uniform* if all its ZC-circuits have the same length and the same signature. In ZC-uniform case, the length of each of the r central circuits (respectively, zigzags) is $\frac{2n}{r}$ (respectively, $\frac{3n}{r}$). For example, for $G = GC_{4,1}(Prism_{12})$, it holds $z = 84^6; 84^{12}_{2,0}$; so, it is not z-uniform. A graph is called *ZC-transitive* if its symmetry group acts transitively on ZC-circuits; clearly, ZC-transitivity implies ZC-uniformity. A graph is called *ZC-knotted* if it has only one ZC-circuit; a graph is called *ZC-balanced* if all its ZC-circuits of the same length and same signature, have identical intersection vectors. We do not know example of a ZC-uniform, but not ZC-balanced, graph. For example, amongst the graphs $GC_{k,l}(G_0 = \text{10-2})$, the first z-unbalanced one occurs for $(k,l) = (7,1)$. The only graphs G_0, which are 3-valent, z-uniform, have at most 14 vertices and such that their leapfrog $GC_{1,1}(G_0)$ are not z-balanced, are Nr.12-6, 14-21, 14-23 and 14-24 on Figure 2.

Above and below we denote by x-y the 3-valent plane graph with x vertices, which appear in y-position, when one uses the generation program Plantri (see [BrMK]); see, for example, Figure 2.

Table 3 present the graphs $GC_{k,l}(G_0)$, which are considered in this paper. In this

Table, r denotes the number of ZC-circuits in $GC_{k,l}(G_0)$. The case $(k,l)=(1,0)$ corresponds to the initial graph G_0. The columns 1–4 give, respectively, the class of graphs, valency d, *p-vector* (i.e. one enumerating the numbers p_i of i-gonal faces) and all realizable symmetry groups for the graphs $GC_{k,l}(G_0)$. The case $k=l=1$ corresponds to the medial graph for 4- and to the leapfrog graph for 3-valent case. The column "r if I" represents the number (conjectured or proved) r of ZC-circuits in the case $k \equiv l \pmod 3$ (for valency 3) or (for valency 4) $k \equiv l \pmod 2$, while the column "r if II" represents the remaining case.

Given a graph G, denote by $Mov(G)$ the permutation group on the set of *directed edges*, which is generated by two *basic permutations*, called *left* L and *right* R; $Mov(G)$ is called the *moving group* of G. *Directed edges* are edges of G_0^* with prescribed direction. We will associate to every pair (k,l) of integers an element of this moving group, which we call (k,l)*-product* of basic permutations, and which encodes the lengths of the ZC-circuits of $GC_{k,l}(G_0)$. For $k=l=1$, this (k,l)-product is, actually, ordinary product in the group $Mov(G_0)$. Take a ZC-circuit of $GC_{k,l}(G_0)$ and fix an orientation on it. It will cross some edges of G_0^*. For any directed edge $\overrightarrow{e}$ of oriented ZC-circuit, there are exactly two possible successors $L(\overrightarrow{e})$ and $R(\overrightarrow{e})$; it is clear for zigzags in 3-valent graph G_0, but for central circuits in 4-valent, it will be obtained from algebraic considerations. The $k+l$ successive left and right choices will define the (k,l)-product. In some cases, the knowledge of normal subgroups of $Mov(G_0)$ will allow an exact computation of the z-vector of $GC_{k,l}(G_0)$ in terms of congruences valid for numbers (k,l). On the other hand, Theorem 4.7 gives a characterization of the graphs G for which $Mov(G)$ is an Abelian group.

Two-faced (i.e. having only p- and q-gonal faces, $2 \le q < p$) 3- and 4-valent plane graphs are studied, for example, in [DeGr01], [DeGr99], [DDF02], [De02], [DeDu02], [DeSt03], [DDS03], [DHL02], for which this work is a follow-up.

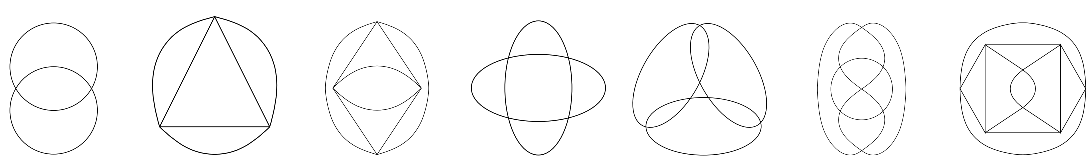
Denote by q_n the class of 3-valent plane graphs having only 6-gonal and q-gonal faces. Euler formula $\sum_{i\ge 1}(6-i)p_i = 12$ for the p-vector of any 3-valent plane graph implies, that the classes 2_n, 3_n, 4_n and 5_n have, respectively, three, four, six and twelve q-gonal faces. 5_n are, actually, the *fullerenes*, well known in Organic Chemistry (see, for example, [FoMa95]).

Call an *i-hedrite* any plane 4-valent graph, such that the number p_j of its j-gonal faces is zero for any j, different from $2, 3$ and 4, and such that $p_2 = 8-i$. So, an n-vertex i-hedrite has $(p_2,p_3,p_4) = (8-i, 2i-8, n+2-i)$. Clearly, $(i;p_2,p_3) = (8;0,8)$, $(7;1,6)$, $(6;2,4)$, $(5;3,2)$ and $(4;4,0)$ are all possibilities.

The *Bundle* is defined as plane 3-valent graph consisting of two vertices with three edges connecting them. A $Foil_m$ is defined as plane 4-valent graph consisting of a m-gon with each edge replaced by a 2-gon; its CC-vector is $2m$, if m is odd, and m^2, if m is even. The medial graph of $Foil_m$ is $Prism_m$, in which m edges, connecting two m-gons, are replaced by 2-gons; its CC-vector is 4^m. Clearly, for $m=2$, 3 and 4, $Foil_m$ are (projections of links) 2_1^2, Trefoil 3_1 and 4_1^2 (see Figure 3).

Class	d	p-vector	Groups	$(k,l)=(1,0)$	$(k,l)=(1,1)$	r if I	r if II
2_n	3	$p_2=3$, p_6	all D_3, D_{3h}	Bundle	tr.Triangle	3	1
3_n	3	$p_3=4$, p_6	all T, T_d	Tetrah.	tr.Tetrahed.	3	3
4_n	3	$p_4=6$, p_6	all O, O_h	Cube	tr.Octahed.	6	4
5_n	3	$p_5=12$, p_6	all I, I_h	Dodecah.	tr.Icosahed.	$6, 10, 15$	
$\mathcal{GP}_m$	3	$p_4=m$, $p_m=2$, p_6 $(m\neq 2,4)$	all D_m, D_{mh}	Prism$_m$	tr.$Prism_m^*$	Conj.6.11	
	4	$p_3=2m$, $p_m=2$, p_4 $(m\neq 3)$	some D_m, D_{md}	$APrism_m$	Med$(APrism_m)$	Conj.6.12	
8-hed.	4	$p_3=8$, p_4	all O, O_h	Octahed.	Cuboctahed.	4	3, 6
4-hed.	4	$p_2=4$, p_4	all D_4, D_{4h}	$Foil_2$	$Foil_4$	2	2
6-hed.	4	$p_2=2$, $p_3=4$, p_4	some D_{2d}, D_2	4_1	$Med(4_1)=8^2_{14}$	2, 4	1, 3
7-hed.	4	$p_2=1$, $p_3=6$, p_4	some C_2, C_{2v}	7^2_6	$Med(7^2_6)$	3, 5, 7	1, 2, 3, 5
5-hed.	4	$p_2=3$, $p_3=2$, p_4	all D_3, D_{3h}	Trefoil 3_1	$Med(3_1)=6^3_1$	3	1

Table 3: Main series of considered graphs $GC_{k,l}(G_0)$



$2^2_1(D_{4h})$ $3_1(D_{3h})$ $4_1(D_{2d})$ $4^2_1(D_{4h})$ $6^3_1(D_{3h})$ $7^2_6(C_{2v})$ $8^2_{14}(C_{2v})$

Figure 3: Minimal plane projections of some alternating links with their symmetry groups

2 The complex rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$

The *root lattice* A_2 is defined by $A_2 = \{x \in \mathbb{Z}^3 : x_0 + x_1 + x_2 = 0\}$. The *square lattice* is denoted by $\mathbb{Z}^2$.

The ring $\mathbb{Z}[\omega]$, where $\omega = e^{\frac{2\pi}{6}i} = \frac{1}{2}(1 + i\sqrt{3})$ of *Eisenstein integers* consists of the complex numbers $z = k + l\omega$ with $k, l \in \mathbb{Z}$ (see also [HaWr96], where ω is replaced by ρ). The norm of such z is denoted by $N(z) = z\overline{z} = k^2 + kl + l^2$ and we will use the notation $t(k,l) = k^2 + kl + l^2$. If one identifies $x = (x_1, x_2, x_3) \in A_2$ with the Eisenstein integer $z = x_1 + x_2\omega$, then it holds $2N(z) = \|x\|^2$.

One has $\mathbb{Z}^2 = \mathbb{Z}[i]$, where $\mathbb{Z}[i]$ consists of the complex numbers $z = k + li$ with $k, l \in \mathbb{Z}$. The norm of such z is denoted by $N(z) = z\overline{z} = k^2 + l^2$ and we will use the notation $t(k,l) = k^2 + l^2$.

Two Eisenstein or two Gaussian integers z and z' are called *associated* if the quotient $\frac{z}{z'}$ is an Eisenstein unit (i.e. ω^k with $0 \le k \le 5$; namely, 1, ω, ω^2, -1, $-\omega$, $-\omega^2$) or a Gaussian unit (i.e. i^k with $0 \le k \le 3$; namely, 1, i, -1, $-i$). They are called *C-associated* if one of the quotients $\frac{z}{z'}$, $\frac{\overline{z}}{z'}$ is an Eisenstein or Gaussian unit. Every Eisenstein or Gaussian integer is associated (respectively, C-associated) to $k + l\omega$ or $k + li$, respectively, with $k, l \ge 0$ (respectively, $0 \le l \le k$).

The lattices A_2 and $\mathbb{Z}^2$ correspond to regular partitions of the plane into regular triangles and squares, respectively. The skeletons of those partitions are infinite graphs; their shortest path metrics are called (in Robot Vision) the *hexagonal distance* and *4-distance*. (The 4-distance is, in fact, a l_1-metric on $\mathbb{Z}^2$.) If $k, l \ge 0$, then the shortest path distance between 0 and $k + l\omega$ (or, respectively, $k + li$) is $k + l$.

Thurston ([Thur98]) developed a global theory of parameter space for sphere triangulations with valency of vertices at most 6. Clearly, our 3-valent two-faced plane graphs q_n are covered by Thurston consideration. Let s denote the number of vertices of valency less than 6; such vertices reflect positive curvature of the triangulation of the sphere S^2. Thurston has built a parameter space with $s - 2$ degrees of freedom (complex numbers). If we restrict ourselves to some particular symmetries of plane graphs, then it restricts the number of parameters needed for a characterization. General fullerenes have 10 degrees of freedom, while those with symmetry I or I_h have just one degree of freedom.

For example, in [FoCrSt87] the fullerenes 5_n with symmetry D_5, D_6, T were described by two complex parameters (or, in other words, by four integer parameters).

We believe, that the hypothesis on valency of vertices (in dual terms, that the graph has no q-gonal faces with $q > 6$) in [Thur98] is unnecessary to his theory of parameter space. Also, we think, that his theory can be extended to the case of quadrangulations instead of triangulations.

In this paper, we focus mainly on the classes of plane graphs, which can be parametrized by *one* complex parameter, namely, by $k + l\omega$ or $k + li$. For those classes, the *GC*-construction, defined below, fully describes them.

Remark 2.1 *(i) A natural number $n = \prod_i p_i^{\alpha_i}$ admits a representation $n = k^2 + l^2$ or $n = k^2 + kl + l^2$ if and only if any α_i is even, whenever $p_i \equiv 3 (mod\, 4)$ (Fermat Theorem) or, respectively, $p_i \equiv 2 \pmod 3$ (see, for example, [CoGu96] and [Con03]).*

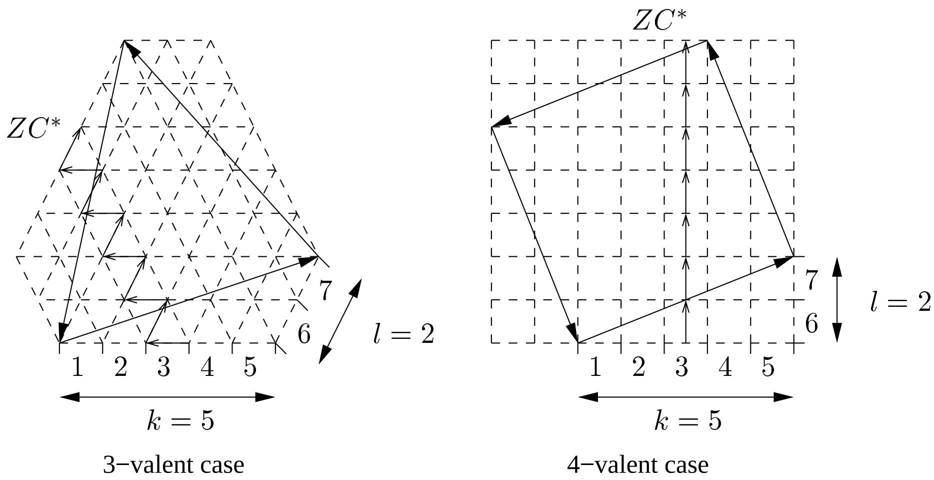


Figure 4: The master polygon and an oriented ZC-circuit for parameters $k = 5$, $l = 2$

(ii) One can have $t(k,l) = t(k',l')$ *with corresponding complex numbers* z, z' *not being C-associated. First cases with* $gcd(k,l) = gcd(k',l') = 1$ *are* $91 = 6^2 + 6 \times 5 + 5^2 = 9^2 + 9 + 1^2$ *and* $65 = 8^2 + 1^2 = 7^2 + 4^2$.

3 The Goldberg-Coxeter construction

First consider the 3-valent case. By duality, every 3-valent plane graph G_0 can be transformed into a *triangulation*, i.e. into a plane graph whose faces are triangles only. The Goldberg-Coxeter construction with parameters k and l consists of subdividing every triangle of this triangulation into another set of faces according to Figure 4, which is defined by two integer parameters k, l. One can see that the obtained faces, if they are not triangles, can be glued with other non-triangle faces (coming from the subdivision of neighboring triangles) in order to form triangles; so, we end up with a new triangulation.

The triangle of Figure 4 has area $\mathcal{A}(k^2 + kl + l^2)$ if $\mathcal{A}$ is the area of a small triangle. By transforming every triangle of the initial triangulation in such way and gluing them, one obtains another triangulation, which we identify with a (dual) 3-valent plane graph and denote by $GC_{k,l}(G_0)$. The number of vertices of $GC_{k,l}(G_0)$ (if the initial graph G_0 has n vertices) is $nt(k,l)$ with $t(k,l) = k^2 + kl + l^2$.

For a 4-valent plane graph G_0, the duality operation transforms it into a quadrangulation and this initial quadrangulation is subdivided according to Figure 4, which is also defined by two integer parameters k, l. After merging the obtained non-square faces, one gets another quadrangulation and the duality operation yields graph $GC_{k,l}(G_0)$ having $nt(k,l)$ vertices with $t(k,l) = k^2 + l^2$.

In both 3- or 4-valent case, the faces of G_0 correspond to some faces of $GC_{k,l}(G_0)$ (see Figure 6 and 11). If $t(k,l) > 1$, then those faces are not adjacent.

The family $GC_{k,l}(Dodecahedron)$ consists of all 5_n having symmetry I_h or I (see [Gold37], [Cox71] and Theorem 5.2). There is large body of literature, where such *icosa-*

(k,l)	symmetry	capsid of virion
$(1,0)$	I_h	*gemini virus*
$(1,1)$	I_h	*turnip yellow mosaic virus*
$(2,0)$	I_h	*hepathite B*
$(2,1)$	I, laevo	*HK97, rabbit papilloma virus*
$(1,2)$	I, dextro	*human wart virus*
$(3,1)$	I, laevo	*rotavirus*
$(4,0)$	I_h	*herpes virus, varicella*
$(5,0)$	I_h	*adenovirus*
$(6,0)$	I_h	*HTLV-1*
$(6,3)$?	I, laevo	*HIV-1*
$(7,7)$?	I_h	*iridovirus*

Table 4: Some capsides of viruses having form of icosahedral dual 5_n, $n = 20t(k,l)$

hedral fullerenes appear as Fuller-inspired *geodesic domes* (in Architecture) and *virus capsides* (protein coats of virions, see [CaKl62]); see, for a survey, [Cox71] and [DDG98]. The Goldberg-Coxeter construction is also used in numerical analysis, i.e. for obtaining good triangulations of the sphere (see, for example [Slo99], [ScSw95]). In Table 4 are listed some examples illustrating present knowledge in this area; in Virology, the number $t(k,l)$ (used for icosahedral fullerenes) is called *triangulation number*. In terms of Buckminster Fuller, the number $k+l$ is called *frequency*, the case $l=0$ is called *Alternate*, and the case $l=k$ is called *Triacon*. He also called the GC-construction *Breakdown* of the initial plane graph G_0.

We will say, that a face has *gonality* q if it has q sides. A q-gonal face of a 3- (or 4-valent) graph G_0 is called *of positive, zero, negative curvature* if $q < 6$ (or 4), $q = 6$ (or 4), $q > 6$ (or 4), respectively, according to the following Euler formula (a discrete analogue of the Gauss-Bonnet formula for surfaces) for 3- or 4-valent plane graphs:

$$\sum_{i\geq 1}(6-i)p_i = 12 \quad \text{or} \quad \sum_{i\geq 1}(4-i)p_i = 8, \text{respectively.}$$

Proposition 3.1 *Let G_0 be a 3- or 4-valent plane graph and denote the graph $GC_{k,l}(G_0)$ also by $GC_z(G_0)$, where $z = k + l\omega$ or $z = k + li$ in 3- or 4-valent case, respectively. The following hold:*

(i) $GC_z(GC_{z'}(G_0)) = GC_{zz'}(G_0)$.

(ii) If z and z' are two associated Eisenstein or Gaussian integers, then $GC_z(G_0) = GC_{z'}(G_0)$.

(iii) $GC_{\overline{z}}(G_0) = GC_z(\overline{G_0})$, where $\overline{G_0}$ denotes the plane graph, which differ from G_0 only by a plane symmetry; if $\overline{G_0} = G_0$ (i.e. $Rot(G_0) \neq Aut(G_0)$) and z, z' are two C-associated Eisenstein or Gaussian integers, then $GC_z(G_0) = GC_{z'}(G_0)$.

(iv) If G_0 has no faces of zero curvature and if $GC_{k,l}(G_0) = GC_{k',l'}(G_0)$ with $0 \leq l \leq k$ and $0 \leq l' \leq k'$, then $(k,l) = (k',l')$.

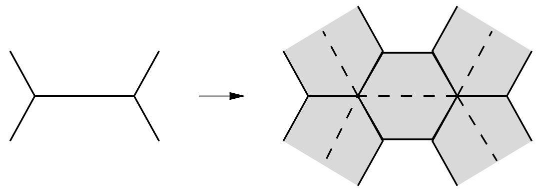


Figure 5: Chamfering seen locally

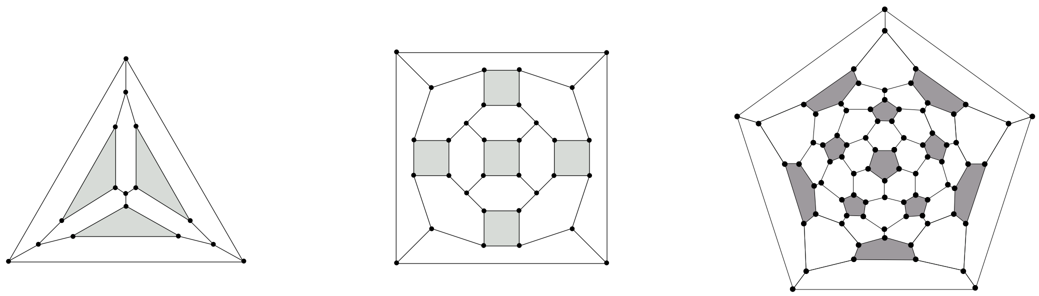


Figure 6: Chamfering $GC_{2,0}(G_0)$ for G_0 being Tetrahedron, Cube and Dodecahedron

Proof. (i) follows from the basic construction depicted in Figure 4, which is extended globally. (ii) also follows from this basic picture. Let G_0 be a 3-valent plane graph, such that $GC_{k,l}(G_0) = GC_{k',l'}(G_0)$. Because of the equality for the numbers of vertices, one obtains $t(k,l) = t(k',l')$. The minimum distance between two faces of non-zero curvature in $GC_{k,l}(G_0)$ is $k+l$; therefore, $k+l = k'+l'$. If one writes $v = k'-k$, then $k' = k+v$ and $l' = l-v$. The equality $t(k,l) = t(k',l')$ yields $v(k-l)+v^2 = 0$ and so, (k',l') is (k,l) or (l,k). Only first case is possible. The 4-valent case can be treated in a similar way. □

In particular, the condition $k \equiv l \pmod 3$ means, that the Eisenstein integer $k+l\omega$ is factorizable by $1+\omega$, i.e. by the complex number corresponding to the leapfrog operation, $GC_{1,1}$. The condition $k \equiv l \pmod 2$ means, that the Gaussian integer $k+li$ is factorizable by $1+i$, i.e. by the complex number corresponding to the medial operation, $GC_{1,1}$. Note that $k \equiv l \pmod 2$ is equivalent to $t(k,l) = (k-l)^2 + 2kl$ being even and $k \equiv l \pmod 3$ is equivalent to $t(k,l) = (k-l)^2 + 3kl$ being divisible by three.

The above Proposition implies, that we can consider only the case $0 \le l \le k$ in computations, since all considered graphs have a symmetry plane.

If $l = 0$, then $GC_{k,l}(G_0)$ is called *k-inflation* of G_0. For $k = 2$, $l = 0$, it is called *chamfering of* G_0 (because Goldberg called the result of his construction for $(k,l) = (2,0)$ on the Dodecahedron, *chamfered dodecahedron*, see Figure 5). Another case, interesting for Chemistry, is *Capra* i.e., $GC_{2,1}$ (see [Diu03]). All symmetries are preserved if $l = 0$ or $l = k$, while only rotational symmetries are preserved if $0 < l < k$. The Goldberg-Coxeter construction can be also defined, similarly, for maps on orientable surfaces. While the notions of medial, leapfrog and k-inflation go over for non-orientable surfaces, the Goldberg-Coxeter construction is not defined on a non-orientable surface.

The Goldberg-Coxeter construction for 3- or 4-valent plane graphs can be seen, in algebraic terms, as the scalar multiplication by Eisenstein or Gaussian integers in the parameter space (see [Sah94]). More precisely, $GC_{k,l}$ corresponds to multiplication by complex number $k+l\omega$ or $k+li$ in the 3- or 4-valent case, respectively.

In Proposition 3.2 and 3.3, we consider the ZC-structure of $GC_{k,0}(G_0)$, i.e. of k-inflation of G_0, in terms of the ZC-structure of G_0 (See example $G_0 = Trefoil$ in Figure 11).

Proposition 3.2 *Let G_0 be a 3-valent plane graph with zigzags $Z_1, \ldots, Z_p$. Choose an orientation on every zigzag.*

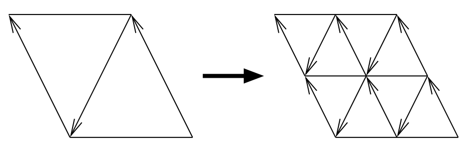
Let G' be the k-inflation of G_0. The graph G' has kp zigzags $Z_{i,j}$ with $1 \le i \le p$ and $1 \le j \le k$; the length of every $Z_{i,j}$ is k times the length of Z_i. The orientation on Z_i induces an orientation on k zigzags $(Z_{i,j})_{1\le j\le k}$.

The intersection between $Z_{i,j}$ and $Z_{i',j'}$ is equal to the intersection between Z_i and $Z_{i'}$, to twice the self-intersection of Z_i, or to the self-intersection of Z_i, respectively, if $i \ne i'$, $i = i$ and $j \ne j'$, or $i = i'$ and $j = j'$, respectively.

In particular, if the z-vector of G_0 is $\ldots, c_v^{n_v}, \ldots; \ldots, d_{v_{\alpha_{v1},\alpha_{v2}}}^{m_v}, \ldots$, then the z-vector of G' is $\ldots, kc_v^{kn_v}, \ldots; \ldots, kd_{v_{\alpha_{v1},\alpha_{v2}}}^{km_v}, \ldots$.

If the intersection vector of Z_i is $(a_i, b_i); i_1^{p_1}, \ldots, i_q^{p_q}$, then the intersection vector of $Z_{i,j}$ is $(a_i, b_i); i_1^{kp_1}, \ldots, i_q^{kp_q}, (2a_i + 2b_i)^{k-1}$.

Proof. Let us consider the 3-valent case. The z-structure of G_0^* differs from the one of G_0 only by reversal of type I and type II. The local structure of zigzags changes according to the rule, which is exemplified by the picture below for the case $k = 2$.



This local picture can be extended to whole graph and we get kp zigzags. The statement about intersections follows easily. □

The 4-valent case is much more complicate. Take a bipartition $\mathcal{C}_1$, $\mathcal{C}_2$ of the face-set of a 4-valent plane graph G_0. This face-set corresponds to a subset of the face-set of $GC_{k,0}$. The graph $(GC_{k,0}(G_0))^*$ is bipartite also; if k is even, then faces corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$ in $GC_{k,0}(G_0)$, are in the same part, while if k is odd, then they are in different parts (see Figure 7 for an example). By convention, we take a bipartition $\mathcal{C}'_1$, $\mathcal{C}'_2$ of the face-set of $GC_{k,0}(G_0)$, such that $\mathcal{C}'_1$ contains $\mathcal{C}_1$ (and also $\mathcal{C}_2$, if k is even).

For a 4-valent plane graph G_0, the graph $GC_{k,0}(G_0)$ coincides with the k-inflation defined in [DeSt03] and [DDS03].

Proposition 3.3 *Let G_0 be a 4-valent plane graph with central circuits $C_1, \ldots, C_p$. Choose an orientation on every central circuit*

Let G' be the k-inflation of G_0. Choose the bipartition $\mathcal{C}'_1$, $\mathcal{C}'_2$ of the faces of G' according to the above rule. The graph G' has kp central circuits $C_{i,j}$ with $1 \le i \le p$ and $1 \le j \le k$; the length of every circuit $C_{i,j}$ is k times the length of C_i.

We define now the orientation on the circuits $C_{i,j}$ in the following way:

- *If $1 \leq j \leq k-1$, then $C_{i,j}$ is oriented in the opposite way of $C_{i,j+1}$.*
- *If k is odd, then the central circuits $C_{i,1}$ and $C_{i,k}$ are oriented in the same direction as the central circuit C_i.*
- *If k is even, then there exist an orientation of all $C_{i,j}$, such that all intersections are of type II.*

With this orientation one obtains that, if the intersection between C_i and $C_{i'}$ is (α_1, α_2) and $i \neq i'$, then the intersection between $C_{i,j}$ and $C_{i',j'}$ is equal to (α_1, α_2) if k is odd and to $(0, \alpha_1 + \alpha_2)$ if k is even. If the self-intersection of C_i is equal to (α_1, α_2), then the self-intersection of $C_{i,j}$ is (α_1, α_2), $(0, \alpha_1 + \alpha_2)$ if k is odd, even, respectively, while the intersection between $C_{i,j}$ and $C_{i,j'}$ is $(2\alpha_1, 2\alpha_2)$, $(0, 2\alpha_1 + 2\alpha_2)$ if k is odd, even, respectively.

In particular, if the CC-vector of G_0 is $\dots, c_v^{n_v}, \dots; \dots, d_{v_{\alpha_{v1}, \alpha_{v2}}}^{m_v}, \dots$, then the CC-vector of G' is $\dots, kc_v^{kn_v}, \dots; \dots, kd_{v_{\alpha_{v1}, \alpha_{v2}}}^{km_v}, \dots$.

If the intersection vector of C_i is $(a_i, b_i); i_1^{p_1}, \dots, i_q^{p_q}$, then the intersection vector of $C_{i,j}$ is $I_i; i_1^{kp_1}, \dots, i_q^{kp_q}, (2a_i + 2b_i)^{k-1}$ with $I_i = (0, a_i + b_i)$ if k is even and $I_{i,i} = (a_i, b_i)$, otherwise.

Proof. By definition of the k-inflation in [DDS03], every central circuit C_i of G_0 corresponds to k central circuits of $GC_{k,0}(G_0)$.

If k is odd, then the central circuits $C_{i,1}$ and $C_{i,k}$ have the orientation of C_i; hence, their pairwise intersection is the same. It is easy to see that the convention of orienting $C_{i,j+1}$ in reverse to $C_{i,j}$, together with the "chess-like" structure of the bipartition $\mathcal{C}'_1$, $\mathcal{C}'_2$, ensures that the intersection between $C_{i,j}$ and $C_{i',j'}$ is independent of j and j'.

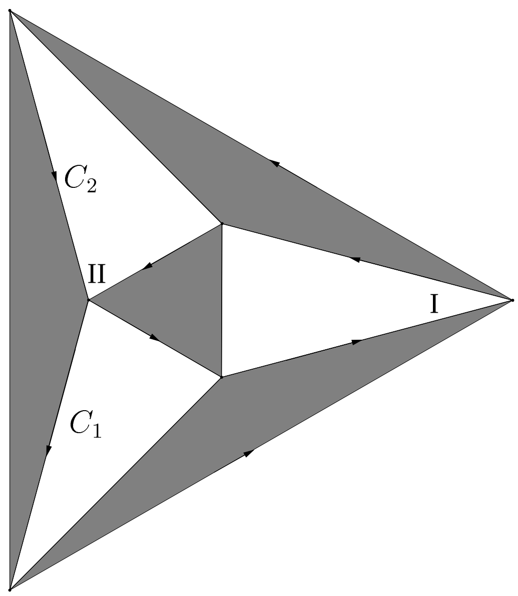
The case of k even is more difficult. Every central circuit C_i corresponds to a set $C_{i,1}$, $\dots$, $C_{i,k}$ of central circuits. By choosing the orientation of $C_{i,1}$, one can assume that it is incident to faces of $\mathcal{C}_1$, $\mathcal{C}_2$ on the left only. The vertices of the intersection between two (possibly, identical) central circuits $C_{i,1}$ and $C_{i',1}$ belong to faces of $\mathcal{C}_1$ or $\mathcal{C}_2$. By the orientation convention, the intersection between $C_{i,1}$ and $C_{i',1}$ are of type II. By the opposition of orientation between $C_{i,j}$ and $C_{i,j+1}$, the type of vertices of intersection between $C_{i,j}$ and $C_{i',j'}$ is independent of j and j'. In particular, $C_{i,k}$ will also be incident on the left only to faces of $\mathcal{C}_1$ and $\mathcal{C}_2$.

The result on intersection vector follow easily. □

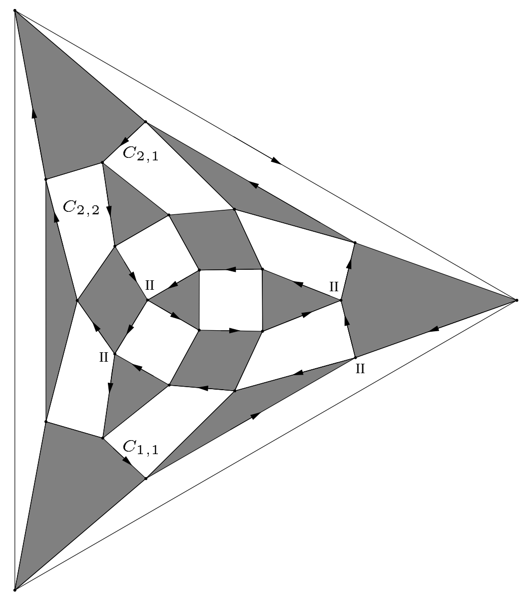
The chosen orientation is necessary for obtaining the above result on intersection vectors; see Figure 7 for an illustration of this point.

4 The moving group and the (k, l)-product

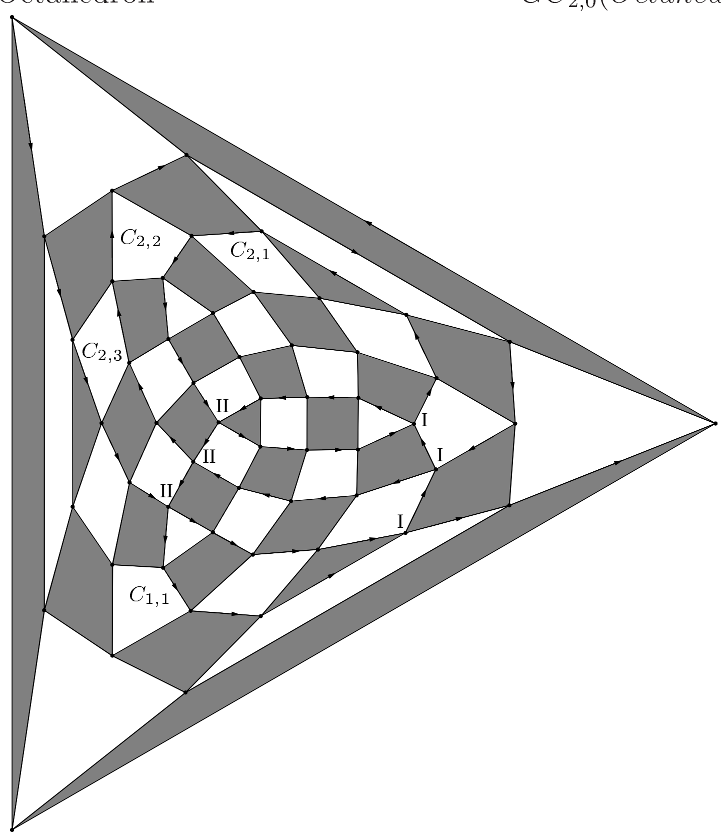
Given a group Γ acting on a set X, the *stabilizer* (also called *isotropy group*) of an element $x \in X$ is the set of elements $g \in \Gamma$, such that $gx = x$. The action is called *transitive* if for every $x, y \in X$ there exist an element $g \in \Gamma$, such that $gx = y$. The *order* of an element $u \in \Gamma$ is the smallest integer $s > 0$, such that $u^s = Id$. The action is called *free* if the stabilizer of each element of X is trivial.



Octahedron



$GC_{2,0}(Octahedron)$



$GC_{3,0}(Octahedron)$

Figure 7: Two central circuits C_1, C_2 in Octahedron and $C_{1,1}$, $(C_{2,i})_{1\leq i\leq k}$ in $GC_{k,0}(Octahedron)$ for $k = 2, 3$

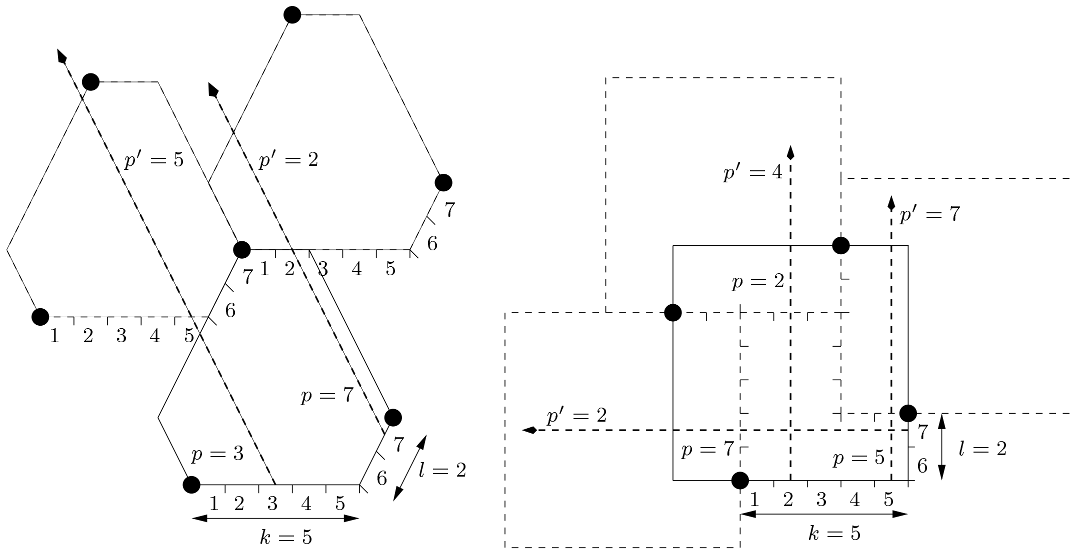


Figure 8: The position mapping $PM(G_0)$

Lemma 4.1 *If $k, l \geq 0$, then the mapping*

$$\begin{cases} \phi_{k,l} : \{1, \dots, k+l\} & \rightarrow & \{1, \dots, k+l\} \\ u & \mapsto & \begin{cases} u+l & \text{if} \quad u \in \{1, \dots, k\} \\ u-k & \text{if} \quad u \in \{k+1, \dots, k+l\} \end{cases} \end{cases}$$

is bijective and periodic with period $k+l$; moreover, the successive images of any $x \in \{1, \dots, k+l\}$ cover entirely the set $\{1, \dots, k+l\}$ of integers.

Proof. If one takes addition modulo $k+l$, then one can write $\phi_{k,l}(u) = u+l$; the lemma follows. $\square$

Let G_0 be a 3- or 4-valent graph. We call *master polygon* a triangle or a square face of G_0^* (see Figure 4). A *directed edge* is an edge of a master polygon with a fixed direction; the set of directed edges is denoted by $\mathcal{DE}$. Given a directed edge $\overrightarrow{e}$, its *reverse* (i.e. the one with the same vertices, but opposite direction) is denoted by $\overleftarrow{e}$.

Any ZC-circuit ZC of $GC_{k,l}(G_0)$, with an orientation, corresponds to a zigzag or a railroad of the dual G_0^*, which we denote ZC^*. If some edges of ZC^* belong to a master polygon, then the orientation of ZC^* determines an entering edge and this entering edge is canonically oriented by ZC^* (see Figure 4).

If $\overrightarrow{e}$ is a directed edge and ZC^* go across $\overrightarrow{e}$, then the *position p of ZC^**, relatively to $\overrightarrow{e}$, is defined as the number of the edge, contained in ZC^*, as numbered in Figure 4; the position of the circuit ZC^*, drawn in Figure 4, is 3. The directed edge, together with its position, determines the circuit ZC^* and its orientation.

Take a circuit ZC^* and a pair $(\overrightarrow{e}, p)$ with $\overrightarrow{e}$ being a directed edge and p being the position of ZC^*. The directed edge $\overrightarrow{e}$ determines a master polygon P, and the next master polygon P' (to which ZC^* belongs) determines a pair $(\overrightarrow{e}', p')$. The following equation is a key to all construction that follows:

$$p' = \phi_{k,l}(p) \; .$$

This equation can be checked on Figure 8 by examining all cases.

The mapping $(\overrightarrow{e}, p) \mapsto (\overrightarrow{e}', p')$ is called the *position mapping* and denoted by $PM(G_0)$.

Since the function $\phi_{k,l}$ is $(k+l)$-periodic by Lemma 4.1, one obtains, for any $(\overrightarrow{e}, p)$, the relation $PM(G_0)^{k+l}(\overrightarrow{e}, p) = (\overrightarrow{e}'', p)$ with $\overrightarrow{e}'' \in \mathcal{DE}$; let us call *iterated p-position mapping* and denote by $IPM_p(G_0, k, l)$ the function

$$\left\{ \begin{array}{rcl} IPM_p(G_0,k,l) : \mathcal{DE} & \to & \mathcal{DE} \\ \overrightarrow{e} & \mapsto & \overrightarrow{e}' \; . \end{array} \right.$$

Given a circuit ZC^*, let $(\overrightarrow{e}, 1)$ be a possible pair of it. Call the *order* of ZC^* and denote by $Ord(ZC)$ the smallest integer s, such that $IPM_1(G_0, k, l)^s \overrightarrow{e} = \overrightarrow{e}$.

Theorem 4.2 *If G_0 is a 3- or 4-valent plane graph without faces of zero curvature and $gcd(k,l) = 1$, then $GC_{k,l}(G_0)$ is tight.*

Proof. Take a ZC-circuit ZC of $GC_{k,l}(G_0)$. The successive pairs of ZC are denoted by $(\overrightarrow{e}_1, p_1)$, ..., $(\overrightarrow{e}_M, p_M)$ with $M = (k+l)Ord(ZC)$. By the computations done above, $p_{i+1} = \phi_{k,l}(p_i)$.

By Lemma 4.1, there exist i_0 and i_1, such that $p_{i_0} = 1$ and $p_{i_1} = k+l$. First case corresponds to an incidence on the left to a face of non-zero curvature, while the second case corresponds to an incidence on the right. $\square$

Remark 4.3 *If amongst faces of G_0 there is one of zero curvature, then, in general, $GC_{k,l}(G_0)$ is not tight if $gcd(k,l) = 1$. For the case of $G_0 = Prism_6$, we expect, that $GC_{k,l}(G_0)$ is tight if and only if $gcd(k,l) = 1$.*

Definition 4.4 *Let G_0 be a 3- or 4-valent plane graph.*

(i) In 3-valent case, define two mappings L and R, which associate to a given directed edge $\overrightarrow{e} \in \mathcal{DE}$ the directed edges $L(\overrightarrow{e})$ and $R(\overrightarrow{e})$, according to Figure 9.

(ii) In 4-valent case, define the mappings g_1, g_2 and g_3, which associate to a given directed edge $\overrightarrow{e} \in \mathcal{DE}$ the directed edges $g_1(\overrightarrow{e})$, $g_2(\overrightarrow{e})$ and $g_3(\overrightarrow{e})$, according to Figure 9. Also define $L = g_1$ and $R = g_3 \circ g_2 \circ g_1^{-1}$, where $\circ$ denotes composition operation.

Fix a directed edge $\overrightarrow{e} \in \mathcal{DE}$ and a position $p \in \{1, \dots, k+l\}$. The following hold:

(i) In 3-valent case, $PM(G_0)(\overrightarrow{e}, p) = (\overrightarrow{e}', \phi_{k,l}(p))$ with $\overrightarrow{e}' = L(\overrightarrow{e})$ or $R(\overrightarrow{e})$, according to $p \in \{1, \dots, k\}$ or $\{k+1, \dots, k+l\}$.

(ii) In 4-valent case, $PM(G_0)(\overrightarrow{e}, p) = (\overrightarrow{e}', \phi_{k,l}(p))$ with $\overrightarrow{e}' = g_1(\overrightarrow{e})$, $g_2(\overrightarrow{e})$ or $g_3(\overrightarrow{e})$, according to $p \in \{1, \dots, k-l\}$, $\{k-l+1, \dots, k\}$ or $\{k+1, \dots, k+l\}$.

Define *directed edge moving group* (in short, *moving group*) $Mov(G_0)$ to be the permutation group of the set $\mathcal{DE}$, which is generated by L and R.

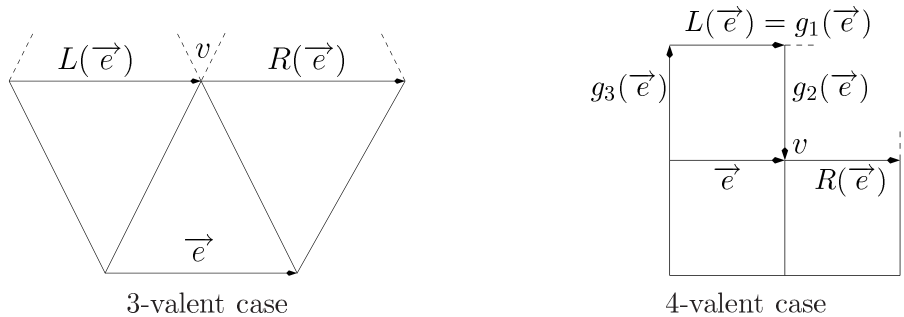


Figure 9: The first and the second mapping

Theorem 4.5 *For any ZC-circuit ZC of $GC_{k,l}(G_0)$ with $gcd(k,l) = 1$, the following hold:*

$$length(ZC) = 2t(k,l)Ord(ZC) \quad \text{if } G_0 \text{ is 3-valent and}$$
$$length(ZC) = t(k,l)Ord(ZC) \quad \text{if } G_0 \text{ is 4-valent.}$$

Proof. Let us consider the 4-valent case. Given a central circuit C, one can consider the sequence of successive pairs $(\overrightarrow{e}_1, p_1)$,...,$(\overrightarrow{e}_M, p_M)$ with $M = (k+l)Ord(C)$. To every directed edge $\overrightarrow{e_i}$, one can associate a master square, say, SQ_i. Moreover, C can be interpreted as a sequence of squares in the dual graph $(GC_{k,l}(G_0))^*$. So, to every pair $(\overrightarrow{e_i}, p_i)$ one can associate the area A_i of the set of squares in SQ_i between pair $(\overrightarrow{e_i}, p_i)$ and pair $(\overrightarrow{e}_{i+1}, p_{i+1})$. The sets, corresponding to area A_1,...,A_{k+l}, can be moved to form a full square of area $t(k,l) = k^2 + l^2$, according to Figure 10. This can be done $Ord(C)$ times. So, the length of C is equal to $t(k,l)Ord(C)$.

In the 3-valent case, the situation is a bit more complicated: for every directed edge $\overrightarrow{e_i}$, we define a master triangle, say, T_i. There is only one triangle $T_{1,i}$, adjacent to T_i and having the directed edge $L(\overrightarrow{e_i})$, and only one triangle $T_{2,i}$, adjacent to T_i and having the directed edge $R(\overrightarrow{e_i})$. The directed edges $L(\overrightarrow{e_i})$ and $R(\overrightarrow{e_i})$ are parallel to the directed edge $\overrightarrow{e_i}$. The area A_i is equal to the area of the set of triangles, which belong to the zigzag going between directed edge $\overrightarrow{e_i}$ and $L(\overrightarrow{e_i})$, $R(\overrightarrow{e_i})$. Those areas can be moved to form a parallelogram (the union of two triangles) of area $2t(k,l)$. So, the length of Z is $2t(k,l)Ord(Z)$. □

We call *partition vectors* and denote by [z], [CC] or, in general, [ZC] the vector obtained from z-vectors and CC-vectors by dividing each length by $2t(k,l)$ and $t(k,l)$, respectively (we remove the subscripts specifying self-intersections of different type). In fact, the sum of the components of [ZC]-vector of any $GC_{k,l}(G_0)$ is the number of edges of G_0.

Theorem 4.6 *Let G_0 be a plane graph; define $s = 6$ or 4 if G_0 is 3- or 4-valent, respectively. The action of $Mov(G_0)$ splits $\mathcal{DE}$ into w orbits of equal size, where w denotes the greatest common divisor of gonalities of all faces of G_0 and of s. The orbit decomposition is as follow:*

(i) If $w = 2$, then for every face F of G_0^*, denote by $DE(F)$ the set of its directed edges having F on the left. G_0^* is bipartite; denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ their corresponding sets of faces. The sets $\mathcal{DE}_i$ of directed edges of faces of $\mathcal{F}_i$ form the orbits of the action of $Mov(G_0)$ on $\mathcal{DE}$.

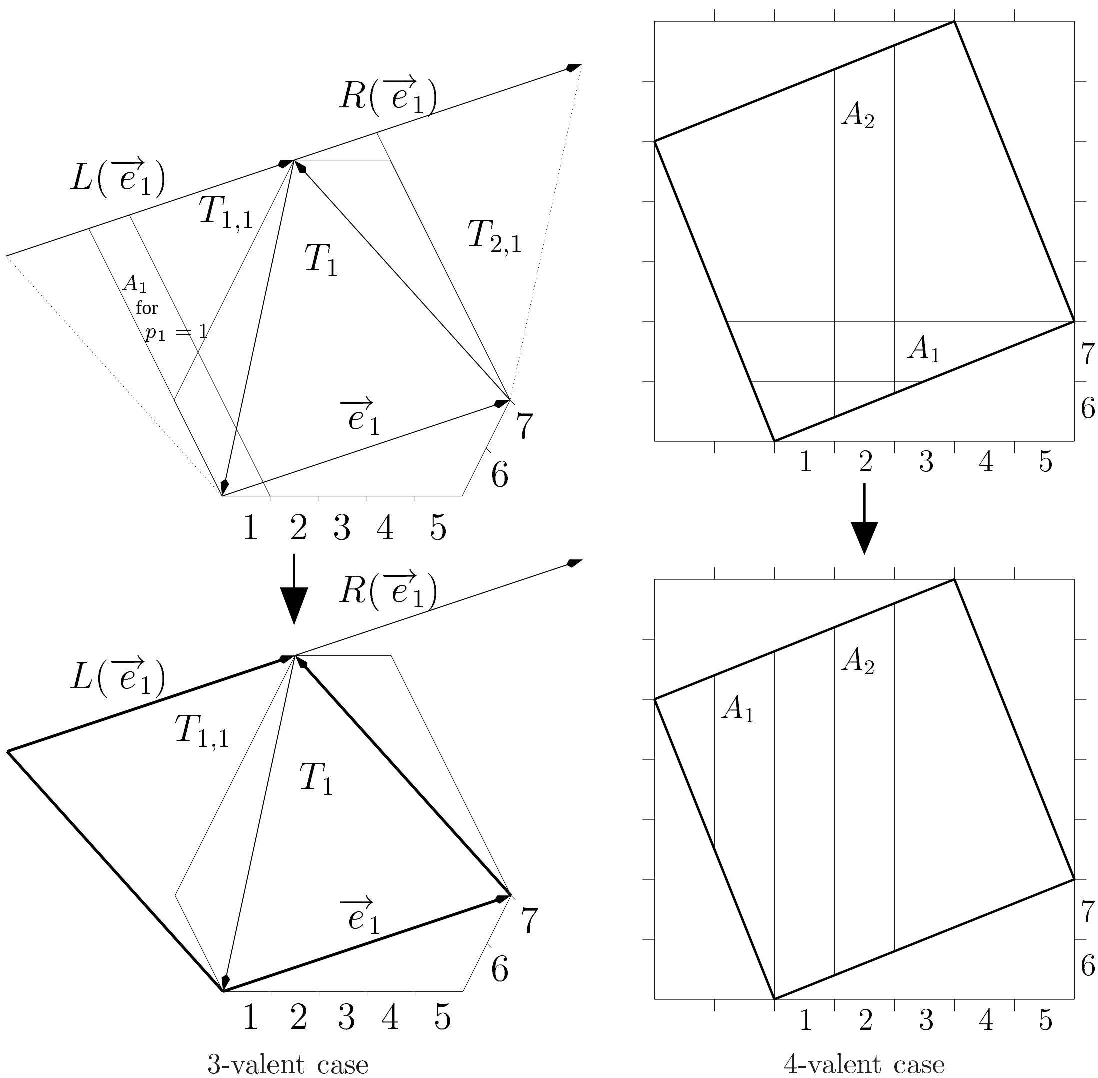


Figure 10: The area covered by ZC-circuits

(ii) If $w = 3$ and G_0 is 3-valent, then there is a tripartition of $\mathcal{DE}$ into 3 orbits O_1, O_2, O_3, such that if $\overrightarrow{e} \in O_i$, then its reverse $\overleftarrow{e}$ is also in O_i.

(iii) In other cases, there is only one orbit.

Proof. We will work in G_0^*. If G_0^* is 4-valent, then fix a square, say, sq of G_0^*. Any directed edge of G_0^* can be moved to a directed edge of sq or its reverse. Moreover, if $\overrightarrow{e}$ has sq on its right, then $L^{-1}(\overrightarrow{e})$ has sq on its left. So, any directed edge is equivalent to a directed edge of $DE(sq)$. Hence, there are at most 4 orbits of directed edges. Any directed edge can be moved using L and R, to a directed edge incident to a fixed vertex v. So, in the case of 4 orbits, the minimal valency is at least 4, which is impossible by Euler formula. Therefore, there is 1 or 2 orbits of directed edges.

If G_0 is 3-valent, then fix a triangle, say, Δ of G_0^*. Any directed edge of G_0^* can be moved to a directed edge of $DE(\Delta)$ or to its reverse. So, there are at most 6 orbits of directed edges. Any directed edge can be moved using L and R to a directed edge incident to a fixed vertex v. So, in the case of 6 orbits, the minimal valency is at least 6, which is impossible by Euler formula. So, there are 1, 2 or 3 orbits of faces.

If all faces have even gonality, then G_0^* is bipartite and the corresponding bipartition of faces $\mathcal{F}_1 = \{F_1, \dots\}$, $\mathcal{F}_2 = \{F'_1, \dots\}$ induces a bipartition of $\mathcal{DE}$ by $\{DE(F_1), \dots\}$ and $\{DE(F'_1), \dots\}$. So, there are two orbits and, given a directed edge $\overrightarrow{e}$, its reverse belongs to the other orbit. If some faces have odd gonality and G_0 is 4-valent, then there is no such bipartition and so, there is only one orbit. In 3-valent case, if $\overrightarrow{e}$ is in orbit, say, O, then its reverse is also in O and one can identify pairs of opposite directed edges with edges.

Take an edge, say, $e = \{v, v'\}$ in G_0^*, and denote by O the orbit, to which it belongs. We will prove, that O contains one third of all edges. By hypothesis, v and v' have valency divisible by 3. By successive iteration of $L \circ R^{-1}$, one gets that at least one third of all edges, incident to v, belong to O. This yields that O contains at least one third of all edges. Now, let us prove that for each vertex, exactly one third of all edges belong to O. Let us take a vertex v incident to two edges $e, e' \in O$, which are adjacent; so, at least two third of all edges, which are incident to v, are in O. By hypothesis, there exists a path of edges $e = e_1, \dots, e_N = e'$, such that e_{i+1}, is obtained by application of L or R, the rotation $L \circ R^{-1}$, $R^{-1} \circ L$, or their inverses. One can assume the path to be of minimal length; this imply that the sequence has no self-intersection. The contradiction arises by application of the Euler formula. So, there are three orbits. □

Theorem 4.7 *If G_0 is a 3- or 4-valent plane graph, then $Mov(G_0)$ is commutative if and only if the graph G_0 is either a 2_n, a 3_n, or a 4-hedrite.*

Proof. In 3-valent case, one can see from Figure 9, that $L \circ R(\overrightarrow{e}) = R \circ L(\overrightarrow{e})$ if and only if v has valency 2, 3 or 6. In dual terms, it corresponds to G_0 having 2-, 3- or 6-gonal faces only. Euler formula $12 = 4p_2 + 3p_3$ for 3-valent plane graphs have solutions $(p_2, p_3)=(3, 0)$ or $(0, 4)$ only.

In 4-valent case, the equality $L \circ R(\overrightarrow{e}) = R \circ L(\overrightarrow{e})$ holds if and only if the vertex v in Figure 9 is 2- or 4-valent. A 4-valent plane graph with all faces being 2- or 4-gons, is exactly a 4-hedrite. □

Remark 4.8 *The problem of generating every graph in the three classes of the above Theorem has been solved:*

- *All 2_n come by the Goldberg-Coxeter construction from the Bundle (see [GrZa74]).*
- *All 3_n are described in [GrünMo63] (see also [DeDu02], where it is recalled).*
- *All 4-hedrites are described in [DeSt03] (see also [DDS03], where it is recalled).*

Moreover, no other general classes of graphs q_n or i-hedrites is known to admit such simple descriptions.

Given a pair $(k,l) \in \mathbb{Z}^2$, define the *residual group* $Res_{k,l}$ to be the quotient of A_2 or $\mathbb{Z}^2$ (seen as a group) by the sub-group generated by complex numbers $k + l\omega$, $\omega(k + l\omega)$ or, respectively, $k + li$, $i(k + li)$.

Conjecture 4.9 *(i) The group $Mov(G_0)$ is isomorphic to a subgroup of $Mov(GC_{k,l}(G_0))$.*

(ii) If $Mov(G_0)$ is commutative, then $Mov(GC_{k,l}(G_0))$ is also commutative and $Mov(GC_{k,l}(G_0))/Mov(G_0)$ is isomorphic to $Res_{k,l}$.

(iii) If G_0 is a graph 3_n (respectively, a 4-hedrite), such that $G_0 \neq GC_{k,l}(G_1)$ for G_1 being any other graph 3_n (respectively, any other 4-hedrite), then $Mov(G_0)$ has $\frac{n^2}{4}$ (respectively, n^2) elements.

(iv) A corollary of (iii): all orders of moving groups are the numbers $\frac{n^2}{4t(k,l)}$ (respectively, $\frac{n^2}{t(k,l)}$) with $t(k,l)$ dividing $\frac{n}{4}$ (respectively, $\frac{n}{2}$) for 3_n (respectively, for 4-hedrites).

Remark 4.10 *The order of the group $Mov(GC_{k,l}(G_0))$ seems to depend on (k,l) in a complicate way and $Mov(G_0)$ is not, in general, a* normal *subgroup of $Mov(GC_{k,l}(G_0))$.*

The following definition of (k,l)-*product* can be considered for any group Γ, but in this paper we used it only for the case, when Γ is a moving group of some 3- or 4-valent plane graph G_0. It seems to us, that the majority of notions of this Section are new in both, combinatorial and algebraic, contexts. However, an analogous expression of this product itself was proposed in [No87], on the Fisher-Griess Monster group.

Definition 4.11 *(the (k,l)-product) Let Γ be a group and g_1, g_2 be two of its elements. Given a pair $(k,l) \in \mathbb{N}^2$ with $gcd(k,l) = 1$, define an element of Γ be their (k,l)-*product *(and denote it by $g_1 \odot_{k,l} g_2$) in the following way:*

Define inductively the sequence $(p_0, \dots, p_{k+l})$ by $p_0 = 1$, $p_i = \phi_{k,l}(p_{i-1})$.

Set $S_i = g_1$ if $p_i - p_{i-1} = l$ and $S_i = g_2$ if $p_i - p_{i-1} = -k$; then set

$$g_1 \odot_{k,l} g_2 = S_{k+l} \dots S_2 S_1 .$$

By convention, set $g_1 \odot_{1,0} g_2 = g_1$ and $g_1 \odot_{0,1} g_2 = g_2$.

In the following Theorem, the above formalism is used to translate the Goldberg-Coxeter construction in terms of representation of permutations as product of cycles.

For an element $u \in Mov(G_0)$, denote by $ZC(u)$ the vector $\dots, c_k^{m_k}, \dots$ with multiplicities m_k being the half of the number of cycles of length c_k in the permutation u acting on the set $\mathcal{DE}$. For S a subset of $Mov(G_0)$, denote by $ZC(S)$ the set of all $ZC(u)$ with $u \in S$.

Theorem 4.12 *Let G_0 be a 3- or 4-valent plane graph. The following hold:*
(i) $IPM_1(G_0, k, l) = L \odot_{k,l} R$ and
(ii) the partition vector [ZC] of $GC_{k,l}(G_0)$ is $ZC(u)$ with $u = IPM_1(G_0, k, l)$.

Proof. In 3-valent case the result follows from the very definition of $IPM_1(G_0, k, l)$. In 4-valent case, the situation is a bit more complicated. Given a sequence of positions $(p_0, p_1, \dots, p_{k+l})$, $S_i = g_1,\ g_2,\ g_3$, according to $p_{i-1} \in \{1, \dots, k-l\}$, $\{k-l+1, \dots, k\}$, $\{k+1, \dots, k+l\}$, and it holds $IPM_1(G_0, k, l) = S_{k+l} \circ \dots \circ S_2 \circ S_1$.

Any multiplication by g_2 is followed by a multiplication by g_3; hence, the relation $IPM_1(G_0, k, l) = g_1 \odot_{k,l} g_3 \circ g_2 \circ g_1^{-1} = L \odot_{k,l} R$.

Take any ZC-circuit ZC and define its *sequence of pairs* as $(\overrightarrow{e}_1, p_1)$, ..., $(\overrightarrow{e}_M, p_M)$.

It holds $M = (k+l)Ord(ZC)$ and the values $p_i = 1,\ k+l$ appear $Ord(ZC)$ times. If one reverses the orientation on ZC, then the corresponding sequence of pairs is $(\overleftarrow{e}_M, k+l+1-p_M)$, ..., $(\overleftarrow{e}_1, k+l+1-p_1)$ with $\overleftarrow{e_i}$ being the reverse directed edge of $\overrightarrow{e_i}$. It implies, that to every ZC-circuit of length $Ord(ZC)$ correspond two cycles:

$$(IPM_1(G_0, k, l)^i \overrightarrow{e}_1)_{0 \le i \le Ord(ZC)-1} \quad \text{and} \quad (IPM_1(G_0, k, l)^i \overleftarrow{e}_M)_{0 \le i \le Ord(ZC)-1},$$

both of length $Ord(ZC)$. □

Remark 4.13 *The following hold:*
(i) $g_1 \odot_{1,1} g_2 = g_2 g_1$, $g_1 \odot_{k,1} g_2 = g_2 g_1^k$ and $g_1 \odot_{k,k-1} g_2 = (g_2 g_1)^{k-1} g_1$.
(ii) $g_1 \odot_{2q+1,2} g_2 = g_1 (g_1^q g_2)^2$ for any integer q.

The Proposition below gives Euclid algorithm formulas, which can be used to compute $g_1 \odot_{k,l} g_2$ in an efficient way.

Proposition 4.14 *If $(k, l) \in \mathbb{N}^2$ with $gcd(k, l) = 1$, then the following hold:*
(i) If q is an integer, then it holds:

$$\begin{cases} g_1 \odot_{k,l} g_2 &= g_1 \odot_{k-ql,l} g_2 g_1^q \quad \text{if } k - ql \ge 0, \\ g_1 \odot_{k,l} g_2 &= g_2^q g_1 \odot_{k,l-qk} g_2 \quad \text{if } l - qk \ge 0. \end{cases}$$

(ii) $\{g_1 \odot_{k,l} g_2\}^{-1} = g_2^{-1} \odot_{l,k} g_1^{-1}$.
(iii) $g_1 \odot_{k,l} g_2 = g_1^k g_2^l$ if g_1 and g_2 commute.
(iv) $g_1 \odot_{k,l} g_2 \neq Id$ if g_1 and g_2 do not commute.

Proof. (i) and (ii) can be obtained by writing down the expressions on both sides and identification. The properties (i) and (ii) allow to compute $g_1 \odot_{k,l} g_2$ by applying the Euclid algorithm to the pair (k, l); at each step of Euclid algorithm, the pair (g_1, g_2) is modified into another pair (g_1', g_2'). It follows from (i) and (ii), that g_1 and g_2 do not commute; so, at any step of the computation, the pair of elements will not commute. Therefore, it is not possible that $g_1 \odot_{k,l} g_2 = Id$, since Id commutes with every element and it yields the commutativity of g_1 and g_2. □

Corollary 4.15 *If partition vector [ZC] of $GC_{k,l}(G_0)$ is 1^p for one pair (k, l), then $Mov(G_0)$ is commutative.*

Proof. The partition vector 1^p corresponds to $IPM_1(G_0, k, l) = Id$; Theorem 4.14 (iv) yields that $Mov(G_0)$ is commutative. $\square$

Remark 4.16 *The only known example of graph G_0, such that partition vector $[ZC]$ of $GC_{k,l}(G_0)$ can be 1^p, is the Bundle.*

Proposition 4.17 *Let $\Gamma = \langle g_1, g_2\rangle$ be a group, generated by two elements g_1 and g_2, and let K be a proper normal subgroup of Γ. Let $(k,l) \in \mathbb{N}^2$ with $gcd(k,l) = 1$. The following hold:*

(i) If Γ/K is non-commutative, then $g_1 \odot_{k,l} g_2 \notin K$.

(ii) If Γ/K is commutative and $\overline{g_2} = \overline{g_1}^{-1}$ (with $\overline{x} = xK$), then $g_1 \odot_{k,l} g_2 \in K$ if and only if $k - l \equiv 0$ modulo the index of K in Γ.

(iii) Denote by n_1 and n_2 the orders of $\overline{g_1}$ and $\overline{g_2}$, respectively, considered as group elements. Assume that the following properties hold:

- *Γ/K is commutative,*
- *$gcd(n_1, n_2) > 1$,*
- *the mapping*

$$\begin{array}{rcl} \Psi : \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} & \to & \Gamma/K \\ (k,l) & \mapsto & \overline{L}^k \overline{R}^l \end{array}.$$

is an isomorphism.

Then $g_1 \odot_{k,l} g_2 \notin K$ for every k, l with $gcd(k,l) = 1$.

Proof. The (k,l)-product goes over to the quotient, i.e. $\overline{g_1 \odot_{k,l} g_2} = \overline{g_1} \odot_{k,l} \overline{g_2}$; so, (i) follows from 4.14 (iii).

If the quotient is commutative, then $\overline{g_1 \odot_{k,l} g_2} = \overline{g_1}^k \overline{g_2}^l = \overline{g_1}^{k-l}$. The quotient is generated by $\overline{g_1}$; so, (ii) follows.

In case (iii), $g_1 \odot_{k,l} g_2 \in K$ if and only if $\overline{g_1}^k \overline{g_2}^l = 1$, i.e. k and l are, respectively, divisible by n_1 and n_2. By the condition $gcd(k,l) = 1$, this implies $k = l = 0$. $\square$

4.1 The stabilizer group

Denote by $\mathcal{P}(G_0)$ the set of all pairs (g_1, g_2) with $g_i \in Mov(G_0)$. Denote by U_{g_1,g_2} the smallest subset of $\mathcal{P}(G_0)$, containing the pair $(g_1, g_2) \in \mathcal{P}(G_0)$, which is stable by the operations $(x, y) \mapsto (x, yx)$ and $(x, y) \mapsto (yx, y)$.

Theorem 4.18 *If G_0 is a 3- or 4-valent plane graph, then it holds:*

(i) The sequence of subsets $U_{i,L,R}$, defined by $U_{0,L,R} = \{(L,R)\}$ and

$$U_{n+1,L,R} = \{(v,w), (v,wv), (wv,w) \text{ with } (v,w) \in U_{n,L,R}\},$$

satisfy to $U_{n,L,R} = U_{L,R}$ for n large enough.

(ii) The set of all possible [ZC]-vectors of $GC_{k,l}(G_0)$ is the set formed by all partition vectors $ZC(v)$ and $ZC(w)$ with $(v,w) \in U_{L,R}$.

Proof. Since G_0 is finite, $Mov(G_0)$ is finite and so, $U_{L,R}$ too. The sequence $(U_{n,L,R})_{n\in\mathbb{N}}$ is increasing and so, by finiteness, there exists an n_0, such that $U_{n_0,L,R} = U_{n_0+1,L,R}$. By construction, the set $U_{n_0,L,R}$ is stable by the operations $(x,y) \mapsto (x,yx)$, (yx,y), which yields (i).

Fix a pair $(k,l) \in \mathbb{N}^2$ with $gcd(k,l) = 1$. By successive applications of Proposition 4.14 (i), one obtains $L \odot_{k,l} R = g_1 \odot_{1,0} g_2$ or $g_1 \odot_{0,1} g_2$ with $(g_1,g_2) \in U_{L,R}$. So, $L \odot_{k,l} R = g_1$ or g_2. Hence, the possible [ZC]-vectors of $GC_{k,l}(G_0)$ are obtained from g_1 or g_2. On the other hand, if $(g_1,g_2) \in U_{L,R}$, then, by reversing the process described in (i), that led to (g_1,g_2), one obtains two pairs $(k_i,l_i) \in \mathbb{N}^2$, $gcd(k_i,l_i) = 1$ (with $i = 1$ or 2), such that $L \odot_{k_i,l_i} R = g_i$. □

The *modular group* $SL_2(\mathbb{Z})$ is the group of all 2×2 integral matrices of determinant 1. This group is generated by the matrices $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. The group $PSL_2(\mathbb{Z})$ is the quotient of $SL_2(\mathbb{Z})$ by its center $\{I_2, -I_2\}$ with I_2 being the identity matrix. The matrices T and U satisfy to $T^2 = -I_2$ and $U^3 = I_2$.

Lemma 4.19 *(i) The group $PSL_2(\mathbb{Z})$ is isomorphic to the group generated by two elements x, y subject to the relations:*

$$x^2 = Id \text{ and } y^3 = Id\,.$$

(ii) The group $SL_2(\mathbb{Z})$ is isomorphic to the group generated by two elements x, y subject to the relations:

$$x^4 = Id,\;\; x^2y = yx^2 \text{ and } y^3 = Id\,.$$

Proof. (i) is proved in [Ne72]. In order to prove (ii), we will use (i) and the surjective mapping

$$\begin{array}{rcl} \phi : SL_2(\mathbb{Z}) & \to & PSL_2(\mathbb{Z}) \\ M & \mapsto & \{M, -M\}\,. \end{array}$$

Let $W = I_2$ be a word in letters T and U. Write $W = S_1^{n_1} \ldots S_N^{n_N}$ with $S_i = T$, U if i is odd, even, respectively. Using the relation $T^4 = I_2$ and $U^3 = I_2$, one can assume that $n_i \in \{1,2,3\}$, $n_i \in \{1,2\}$ if i is odd, even, respectively. Using the relation $T^2U = UT^2$, one can reduce ourselves to the case of $n_i = 1$ if i odd and greater than 1. Using the morphism ϕ and the property (i), one obtains $m_i = 0$ if i is even. So, the expression can be rewritten as $T^h = I_2$. □

Definition 4.20 *Let Γ be a group.*

(i) Let $\mathcal{P}(\Gamma)$ be the set of all pairs (g_1,g_2) of elements of Γ.

(ii) The derived group *$D(\Gamma)$ is defined as the group generated by all $uvu^{-1}v^{-1}$ with $u,v \in \Gamma$; it is a normal subgroup of Γ and it is trivial if and only if Γ is commutative.*

(iii) The group $D(\Gamma)$ acts on Γ and $\mathcal{P}(\Gamma)$ in the following way:

$$\left\{\begin{array}{rcl} Int : D(\Gamma) \times \Gamma & \to & \Gamma \\ (a,g) & \mapsto & Int_a(g) = aga^{-1}, \end{array}\right.$$
$$\left\{\begin{array}{rcl} Int : D(\Gamma) \times \mathcal{P}(\Gamma) & \to & \mathcal{P}(\Gamma) \\ (a,(g_1,g_2)) & \mapsto & Int_a(g_1,g_2) = (Int_a(g_1), Int_a(g_2))\,. \end{array}\right.$$

The set of equivalence classes of $\mathcal{P}(\Gamma)$ under this action is denoted by $\mathcal{CP}(\Gamma)$.

The mappings Int_a are automorphisms, which are usually called *interior*.

Theorem 4.21 *There exists a group action*

$$\begin{array}{rcl} \phi : SL_2(\mathbb{Z}) \times \mathcal{CP}(\Gamma) & \to & \mathcal{CP}(\Gamma) \\ (M, c) & \mapsto & \phi(M)c \ , \end{array}$$

such that if $\phi(M)(g_1, g_2) = (h_1, h_2)$, *then* $g_1 \odot_{(k,l)M} g_2$ *is conjugated to* $h_1 \odot_{k,l} h_2$, *where* $(k, l)M = (ak + cl, bk + dl)$ *for* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof. Let us define:

$$\begin{array}{rcl} \phi(T)(g_1, g_2) & = & (g_2, g_2 g_1^{-1} g_2^{-1}) = Int_{g_2}(g_2, g_1^{-1}) \text{ and} \\ \phi(U)(g_1, g_2) & = & (g_2, g_2 g_1^{-1} g_2^{-2}) = Int_{g_2}(g_2, g_1^{-1} g_2^{-1}) \ . \end{array}$$

This defines mappings from $\mathcal{P}(\Gamma)$ to $\mathcal{P}(\Gamma)$ and so, mappings from $\mathcal{CP}(\Gamma)$ to $\mathcal{CP}(\Gamma)$.

If $M \in SL_2(\mathbb{Z})$ then one can find an expression $M = S_1 \ldots S_N$ with $S_i = T$ or U and define:

$$\begin{array}{rcl} \phi(M) : \mathcal{CP}(\Gamma) & \to & \mathcal{CP}(\Gamma) \\ c & \mapsto & \phi(M)c = \phi(S_1) \ldots \phi(S_N)c \ . \end{array}$$

In order to prove, that ϕ is well defined, one needs to prove the independence of $\phi(M)$, over the different expressions of M, in terms of T and U. By standard, but tedious, computations one gets, using the definition of $\phi(T)$ and $\phi(U)$:

$$\left\{ \begin{array}{rcl} \phi(T)^4(g_1, g_2) & = & \phi(U)^3(g_1, g_2) = Int_{g_2 g_1^{-1} g_2^{-1} g_1}(g_1, g_2), \\ \phi(U)\phi(T)^2(g_1, g_2) & = & \phi(T)^2 \phi(U)(g_1, g_2) \ . \end{array} \right.$$

The above computations prove the independence of $\phi(M)$ over the different possible expressions of M, since, by Lemma 4.19, all relations, satisfied by T and U, are generated by $T^4 = U^3 = I_2$ and $T^2 U = U T^2$. One obtains the relation $\phi(MM') = \phi(M)\phi(M')$ by concatenating two expressions of M and M' in terms of T and U.

One gets $\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (g_2 g_1, g_2)$ and $\phi \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (g_1, g_2 g_1)$, which yields the asked relation for the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since those matrices generate $SL_2(\mathbb{Z})$, the relation is always true. $\square$

Note that the (k, l)-product $g_1 \odot_{k,l} g_2$ is defined for every pair (k, l) with $k \geq 0$, $l \geq 0$ and $gcd(k, l) = 1$. Using the matrices T or U, one can extend it for every pair (k, l) with $gcd(k, l) = 1$, keeping in mind the important fact, that it is defined only up to conjugacy. The obtained extension still denoted $g_1 \odot_{k,l} g_2$ satisfy formula (i) of Proposition 4.14 up to conjugacy without restriction on signs.

For a given 3- or 4-valent plane graph, denote $\mathcal{CP}(G_0)$ the set of equivalence classes of $\mathcal{P}(G_0)$ under the action of $D(Mov(G_0))$. Also, denote by $Stab(G_0)$ the stabilizer of the pair $(L, R) \in \mathcal{CP}(G_0)$ under the action of $SL_2(\mathbb{Z})$ on $\mathcal{CP}(G_0)$.

Proposition 4.22 *If G_0 be a 3- or 4-valent plane graph, then the following hold:*

(i) $Stab(G_0)$ is a finite index subgroup of $SL_2(\mathbb{Z})$, whose index I is equal to the size of the orbit of $(L,R) \in \mathcal{CP}(G_0)$ under the action of $SL_2(\mathbb{Z})$.

(ii) If $(k_1,l_1) = (k_0,l_0)M$ with $M \in Stab(G_0)$, then $GC_{k_0,l_0}(G_0)$ and $GC_{k_1,l_1}(G_0)$ have the same [ZC]-vector.

(iii) There exist a finite set $\{(k_1,l_1),\dots,(k_I,l_I)\}$ with $gcd(k_i,l_i) = 1$, such that, denoting by P_i the [ZC]-vector of $GC_{k_i,l_i}(G_0)$, the following hold: for every (k,l) with $gcd(k,l) = 1$, there is an $i_0 \in \{1,\dots,I\}$ and an $M \in Stab(G_0)$, such that $(k,l)M = (k_{i_0},l_{i_0})$ and $GC_{k,l}(G_0)$ has [ZC]-vector P_i.

Proof. (i) The group $Mov(G_0)$ is finite; so, $\mathcal{P}(G_0)$ and $\mathcal{CP}(G_0)$ are finite and the orbit of (L,R) is finite also. This implies the finite index property by elementary group theory.

(ii) If $(k_1,l_1) = (k_0,l_0)M$, then $L \odot_{k_0,l_0} R$ and $L \odot_{k_1,l_1} R$ are equal, up to a conjugacy. Since conjugacy does not change the cyclic structure, it does not change the corresponding [ZC]-vector. So, $GC_{k_0,l_0}(G_0)$ has the same [ZC]-vector as $GC_{k_1,l_1}(G_0)$.

(iii) Since a partition vector partitions a finite set, there exist a finite number of possibilities for it. Denote by $M_1,\dots,M_I$ the set of coset representatives of $Stab(G_0)$ in $SL_2(\mathbb{Z})$. The group $SL_2(\mathbb{Z})$ is transitive on the set of pairs $(k,l) \in \mathbb{Z}^2$ with $gcd(k,l) = 1$. So, for any $(k,l) \in \mathbb{Z}^2$ with $gcd(k,l) = 1$, there exists $P \in SL_2(\mathbb{Z})$, such that $(k,l)P = (1,0)$. Write $P = MM_i$ with $M \in Stab(G_0)$ and one obtains $(k,l)M = (k_i,l_i)$ with $(k_i,l_i) = (1,0)M_i^{-1}$. □

Remark 4.23 *(i) The hexagonal (or square) lattice have a point group of isometry of order 6 (or 4) of rotation of angle $\frac{\pi}{3}$ (or $\frac{\pi}{2}$). So, $GC_{k,l}(G_0)$ is isomorphic to $GC_{-l,k+l}(G_0)$ (or to $GC_{-l,k}(G_0)$). One would expect, that $Stab(G_0)$ contains a subgroup, which is isomorphic to this point group. In fact, this is the case of Dodecahedron and Octahedron, but not of Tetrahedron, for which $-I_2 \notin Stab(Tetrahedron)$. It may be possible, that our definition of membership in $Stab(G_0)$ is too strict and that with another definition, one will get this point group as subgroup.*

(ii) It seems, that there are no constraints on the values of the coefficients of elements of $Stab(G_0)$.

Conjecture 4.24 *(i) $Stab(Dodecahedron)$ is generated by*

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -4 & -3 \\ 3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix};$$

(ii) $Stab(Cube)$ is generated by

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix};$$

(iii) $Stab(Octahedron)$ is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -4 & -3 \\ 3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conjecture 4.25 *For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one defines A' by:*

(i) if either $a \neq d$, or $a = d = 0$, then $A' = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$;

(ii) otherwise, $A' = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Let G_0 be a 3-valent graph. If $A \in Stab(G_0)$, then $A' \in Stab(G_0)$.

5 Classes of graphs

Theorem 5.1 *Every graph 2_n comes as $GC_{k,l}(Bundle)$; its symmetry group is D_{3h} if $l = 0$, k and D_3, otherwise.*

Proof. It is given implicitly in [GrZa74]. □

The complete list of all possible symmetry groups of graphs q_n and i-hedrites were found: for 5_n in [FoMa95], for 3_n in [FoCr97], for 4_n in [DeDu02] and for i-hedrites in [DDS03].

Part (iv) of Theorem below is proved in [Gold37] and (i), (ii) are only indicated there.

Theorem 5.2 *(i) Any graph 3_n with symmetry T or T_d is $GC_{k,l}(Tetrahedron)$,*
(ii) any graph 4_n with symmetry O or O_h is $GC_{k,l}(Cube)$,
(iii) any graph 4_n with symmetry D_6 or D_{6h} is $GC_{k,l}(Prism_6)$,
(iv) any graph 5_n with symmetry I or I_h is $GC_{k,l}(Dodecahedron)$,
(v) any 4-hedrite with symmetry D_4 or D_{4h} is $GC_{k,l}(Foil_2)$,
(vi) any 5-hedrite of symmetry D_3 or D_{3h} is $GC_{k,l}(Trefoil)$,
(vii) any 8-hedrite of symmetry O or O_h is $GC_{k,l}(Octahedron)$.

Proof. Take a graph 3_n of symmetry T or T_d. Given a face F, the size of its orbit (under the action of the group T) is 4 if F lies on an axis of rotation of order 3, 6 if F lies on an axis of rotation of order 2, or 12 if F is in general position. This implies that all four triangles are on axis of order 3. Take a triangle, say, T_1; after adding p rings of hexagons, one finds a triangle and so, three triangles, say, T_2, T_3 and T_4. The position of triangle T_2 relatively to T_1 defines the Eisenstein integer, corresponding to this graph. One can see easily, that this graph is $GC_{k,l}(Tetrahedron)$.

Take a graph 4_n of symmetry O or O_h. One 4-fold symmetry axis goes through a square, say, sq_1. After adding p rings of hexagons around sq_1, one finds a square and so, by symmetry, four squares, say, sq_2, sq_3, sq_4, sq_5. The position of the square sq_2 relatively to sq_1 defines an Eisenstein integer $z = k + l\omega$. The graph can be completed in an unique way and this proves, that it is $GC_{k,l}(Cube)$.

Take a graph 5_n of symmetry I or I_h. Any 5-fold axis must go though two pentagons. Since the group I contains six 5-fold axises, this means that every pentagon belongs to one 5-fold axis. Take a pentagon, say, P_1; after adding p rings of hexagons around P_1, one finds five pentagons, in cyclic order, say, P_2, P_3, P_4, P_5, P_6. The position of pentagon P_2 relatively to P_1 defines an Eisenstein integer $k + l\omega$, which is equal to the position of P_3 relatively to P_2 and to the position of P_1 relatively to P_3. The figure formed by P_1,

P_2, P_3 is reproduced all over the graph, thanks to the six 5-fold axes. So, the Eisenstein integer defines entirely the graph.

Take a 4-hedrite G with symmetry D_4 or D_{4h}. The 4-fold axis must go through two vertices, say, v_1, v_2 or two 4-gonal faces, say, sq_1, sq_2. After adding p rings of squares around v_1 or sq_1, one finds a 2-gon and so, by symmetry, four 2-gons, say, Δ_i, $1 \leq i \leq 4$. The position of Δ_2, relatively to Δ_1, determines a Gaussian integer $k + li$, such that $G = GC_{k,l}(Foil_2)$.

Take an 8-hedrite of symmetry O or O_h. Any 3-fold axis must go through two triangles. Since there are four 3-fold axis of symmetry, this implies that any triangle contains a 3-fold axis of symmetry. The proof is then similar to the case of 5_n with symmetry I or I_h.

The proofs of (iii) and (vi) are special cases of, respectively, (i) and (ii) of Proposition 5.3. $\square$

For other classes of graphs, the description should be done in terms of several complex parameters. For them, it is not possible to obtain a description in terms of Goldberg-Coxeter construction of basic graphs, even a finite number of such graphs.

Proposition 5.3 *(i) Let $\mathcal{GP}_m$ (for $m \neq 2, 4$) denote the class of* 3*-valent plane graphs with two m-gonal faces, m* 4*-gons and p_6* 6*-gonal faces. Every such graph, having a m-fold axis, comes as $GC_{k,l}(Prism_m)$ and has symmetry group D_m or D_{mh}.*

(ii) Let $\mathcal{GF}_m$ (for $m \neq 2, 3$) denote the class of 4*-valent plane graphs with two m-gonal faces, m* 2*-gons and p_4* 4*-gonal faces. Every such graph, having a m-fold axis, comes as $GC_{k,l}(Foil_m)$ and has symmetry group D_m or D_{mh}.*

Proof. Take a graph $\mathcal{GP}_m$ with an m-fold axis; the m-fold axis goes through two m-gonal faces, say, F_1 and F_2. After adding p rings of hexagons around F_1, one finds a square, say, sq_1 and so, by symmetry, m squares, say, $sq_1, \ldots, sq_m$. The position of sq_1 relatively to F_1 defines an Eisenstein integer $k + l\omega$, such that the graph is $GC_{k,l}(Prism_m)$.

Take a graph $\mathcal{GF}_m$ with an m-fold axis. The m-fold axis must go through the two m-gonal faces, say, F_1 and F_2. After adding p rings of squares around F_1, one finds a 2-gon and so, by symmetry, m 2-gons, say, $D_1, \ldots, D_m$. The position of D_1 relatively to F_1 defines a Gaussian integer $k + li$. Once the position of the digons D_i is found, the graph is uniquely determined and so, it is $GC_{k,l}(Foil_m)$. $\square$

6 The ZC-structure of the Goldberg-Coxeter construction of basic plane graphs

Consider the Goldberg-Coxeter construction $GC_{k,l}(G)$ for some two-faced plane graphs of high symmetry. Observe that if $gcd(k, l) = u$, then one can decompose, using Proposition 3.1, the action as two consecutive ones: $GC_{\frac{k}{u},\frac{l}{u}}(G)$ and u-inflation. So, using Proposition 3.2 and 3.3, it suffices to consider only the case $gcd(k, l) = 1$.

We will consider below the following problems:

- what are the possible [ZC]-vectors of $GC_{k,l}(G_0)$?
- how can those [ZC]-vectors be expressed in terms of (k, l)?

Given a graph G_0, the first problem can be solved by using Theorem 4.18.

For the second problem, one can prove in some cases (see Theorem 6.7) simple congruence conditions which determine the [ZC]-vector, by using the normal subgroups of the moving group and Proposition 4.17.

While the moving group allows us to prove most of the results below, in some cases (see Theorem 6.5) the geometric considerations are sufficient. An important case, considered in Theorem 6.1 and 6.2, is the one, when $Rot(G_0)$ is transitive on $\mathcal{DE}$. Given a group Γ, the enumeration of 3-valent maps M with $Rot(G_0) = \Gamma$ being transitive on $\mathcal{DE}$, is carried on in [Jo85].

Theorem 6.1 *If G_0 is a 3- or 4-valent plane graph, then the following hold:*

(i) The actions of $Rot(G_0)$ and $Mov(G_0)$ on $\mathcal{DE}$ commute.

(ii) The action of $Rot(G_0)$ on $\mathcal{DE}$ is free.

(iii) If the action of $Rot(G_0)$ on $\mathcal{DE}$ is transitive, then:

(iii.1) the action of $Mov(G_0)$ on $\mathcal{DE}$ is free,

(iii.2) every directed edge $\overrightarrow{e} \in \mathcal{DE}$ defines an injective group morphism

$$\left\{\begin{array}{rcl} \phi_{\overrightarrow{e}} : Mov(G_0) & \to & Rot(G_0) \\ u & \mapsto & \phi_{\overrightarrow{e}}(u) \end{array}\right. \quad \textit{with} \;\; u^{-1}(\overrightarrow{e}) = \phi_{\overrightarrow{e}}(u)(\overrightarrow{e}),$$

(iii.3) if $\overrightarrow{e}, \overrightarrow{e}' \in \mathcal{DE}$, then there is a $w \in Rot(G_0)$, such that $\phi_{\overrightarrow{e}'}(u) = w^{-1} \circ \phi_{\overrightarrow{e}}(u) \circ w$,

(iii.4) for any $\overrightarrow{e} \in \mathcal{DE}$, $\phi_{\overrightarrow{e}}(Mov(G_0))$ is the normal subgroup of $Rot(G_0)$, formed by all elements preserving the orbit partition of $\mathcal{DE}$ under the action of $Mov(G_0)$.

Proof. (i) The action of $Mov(G_0)$ is defined, in geometric terms, on Figure 9; so, any rotation of G_0 preserves this picture and two actions commute.

(ii) The only rotation, preserving a directed edge, is, clearly, identity.

(iii.1) Let $\overrightarrow{e}$ be a directed edge and u be an element stabilizing $\overrightarrow{e}$. It implies the equality $u(\overrightarrow{e}) = \overrightarrow{e}$. If $\overrightarrow{e}'$ is another directed edge, then, by transitivity, there exists a $w \in Rot(G_0)$, such that $\overrightarrow{e} = w(\overrightarrow{e}')$. One gets $w^{-1} \circ u \circ w(\overrightarrow{e}') = \overrightarrow{e}'$ and, by commutativity, $u(\overrightarrow{e}') = \overrightarrow{e}'$. So, u is the identity.

(iii.2) If $\overrightarrow{e}$ is a directed edge of G_0 and $u \in G_0$, then, by transitivity and (ii), there is an unique $v \in Rot(G_0)$, such that $u^{-1}(\overrightarrow{e}) = v(\overrightarrow{e})$. If v denotes $\phi_{\overrightarrow{e}}(u)$, then the following hold:

$$\begin{array}{rcl} \phi_{\overrightarrow{e}}(u) \circ \phi_{\overrightarrow{e}}(u')\overrightarrow{e} & = & \phi_{\overrightarrow{e}}(u) \circ u'^{-1}(\overrightarrow{e}) \\ & = & u'^{-1} \circ \phi_{\overrightarrow{e}}(u)(\overrightarrow{e}), \text{ by commutativity of } Rot(G_0) \text{ and } Mov(G_0), \\ & = & u'^{-1} \circ u^{-1}(\overrightarrow{e}) = (u \circ u')^{-1}(\overrightarrow{e}) \\ & = & \phi_{\overrightarrow{e}}(u \circ u')(\overrightarrow{e}), \text{ by the definition of } \phi_{\overrightarrow{e}} . \end{array}$$

Therefore, (iii.1) yields equality $\phi_{\overrightarrow{e}}(u \circ u') = \phi_{\overrightarrow{e}}(u) \circ \phi_{\overrightarrow{e}}(u')$ and injectivity of $\phi_{\overrightarrow{e}}$.

(iii.3) If $\overrightarrow{e}'$ is another directed edge, then there is an unique w, such that $\overrightarrow{e} = w(\overrightarrow{e}')$. So, one gets again, by commutativity, $u(\overrightarrow{e}') = w^{-1} \circ v \circ w(\overrightarrow{e}')$, i.e. $\phi_{\overrightarrow{e}'}(u) = w^{-1} \circ \phi_{\overrightarrow{e}}(u) \circ w$.

(iii.4) It can be checked, using the construction of orbit done in Theorem 4.6, that any element of $Rot(G_0)$, which leaves invariant one orbit, say, O_1, will leave invariant other orbits. By construction, any element u of the form $\phi_{\overrightarrow{e}}(u)$ will leave invariant the orbit of $\overrightarrow{e}$ and so, any orbit. Moreover, using freeness of the action, one proves, that if $f \in Rot(G_0)$ preserves the partition of $\mathcal{DE}$ into orbits under the action of $Mov(G_0)$, then there exists an $u \in Mov(G_0)$, such that $\phi_{\overrightarrow{e}}(u) = f$. So, $\phi_{\overrightarrow{e}}(Mov(G_0))$ is the group of transformations preserving the partition of $\mathcal{DE}$ into orbits and it is normal by (iii.3). $\square$

Theorem 6.2 *Let G_0 be a 3- or 4-valent n-vertex plane graph, such that $Rot(G_0)$ is transitive on $\mathcal{DE}$. Let (k,l) with $gcd(k,l) = 1$ and let r denote the number of ZC-circuits of $GC_{k,l}(G_0)$. The following hold:*

(i) $GC_{ku,lu}(G_0)$ is ZC-uniform and it holds:

(i.1) if u is even, then there are $\frac{u}{2}$ orbits of ZC-circuits of size $2r$ each,

(i.2) if u is odd, then there are $\frac{u-1}{2}$ orbits of ZC-circuits of size $2r$ and one orbit of size r,

(i.3) $GC_{ku,lu}(G_0)$ is ZC-transitive if and only if $u = 1$ or 2.

(ii) If i_0 denotes the number of faces of non-zero curvature, which are incident to a fixed ZC-circuit ZC of $GC_{k,l}(G_0)$ with $gcd(k,l) = 1$, then:

(ii.1) i_0 is even, $r = \frac{|\mathcal{S}(G_0)|}{i_0}$ and the stabilizer of ZC is the point subgroup $D_{i_0/2}$ (or C_2) of $Rot(G_0)$, if $i_0 > 2$ (or $i_0 = 2$, respectively),

(ii.2) r is equal to:

$$\begin{cases} \frac{3n}{2Ord(IPM_1(G_0,k,l))} & \text{in the 3-valent case,} \\ \frac{2n}{Ord(IPM_1(G_0,k,l))} & \text{in the 4-valent case.} \end{cases}$$

Proof. We consider only the 3-valent case, since a proof for the 4-valent case is similar.

Not all faces are 6-gonal, since we consider finite plane graphs. The transitivity of $Rot(G_0)$ on $\mathcal{DE}$ implies transitivity on the set of faces; so, all faces have the same number q of edges, where $q < 6$. This yields $GC_{k,l}(G_0)$ being tight if $gcd(k,l) = 1$. Since $G_1 = GC_{k,l}(G_0)$ is tight, every zigzag Z is incident on the right to a non 6-gonal face F; this incidence corresponds to a directed edge $\overrightarrow{e} \in \mathcal{DE}$. The directed edge $\overrightarrow{e}$ belongs to F and comes, in fact, from G_0. The transitivity of $Rot(G_0)$ on $\mathcal{DE}$ yield the transitivity on the set of zigzags of G_1, since $\overrightarrow{e}$ defines the zigzag Z.

Now denote $G_2 = GC_{ku,lu}(G_0) = GC_{u,0}(G_1)$. Every zigzag Z of G_1 corresponds to a set of zigzags $Z_1, \ldots, Z_u$ of G_2. If Z has positions $(\overrightarrow{e}, 1)$ and $(\overrightarrow{e}', k+l)$, then there exists a transformation $g \in Rot(G_0)$, such that $g(\overrightarrow{e}) = \overleftarrow{e}'$ with $\overleftarrow{e}'$ being the reverse of $\overrightarrow{e}'$. This transformation reverses the orientation of Z and maps Z_1 to Z_u in G_2 and, more generally, Z_s to Z_{u+1-s}.

(ii.1) Suppose that ZC has the right incidences $\overrightarrow{e}_1, \ldots, \overrightarrow{e}_s$ and the left incidences $\overrightarrow{e}'_1, \ldots, \overrightarrow{e}'_{s'}$ with $i_0 = s + s'$. By transitivity on $\mathcal{DE}$, there exists an element $g_0 \in Rot(G_0)$, such that $g_0(\overrightarrow{e}_1) = \overrightarrow{e}'_1$. This yields $s = s'$ and $i_0 = 2s$. Consider now the group $Stab_2$ of transformations preserving the set $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_{i_0/2}\}$. $Stab_2$ is a normal subgroup of the stabilizer $Stab_1$. The stabilizer $Stab_2$ can do only cyclic shifts on the right incidences $\overrightarrow{e}_1, \ldots, \overrightarrow{e}_{i_0/2}$ and so, it is isomorphic to $C_{i_0/2}$ and $Stab_1$ is isomorphic to $D_{i_0/2}$.

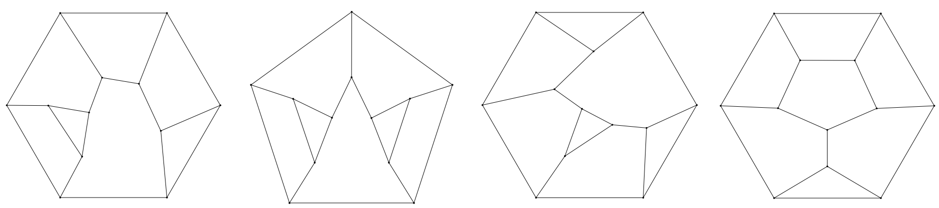
(ii.2) Take a zigzag Z of $GC_{k,l}(G_0)$ and define the sequence $\overrightarrow{e_1}, \dots, \overrightarrow{e}_{Ord(Z)}$ by $\overrightarrow{e}_{i+1} = IPM_1(G_0,k,l)\overrightarrow{e_i}$. By z-transitivity, all zigzags have the same length; so, $Ord(Z) = Ord(IPM_1(G_0,k,l))$. The length of Z is $Ord(Z)2t(k,l)$. Since $Rot(G)$ is z-transitive, one obtains, by direct enumeration and using that every edge is covered two times, $rOrd(Z)2t(k,l) = 3nt(k,l)$ and so, $r = \frac{3n}{2Ord(IPM_1(G_0,k,l))}$. □

Remark 6.3 *(i) Every element of $Rot(G_0)$ yields a restriction on the possibilities for $Mov(G_0)$ by Theorem 6.1 (i).*

(ii) A 3-valent (respectively, 4-valent) plane graph G_0 with n vertices has $3n$ (respectively, $4n$) directed edges. The generators L and R of $Mov(G_0)$ are even permutations of those directed edges by Proposition 4.12. So, $Mov(G_0)$ is isomorphic to a subgroup of $Alt(3n)$ (respectively, of $Alt(4n)$).

(iii) In the extreme case of $Rot(G_0)$ being transitive, the group $Mov(G_0)$ is isomorphic to a subgroup of $Rot(G_0)$; so, it has at most $3n$ (or $4n$) elements.

(iv) The smallest 3-valent plane graphs, for which $Mov(G_0)$=$Alt(3n)$, are given in the picture below with their symmetry groups.



12-1 (C_1) *12-2* (C_s) *12-4* (C_1) *12-9* (C_s)

Does there exist an example of a 4-valent plane graphs with $Mov(G_0) = Alt(4n)$?

A face F of a 3- (or 4-valent) plane graph is called 1-*colored* if all its vertices (or, respectively, edges) belong to one ZC-circuit.

Lemma 6.4 *If G is a 3- or 4-valent tight plane graphs, whose faces of non-zero curvature are all 1-colored, then it is ZC-knotted.*

Proof. Let G be a 4-valent tight plane graph with all faces of non-zero curvature being 1-colored. Let $C_1, \dots, C_r$ be the central circuits of G.

If two central circuits C_i and C_j have opposite edges of a square, then they define a road, which is a pseudo-road, since G is tight and finish on a q-gonal face with $q \neq 4$. The 1-coloring property yields $C_i = C_j$.

Assume that two central circuits C_i and C_j intersect in one vertex, say, v. If v belongs to a non-square face, then one obtains $C_i = C_j$ by 1-coloring property. If not, then one can find a vertex v', which is adjacent to v, such that $\{v, v'\}$ belongs to a square. Using the above reasoning, one finds that C_i and C_j intersect in v'. Since G is connected, C_i and C_j intersect in a vertex of a q-gonal face with $q \neq 4$; so, $C_i = C_j$.

The proof in 3-valent case is similar. □

Theorem 6.5 *If $0 \le l \le k$ with $gcd(k,l) = 1$, then in 8 cases below only following [ZC]-vectors of $GC_{k,l}(G_0)$ occurs. The last column gives the index of $Stab(G_0)$ in $SL_2(\mathbb{Z})$:*

G_0	*possible [ZC]*	*index*
Tetrahedron	2^3	*6*
Dodecahedron	5^6 *or* 3^{10}, *or* 2^{15}	*10*
Bundle	1^3 *or* 3	*8*
Klein map $\{3^7\}$	3^{28} *or* 4^{21}	*7*
Cube	3^4 *or* 2^6	*8*
Octahedron	4^3 *or* 3^4, *or* 2^6	*9*
link 2_1^2	2^2	*6*
Trefoil 3_1	2^3 *or* 6	*6*

Proof. The group $Rot(G_0)$ is transitive on $\mathcal{DE}$ for all cases, considered here, except Trefoil 3_1. So, by Theorem 6.2, the partition vector has form l^r. Now, Theorem 6.2 gives that the number r of ZC-circuits is equal to $\frac{3n}{2g}$ or $\frac{2n}{g}$, with g being the order of an element of $Mov(G_0)$, which using embedding $\phi_{\overrightarrow{e}}$ of Theorem 6.1 is an element of $Rot(G_0)$.

If G_0 is the *Bundle*, then the orders of elements of $Rot(G_0)$ are 1, 2 or 3, which yields $r = 1$ or 3 as the only possibilities; those values of r are attained for (k,l)=$(1,0)$ and $(1,1)$, respectively.

If G_0 is Tetrahedron or link 2_1^2, then $Mov(G_0) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and, using Theorem 4.17 (iii) with $K = \{Id\}$, one proves that $L \odot_{k,l} R \neq Id$ and so, $L \odot_{k,l} R$ is necessarily of order 2, which proves the required results. Another possibility is to use Theorem 5.2 from [DeDu02] (respectively, Theorem 5 of [DDS03]), which gives that a tight 3_n (respectively, 4-hedrite) has exactly three zigzags (respectively, two central circuits).

In all other cases $Mov(G_0)$ is non-commutative and so, by Corollary 4.15, it holds $l > 1$.

If G_0 is Dodecahedron, then $r = 6$, 10 or 15, which are attained for $(k,l) = (1,0)$, $(1,1)$ and $(2,1)$, respectively.

If G_0 is Octahedron, then $r = 3$, 4 or 6, which are attained for (k,l)=$(1,0)$, $(1,1)$ and $(2,1)$, respectively.

If G_0 is Cube, then $r = 6$, 4 or 3. Assume that $r = 3$; then, by Theorem 6.2, the stabilizer of any zigzag Z is D_4, with the 4-fold axis going through two squares, say, sq_1 and sq_6. Z cannot be incident to sq_1 or sq_6, since it would yield 1-coloring property and so, G being z-knotted. So, Z is incident exactly once to each of the squares, say, $sq_2,\dots,sq_5$. One can construct a zigzag Z', which is parallel to Z and incident to both, sq_1 and sq_2. Either $\{sq_2, sq_4\}$, or $\{sq_3, sq_5\}$ form the 4-fold axis of Z; so, either sq_2, or sq_3 are 1-colored. Therefore, $r = 3$ is not possible and the values $r = 4$, 6 are attained for (k,l)=$(1,0)$ and $(1,1)$.

If G_0 is Klein map, then the orders of non-zero element of $Rot(G_0)$ are 2, 3, 4 or 7. In order to show the impossibility of 2 and 7, we use Theorem 4.18.

$GC_{k,l}(Trefoil)$ is tight; so, by Theorem 4 in [DDS03], there are at most three central circuits. Assume that $GC_{k,l}(Trefoil)$ has two central circuits, say, C_1 and C_2. Since $GC_{k,l}(Trefoil)$ has a 3-fold rotation axis, by going through triangles, say, T_1 and T_2, one obtains, that those two triangles are 1-colored. Two parallel edges, say, e_1 and e_2 of a square will define a pseudo-road, which finish either on a 2-gon, giving e_1, e_2 in the same central circuit, or on a 3-gon, giving also e_1, e_2 in the same central circuit. The proof goes in the same way, as in Lemma 6.4, and one obtains, that $GC_{k,l}(Trefoil)$ has one central circuit. CC-transitivity is trivial in the case $r = 1$; in the case $r = 3$, the 3-fold axis of symmetry around T_1, T_2 gives CC-transitivity and so, $[CC] = 2^3$. □

Remark 6.6 *The above proof of Theorem 6.5 uses Corollary 4.15 (iii). Another, more combinatorial, method is possible: the maximum number of zigzags (or central circuits) of a tight graph q_n (or i-hedrite, respectively) is bounded (see [DeDu02] and [DDS03]). For example, the maximal number of central circuits of a tight 8-hedrite is 6, while a tight graph 4_n has at most 9 zigzags (we expect 8 to be the maximal value).*

For the link 7^2_6, the index is 1764, and all possibilities of [ZC]-vectors are, with their first appearance (k, l):

14	(4, 1)	$1^2, 12$	(9, 7)	$1^2, 2^2, 8$	(5, 1)	$1^4, 2, 4^2$	(21, 19)
$1^4, 3^2, 4$	(7, 5)	$2, 12$	(10, 3)	$2, 6^2$	(5, 2)	$2^2, 10$	(2, 1)
$2^2, 4, 6$	(11, 2)	$2^3, 4^2$	(5, 3)	$2^4, 6$	(9, 2)	2^7	(29, 21)
$3^2, 8$	(1, 1)	$4, 10$	(1, 0)	$4, 5^2$	(3, 1)	$4^2, 6$	(9, 1)
$6, 8$	(3, 2)						

We expect also, that if $GC_{k,l}(7^2_6)$ has two central circuits, then the closest integer to $\frac{|C_1 \cap C_2|}{t(k,l)}$ is 3.

Theorem 6.7 *The [ZC]-vectors of $GC_{k,l}(G_0)$ are distributed in the following way (cf. Table 3 above):*

G_0	*[ZC] if I*	*[ZC] if II*	*index*
Bundle	1^3	3	*8*
Cube	2^6	3^4	*8*
Dyck map $\{3^8\}$	4^{12}	3^{16}	*8*
trunc. Cube	$2^{12}, 3^4$ *or* 2^{18}	6^6 *or* 9^4	*64*
trunc. Dodecahedron	$2^{30}, 3^{10}$ *or* $2^{30}, 5^6$ *or* 2^{45}	15^6 *or* 6^{15} *or* 9^{10}	*80*
trunc. Cuboctahedron	$2^{12}, 4^{12}$ *or* $2^{24}, 3^8$ *or* 2^{36} *or* $3^8, 4^{12}$	6^{12} *or* 9^8	*256*
Trefoil 3_1	2^3	6	*6*
Octahedron	3^4	4^3 *or* 2^6	*9*
knot 4_1	$1^2, 3^2$ *or* $2, 6$	$2^2, 4$ *or* 8	*72*

Proof. All those results follow from repeated application of Proposition 4.17. The groups were computed using GAP [GAP] and PlanGraph [Dut02].

The group $Mov(Bundle) = \mathbb{Z}_3$ is commutative. Using 4.17 (ii) with $K = \{Id\}$ (i.e. the trivial normal subgroup), the result follows.

The group $Mov(Cube)$ is isomorphic to the non-commutative group $Alt(4)$ and has the normal subgroup $K = \langle (1,2)(3,4), (1,3)(2,4) \rangle$. So, $L \odot_{k,l} R \neq Id$ and $L \odot_{k,l} R \in K$ if and only if $k \equiv l \pmod 3$. The set of elements of order 3 of $Alt(4)$ is exactly $Alt(4) - K$ and the set of elements of order 2 of $Alt(4)$ is $K - \{Id\}$. This yields the required result.

The group $Mov(Trefoil)$ has order 36 and has one normal subgroup K_1 of order 9, for which $Mov(Trefoil)/K_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$. So, applying 4.17 (ii), one obtains $L \odot_{k,l} R \notin K_1$.

$\{Id, \overline{LR}\}$ is a normal subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which corresponds to a normal subgroup K_2 of $Mov(G_0)$ of order 18. Using 4.17 (ii), one obtains $L \odot_{k,l} R \in K_2$ if and only if $k \equiv l \pmod 2$. The elements of $Mov(G_0) - K_2$ correspond to $GC_{k,l}(Trefoil)$ having one central circuit, while the elements of $K_2 - K_1$ correspond to $GC_{k,l}(Trefoil)$ having three central circuits. So, the result follows.

The group $Mov(Octahedron)$ is isomorphic to $Sym(4)$, which possess normal subgroups $K_1 = Alt(4)$ and $K_2 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$. $Mov(Octahedron)/_{K_2}$ is non-commutative; so, $L \odot_{k,l} R \notin K_2$. By 4.17 (ii), it holds $L \odot_{k,l} R \in K_1$ if and only if $k \equiv l \pmod 2$. The elements of $K_1 - K_2$ have order 3, while the elements of $Mov(G_0) - K_1$ have order 2 or 4. So, $r = 4$ if and only if $k \equiv l \pmod 2$ and $r \in \{3, 6\}$ if and only if $k - l \equiv 1 \pmod 2$.

The group $Mov(Dyck\ map)$ has 48 elements and two normal subgroups, K_1 and K_2, of order 4 and 16, respectively. The quotient $Mov(Dyck\ map)/_{K_1}$ is non-commutative; so, $L \odot_{k,l} R \notin K_1$. The quotient $Mov(Dyck\ map)/_{K_2}$ is commutative and $\overline{L} = \overline{R}^{-1}$. So, $L \odot_{k,l} R \in K_2$ if and only if $k \equiv l \pmod 3$. However, elements of $Mov(G_0) - K_1$ correspond to $[z] = 3^{16}$, while elements of $K_1 - K_2$ correspond to $[z] = 4^{12}$. So, the result follows.

For the remaining cases of knot 4_1, truncated Cube, truncated Dodecahedron and truncated Cuboctahedron, the technique was always the same:

- first compute the set $\mathcal{S}$ of possibilities for [ZC], using Theorem 4.18,
- find a normal subgroup K of index 2 or 3 in $Mov(G_0)$,
- the sets $ZC(K) \cap \mathcal{S}$ and $ZC(Mov(G_0) - K) \cap \mathcal{S}$ are disjoint, which yield the required result.

Those computer computations had to deal with the size of the moving groups; for example $Mov(trunc.\ Cuboctahedron)$ has 1327104 elements. □

Remark 6.8 *(i) One can prove easily, that for pairs $(k, l) = (2l - 1, 1)$, $(2l - 7, l)$, $(2l - 17, l)$ the graph $GC_{k,l}(Octahedron)$ has 3 central circuits for every l. Also the graph $GC_{2l-3,l}(Octahedron)$ has 6 central circuits for every l. We expect that for other values of i the number of central circuits of $GC_{2l-i,l}(Octahedron)$ depends on l.*

(ii) $GC_{k,3}(Octahedron)$ with $k \equiv 1, 2 \pmod 3$ has 6 central circuits for every k and we expect, that for other values of l, the number of central circuits depends on k.

Examples of 3-valent z-uniform graphs are Tetrahedron, $Prism_3$, Cube, 10-2, 10-3, $Prism_5$ (see Figure 2)). In Tables 5 and 6 we present the [z]- and [CC]-vectors of such graphs for pairs (k, l) with $t(k, l) \leq 200$. We add * to k, l in the first column if $k \equiv l \pmod 3$ or $k \equiv l \pmod 2$ in 3- or 4-valent case, respectively. For Cube, Dodecahedron, Trefoil and Octahedron we also indicate the intersection vectors.

Conjecture 6.9 *(i) For $GC_{k,l}(Icosidodecahedron)$, $[CC]$-vector is:*

(2^{30}), (3^{20}) *or* (5^{12}) *if* $k \equiv l \pmod 2$,

(10^6), (4^{15}) *or* (6^{10}), *otherwise.*

(ii) For $GC_{k,l}(truncated\ \ Icosidodecahedron)$, $[z]$-vector is:

$2^{30}, 3^{40}$ *or* $2^{30}, 5^{24}$ *or* $3^{20}, 5^{24}$ *or* $2^{60}, 3^{20}$ *or* $2^{60}, 5^{12}$ *or* $3^{40}, 5^{12}$ *or* 2^{90} *or* 3^{60} *or* 5^{36} *if* $k \equiv l \pmod 2$,

9^{20}, 6^{30} *or* 15^{12}, *otherwise.*

Theorem 4.18 yield the list of all possible [ZC]-vectors. The Proposition 4.17 does not yield the expected partition of [CC]-vectors for Icosidodecahedron, while truncated Icosidodecahedron was too complex to be treated.

In the remainder of this Section, we indicate the properties, which we expect to hold for ZC-structure and moving group of $Foil_m$, $Prism_m$ and $APrism_m$. We extracted those conjectures from extensive computation and expect that the proofs will come from better understanding of the moving group and the (k, l)-product.

Conjecture 6.10 *For* $GC_{k,l}(Foil_m)$ *with* $gcd(k, l) = 1$ *holds:* $[CC]$ *is* 2^m *if* $k - l$ *is odd and, otherwise, it is* m *or* $(\frac{m}{2})^2$ *for* m *odd or even, respectively.*

This conjecture was checked for $m \leq 20$; in the computer proof were used normal subgroups of $Mov(Foil_m)$ and Proposition 4.17. However, doing a proof for general m pose several problems: there are many normal subgroups in $Mov(G_0)$ and the computer proof came from the use of all of them.

Conjecture 6.11 *On z-structure of* $GC_{k,l}(Prism_m)$ *with* $gcd(k, l) = 1$*, we conjecture:*
(i) $GC_{k,l}(Prism_m)$ *is z-balanced and tight.*
(ii) All possible [z]-vectors for $GC_{k,l}(Prism_m)$ *are:*
(ii.1) if $k \equiv l \pmod 3$*: all* $2^m, (\frac{m}{j})^j$,
where j *is any divisor of* m*, such that* $j \equiv 2 \pmod 4$*, if* $m \equiv 0 \pmod 2$.
(ii.2) if $k - l \equiv 1, 2 \pmod 3$*: all* $(\frac{3m}{j})^j$,
where j *is any divisor of* m*, such that* $j \equiv m \pmod 4$*, if* $m \equiv 0 \pmod 2$.
(iii) Denoting $m^* = \frac{m}{gcd(m,4)}$*, the following hold:*
(iii.1) $[z] = 3^m$ *in the case* $l = k - 1$ *if and only if* $k = 2, 2m^* - 1 \pmod{2m^*}$*;*
$[z] = 3^m$ *in the case* $l = 1$ *if and only if* $k = 2, 3m^* - 3 \pmod{3m^*}$,
(iii.2) $[z] = 2^m, (\frac{m}{2})^2$ *in the case* $m \equiv 0 \pmod 4$, $k \equiv l \pmod 3$,
(iii.3) if $m \equiv 1, 2, 3 \pmod 4$*, then:*

- $[z] = 2^m, 1^m$ *in the case* $l = k-3$ *if and only if* $k = 3m^*-5, 3m^*-1, 3m^*+4, 3m^*+8 \pmod{6m^*}$*;*
- $[z] = 2^m, 1^m$ *in the case* $l = 1$ *if and only if* $k = \frac{m^*-1}{2} \pmod{m}^*$,

(iii.4) in the case $(k, l) = (1, 1)$, $[z] = 2^m, (\frac{m}{2})^2$ *or* $2^m, m$*, if* m *is even or odd, respectively.*
(iv) The order of $Mov(Prism_m)$ *is* $12(m^*)^3$ *and its largest normal subgroup has index* 3*. The orders of all other normal subgroups are exactly the numbers of the form* $2^i q^3$*, where* $0 \leq i \leq max(3t - 6, 0)$*,* t *is the exponent of* 2 *in the factorization of* m *and* q *is any odd divisor of* m.

(v) $\begin{pmatrix} 2m+1 & -2m \\ 2m & 1-2m \end{pmatrix} \in Stab(Prism_m)$.

(vi) the index of $Stab(Prism_m)$ *is* $\frac{64}{9}(m^*)^2$ *if* $m \equiv 0 \pmod 3$ *and* $8(m^*)^2$*, otherwise.*

Conjecture 6.12 *On z-structure of* $GC_{k,l}(APrism_m)$ *with* $gcd(k,l) = 1$*, we conjecture:*

(i) $GC_{k,l}(APrism_m)$ *is z-balanced and tight.*

(ii) All possible [CC]-vectors for $GC_{k,l}(APrism_m)$ *are:*

(ii.1) if $k - l \equiv 1 \pmod 2$*, then* $[CC] = 2^m, (\frac{2m}{j})^j$ *and* $(\frac{4m}{j})^j$*, where* j *is any odd divisor of* m*, such that* $j \equiv 0 \pmod 3$ *if* $m \equiv 0 \pmod 3$.

(ii.2) if $k \equiv l \pmod 2$*, then* $[CC]=(\frac{m}{i})^i, (\frac{3m}{j})^j$*, where* i*,* j *are any divisors of* m*, such that:*

- $j \equiv 0 \pmod 3$ *if* $m \equiv 0 \pmod 3$ *and*
- *either* i*,* j *are odd and* $gcd(i,j) = 1$*, or* $gcd(i,j) = 2$ *and* $i + j \equiv 2 \pmod 4$.

(iv) Denote $m^* = \frac{m}{gcd(m,3)}$*. The order of* $Mov(APrism_m)$ *is* $24\frac{(m^*)^4}{gcd(m,2)}$.

Let $m^* = \Pi_{t=1}^T p_t$ *with* $2 \le p_1 \le p_2 \le \cdots \le p_T$ *and all* p_t *are prime.*

(iv.1) If m^* *is odd, then the orders of normal subgroups are all numbers of form* $\Pi_{t=1}^T p_t^{j_t}$ *with* $j_t \in \{0,1,3,4\}$ *and* $4\Pi_{t=1}^T p_t^{j_t}$ *or* $12\Pi_{t=1}^T p_t^{j_t}$ *with* $j_t \in \{3,4\}$.

(iv.2) If m^* *is even, then the same expressions hold, but* $j_1 \neq 4$.

In terms of index: the indexes of the above groups are:

if m^* *is odd:* $2g$*,* $6g$ *for any divisor* g *of* m^* *and any* $24\Pi_{t=1}^T p_t^{j_t}$ *for* $j_t \in \{0,1,3,4\}$*;*

if m^* *is even:* $2g$*,* $6g$ *for any divisor* g *of* $\frac{m^*}{2}$ *and any* $24\Pi_{t=2}^T p_t^{j_t}$, $96\Pi_{t=2}^T p_t^{j_t}$, $192\Pi_{t=2}^T p_t^{j_t}$ *for* $j_t \in \{0,1,3,4\}$.

(v) It holds $\begin{pmatrix} 3m+1 & 3m \\ -3m & -3m+1 \end{pmatrix} \in Stab(APrism_m)$ *and* $Stab(APrism_m)$ *is stable by transposition.*

(vii) the index of $Stab(APrism_m)$ *in* $SL_2(\mathbb{Z})$ *is* $gcd(m,4)m^2$ *if* $m \equiv 0 \pmod 3$ *and* $9gcd(m,4)m^2$*, otherwise.*

7 Projections of ZC-transitive $GC_{k,l}(G_0)$ for some graphs G_0

We consider in this Section the case, when $GC_{k,l}(G_0)$ is ZC-transitive if $gcd(k,l) = 1$. Such situation occurs if $Rot(G_0)$ is transitive on the set $\mathcal{DE}$ of directed edges and in some other cases, for example, for G_0 being Trefoil 3_1.

By transitivity of $Aut(G_0)$ on the set of ZC-circuits (apropos, transitivity of $Rot(G_0)$ on $\mathcal{DE}$ implies ZC-transitivity by Theorem 6.2), all ZC-circuits have the same signature, which we denote by (α_1, α_2).

Definition 7.1 *Let* G_0 *be* 3- *or* 4*-valent plane graph, such that* $GC_{k,l}(G_0)$ *is ZC-transitive. Call* projection of G *and denote by* $Proj_{k,l}(G_0)$ *the plane graph, obtained by the deletion of all but one central circuits of* $Med(GC_{k,l}(G_0))$ *(or* $GC_{k,l}(G_0)$*). It has* $\alpha_1 + \alpha_2$ *vertices.*

$G_0 =$		*Cube*		$Prism_3$	$Prism_6$	10-2	10-3	*Dodecahedron*	
k,l	$t(k,l)$	$[z]$	*Int*	$[z]$	$[z]$	$[z]$	$[z]$	$[z]$	*Int*
$1,0$	1	3^4	$(0,0);2^3$	9	9^2	15	15	5^6	$(0,0);2^5$
$1,1^*$	3	2^6	$(0,0);2^4,4$	$2^3;3$	$2^6,3^2$	$2^2;3,8$	$2^3;9$	3^{10}	$(0,0);2^9$
$2,1$	7	3^4	$(3,0);12^3$	3^3	3^6	15	$3,4^3$	2^{15}	$(0,0);2^{14}$
$3,1$	13	3^4	$(9,0);20^3$	9	9^2	$6,9$	$2^3,3^3$	3^{10}	$(0,3);8^9$
$3,2$	19	3^4	$(9,0);32^3$	9	9^2	$6,9$	5^3	5^6	$(0,15);32^5$
$4,1^*$	21	2^6	$(4,0);14^4,20$	$1^3,2^3$	$1^6;2^6$	$2^2,3^2,5$	15	5^6	$(5,10);36^5$
$5,1$	31	3^4	$(18,0);50^3$	9	9^2	15	5^3	5^6	$(15,10);52^5$
$4,3$	37	3^4	$(19,0);62^3$	9	9^2	$6,9$	15	5^6	$(0,25);64^5$
$5,2^*$	39	2^6	$(12,0);26^4,28$	$2^3,3$	$2^6,3^2$	$1^2,2^5,3$	15	5^6	$(0,25);68^5$
$6,1$	43	3^4	$(30,0);66^3$	3^3	3^6	15	$2^3,9$	3^{10}	$(0,9);24^3,28^6$
$5,3$	49	3^4	$(36,0);74^3$	9	9^2	15	$2^3,9$	3^{10}	$(0,9);28^3,32^6$
$7,1^*$	57	2^6	$(12,0);38^4,52$	$2^3,3$	$2^6,3^2$	$1^3,2^5$	$1^3,4^3$	2^{15}	$(0,4);14^4,18^{10}$
$5,4$	61	3^4	$(30,0);102^3$	3^3	3^6	15	$1^3,4^3$	2^{15}	$(0,4);14^4,18^{10}$
$7,2$	67	3^4	$(45,0);104^3$	9	9^2	15	$3,4^3$	2^{15}	$(0,8);18^{14}$
$8,1$	73	3^4	$(45,0);116^3$	9	9^2	15	$2^3,9$	3^{10}	$(0,18);42^3,46^6$
$7,3$	79	3^4	$(54,0);122^3$	9	9^2	15	$2^3,9$	3^{10}	$(0,18);46^3,50^6$
$6,5$	91	3^4	$(45,0);152^3$	9	9^2	15	15	5^6	$(0,70);154^5$
$9,1$	91	3^4	$(63,0);140^3$	9	9^2	15	5^3	5^6	$(30,40);154^5$
$7,4^*$	93	2^6	$(24,0);62^4,76$	$2^3,3$	$2^6,3^2$	$2^2,3^2,5$	15	5^6	$(0,70);158^5$
$8,3$	97	3^4	$(72,0);146^3$	9	9^2	15	15	5^6	$(10,70);162^5$
$9,2$	103	3^4	$(63,0);164^3$	9	9^2	$3,6^2$	$2^3,9$	3^{10}	$(12,12);58^3,66^6$
$7,5$	109	3^4	$(81,0);164^3$	9	9^2	$3,6^2$	15	5^6	$(40,40);186^5$
$10,1^*$	111	2^6	$(24,0);74^4,100$	$2^3,3$	$2^6,3^2$	$2^2,3^2,5$	15	5^6	$(50,40);186^5$
$7,6$	127	3^4	$(63,0);212^3$	9	9^2	15	5^3	5^6	$(0,90);218^5$
$8,5^*$	129	2^6	$(40,0);86^4,92$	$1^3,2^3$	$1^6,2^6$	$2,5,8$	15	5^6	$(20,80);218^5$
$9,4$	133	3^4	$(99,0);200^3$	9	9^2	15	5^3	5^6	$(0,90);230^5$
$11,1$	133	3^4	$(84,0);210^3$	3^3	3^6	$6,9$	$2^3,3^3$	3^{10}	$(0,30);74^3,86^6$
$10,3$	139	3^4	$(102,0);210^3$	3^3	3^6	15	15	5^6	$(10,100);234^5$
$11,2^*$	147	2^6	$(48,0);98^4,100$	$2^3,3$	$2^6,3^2$	$2^2,3^2,5$	$3,4^3$	2^{15}	$(4,12);38^8,42^6$
$9,5$	151	3^4	$(99,0);236^3$	9	9^2	15	$2^3,3^3$	3^{10}	$(0,30);86^3,98^6$
$12,1$	157	3^4	$(108,0);242^3$	9	9^2	15	$3,4^3$	2^{15}	$(0,12);38^8,50^6$
$11,3$	163	3^4	$(108,0);254^3$	9	9^2	15	$2^3,9$	3^{10}	$(6,42);98^9$
$8,7$	169	3^4	$(84,0);282^3$	3^3	3^6	15	$3,4^3$	2^{15}	$(0,12);38^4,50^{10}$
$11,4$	181	3^4	$(135,0);272^3$	9	9^2	15	$3,4^3$	2^{15}	$(0,20);46^4,50^{10}$
$13,1^*$	183	2^6	$(40,0);122^4,164$	$1^3,2^3$	$1^6,2^6$	$2^2,3,8$	$2^3,9$	3^{10}	$(0,45);104^3,116^6$
$9,7$	193	3^4	$(144,0);290^3$	9	9^2	15	$2^3,9$	3^{10}	$(0,45);108^3,124^6$
$13,2$	199	3^4	$(135,0);308^3$	9	9^2	15	15	5^6	$(20,125);340^5$

Table 5: z-structure of $GC_{k,l}(G_0)$, $t(k,l) \leq 200$, for some 3-valent graphs G_0.

$G_0 =$		$Trefoil\ 3_1$		4_1	7^2_6	$Octahedron$		$APrism_4$
k,l	$t(k,l)$	$[CC]$	Int	$[CC]$	$[CC]$	$[CC]$	Int	$[CC]$
$1,0$	1	6	$(3,0)$	8	$4;10$	4^3	$(0,0);2^2$	16
$1,1^*$	2	2^3	$(0,0);2^2$	$2,6$	$3^2;8$	3^4	$(0,0);2^3$	$4;12$
$2,1$	5	6	$(15,0)$	$2^2;4$	$2^2;10$	2^6	$(0,0);2^5$	$2^4;8$
$3,1^*$	10	2^3	$(2,0);8^2$	$1^2;3^2$	$4,5^2$	3^4	$(3,0);8^3$	$2^2;3^4$
$3,2$	13	6	$(39,0)$	8	$6,8$	4^3	$(4,4);18^2$	16
$4,1$	17	6	$(51,0)$	8	14	4^3	$(4,4);26^2$	16
$4,3$	25	6	$(75,0)$	8	14	4^3	$(8,8);34^2$	16
$5,1^*$	26	2^3	$(8,0);18^2$	$1^2,3^2$	$1^2;2^2,8$	3^4	$(9,0);20^3$	$1^4;6^2$
$5,2$	29	6	$(87,0)$	8	$2,6^2$	4^3	$(8,8);42^2$	16
$5,3^*$	34	2^3	$(8,0);26^2$	$1^2,3^2$	$2^3,4^2$	3^4	$(9,0);28^3$	$1^4;6^2$
$6,1$	37	6	$(111,0)$	$2^2,4$	14	2^6	$(2,2);10,14^4$	$2^4,8$
$5,4$	41	6	$(123,0)$	$2^2,4$	14	2^6	$(2,2);14^4,18$	$2^4,8$
$7,1^*$	50	2^3	$(16,0);34^2$	$2,6$	$2^2,10$	3^4	$(18,0);38^3$	$4,12$
$7,2$	53	6	$(159,0)$	$2^2,4$	14	2^6	$(4,4);18^5$	$2^4,8$
$7,3^*$	58	2^3	$(16,0);42^2$	$2,6$	$4,5^2$	3^4	$(18,0);46^3$	$4,12$
$6,5$	61	6	$(183,0)$	8	$2^2,10$	4^3	$(20,20);82^2$	16
$7,4$	65	6	$(195,0)$	8	$6,8$	4^3	$(20,20);90^2$	16
$8,1$	65	6	$(195,0)$	8	$4,10$	4^3	$(16,16);98^2$	16
$8,3$	73	6	$(219,0)$	8	$4,10$	4^3	$(24,24);98^2$	16
$7,5^*$	74	2^3	$(18,0);56^2$	$2,6$	$1^4,3^2,4$	3^4	$(24,0);58^3$	$4,12$
$9,1^*$	82	2^3	$(26,0);56^2$	$2,6$	$4^2,6$	3^4	$(30,0);62^3$	$4,12$
$7,6$	85	6	$(255,0)$	8	$4,10$	4^3	$(28,28);114^2$	16
$9,2$	85	6	$(255,0)$	$2^2,4$	$2^4,6$	2^6	$(6,6);26,30^4$	$2^4,8$
$8,5$	89	6	$(267,0)$	8	$2^2,10$	4^3	$(28,28);122^2$	16
$9,4$	97	6	$(291,0)$	8	$4,10$	4^3	$(28,28);138^2$	16
$10,1$	101	6	$(303,0)$	$2^2,4$	$4,10$	2^6	$(6,6);26,38^4$	$2^4,8$
$9,5^*$	106	2^3	$(34,0);72^2$	$2,6$	$3^2,8$	3^4	$(30,0);86^3$	$4,12$
$10,3$	109	6	$(327,0)$	8	$2,12$	4^3	$(36,36);146^2$	16
$8,7$	113	6	$(339,0)$	$2^2,6$	14	2^6	$(6,6);38^4,50$	$2^4,8$
$11,1^*$	122	2^3	$(40,0);82^2$	$1^2,3^2$	$4^2,6$	3^4	$(45,0);92^3$	$1^4,6^2$
$11,2$	125	6	$(375,0)$	8	$2^2,4,6$	4^3	$(40,40);170^2$	16
$9,7^*$	130	2^3	$(32,0);98^2$	$1^2,3^2$	$1^2,12$	3^4	$(45,0);100^3$	$1^4,6^2$
$11,3^*$	130	2^3	$(40,0);90^2$	$2,6$	$2^4,6$	3^4	$(48,0);98^3$	$4,12$
$11,4$	137	6	$(411,0)$	$2^2,4$	$2^2,10$	2^6	$(10,10);46^4,50$	$2^4,8$
$9,8$	145	6	$(435,0)$	8	14	4^3	$(48,48);194^2$	16
$12,1$	145	6	$(435,0)$	8	$4,10$	4^3	$(36,36);218^2$	16
$11,5^*$	146	2^3	$(48,0);98^2$	$1^2,3^2$	$3^2,8$	3^4	$(45,0);116^3$	$1^4,6^2$
$10,7$	149	6	$(447,0)$	$2^2,4$	$4^2,6$	2^6	$(12,12);50^5$	$2^4,8$
$11,6$	157	6	$(471,0)$	8	$4,10$	4^3	$(48,48);218^2$	16
$12,5$	169	6	$(507,0)$	8	$4,10$	4^3	$(52,52);234^2$	16
$11,7^*$	170	2^3	$(50,0);120^2$	$1^2,3^2$	$3^2,8$	3^4	$(63,0);128^3$	$2^2,3^4$
$13,1^*$	170	2^3	$(56,0);114^2$	$1^2,3^2$	$2^2,10$	3^4	$(63,0);128^3$	$2^2,3^4$
$13,2$	173	6	$(519,0)$	8	14	4^3	$(52,52);242^2$	16
$13,3^*$	178	2^3	$(56,0);122^2$	$2,6$	$1^2,12$	3^4	$(66,0);134^3$	$4,12$
$10,9$	181	6	$(543,0)$	8	$6,8$	4^3	$(60,60);242^2$	16
$11,8$	185	6	$(555,0)$	$2^2,4$	14	2^6	$(14,14);62^4,66$	$2^4,8$
$13,4$	185	6	$(555,0)$	8	14	4^3	$(60,60);250^2$	16
$12,7$	193	6	$(579,0)$	8	$2^2,10$	4^3	$(64,64);258^2$	16
$13,5^*$	194	2^3	$(56,0);138^2$	$1^2,3^2$	$2^2,10$	3^4	$(69,0);148^3$	$1^4,6^2$
$14,1$	197	6	$(591,0)$	$2^2,4$	14	2^6	$(12,12);50,74^4$	$2^4,8$

Table 6: CC-structure of $GC_{k,l}(G_0)$, $t(k,l) \leq 200$, for some 4-valent graphs G_0

		$(k,k-1)$			$(k,1)$		
G_0	[ZC]	k	α_1	α_2	$(k,1)$	α_1	α_2
Dodec.	2^{15}	$2 \pmod 3$	0	$4\binom{\lfloor \frac{k}{3}+1 \rfloor}{2}$	$2 \pmod 5$	0	$4\binom{\lfloor \frac{k}{5}+1 \rfloor}{2}$
	3^{10}	none	—	—	$1,3 \pmod 5$	0	$3\binom{\lceil (k-1)/2 \rceil}{2}$
	5^6	$1,3 \pmod 3$	0	?	$0,4 \pmod 5$	$5\binom{\lceil \frac{2}{5}(k+1) \rceil}{2}$	$10\lceil \frac{k}{5} \rceil^2$
Octah.	2^6	$2 \pmod 3$	$\frac{(k-2)(k-1)}{9}$	$\frac{(k-2)(k-1)}{9}$	$2 \pmod 4$	$2\binom{\frac{k+2}{4}}{2}$	$2\binom{\frac{k+2}{4}}{2}$
	3^4	none	—	—	$1,3 \pmod 4$	$3\binom{\frac{k+1}{2}}{2}$	0
	4^3	$0,1 \pmod 3$	$\equiv 0 \pmod 4$	$\equiv 0 \pmod 4$	$0 \pmod 4$	$\frac{k^2}{4}$	$\frac{k^2}{4}$
Cube	2^6	none	—	—	$1 \pmod 3$	$4\binom{\lceil (k+l)/3 \rceil}{2}$	0
	3^4	all	$3\binom{k}{2}$	0	$0,2 \pmod 3$	$3\binom{k-\lfloor (k-1)/3 \rfloor}{2}$	0

Table 7: Conjectured [ZC]-vector and signature for $GC_{k,l}(G_0)$ with $l = k-1, 1$ and G_0 being a Platonic polyhedron

Tables 8 and 9 represent the projections of $GC_{k,l}(G_0)$ with G_0 being, respectively, Cube, Dodecahedron and Trefoil, Octahedron. The first column contains (k,l) and mark * if $k \equiv l \pmod 3$ (respectively, $k \equiv l \pmod 2$). For each graph G_0 and considered pair (k,l) we indicate the [ZC]-vector, the number **Nr** of its projection, its symmetry group and p-vector. The Figures 12, 13 and 14, 15 present pictures of projections given in Tables 8 and 9, respectively, by their numbers in Figures.

Remark, that projections **Nr.1**, **2**, **9**, **11**, **12**, **13**, **14** of $GC_{k,l}(Cube)$ coincide with projections **Nr.1**, **3**, **6**, **7**, **8**, **9**, **10** of $GC_{k,l}(Dodecahedron)$. Remark also, that in Table 9, for Trefoil, we omit projections in the CC-knotted case, since it coincides with the graph itself.

The plane graph $Proj_{k,l}(G_0)$ is 4-valent with one central circuit; hence, one can use the notion of type of intersection defined in 1.1. However, this intersection does not correspond to the self-intersection of the corresponding central circuit in $GC_{k,l}(G_0)$. For instance, central circuits of $GC_{13,3}(Octahedron)$ have self-intersection $(66,0)$, while their projection have self-intersection $(33,33)$.

Proposition 7.2 *If G_0 is a 3-valent plane graph, then $Med(G_0)$ appears as a projection of $Med(GC_{k,0}(G_0))$.*

Proof. Take the zigzags $(Z_i)_{1\le i\le p}$ of G_0; they correspond to the set of central circuits $(C_i)_{1\le i\le p}$ in $Med(G_0)$. Let the set of zigzags of $GC_{k,0}(G_0)$ be $(Z_{i,j})_{1\le i\le p\ 1\le j\le k}$. Those zigzags $Z_{i,j}$ become central circuits $C_{i,j}$ in $Med(GC_{k,0}(G_0))$. The central circuit C_i correspond to the set of central circuits $(C_{i,j})_{1\le j\le k}$ forming a parallel class. So, after removing the central circuits $C_{i,j}$ with $1 \le i \le p$ and $2 \le j \le k$, one obtains $Med(G_0)$. $\square$

The Proposition 7.2 means, that one can consider projection only for $GC_{k,l}(G_0)$ with $gcd(k,l) = 1$. Every symmetry preserving a ZC-circuit in $GC_{k,l}(G_0)$ yields a symmetry of the projection graph. This symmetry group is denoted by $Rot_{k,l}(G_0)$. Note, that the group of all symmetries of $Proj_{k,l}(G_0)$ can be larger than $Rot_{k,l}(G_0)$. We expect equality $Rot_{k,l}(G_0)$=$Aut(Proj_{k,l}(G_0))$ in all, but a finite number, of cases.

If G_0 is Cube, Dodecahedron or Octahedron, then one can apply Theorem 6.2 and get that $Rot_{k,l}(G_0) = D_m$. The group $Rot(Trefoil) = D_3$ is not transitive on directed

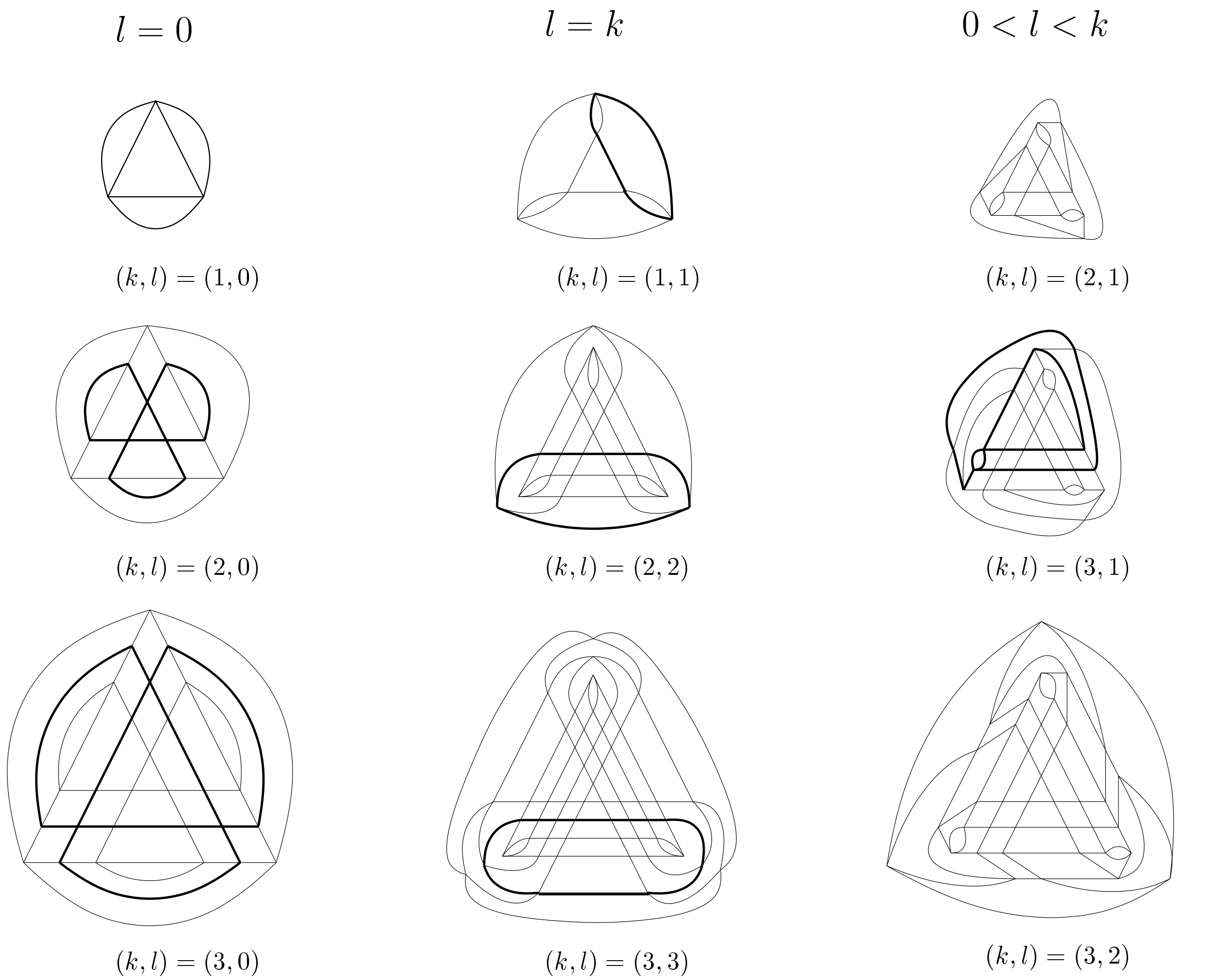


Figure 11: Graphs $GC_{k,l}(G_0)$ with $G_0 = Trefoil$ for $0 \leq l \leq k \leq 3$; in non-knotted case a projection is marked by double line

edges. If the graph $GC_{k,l}(Trefoil)$ has 3 central circuits, then the stabilizer of a central circuit has order 2 and the group itself is C_2.

See on Figure 11 a list of first 5-hedrites of symmetry D_{3h} and D_3 with their projections marked by double lines.

The following proposition is to compare with Theorem 4.6.

Proposition 7.3 *If G_0 is a 3- or 4-valent plane graph, whose faces have gonality divisible by 3 or 2, respectively, then all ZC-circuits of $GC_{k,l}(G_0)$ are simple.*

Proof. If G_0 satisfy this property, then $GC_{k,l}(G_0)$ satisfy it too. The 3-valent case was proved in [Mo64]. Let us consider the 4-valent case.

In fact, if a central circuit of G_0 self-intersects, then, in terms of [DDS03], one gets an 1-*gonal regular patch* P (i.e. a patch with an angle $\frac{\pi}{2}$, see [DDS03] for details).

By applying the local Euler formula (proved in [DeSt03]), one obtains:

$$3 = 4 - t = \sum_i (4-i)p'_i$$

with p'_i being the number of i-gonal faces in P, and obtains a contradiction, since the right hand side is even. □

Proposition 7.4 *For $GC_{k,l}(Dodecahedron)$ with $r = 6$, for $GC_{k,l}(Cube)$ with $r = 4$ and for $GC_{k,l}(Octahedron)$ with $r = 3$ or 4, the symmetry group is transitive on pairs of ZC-circuits and their pairwise intersection has the same size for every two different ZC-circuits.*

Proof. The stabilizers of ZC-circuits are point groups D_m by Theorem 6.2.

If $GC_{k,l}(Dodecahedron)$ has 6 zigzags $Z_1, \dots, Z_6$, then $Stab(Z_1) = D_5$. The conjugacy class of D_5 in $Rot(Dodecahedron) = Alt(5)$ has 6 elements. The pairwise intersection of those subgroups has size 2. So, the action of $Stab(Z_1)$ on Z_2 yields five zigzags $Z_2, \dots, Z_6$, i.e. G is transitive on pairs of zigzags.

If $GC_{k,l}(Cube)$ has 4 zigzags $Z_1, \dots, Z_4$, then $Stab(Z_1) = D_3$. The conjugacy class of D_3 in $Rot(Cube) = Sym(4)$ has 4 elements. The pairwise intersection of those subgroups has size 2 and the proof is as above.

If $GC_{k,l}(Octahedron)$ has 3 central circuits C_1, C_2, C_3, then pairs of central circuits correspond to central circuits and so, we get again transitivity. If it has 4 central circuits, then the proof is the same as for $GC_{k,l}(Cube)$. □

Conjecture 7.5 *(i) Is it true that if G_0, G_1 are two 4-valent plane graphs, then the set of pairs (k,l) with $gcd(k,l) = 1$, such that $G_0 = Proj_{k,l}(G_1)$, is finite?*

(ii) Is it true that if G_0 is a 4-valent plane graph and G_1 a 3-valent plane graph, then the set of pairs (k,l) with $gcd(k,l) = 1$, such that $G_0 = Proj_{k,l}(G_1)$, is finite?

A 4-valent plane graph can have central circuits of the same length, but with different number of self-intersections. For example, $GC_{5,3}(G_0 = 7^2_6)$ (see Table 6) has one central circuit of length 68 with self-intersection 2, while any of two other central circuits of length 68 have self-intersection 4.

Conjecture 7.6 *(i) Each central circuit of $GC_{k,l}(Trefoil)$ has self-intersection of the form $(x,0)$.*

(ii) If $gcd(k,l) = 1$, then $Proj_{k,l}(Trefoil)$ is a 5-hedrite, except of the cases (k,l)=$(1,1)$ or $(3,1)$.

Remark, that for $GC_{k,l}(4_1)$ all central circuits satisfy to $\alpha_2 = 0$ if $k \equiv l \pmod 2$ and $\alpha_1 = \alpha_2$, otherwise.

Conjecture 7.7 *(i) The 2-fold axis of the point group $Rot_{k,l}(G_0)$ do not go through vertices of $Proj_{k,l}(Cube)$ or $Proj_{k,l}(Dodecahedron)$, if the rotation group is D_2.*

(ii) $Proj_{k,l}(Cube)$ and $Proj_{k,l}(Dodecahedron)$ do not have q-gonal faces with $q > 6$.

(iii) Denote by p_2 the number of 2-gons, for a projection of $GC_{k,l}(Cube)$ it holds:

(iii.1) if $r = 6$, then $p_2 = 0$ or 2,

(iii.2) if $r = 4$, then $p_2 = 0$ or 6, except of $Proj_{2,1}(Cube)$, for which $p_2 = 3$.

(iv) For a projection of $GC_{k,l}(Dodecahedron)$, one can have $p_2 > 0$ only in case $[z] = 2^{15}$, for which $p_2 = 2$; in this case α_1 and α_2 are divisible by 4.

The projections, considered in this Section, are often one of the following forms:

(i) The *Conway graph* $(k \times m)^*$ (see, for example, [Kaw96]) is, for $k = 2$, m-sided antiprism; for $k > 2$, it comes from $((k-1) \times m)^*$ by inscribing an m-gon in the first of its two m-gons. In particular, $(2 \times 2)^* = 4_1$, $(2 \times 4)^* = 8_{18}$, $(3 \times 3)^* = 9_{40}$.

(ii) The D_m-*spiral* alternating knot is a 4-valent plane graph with symmetry D_m having p-vector ($p_m = 2$, $p_3 = 2m$, p_4, other $p_i = 0$) and only one central circuit.

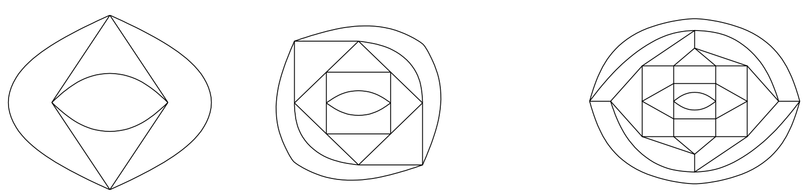
Conjecture 7.8 *If (k, l) has the form $(3b + 4, 1)$, then $G = Proj_{k,l}(Cube)$ is a D_2*-spiral *alternating knot. Moreover, we expect the following:*

(i) 4 triangles of G occur in two pairs of adjacent ones,

(ii) there are four pseudo-roads, linking each 2-gons to triangles, and having the same length b,

(iii) G has $4\binom{b+2}{2}$ vertices.

See the cases $b = 0$, 1 and 2 on the picture below.



$b = 0$ $b = 1$ $b = 2$

Conjecture 7.9 *If d is the length of each central circuit, r the number of central circuits and (α_1, α_2) the signature of each central circuit in $GC_{k,l}(Octahedron)$, then we conjecture:*

(i) $\alpha_1 \equiv 0 \pmod 3$ and $\alpha_2 = 0$ if $k \equiv l \pmod 2$; otherwise, $\alpha_1 = \alpha_2$.

(ii) $\alpha_1 = \alpha_2 \geq \frac{d-4}{16}$ if $r = 3$ with equality if and only if (k, l)=$(4p, 1)$;

$\alpha_1 \leq \frac{d-6}{8}$ if $r = 4$ with equality if $l = 1$.

	$GC_{k,l}(Cube)$				$GC_{k,l}(Dodecahedron)$			
k,l	$[z]$	$Projection$	$Group$	$p_2,\dots,p_6$	$[z]$	$Projection$	$Group$	$p_2,\dots,p_6$
$1,0$	3^4	0_1	$D_{\infty h}$	$0,0,0,0,0$	5^6	0_1	$D_{\infty h}$	$0,0,0,0,0$
$1,1^*$	2^6	0_1	$D_{\infty h}$	$0,0,0,0,0$	3^{10}	0_1	$D_{\infty h}$	$0,0,0,0,0$
$2,1$	3^4	$9=Trefoil$	D_{3h}	$3,2,0,0,0$	2^{15}	0_1	$D_{\infty h}$	$0,0,0,0,0$
$3,1$	3^4	10	D_{3h}	$6,2,0,0,3$	3^{10}	$6=Trefoil$	D_{3h}	$3,2,0,0,0$
$3,2$	3^4	$11=(3\times 3)^*$	D_{3h}	$0,8,3,0,0$	5^6	$13=(3\times 5)^*$	D_{5h}	$0,10,5,2,0$
$4,1^*$	2^6	$1=(2\times 2)^*$	D_{2d}	$2,4,0,0,0$	5^6	$13=(3\times 5)^*$	D_{5h}	$0,10,5,2,0$
$5,1$	3^4	12	D_3	$0,8,12,0,0$	5^6	$14=(5\times 5)^*$	D_{5h}	$0,10,15,2,0$
$4,3$	3^4	12	D_3	$0,8,12,0,0$	5^6	$14=(5\times 5)^*$	D_{5h}	$0,10,15,2,0$
$5,2^*$	2^6	5	D_2	$2,8,0,4,0$	5^6	$14=(5\times 5)^*$	D_{5h}	$0,10,15,2,0$
$6,1$	3^4	25	D_3	$6,12,0,12,2$	3^{10}	$7=(3\times 3)^*$	D_{3h}	$0,8,3,0,0$
$5,3$	3^4	17	D_3	$6,14,6,6,6$	3^{10}	$7=(3\times 3)^*$	D_{3h}	$0,8,3,0,0$
$7,1^*$	2^6	2	D_2	$2,4,8,0,0$	2^{15}	$1=(2\times 2)^*$	D_{2d}	$2,4,0,0,0$
$5,4$	3^4	13	D_3	$0,8,24,0,0$	2^{15}	$1=(2\times 2)^*$	D_{2d}	$2,4,0,0,0$
$7,2$	3^4	26	D_3	$0,24,12,6,5$	2^{15}	$2=(2\times 4)^*$	D_{4d}	$0,8,2,0,0$
$8,1$	3^4	18	D_3	$0,14,27,6,0$	3^{10}	8	D_3	$0,8,12,0,0$
$7,3$	3^4	19	D_3	$0,20,24,12,0$	3^{10}	8	D_3	$0,8,12,0,0$
$6,5$	3^4	14	D_3	$0,8,39,0,0$	5^6	15	D_5	$0,10,60,2,0$
$9,1$	3^4	20	D_3	$6,26,9,18,6$	5^6	18	D_5	$0,40,10,12,10$
$7,4^*$	2^6	6	D_2	$2,8,12,4,0$	5^6	15	D_5	$0,10,60,2,0$
$8,3$	3^4	28	D_3	$0,36,12,24,2$	5^6	19	D_5	$0,20,50,12,0$
$9,2$	3^4	27	D_3	$0,24,27,12,2$	3^{10}	11	D_3	$0,14,6,6,0$
$7,5$	3^4	29	D_3	$6,50,0,0,27$	5^6	20	D_5	$0,50,0,22,10$
$10,1^*$	2^6	3	D_2	$2,4,20,0,0$	5^6	21	D_5	$0,40,30,12,10$
$7,6$	3^4	15	D_3	$0,8,57,0,0$	5^6	16	D_5	$0,10,80,2,0$
$8,5^*$	2^6	7	D_2	$2,24,0,12,4$	5^6	22	D_5	$0,30,30,50,22$
$9,4$	3^4	32	D_3	$6,48,6,30,11$	5^6	16	D_5	$0,10,80,2,0$
$11,1$	3^4	30	D_3	$0,24,48,12,2$	3^{10}	9	D_3	$0,8,24,0,0$
$10,3$	3^4	33	D_3	$6,54,12,6,26$	5^6	17	D_5	$0,20,80,12,0$
$11,2^*$	2^6	8	D_2	$2,28,4,8,8$	2^{15}	$4=(4\times 4)^*$	D_{4d}	$0,8,10,0,0$
$9,5$	3^4	31	D_3	$0,42,30,24,5$	3^{10}	9	D_3	$0,8,24,0,0$
$12,1$	3^4	22	D_3	$6,44,24,24,12$	2^{15}	3	D_2	$2,4,8,0,0$
$11,3$	3^4	21	D_3	$0,38,48,18,6$	3^{10}	12	D_3	$0,14,30,6,0$
$8,7$	3^4	16	D_3	$0,8,78,0,0$	2^{15}	3	D_2	$2,4,8,0,0$
$11,4$	3^4	34	D_3	$6,72,18,6,35$	2^{15}	5	D_2	$0,8,14,0,0$
$13,1^*$	2^6	4	D_2	$2,4,36,0,0$	3^{10}	10	D_3	$0,8,39,0,0$
$9,7$	3^4	24	D_3	$6,80,12,12,36$	3^{10}	10	D_3	$0,8,39,0,0$
$13,2$	3^4	23	D_3	$0,56,51,12,18$	5^6	23	D_5	$0,50,65,22,10$

Table 8: Projections of $GC_{k,l}(G_0)$, $t(k,l)\leq 200$, with G_0 being Cube or Dodecahedron

Final remarks This research leaves many open questions, for examples:

- to extend Thurston's idea to classes of plane graphs, defined by more than one parameter,
- to consider the self-intersection number of ZC-circuits in $GC_{k,l}(G_0)$,
- to prove the conjectures of expression of [ZC] for $Foil_m$, using another idea than the moving group formalism,
- to prove that one can have $[ZC]=1^p$ only for Bundle,
- to extend the Goldberg-Coxeter construction to higher dimension and, more precisely, to simplicial and cubical complexes.

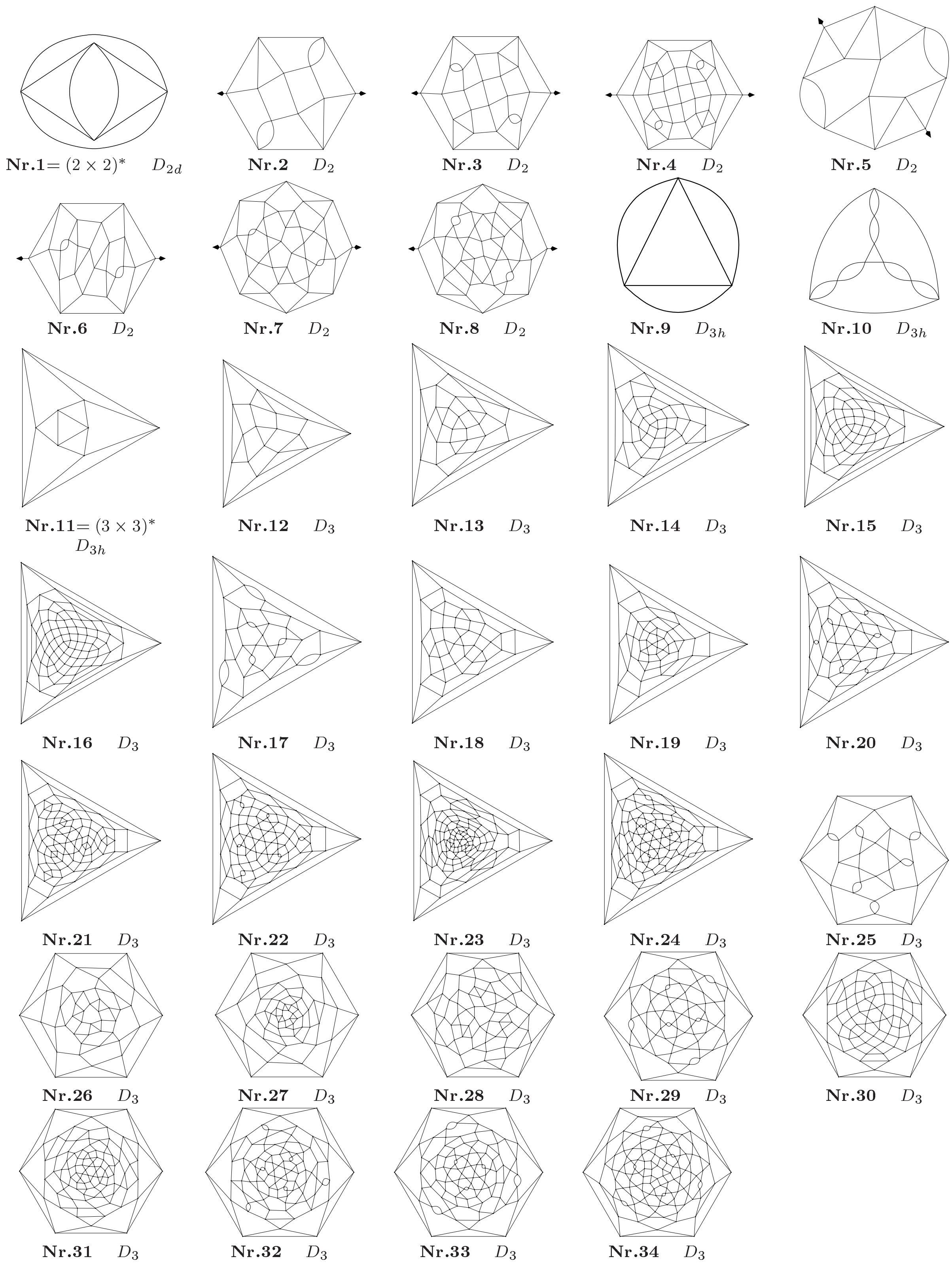


Figure 12: **Projections of $GC_{k,l}(Cube)$ from Table 8**

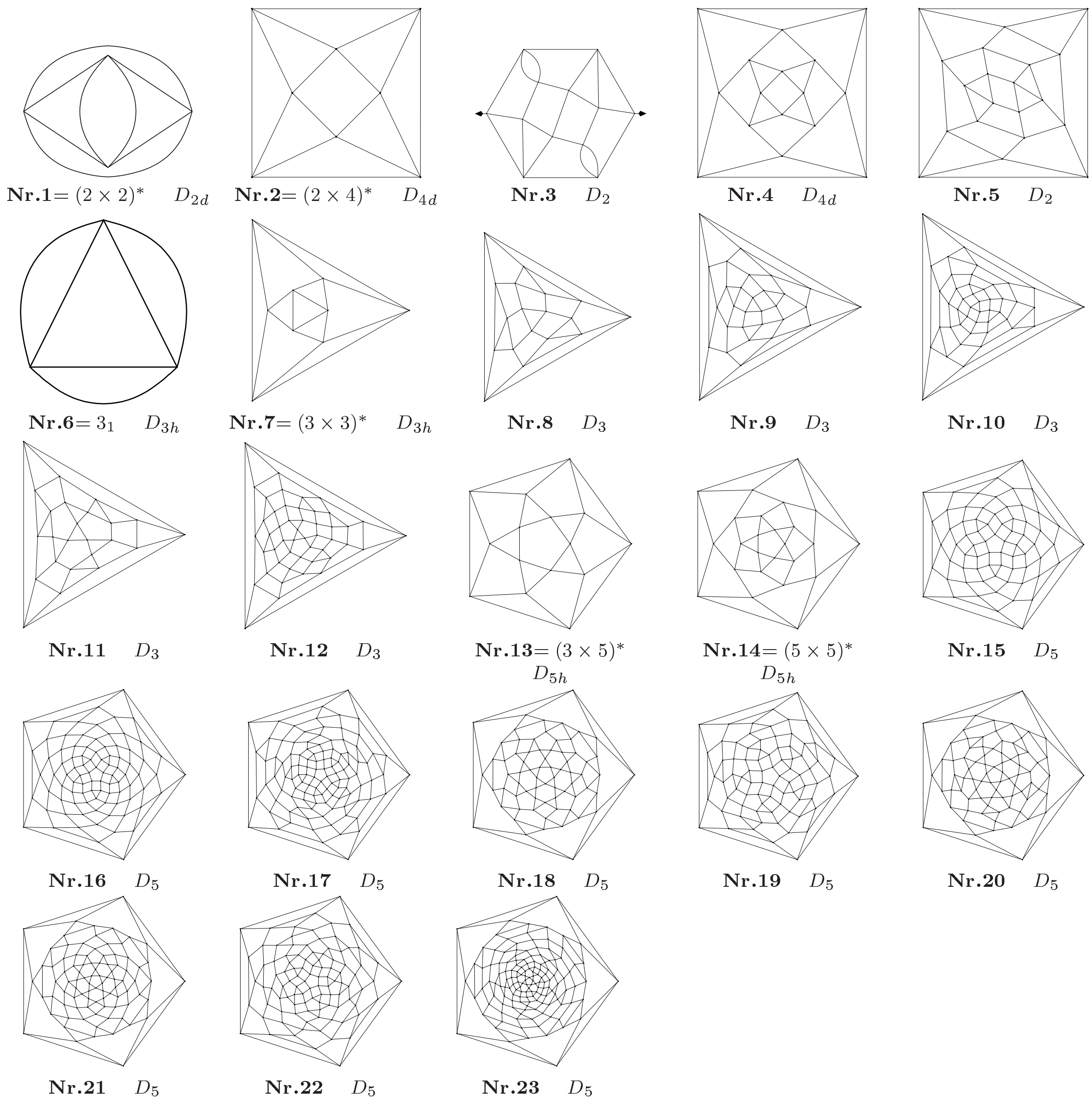


Figure 13: **Projections of $GC_{k,l}(Dodecahedron)$ from Table 8**

	$GC_{k,l}(Trefoil)$				$GC_{k,l}(Octahedron)$			
k,l	$[CC]$	$Projection$	$Group$	p_1,p_2,p_3,p_4	$[CC]$	$Projection$	$Group$	p_2,p_3,p_4
$1,0$	6		D_{3h}	$0,3,2,0$	4^3	0_1	$D_{\infty h}$	$0,0,0$
$1,1^*$	2^3	0_1	D_{∞}	$0,0,0,0$	3^4	0_1	$D_{\infty h}$	$0,0,0$
$2,1$	6		D_3	$0,3,2,12$	2^6	0_1	$D_{\infty h}$	$0,0,0$
$3,1^*$	2^3	1	C_{2v}	$2,1,0,1$	3^4	8	D_{3h}	$3,2,0$
$3,2$	6		D_3	$0,3,2,36$	4^3	20	D_{4d}	$0,8,2$
$4,1$	6		D_3	$0,3,2,48$	4^3	20	D_{4d}	$0,8,2$
$4,3$	6		D_3	$0,3,2,72$	4^3	21	D_{4d}	$0,8,10$
$5,1^*$	2^3	2	C_2	$0,3,2,5$	3^4	9	D_{3h}	$0,8,3$
$5,2$	6		D_3	$0,3,2,84$	4^3	21	D_{4d}	$0,8,10$
$5,3^*$	2^3	2	C_2	$0,3,2,5$	3^4	9	D_{3h}	$0,8,3$
$6,1$	6		D_3	$0,3,2,108$	2^6	1	D_{2d}	$2,4,0$
$5,4$	6		D_3	$0,3,2,120$	2^6	1	D_{2d}	$2,4,0$
$7,1^*$	2^3	3	C_2	$0,3,2,13$	3^4	10	D_3	$0,8,12$
$7,2$	6		D_3	$0,3,2,156$	2^6	20	D_{4d}	$0,8,2$
$7,3^*$	2^3	3	C_2	$0,3,2,13$	3^4	10	D_3	$0,8,12$
$6,5$	6		D_3	$0,3,2,180$	4^3	22	D_4	$0,8,34$
$7,4$	6		D_3	$0,3,2,192$	4^3	22	D_4	$0,8,34$
$8,1$	6		D_3	$0,3,2,192$	4^3	24	D_4	$0,8,26$
$8,3$	6		D_3	$0,3,2,216$	4^3	25	D_4	$0,8,42$
$7,5^*$	2^3	4	C_2	$0,3,2,15$	3^4	11	D_3	$0,8,18$
$9,1^*$	2^3	5	C_2	$0,3,2,23$	3^4	12	D_3	$0,8,24$
$7,6$	6		D_3	$0,3,2,252$	4^3	23	D_4	$0,8,50$
$9,2$	6		D_3	$0,3,2,252$	2^6	4	D_2	$0,8,6$
$8,5$	6		D_3	$0,3,2,264$	4^3	26	D_4	$0,8,50$
$9,4$	6		D_3	$0,3,2,288$	4^3	23	D_4	$0,8,50$
$10,1$	6		D_3	$0,3,2,300$	2^6	2	D_2	$2,4,8$
$9,5^*$	2^3	6	C_2	$0,3,2,31$	3^4	12	D_3	$0,8,24$
$10,3$	6		D_3	$0,3,2,324$	4^3	28	D_4	$0,8,66$
$8,7$	6		D_3	$0,3,2,336$	2^6	2	D_2	$2,4,8$
$11,1^*$	2^3	8	C_2	$0,3,2,37$	3^4	14	D_3	$0,8,39$
$11,2$	6		D_3	$0,3,2,372$	4^3	29	D_4	$0,8,74$
$9,7^*$	2^3	7	C_2	$0,3,2,29$	3^4	13	D_3	$0,8,39$
$11,3^*$	2^3	9	C_2	$0,3,2,37$	3^4	15	D_3	$0,8,42$
$11,4$	6		D_3	$0,3,2,408$	2^6	5	D_2	$0,8,14$
$9,8$	6		D_3	$0,3,2,432$	4^3	27	D_4	$0,8,90$
$12,1$	6		D_3	$0,3,2,432$	4^3	30	D_4	$0,8,66$
$11,5^*$	2^3	10	C_2	$0,3,2,45$	3^4	14	D_3	$0,8,39$
$10,7$	6		D_3	$0,3,2,444$	2^6	6	D_2	$0,8,18$
$11,6$	6		D_3	$0,3,2,468$	4^3	27	D_4	$0,8,90$
$12,5$	6		D_3	$0,3,2,504$	4^3	31	D_4	$0,8,98$
$11,7^*$	2^3	11	C_2	$0,3,2,47$	3^4	16	D_3	$0,8,57$
$13,1^*$	2^3	12	C_2	$0,3,2,53$	3^4	17	D_3	$0,8,57$
$13,2$	6		D_3	$0,3,2,516$	4^3	32	D_4	$0,8,98$
$13,3^*$	2^3	13	C_2	$0,3,2,53$	3^4	19	D_3	$0,8,60$
$10,9$	6		D_3	$0,3,2,540$	4^3	33	D_4	$0,8,114$
$11,8$	6		D_3	$0,3,2,552$	2^6	7	D_2	$0,8,22$
$13,4$	6		D_3	$0,3,2,552$	4^3	34	D_4	$0,8,114$
$12,7$	6		D_3	$0,3,2,576$	4^3	35	D_4	$0,8,122$
$13,5^*$	2^3	14	C_2	$0,3,2,53$	3^4	18	D_3	$0,8,63$
$14,1$	6		D_3	$0,3,2,588$	2^6	3	D_2	$2,4,20$

Table 9: Projections of $GC_{k,l}(G_0)$, $t(k,l) \leq 200$, with G_0 being Trefoil or Octahedron

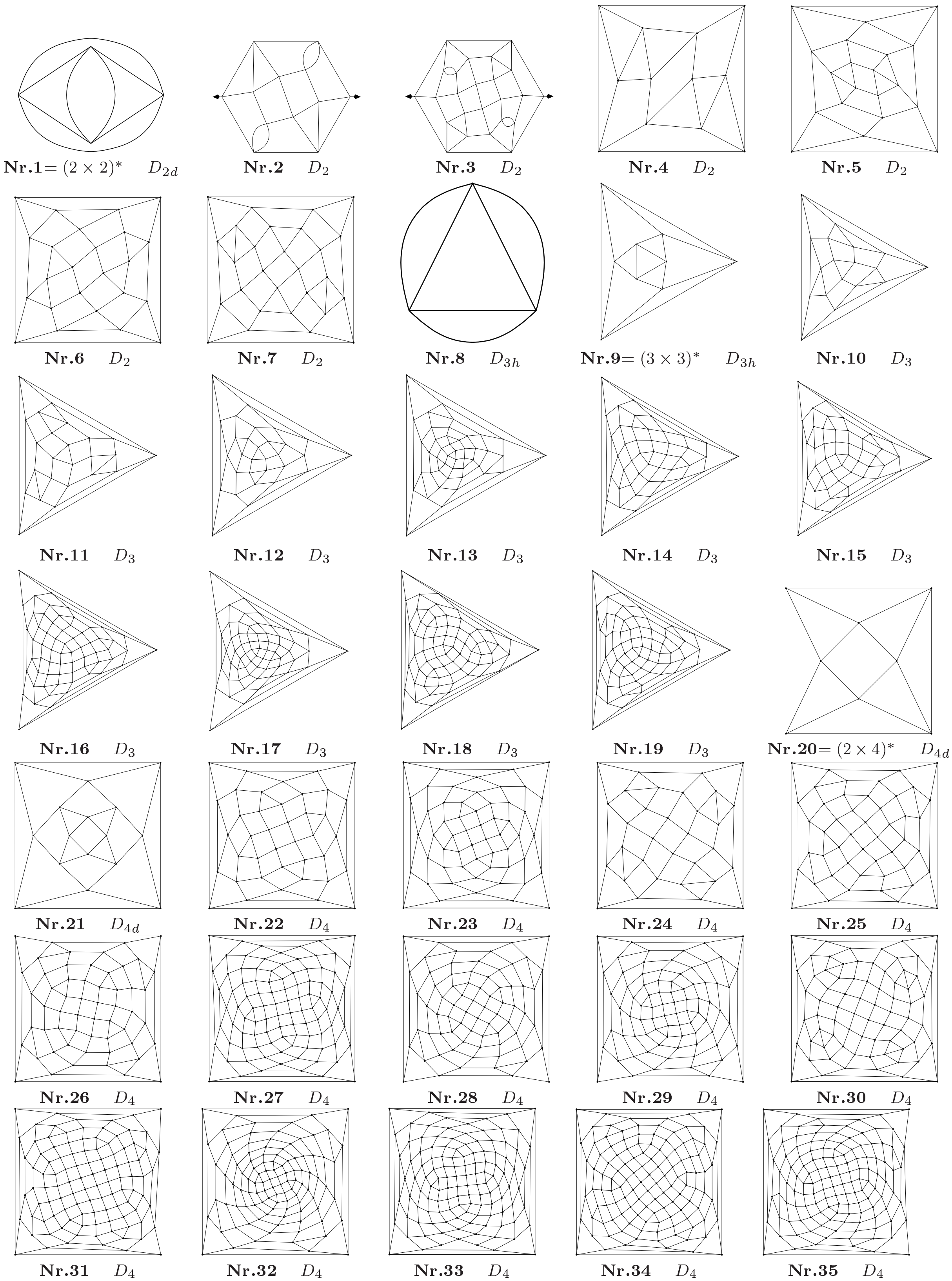


Figure 14: **Projections of $GC_{k,l}(Octahedron)$ from Table 9**

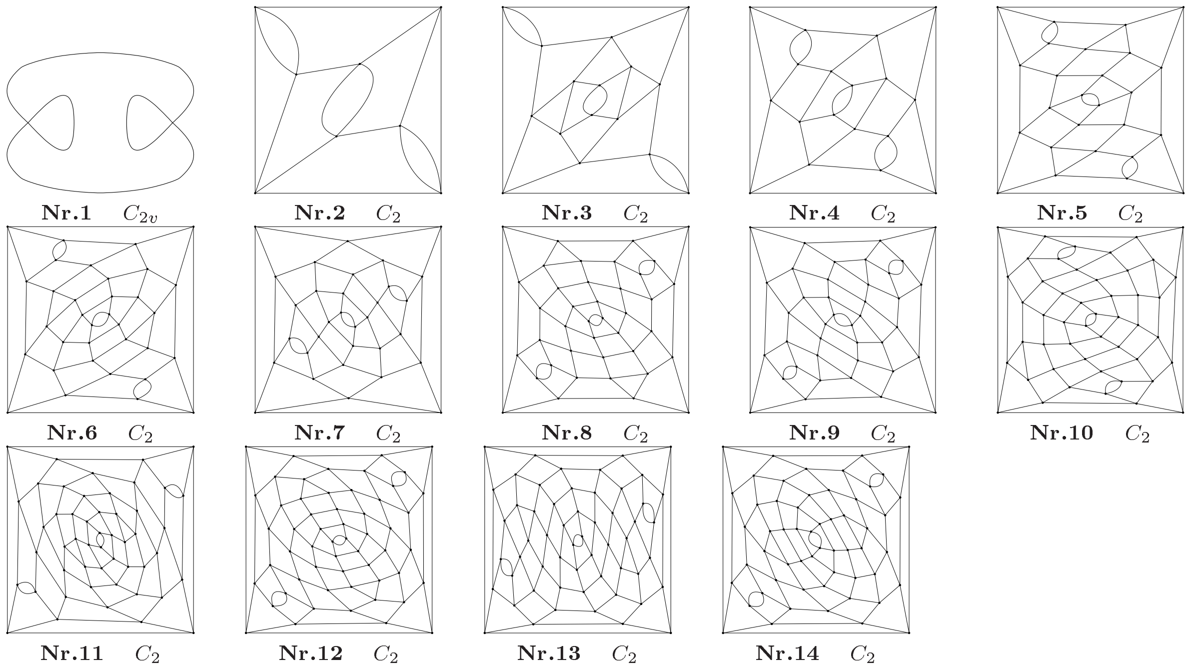


Figure 15: **Projections of $GC_{k,l}(Trefoil)$ from Table 9**

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