Class-Uniformly Resolvable Group Divisible Structures II: Frames.

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Abstract

We consider Class-Uniformly Resolvable frames (CURFs), which are group divisible designs with partial resolution classes subject to the class-uniform condition. We derive the necessary conditions, including extremal bounds, build the foundation for general CURF constructions, including a frame variant of the λ blow-up construction from part I. We also establish a PBD-closure result. For CURFs with blocks of size two and three we determine the existence of CURFs of type g^u, completely for $g = 3$, with a small list of exceptions for $g = 6$, asymptotically for $g = 4, 5$ and give some other infinite families.

1 Introduction

A Class-Uniformly Resolvable incidence structure is one where each resolution class has the same number of blocks of each size. Class-uniformly resolvable designs (CURDs) and class-uniformly resolvable group divisible designs (CURGDDs) have been studied in [7, 4, 11, 14], which contain motivations, applications and general discussions of these

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and related objects. We will assume that the reader is acquainted with the design theory terminology and we refer them to [2].

An idea which has proved productive in the investigation of resolvable structures is that of a frame [8, 13]. Frames can be generalized to admit class-uniform partial resolutions.

Definition 1.1. A *Class-Uniformly Resolvable Frame*, $CURF_\lambda$, of type g^u with partition $\prod k^{p_k}$ is a GDD_λ with the additional property that the blocks can be partitioned into partial resolution classes with partition $\prod k^{p_k}$ and the complement of a partial resolution class is exactly one group. We say that the class *misses* the complementary group.

We call such a CURF *Homogeneous*, HCURF, if the number of partial resolution classes missing each group is a fixed number.

In this definition, and most of the rest of the rest of the paper, we assume that all groups have the same size. If this is not the case, it would be necessary to allow different partitions that correspond to the partial parallel classes missing groups of different sizes. CURFs were first introduced by Lamken *et al.* [11], where they considered a number of CURD constructions involving CURFs. A CURF with all $g = 1$ is a CURD with blocks of size one. If $\lambda = 1$ we drop the subscript λ.

The homogeneous condition is equivalent to requiring that each point in the design is on a fixed number of partial resolution classes. In this case we use $m = r/u$ to denote the number of partial resolution classes missing each group. All of the CURFs that we construct will be homogeneous, but the following example shows that homogeneity is not necessary, even in the case of equal group sizes and only two block sizes.

Example 1.2. A CURF of type 3^6 with partition $2^6 3^1$, where groups 0, 1 and 2 are missed by three classes each, groups 3, 4 and 5 are missed by two classes each. The point set will be $\mathbb{Z}_{18}$ and $r_x = 12$ for points $0 \leq x \leq 8$, whereas $r_x = 13$ for points $9 \leq x \leq 17$. The groups are defined as follows:

group 0: $\{0, 1, 2\}$	group 1: $\{3, 4, 5\}$	group 2: $\{6, 7, 8\}$
group 3: $\{9, 10, 11\}$	group 4: $\{12, 13, 14\}$	group 5: $\{15, 16, 17\}$

Each row in Table 1 is a resolution class, the group it misses is indicated on the left. This will be our standard notation throughout this paper.

In Section 2 we derive the necessary conditions for the existence of class-uniformly resolvable frames. When there are only two block sizes we develop tighter conditions and bounds on the admissible sizes of a CURF. In Section 3 we present constructions for CURFs: Wilson's construction, a general multiplicative construction and a construction which allows the reduction of the index. Wilson's construction also provides us with a PBD closure result for CURFs which is notable because closure results have been elusive for CURDs and CURGDDs. In Section 4 we use these constructions to produce HCURFs.

0 :	$\{3,6,9\}$	$\{7,14\}$	$\{13,11\}$	$\{5,12\}$	$\{16,10\}$	$\{17,4\}$	$\{15,8\}$
0 :	$\{4,7,10\}$	$\{9,14\}$	$\{8,13\}$	$\{17,11\}$	$\{16,6\}$	$\{3,12\}$	$\{5,15\}$
0 :	$\{11,12,16\}$	$\{9,7\}$	$\{13,15\}$	$\{14,5\}$	$\{8,17\}$	$\{10,3\}$	$\{4,6\}$
1 :	$\{0,6,12\}$	$\{15,9\}$	$\{10,2\}$	$\{7,11\}$	$\{16,13\}$	$\{8,1\}$	$\{14,17\}$
1 :	$\{1,7,13\}$	$\{0,10\}$	$\{12,17\}$	$\{14,16\}$	$\{9,8\}$	$\{15,6\}$	$\{11,2\}$
1 :	$\{9,13,17\}$	$\{6,10\}$	$\{8,16\}$	$\{14,1\}$	$\{11,0\}$	$\{7,12\}$	$\{15,2\}$
2 :	$\{0,3,15\}$	$\{2,4\}$	$\{9,12\}$	$\{5,16\}$	$\{10,13\}$	$\{14,11\}$	$\{1,17\}$
2 :	$\{2,5,17\}$	$\{13,3\}$	$\{4,14\}$	$\{10,12\}$	$\{16,0\}$	$\{1,9\}$	$\{15,11\}$
2 :	$\{10,14,15\}$	$\{1,12\}$	$\{3,17\}$	$\{4,11\}$	$\{16,9\}$	$\{0,5\}$	$\{2,13\}$
3 :	$\{2,8,14\}$	$\{0,13\}$	$\{12,4\}$	$\{1,15\}$	$\{7,5\}$	$\{6,17\}$	$\{3,16\}$
3 :	$\{1,4,16\}$	$\{0,14\}$	$\{8,3\}$	$\{15,12\}$	$\{13,5\}$	$\{2,6\}$	$\{7,17\}$
4 :	$\{1,5,6\}$	$\{17,0\}$	$\{3,11\}$	$\{10,8\}$	$\{4,15\}$	$\{2,9\}$	$\{16,7\}$
4 :	$\{0,4,8\}$	$\{1,3\}$	$\{15,7\}$	$\{6,11\}$	$\{16,2\}$	$\{17,10\}$	$\{5,9\}$
5 :	$\{2,3,7\}$	$\{4,13\}$	$\{9,0\}$	$\{12,8\}$	$\{10,5\}$	$\{11,1\}$	$\{14,6\}$
5 :	$\{5,8,11\}$	$\{12,2\}$	$\{4,9\}$	$\{7,0\}$	$\{3,14\}$	$\{1,10\}$	$\{13,6\}$

Table 1: A non-homogeneous CURF of type 3^6 with partition $2^6 3^1$.

2 Necessary Conditions

2.1 General Necessary Conditions

We now give the general necessary conditions for the existence of a CURF$_\lambda$ of type g^u with partition $\prod k^{p_k}$ containing r partial resolution classes, in which the ith group, G_i is missed by m_i resolution classes. Standard counting arguments give the following:

$$u \geq \max_{k\in K}\{k\} + 1, \tag{1}$$

$$g(u-1) = \sum_{k\in K} k p_k, \tag{2}$$

$$r \sum_{k\in K} k(k-1)p_k = \lambda u(u-1)g^2, \tag{3}$$

$$r = \sum_{i=1}^{u} m_i. \tag{4}$$

If the CURF is homogeneous then each group is missed by $m = r/u$ partial resolution classes and Equation 3 reduces to

$$m \sum_{k\in K} k(k-1)p_k = \lambda(u-1)g^2. \tag{5}$$

Let $r_k(x)$ be the number of blocks of size k which contain the point x. It is not generally true that r_k is independent of x as illustrated in Example 1.2. The following example illustrates that even homogeneity is not sufficient to guarantee independence of $r_k(x)$ from x.

Example 2.1. The following is an HCURF_8 of type 1^9 with partition $1^3 2^1 3^1$. Each group is missed by 8 partial resolution classes. The point set is $\mathbb{Z}_8 \cup \{\infty\}$. Consider the following 9 partial resolution classes. Develop the classes additively modulo 8.

$$\begin{array}{rlllll}
\infty: & \{0,1,3\} & \{4,5\} & \{2\} & \{6\} & \{7\} \\
2: & \{0,1,3\} & \{4,5\} & \{6\} & \{7\} & \{\infty\} \\
2: & \{0,1,3\} & \{4,5\} & \{6\} & \{7\} & \{\infty\} \\
5: & \{0,1,3\} & \{2,4\} & \{6\} & \{7\} & \{\infty\} \\
5: & \{0,1,3\} & \{2,4\} & \{6\} & \{7\} & \{\infty\} \\
0: & \{\infty,3,7\} & \{2,4\} & \{1\} & \{5\} & \{6\} \\
1: & \{\infty,3,7\} & \{2,5\} & \{1\} & \{4\} & \{6\} \\
1: & \{\infty,3,7\} & \{2,5\} & \{1\} & \{4\} & \{6\} \\
1: & \{\infty,3,7\} & \{2,5\} & \{1\} & \{4\} & \{6\}
\end{array}$$

This example has high λ, but it satisfies the hypotheses of Theorem 3.4 and so can be used to produce an HCURF_1 of type 8^9 with partition $1^{24} 2^8 3^8$ which contains points that appear in no pairs.

We call a CURF *point-block regular* if every point appears the same number of times in blocks of each size, i.e. $r_k(x) = r_k$ is constant across x. A CURF is called *point regular* if all points are in the same number of blocks. Counting the number of blocks that contain a point x yields

$$\sum_{k \in K} r_k(x) = r - m_i, \tag{6}$$

where x is in the group G_i. Thus the condition of homogeneity, $m_i = m$ for every i, is equivalent to point regularity. Further, if a CURF is point-block regular then it is homogeneous. The falsity of the converse is demonstrated by Example 2.1. Counting the number of pairs containing x in every block through x gives other necessary conditions:

$$\sum_{k \in K} r_k(x)(k-1) = \lambda g(u-1). \tag{7}$$

Assuming that there are no blocks of size 1 and our block sizes are enumerated in order, we get

$$k_1 - 1 \leq \frac{\lambda g(u-1)}{r - m_i} \leq k_n - 1. \tag{8}$$

Taking $m_{\min}$ and $m_{\max}$ to be the minimum and maximum m_i's respectively and $\overline{m}$ to be the average of the m_i's, we have

Theorem 2.2. *If $m_{\min} > 0$ and $n > 1$ then there is no CURF_1 of type g^u with partition $\prod_i k_i^{p_{k_i}}$, $1 < k_1 < \ldots < k_n$, for any $g < k_1$.*

Proof. Suppose that such a CURF exists. We first consider the case $g < k_1 - 1$, Equation 8 gives $m_{\min} \leq g/(k_1 - 1) < 1$.

When $g = k_1 - 1$, Equation 8 then gives $1 \leq m_{\min} \leq \overline{m} \leq g/(k_1 - 1) = 1$, so $m_{\min} = \overline{m} = 1$. Thus $m_i = 1$ for all i and the CURF is homogeneous and so Equations 2 and 5 give

$$\sum_i k_i(k_i - 1)p_{k_i} = (k_1 - 1)\sum_i k_i p_{k_i}.$$

But since there are at least two block sizes and k_1 is the smallest block size

$$\sum_i k_i(k_i - 1)p_{k_i} > (k_1 - 1)\sum_i k_i p_{k_i}.$$

□

The existence of k-frames with group size $k-1$ demonstrates the necessity of at least two block sizes in the hypotheses of the theorem. Example 2.3 illustrates that the condition $m_{\min} > 0$ is indeed necessary. Example 2.4 illustrates that the bound $g < k_1$ is the strongest possible.

Example 2.3. A CURF of type 1^{25} with partition $2^9 3^2$, with 20 resolution classes, the groups are individual points. The group $\{0\}$ is missed by no partial resolution class.

1 :	$\{22,6,9\}$	$\{15,8,20\}$	$\{12,21\}$	$\{11,4\}$	$\{3,10\}$	$\{14,18\}$	$\{16,19\}$	$\{24,13\}$	$\{17,23\}$	$\{5,7\}$	$\{0,2\}$
2 :	$\{11,16,3\}$	$\{9,4,20\}$	$\{12,22\}$	$\{24,23\}$	$\{0,5\}$	$\{13,6\}$	$\{14,17\}$	$\{10,21\}$	$\{1,19\}$	$\{7,15\}$	$\{8,18\}$
3 :	$\{2,13,4\}$	$\{23,15,18\}$	$\{20,21\}$	$\{6,10\}$	$\{8,12\}$	$\{11,19\}$	$\{9,17\}$	$\{24,7\}$	$\{16,14\}$	$\{0,1\}$	$\{22,5\}$
4 :	$\{5,12,6\}$	$\{11,9,1\}$	$\{8,24\}$	$\{18,3\}$	$\{17,2\}$	$\{20,19\}$	$\{7,14\}$	$\{15,22\}$	$\{13,16\}$	$\{21,23\}$	$\{10,0\}$
5 :	$\{4,12,3\}$	$\{10,19,2\}$	$\{8,16\}$	$\{6,18\}$	$\{9,15\}$	$\{11,13\}$	$\{20,0\}$	$\{24,22\}$	$\{21,14\}$	$\{17,1\}$	$\{23,7\}$
6 :	$\{8,14,19\}$	$\{5,18,4\}$	$\{7,9\}$	$\{13,20\}$	$\{1,15\}$	$\{3,17\}$	$\{11,22\}$	$\{16,2\}$	$\{10,12\}$	$\{0,23\}$	$\{21,24\}$
7 :	$\{9,21,19\}$	$\{18,10,20\}$	$\{11,2\}$	$\{4,14\}$	$\{8,13\}$	$\{0,15\}$	$\{1,22\}$	$\{16,5\}$	$\{3,24\}$	$\{12,23\}$	$\{6,17\}$
8 :	$\{24,0,11\}$	$\{16,12,18\}$	$\{6,23\}$	$\{10,14\}$	$\{7,20\}$	$\{15,2\}$	$\{5,21\}$	$\{4,22\}$	$\{9,3\}$	$\{17,19\}$	$\{1,13\}$
9 :	$\{1,6,14\}$	$\{18,24,17\}$	$\{12,13\}$	$\{10,11\}$	$\{21,4\}$	$\{23,20\}$	$\{19,5\}$	$\{0,16\}$	$\{3,15\}$	$\{8,2\}$	$\{22,7\}$
10 :	$\{7,21,11\}$	$\{22,0,13\}$	$\{9,24\}$	$\{2,3\}$	$\{12,1\}$	$\{14,5\}$	$\{18,19\}$	$\{6,16\}$	$\{8,23\}$	$\{17,20\}$	$\{15,4\}$
11 :	$\{4,10,23\}$	$\{20,1,5\}$	$\{19,13\}$	$\{14,9\}$	$\{6,0\}$	$\{18,21\}$	$\{12,24\}$	$\{2,7\}$	$\{16,22\}$	$\{15,17\}$	$\{3,8\}$
12 :	$\{22,23,19\}$	$\{2,20,6\}$	$\{24,10\}$	$\{17,13\}$	$\{9,16\}$	$\{11,14\}$	$\{15,5\}$	$\{1,8\}$	$\{21,3\}$	$\{7,18\}$	$\{0,4\}$
13 :	$\{2,5,24\}$	$\{14,3,0\}$	$\{15,10\}$	$\{9,23\}$	$\{4,17\}$	$\{7,12\}$	$\{20,16\}$	$\{21,1\}$	$\{6,19\}$	$\{8,22\}$	$\{18,11\}$
14 :	$\{24,15,16\}$	$\{7,10,17\}$	$\{12,2\}$	$\{1,18\}$	$\{20,22\}$	$\{0,19\}$	$\{9,5\}$	$\{11,23\}$	$\{4,8\}$	$\{3,6\}$	$\{13,21\}$
15 :	$\{17,12,11\}$	$\{8,6,7\}$	$\{9,0\}$	$\{19,24\}$	$\{3,5\}$	$\{16,4\}$	$\{10,1\}$	$\{22,21\}$	$\{14,20\}$	$\{2,18\}$	$\{23,13\}$
16 :	$\{17,8,5\}$	$\{15,13,14\}$	$\{0,18\}$	$\{24,1\}$	$\{23,2\}$	$\{9,12\}$	$\{22,10\}$	$\{6,21\}$	$\{7,4\}$	$\{3,19\}$	$\{11,20\}$
17 :	$\{1,7,16\}$	$\{9,13,10\}$	$\{18,22\}$	$\{15,6\}$	$\{2,21\}$	$\{12,0\}$	$\{3,20\}$	$\{23,5\}$	$\{11,8\}$	$\{14,24\}$	$\{4,19\}$
18 :	$\{14,22,2\}$	$\{1,23,3\}$	$\{11,6\}$	$\{20,12\}$	$\{4,24\}$	$\{8,9\}$	$\{17,0\}$	$\{10,16\}$	$\{7,19\}$	$\{13,5\}$	$\{21,15\}$
19 :	$\{3,13,7\}$	$\{8,0,21\}$	$\{1,2\}$	$\{4,6\}$	$\{14,12\}$	$\{20,24\}$	$\{11,15\}$	$\{17,22\}$	$\{16,23\}$	$\{9,18\}$	$\{5,10\}$
20 :	$\{15,19,12\}$	$\{21,16,17\}$	$\{13,18\}$	$\{23,14\}$	$\{24,6\}$	$\{3,22\}$	$\{9,2\}$	$\{1,4\}$	$\{8,10\}$	$\{5,11\}$	$\{7,0\}$

Example 2.4. A CURF of type 4^{15} with partition $2^{16}3^8$. One group has $m_i = 14$ and all other groups are missed by exactly one partial resolution class. Start with a CURF_2 of type 1^{15} with partition $2^4 3^2$, with 21 resolution classes. The point set is $\mathbb{Z}_7 \times \mathbb{Z}_2 \cup \{\infty\}$. Develop the following resolution classes modulo 7 in the first coordinate.

$$\begin{array}{rllllll}
(4,1): & \{\infty,(0,0),(0,1)\} & \{(1,0),(2,0),(5,1)\} & \{(1,1),(2,1)\} & \{(3,1),(6,1)\} & \{(3,0),(5,0)\} & \{(4,0),(6,0)\} \\
(6,0): & \{\infty,(0,0),(0,1)\} & \{(2,1),(4,1),(3,0)\} & \{(1,1),(3,1)\} & \{(5,1),(6,1)\} & \{(1,0),(4,0)\} & \{(2,0),(5,0)\}. \\
\infty: & \{(0,1),(3,1),(5,0)\} & \{(2,1),(4,1),(3,0)\} & \{(0,0),(6,1)\} & \{(1,1),(6,0)\} & \{(2,1),(4,0)\} & \{(3,0),(4,1)\}
\end{array}$$

A CURF_4 satisfying the conditions of Theorem 3.4 is obtained by doubling these blocks and so produces the desired CURF_1.

2.2 Necessary Conditions for CURFs with 2 Block Sizes

In this section we consider the case where there are exactly two block sizes k and l; we will consider k and l to be interchangeable. The bounds here are derived in a similar manner to those in [4].

$$\begin{aligned} lp_l &\equiv g(u-1) \bmod k & (9)\\ r_l(x) &= \frac{\lambda g(u-1)-(r-m_i)(k-1)}{l-k}. & (10)\end{aligned}$$

Thus for two block sizes, homogeneity implies point-block regularity and thus the two are equivalent (see Equation 6). To illustrate that these conditions do not generalize, Example 1.2 is a CURF with two block sizes which is neither homogeneous nor point-block regular. Example 2.1 gives an HCURF with three block sizes which is not point-block regular.

Letting $\alpha_l = l(l-k)p_l$ and defining $d = \gcd(r(k-1)^2, \lambda((k-1)gu - \alpha_l))$, we get

$$g(u-1) \le \lambda \left| \frac{\alpha_l}{d}\left(\frac{\alpha_l}{k-1} - g\right)\right| - \frac{\alpha_l}{k-1} \qquad (11)$$

When $g(k-1) = \alpha_l$, which requires that $l > k$, we obtain no bound on $g(u-1) = r(k-1)/\lambda$. When the CURF is homogeneous, and defining $d = \gcd(m(u-1)(k-1)^2, \lambda((k-1)g(u-1) - \alpha_l))$, we get

$$g(u-1) \le \frac{\alpha_l(\lambda\alpha_l - d)}{d(k-1)} \qquad (12)$$

When the blocks are of size 2 and 3 we get

$$\begin{aligned} p_3 &\equiv g(u-1) \bmod 2, & (13)\\ 2p_2 &\equiv g(u-1) \bmod 3, & (14)\\ \lambda p_3 &\equiv 0 \bmod 2, & \\ r_3(x) &= \lambda g(u-1) - r + m_i, & (15)\\ r_2(x) &= 2(r-m_i) - \lambda g(u-1), & (16)\\ & & (17)\end{aligned}$$

and assuming homogeneity since it is the most useful case,

$$\begin{aligned} g(u-1) &\le 3p_3(\lambda\frac{3p_3}{d_3} - 1), & (18)\\ g(u-1) &\le p_2(\lambda\frac{p_2}{d_2} + 1), & (19)\end{aligned}$$

where $d_3 = \gcd(m(u-1), \lambda(g(u-1) - 3p_3))$ and $d_2 = \gcd(2m(u-1), \lambda(g(u-1) + p_2))$.

3 General Frame Constructions

In this section we give some general constructions for creating CURFs.

Theorem 3.1 (Wilson's Fundamental Frame Construction). *If there exists a K-GDD of type g^u, and for every $k \in K$ there exists an (H)CURF of type n^k with partition $\prod i^{(k-1)q_i}$ (which implies a constant ratio r/k), then there exists an (H)CURF of type $(ng)^u$ and partition $\prod i^{(u-1)q_i}$.*

Proof. This is Wilson's fundamental frame construction [8]. We blow up the master K-GDD giving each point weight n. Place the ingredient (H)CURFs on the blown up blocks. We recover the partial parallel classes missing the group $G \times \mathbb{Z}_n$, where G is a group of the master K-GDD, by considering the blocks of the master design through each point $x \in G$. The union of one partial parallel class missing $\{x\} \times \mathbb{Z}_n$ from each of the the frames placed on the blow up of these blocks form a partial parallel class missing $G \times \mathbb{Z}_n$. □

This theorem gives a PBD closure result for CURFs. For a fixed $\overline{m}$ and g, modular conditions for admissible u can be found. There is another variation of Wilson's Construction.

Theorem 3.2. *If there exists an (H)CURF of type g^u with partition $\prod k^{p_k}$ and there exists an CURGDD of type n^k for each k that appears with partition $\prod i^{q_{i,k}}$, each with the same r then there exists an (H)CURF of type $(ng)^u$ with partition $\prod i^{\sum_k q_{i,k}p_k}$.*

Proof. Blow up each point of the master CURF by n and fill the blown up blocks with the ingredient CURGDDs [8]. □

We now include a standard construction for breaking up the groups of a CURF.

Theorem 3.3 (Breaking up Groups). *If there exists an CURF of type $(ng)^u$ with partition $\prod k^{p_k}$ with r partial resolution classes and for each group G_i there is an CURF of type g^n and partition $\prod k^{q_k}$ with m_i resolution classes, then there exists an CURF of type g^{nu} with partition $\prod k^{p_k+q_k}$.*

Proof. Replace groups by smaller structure. □

We now present a notable construction which allows us to reduce the index of a CURF$_\lambda$, whilst increasing the group size, similar to that found in [7]. We refer the reader there for definitions, notation and proof. The details of checking that the frame property is preserved are straightforward.

Theorem 3.4 (λ Blow-up). *If there exists a (H)CURF$_\lambda$ of type g^u with partition $\prod k^{p_k}$ 1-maximally contained in $(X, \mathcal{M})$ and for each $M \in \mathcal{M}$ there exists an RTD$_{\lambda_1}(|M|, \lambda)$ then there exists a (H)CURF$_{\lambda_1}$ of type $(\lambda g)^u$ and partition $\prod k^{\lambda p_k}$.* □

When the only block sizes in $\mathcal{B}$ are 1, 2 and k and the blocks of size k (repetitions ignored) form a packing of index 1 then the hypotheses of Theorem 3.4 are satisfied. We illustrate this construction with the following example.

Example 3.5. We present an HCURF_{10} of type 1^8 with partition $2^2 3^1$ with $r = 56$ ($m = 7$) resolution classes. The blocks of size 3 form the blocks of a packing on 8 points repeated seven times. Develop the following seven base blocks cyclically over $\mathbb{Z}_8$.

$$\begin{array}{llll}
5: & \{0,1,3\} & \{2,6\} & \{4,7\} \\
5: & \{0,1,3\} & \{2,6\} & \{4,7\} \\
5: & \{0,1,3\} & \{2,6\} & \{4,7\} \\
4: & \{0,1,3\} & \{2,6\} & \{5,7\} \\
7: & \{0,1,3\} & \{2,6\} & \{4,5\} \\
7: & \{0,1,3\} & \{2,4\} & \{5,6\} \\
7: & \{0,1,3\} & \{2,4\} & \{5,6\}
\end{array}$$

We take

$$\begin{aligned}
\mathcal{M} \;=\; & \{\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,0\}, \\
& \{6,7,1\},\{7,0,2\},\{0,4\},\{1,5\},\{2,6\},\{3,7\}\}.
\end{aligned}$$

Applying Theorem 3.4 produces an HCURF_1 of type 10^8 with partition $2^{20}3^{10}$.

4 Existence of HCURFs

We now demonstrate the utility of these constructions. For blocks of size 2 and 3, we use PBD closure to establish the existence of all HCURFs with $g = 3$, asymptotic existence $g = 4, 5$, all but a small finite number for $g = 6$ for all m and $g = 7$ for $m = 4$ and asymptotically for $g = 8$ for $m = 7$. Following this, we apply Theorems 3.1 and 3.2 with different kinds of ingredients. Finally we end with some prime power difference constructions. We will make frequent use of Theorem 3.4 in these solutions.

4.1 PBD Closure Constructions

For block sizes 2 and 3 if either p_2 or p_3 are zero then the HCURFs are 2-frames or 3-frames respectively and their existence has been completely solved [2]. Therefore, in what follows we will assume that $p_2, p_3 > 0$. For each g there is a range of possible values of m, obtained from Equation 8. for each of these values of m we obtain modular conditions on u which are amenable to PBD-closure. For each value of g, Theorem 3.2 with RTDs as ingredients give existence results for HCURFs with group size gn for any $n \neq 2, 6$.

4.1.1 $g = 2$

If $g = 2$ there are no HCURFs with $p_2, p_3 > 0$ by Equation 8.

4.1.2 $g = 3$

When $g = 3$ the admissible value of m that permits both block sizes is $m = 2$. In this case the HCURFs have parameters 3^{4t+1} and partition $2^{3t}3^{2t}$ for $t > 0$; we solve this case completely.

Theorem 4.1. *An HCURF of type $(3n)^u$ with $u = 4t+1$ partition $2^{3t}3^{2t}$, for all $n \neq 2, 6$ and all u.*

Proof. ¿From the equations in Section 2, the conditions are necessary. In particular Inequality 8 implies that $m = 2$. The case $m = 3$ corresponds to a 2-frame ($p_3 = 0$). We now establish the existence for $m = 2$ and $u \equiv 1 \bmod 4$.

In [9] (see also [2]) it is shown that $\{5, 9, 13, 17, 29, 33\}$ is a generating set for $\{x \mid x \in \mathbb{Z},\, x > 4,\, x \equiv 1 \bmod 4\}$. Thus there is a PBD with block sizes from $\{5, 9, 13, 17, 29, 33\}$ on v points for every integer $v \equiv 1 \bmod 4$, larger than 4. Thus by establishing the existence of HCURFs of type 3^u for $u \in \{5, 9, 13, 17, 29, 33\}$ and applying Theorem 3.1, giving each point weight 3, we produce all the desired HCURFs.

When $u \equiv 1, 5 \mod 12$ we construct HCURFs directly using a difference construction. The point set is $\mathbb{Z}_3 \times \mathbb{Z}_u$, the groups are $\{(0,i), (1,i), (2,i)\}$ for $0 \leq i < u$. Consider the following two partial classes, each missing the group $\{(0,0), (1,0), (2,0)\}$. The first class is

$$\begin{array}{ll} \pm\{(0,i), (1,2i), (2,3i)\} & 1 \leq i \leq \frac{u-1}{4} \\ \{\{(0,i), (0,-i)\}, \{(1,2i), (1,-2i)\}, \{(2,3i), (2,-3i)\}\} & \frac{u+3}{4} \leq i \leq \frac{u-1}{2}. \end{array}$$

The second class is

$$\begin{array}{ll} \pm\{(0,i), (1,2i), (2,3i)\} & \frac{u+3}{4} \leq i \leq \frac{u-1}{2} \\ \{\{(0,i), (0,-i)\}, \{(1,2i), (1,-2i)\}, \{(2,3i), (2,-3i)\}\} & 1 \leq i \leq \frac{u-1}{4}. \end{array}$$

Develop these classes modulo u to get all the classes. In particular, this constructs HCURFs for $u = 5,\ 13,\ 17$ and 29.

We now present an HCURF of type 3^9 with partition $2^6 3^4$. The point set is $\mathbb{Z}_3 \times \mathbb{Z}_9$ and the groups are $\{(0,i), (1,i), (2,i)\}$ for $0 \leq i \leq 8$. Develop the following classes, which miss the group $\{(0,6), (1,6), (2,6)\}$, $\bmod(-, 9)$.

$\{(0,0), (1,1), (2,2)\}$, $\{(0,1), (1,3), (2,5)\}$, $\{(0,2), (1,5), (2,8)\}$, $\{(0,4), (1,8), (2,3)\}$, $\{(0,3), (0,5)\}$, $\{(0,7), (0,8)\}$, $\{(1,0), (1,2)\}$, $\{(1,4), (1,7)\}$, $\{(2,0), (2,4)\}$, $\{(2,1), (2,7)\}$

$\{(0,0), (1,5), (2,1)\}$, $\{(0,1), (1,7), (2,4)\}$, $\{(0,2), (1,0), (2,7)\}$, $\{(0,4), (1,3), (2,2)\}$, $\{(0,3), (0,7)\}$, $\{(0,5), (0,8)\}$, $\{(1,1), (1,2)\}$, $\{(1,4), (1,8)\}$, $\{(2,0), (2,8)\}$, $\{(2,3), (2,5)\}$

We now present an HCURF of type 3^{33} with partition $2^{24}3^{16}$. The point set is $\mathbb{Z}_3 \times \mathbb{Z}_{33}$ and the groups are $\{(0,i), (1,i), (2,i)\}$ for $0 \leq i \leq 32$. Develop the following classes, which miss the group $\{(0,18), (1,18), (2,18)\}$, $\bmod(-, 33)$.

$\{(0,0), (1,1), (2,2)\}$, $\{(0,1), (1,3), (2,5)\}$, $\{(0,2), (1,5), (2,8)\}$, $\{(0,3), (1,7), (2,11)\}$,
$\{(0,4), (1,9), (2,14)\}$, $\{(0,5), (1,11), (2,17)\}$, $\{(0,6), (1,13), (2,20)\}$,
$\{(0,7), (1,15), (2,23)\}$, $\{(0,8), (1,17), (2,26)\}$, $\{(0,9), (1,19), (2,29)\}$,
$\{(0,10), (1,21), (2,32)\}$, $\{(0,12), (1,24), (2,3)\}$, $\{(0,13), (1,26), (2,6)\}$,
$\{(0,14), (1,28), (2,9)\}$, $\{(0,15), (1,30), (2,12)\}$, $\{(0,16), (1,32), (2,15)\}$,
$\{(0,11), (0,22)\}$, $\{(0,17), (0,31)\}$, $\{(0,19), (0,32)\}$, $\{(0,20), (0,27)\}$, $\{(0,21), (0,30)\}$,
$\{(0,23), (0,28)\}$, $\{(0,24), (0,26)\}$, $\{(0,25), (0,29)\}$, $\{(1,0), (1,6)\}$, $\{(1,2), (1,10)\}$,
$\{(1,4), (1,8)\}$, $\{(1,12), (1,23)\}$, $\{(1,14), (1,31)\}$, $\{(1,16), (1,29)\}$, $\{(1,20), (1,27)\}$,

$\{(1,22),(1,25)\}$, $\{(2,0),(2,1)\}$, $\{(2,4),(2,7)\}$, $\{(2,10),(2,16)\}$, $\{(2,13),(2,21)\}$, $\{(2,19),(2,28)\}$, $\{(2,22),(2,24)\}$, $\{(2,25),(2,30)\}$, $\{(2,27),(2,31)\}$

$\{(0,0),(1,17),(2,1)\}$, $\{(0,1),(1,19),(2,4)\}$, $\{(0,2),(1,21),(2,7)\}$, $\{(0,3),(1,23),(2,10)\}$, $\{(0,4),(1,25),(2,13)\}$, $\{(0,5),(1,27),(2,16)\}$, $\{(0,6),(1,29),(2,19)\}$, $\{(0,7),(1,31),(2,22)\}$, $\{(0,8),(1,0),(2,25)\}$, $\{(0,9),(1,2),(2,28)\}$, $\{(0,10),(1,4),(2,31)\}$, $\{(0,12),(1,7),(2,2)\}$, $\{(0,13),(1,9),(2,5)\}$, $\{(0,14),(1,11),(2,8)\}$, $\{(0,15),(1,13),(2,11)\}$, $\{(0,16),(1,15),(2,14)\}$, $\{(0,11),(0,27)\}$, $\{(0,17),(0,32)\}$, $\{(0,19),(0,31)\}$, $\{(0,20),(0,30)\}$, $\{(0,21),(0,29)\}$, $\{(0,22),(0,28)\}$, $\{(0,23),(0,26)\}$, $\{(0,24),(0,25)\}$, $\{(1,1),(1,10)\}$, $\{(1,3),(1,8)\}$, $\{(1,5),(1,6)\}$, $\{(1,12),(1,26)\}$, $\{(1,14),(1,32)\}$, $\{(1,16),(1,28)\}$, $\{(1,20),(1,30)\}$, $\{(1,22),(1,24)\}$, $\{(2,0),(2,12)\}$, $\{(2,3),(2,20)\}$, $\{(2,6),(2,26)\}$, $\{(2,9),(2,24)\}$, $\{(2,15),(2,29)\}$, $\{(2,17),(2,27)\}$, $\{(2,21),(2,32)\}$, $\{(2,23),(2,30)\}$

We have constructed an HCURF of type 3^u with partition $2^{3(u-1)/4}3^{u-1/2}$ for each $u \in \{5,9,13,17,29,33\}$ hence the result follows from Theorem 3.1. □

4.1.3 $g = 4$

We now consider the case where $g = 4$. The admissible value of m that permits both block sizes is $m = 3$. In this case the HCURFs have parameters 4^{9t+1} and partition $2^{12t}3^{4t}$ for $t > 0$.

Theorem 4.2. *There exists an HCURF of type $(4n)^u$ with $u = 9t+1$ partition $2^{12nt}3^{4nt}$ for all u of the form*

$$u = \prod_{m \in D} m^{e_m},$$

where D is the set of integers defined below, and for all $u \equiv 1 \bmod 9$ sufficiently large.

Proof. Let $C = \{9t+1 \mid 1 \le t \le 100\}$, in [5] it is shown that an HCURF_{4n} of type 1^u partition $2^{(u-1)/3}3^{(u-1)/9}$ exists satisfying the conditions of Theorem 3.4 for all $u \in C$, the first 5 are given in the appendix. Let $A = \{x^2+x+1 \mid x \in C,\ x = q+1,\ q \text{ a prime power}\}$ and let $B = \{x^2 \mid x \in C,\ x = q,\ q \text{ a prime power}\}$. Using the projective and affine planes in Theorem 3.1 we get an HCURF of type 4^u for $u \in A \cup B$. Let $D = A \cup B \cup C$, by repeated application of Theorems 3.2 and 3.3 we get the first result.

By Wilson's theorem the necessary conditions, $v \equiv 1 \bmod 9$, for the existence of a $\mathrm{PBD}(v,\{10,19\},1)$ are asymptotically sufficient [15]. Applying Theorem 3.1 gives the asymptotic result. □

4.1.4 $g = 5$

For $g = 5$ there are two values: $m = 3, 4$; we show their asymptotic existence in order.

Theorem 4.3. *There exists an HCURF of type $(5n)^u$ with $u = 18t+1$, partition $2^{15nt}3^{20nt}$ for all $n \neq 2, 6$, $u = 19^e 37^f 55^g$ and asymptotically for all $u \equiv 1 \bmod 18$.*

Proof. For $t = 1, 2, 3$ the HCURFs are given in Appendix A.2. Repeated application of Theorems 3.2 and 3.3 gives the first result. By Wilson's theorem the necessary conditions for the existence of a PBD$(v, \{19, 37, 55\}, 1)$, $v \equiv 1 \bmod 18$, are asymptotically sufficient [15]. Applying Theorem 3.1 gives the result. $\square$

Similarly the HCURFs given in Appendix A.3 give

Theorem 4.4. *There exists an HCURF of type* $(5n)^u$ *with* $u = 24t+1$, *partition* $2^{45nt}3^{10nt}$ *for all* $n \neq 2, 6$, $u = 25^e 49^f$ *and asymptotically for all* $u \equiv 1 \bmod 24$. $\square$

4.1.5 $g = 6$

For $g = 6$ there are two values: $m = 4, 5$; in each case we find all but a small number of exceptions.

Theorem 4.5. *There exists an HCURF of type* $(6n)^u$ *with* $u = 2t + 1$, *partition* $2^{3nt}3^{2nt}$ *for all* $n \neq 2, 6$,

$$u \notin \{43, 51, 59, 71, 75, 83, 87, 95, 107, 111, 115, 119, 131, 135, 139, 167, 179, 183, 191, 195\}.$$

Proof. By PBD closure [1], it is sufficient to find HCURFs for $u \in \{5, 7, 9, 11, 13\}$. We present the first of these here, the remaining are given in Appendix A.4.

The point set is $\mathbb{Z}_6 \times \mathbb{Z}_5$. The following class misses the group $\{(i, 4)|0 \leq i \leq 5\}$:

$$\begin{array}{ccccc} \{(0,0),(1,1),(2,2)\} & \{(0,2),(0,3),(3,0)\} & \{(4,0),(5,1)\} & \{(3,1),(5,3)\} & \{(3,2),(4,3)\} \\ \{(1,2),(1,3),(5,0)\} & \{(2,1),(2,3),(4,2)\} & \{(0,1),(5,2)\} & \{(1,0),(4,1)\} & \{(2,0),(3,3)\} \end{array}.$$

Multiply this class by $(1, 1)$, $(-1, 1)$, $(1, 2)$, and $(-1, 2)$ and develop the resulting classes modulo (-,5). $\square$

In [6] the existence of an HCURF of type 6^{5t+1} partition $2^{12t}3^{2t}$ for $5t + 1 \in \{6, 11, 16, 21, 26, 36, 41\}$ is established. These values represent a generating set for $u \equiv 1 \bmod 5$ [2] with 14 exceptions, six of which are also in [6]. This leaves $u \in \{56, 86, 116, 191, 236, 291, 311, 321\}$ open. Thus we may apply Theorem 3.1 and Corollary 3.2 to give the following, the case $n = 6$ is given in [11].

Theorem 4.6. *There exists an HCURF of type* $(6n)^u$ *with* $u = 5t + 1$ *partition* $2^{12nt}3^{2nt}$ *for every* $n \neq 2$, *with 8 possible exceptions in* u. $\square$

4.1.6 $g = 7$

For $g = 7$, $m = 4, 5, 6$ are possible; we show existence for the family $m = 4$ using the PBD-closure results known for $u \equiv 1 \bmod 8$ [2].

Theorem 4.7. *There exists an HCURF of type* $(7n)^u$ *with* $u = 8t + 1$, *partition* $2^{7nt}3^{14nt}$ *for all* $n \neq 2, 6$, *with 66 possible exceptions in* u.

Proof. Theorem 4.14 gives $u = 9$. In Appendix A.5 we give these HCURFs for $u \in \{17, 25, 33, 41, 49, 57, 65, 89, 97, 105, 113, 121, 129\}$. This corresponds to the generating set from [2] (p. 212) for $1 \bmod 8$, we refer the reader there for a list of the 66 possible exceptions. $\square$

4.1.7 $g = 8$

For $g = 8$, $m = 5, 6, 7$ are possible; we show asymptotic existence for the family $m = 7$.

Theorem 4.8. *There exists an HCURF of type* $(8n)^u$ *with* $u = 21t+1$, *partition* $2^{72tn}3^{8tn}$ *and* $m = 7n$, *for all* $n \geq 1$ *and* $u = 22^e 43^f$ *and asymptotically for all* $u \equiv 1 \bmod 21$.

Proof. For $u \in \{22, 43\}$ we present an HCURF_8 of type 1^u with partition $2^{9(u-1)/21}3^{(u-1)/21}$ which satisfies the conditions of Theorem 3.4 in Appendix A.6. For each n apply Theorem 3.4 to n copies of the HCURF. Then by repeated application of Theorems 3.2 and 3.3 we get the result. By Wilson's theorem the necessary conditions for the existence of a $\mathrm{PBD}(v, \{22, 43\}, 1)$, $v \equiv 1 \bmod 21$, are asymptotically sufficient [15]. Applying Theorem 3.1 gives the result. □

4.2 Product Constructions

Up to this point we have used closure results, we now give some other types of existence proofs. Since the existence of 4-GDDs has been completely solved finding just one HCURF with $u = 4$ produces spectacular results.

Lemma 4.9. *[10] Let g and u be positive integers. Then there exists a 4-GDD of type g^u if and only if the following conditions are satisfied:*

- $g \equiv 1, 5 \bmod 6$, $u \equiv 1, 4 \bmod 12$ *and* $u \geq 4$*;*
- $g \equiv 2, 4 \bmod 6$, $u \equiv 1 \bmod 3$, $u \geq 4$ *and* $(g, u) \neq (2, 4)$*;*
- $g \equiv 3 \bmod 6$, $u \equiv 0, 1 \bmod 4$ *and* $u \geq 4$*; and*
- $g \equiv 0 \bmod 6$, $u \geq 4$ *and* $(g, u) \neq (6, 4)$*;*

Lemma 4.10. *There exists an HCURF of type* 12^4 *with partition* $2^{12}3^4$.

Proof. We exhibit an HCURF_4 of type 3^4 with partition $2^3 3^1$ satisfying the conditions of Theorem 3.4. The set of point is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. The groups are $\{i\} \times \mathbb{Z}_3$ where $i \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Develop the following three partial parallel classes over $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$$\begin{array}{llll}
\{(0,1,2),(1,0,1),(1,1,0)\} & \{(0,1,1),(1,0,2)\} & \{(0,1,0),(1,1,2)\} & \{(1,0,0),(1,1,1)\} \\
\{(0,1,2),(1,0,1),(1,1,0)\} & \{(0,1,1),(1,1,1)\} & \{(0,1,0),(1,0,0)\} & \{(1,0,2),(1,1,2)\} \\
\{(0,1,2),(1,0,1),(1,1,0)\} & \{(0,1,1),(1,1,1)\} & \{(0,1,0),(1,0,0)\} & \{(1,0,2),(1,1,2)\}
\end{array}.$$

□

Theorem 4.11. *Let g and u be positive integers. Then there exists an HCURF of type* $(12g)^u$ *with partition* $2^{4g(u-1)}3^{4g(u-1)/3}$ *if and only if the following conditions are satisfied:*

- $g \equiv 1, 5 \bmod 6$, $u \equiv 1, 4 \bmod 12$ *and* $u \geq 4$*;*
- $g \equiv 2, 4 \bmod 6$, $u \equiv 1 \bmod 3$, $u \geq 4$ *and* $(g, u) \neq (2, 4)$*;*

- $g \equiv 3 \bmod 6$, $u \equiv 0, 1 \bmod 4$ *and* $u \geq 4$*; and*
- $g \equiv 0 \bmod 6$, $u \geq 4$ *and* $(g, u) \neq (6, 4)$*;*

We note that this gives asymptotic existence for HCURFs with $g = 12$ and $m = 9$.

Similarly, since the existence of 4-frames is mostly known, CURGDDs may be used to obtain CURFs.

Lemma 4.12. *[7] There exists a CURGDD of type* $(3n)^4$ *with partition* $2^{3n}3^{2n}$ *and* $r = 6n$ *resolution classes for all* $n \geq 1$*.*

Theorem 4.13. *If* $u \geq 5$, $g \equiv 0 \bmod 3$ *and* $g(u-1) \equiv 0 \bmod 4$*, there exists an HCURF of type* $(3gn)^u$ *with partition* $2^{3ng(u-1)/4}3^{ng(u-1)/2}$ *for every* $n > 0$*, with a small finite number of possible exceptions for each* g*.*

Proof. Apply Theorem 3.2 taking a 4-frame as the master and the CURGDDs from Lemma 4.12 as ingredients. The existence of 4-frames is known [12]; for a list of the possible exceptions we refer the reader to [3, 12] (see also [2]). □

Next we give a construction based on breaking up an RBIBD in a systematic way.

Theorem 4.14. *There exists an HCURF of type* $(6t+1)^{6t+3}$ *with partition* $2^{6t+1}3^{2t(6t+1)}$ *and* $m = 3t + 1$ *for all* $t \geq 0$*.*

Proof. We establish the existence of an HCURF_{6t+1} of type 1^{6t+3} and partition $2^1 3^{2t}$ for all $t \geq 0$ that satisfy the conditions of Theorem 3.4. Start with a RBIBD$(6t+3, 3, 1)$ (KTS$(6t+3)$). For each resolution class of this design we will produce $6t+3$ copies of it and modify them in a systematic way. For each block of the resolution class (there are $2t+1$ blocks) we will modify three of the copies of the resolution class by replacing this block by the three resolution classes of a CURD with partition $1^1 2^1$. Doing this for each resolution class gives us $6t+3$ resolution classes of the form $1^1 2^1 3^{2t}$ which cover each edge from the original resolution class of the KTS exactly $6t+1$ times. Each point appears as a block of size one the same number of times, independent of the choice of the point and so this is really an HCURF_{6t+1} of type 1^{6t+3} and partition $2^1 3^{2t}$. Application of Theorem 3.4 gives the result. □

4.3 Prime Power Constructions

We now present a number of constructions that use Theorem 3.4 to construct infinite families of CURFs.

Theorem 4.15. *If* q *is a prime power and there exists integers* d, k, m *such that* $d \leq (q-1-k)/2$ *and* mk *divides* $d - (mk - k + 1)$*, then there exists a* $\text{HCURF}_{2(d+k-1)/(mk)}$ *of type* $(mk)^q$ *with partition*

$$1^{mk(q-2d-k-1)}2^{dk+m\binom{k}{2}}k^m$$

satisfying the conditions of Theorem 3.4.

Proof. On q points we make a class with one block of size k and d disjoint pairs. label each of the points uniquely but arbitrarily from the set $\{1, 2, \ldots, k\}$. Label one of the pairs 0, one of the pairs j for each $j \in \{0, 1, \ldots, mk-1\} \setminus \{1, 2, \ldots, k-1\}$ and all of the remaining pairs from the set $\{0, 1, \ldots, mk-1\}$ each label exactly $(d-(mk-k+1))/mk$ times.

Develop this class through the group of affine functions on GF(q) (functions of the form $x \rightarrow ax+b$ where $a, b \in GF(q)$ and $a \neq 0$). Blow up each point by mk. Lift the block of size k to m blocks of size k. The point labeled j in the ith such block will be lifted to the new point $im+j$ where $0 \leq i < m$. If $\{(x, y), (x', y')\}$ is an edge contained in a block of size k then add the pair $\{(x, y'), (x', y)\}$. Now blow up each edge $\{x, y\}$ to a set of disjoint edges resolving groups x and y: $\{(x, l), (y, l+m)\}$ where $l \in \mathbb{Z}_{mk}$, m is the label on that edge and addition is mod mk. □

Applying Theorem 3.4 we get:

Corollary 4.16. *If q is a prime power and there exists integers d, k, m such that $d \leq (q-1-k)/2$ and mk divides $d-(mk-k+1)$, then there exists a HCURF of type $(2(d+k-1))^q$ with partition*

$$1^{2(d+k-1)(q-2d-k-1)} 2^{2(d+k-1)(dk+m\binom{k}{2})/mk} k^{2(d+k-1)/k}.$$

Using difference sets gives the following.

Theorem 4.17. *If $q = 2^n$ and there exists an (H)CURF of type 1^{q+1} with partition $\prod k^{p_k}$ and r resolution classes then there exists an (H)CURF of type*

$$\left(\frac{r(q^2-1)}{\gcd(r, q^2-1)}\left(\frac{q^2-q}{q^2-1}+\frac{1}{r}\right)\right)^{q^2+q+1}$$

with partition

$$2^{\left(\frac{r(q^2-1)}{\gcd(r,q^2-1)}\left(\frac{q^2-q}{q^2-1}+\frac{1}{r}\right)\right)q^2/2} \prod k^{\left(\frac{r(q^2-1)}{\gcd(r,q^2-1)}\left(\frac{q^2-q}{q^2-1}+\frac{1}{r}\right)\right)p_k}.$$

Proof. There exists a $(q^2+q+1, q+1, 1)$ difference set, let B be the base block. We note that B^c is a (q^2+q+1, q^2, q^2-q) difference set. Place $(q^2-1)/\gcd(r, q^2-1)$ copies of the (H)CURF on the points of B. Place $r/\gcd(r, q^2-1)$ copies of any 1-factorization on B^c. This gives $r(q^2-1)/\gcd(r, q^2-1)$ resolution classes. Develop these classes in $\mathbb{Z}_{q^2+q+1}$ to get a (H)CURF$_\lambda$ on q^2+q+1 points with partition $2^{q^2/2}\prod k^{p_k}$ which satisfies the conditions of Theorem 3.4, where

$$\lambda = \frac{r(q^2-1)}{\gcd(r, q^2-1)}\left(\frac{q^2-q}{q^2-1}+\frac{1}{r}\right).$$

The result follows from Theorem 3.4. □

Theorem 4.18. *If q is an odd prime power and there exists a CURD on $q+1$ points with partition $\prod k^{p_k}$ and r resolution classes, then there exists a HCURF of type*

$$\left(\frac{rq^2}{\gcd(r,q^2)}\left(\frac{q^2-q}{q^2}+\frac{1}{r}\right)\right)^{q^2+q+1}$$

with partition

$$2^{\left(\frac{rq^2}{\gcd(r,q^2)}\left(\frac{q^2-q}{q^2}+\frac{1}{r}\right)\right)(q^2-1)/2}\prod k^{\left(\frac{rq^2}{\gcd(r,q^2)}\left(\frac{q^2-q}{q^2}+\frac{1}{r}\right)\right)p_k}.$$

Proof. There exists a $(q^2+q+1, q+1, 1)$ difference set, let B be the base block. We note that B^c is a (q^2+q+1, q^2, q^2-q) difference set. Place $q^2/\gcd(r,q^2)$ copies of the CURD on the points of B. Place $r/\gcd(r,q^2)$ copies of any near 1-factorization on B^c. This gives $rq^2/\gcd(r,q^2)$ partial resolution classes. Develop these classes in $\mathbb{Z}_{q^2+q+1}$ to get a (H)CURF$_\lambda$ on q^2+q+1 points with partition $2^{q^2/2}\prod k^{p_k}$ which satisfies the conditions of Theorem 3.4, where

$$\lambda = \frac{rq^2}{\gcd(r,q^2)}\left(\frac{q-1}{q}+\frac{1}{r}\right).$$

The result follows from Theorem 3.4. □

By using a CURD$(q+1,(q+1)/2,1,(q-1)/3)$ (3-frames of type $2^{(q+1)/2}$) where $q \equiv 1 \bmod 6$ and is a prime power as ingredients in Theorem 4.18 we have

Corollary 4.19. *For all $q \equiv 1 \bmod 6$ a prime power there exists an HCURF of type $\left(\frac{q(q^2+2q-1)}{2}\right)^{q^2+q+1}$ with partition*

$$2^{q(q^2+2q-1)(q^2+1)/4}3^{q(q^2+2q-1)(q-1)/6}.$$

Using Theorem 3.2 with CURGDDs produced by Corollary 4.1 and Theorem 4.2 from [7] (with three or four groups) as ingredients and frames with block sizes three or four [2] as the master, we get the following theorems.

Corollary 4.20. *There exists an HCURF of type $(hq(q-m))^u$ with partition*

$$2^{hm(u-1)(q-m)}3^{h(u-1)(q-m)(q-2m)/3}$$

for all $q \geq 3$ a prime power, $0 \leq m \leq q/2$, $u \geq 4$, $h \equiv 0 \bmod 2$ and $h(u-1) \equiv 0 \bmod 3$.

Corollary 4.21. *When a 4-frame of type h^u exists, $q \geq 3$ a prime power, $0 \leq m \leq q/2$ and for all $g \equiv 0 \bmod 6$ then there exists an HCURF of type $(hgq(q-m))^u$ with partition $2^{hmg(u-1)(q-m)}3^{hg(u-1)(q-m)(q-2m)/3}$*

Corollary 4.22. *When a 4-frame of type h^u exists, then there exists an HCURF of type $(3h)^u$ and partition $2^{3h(u-1)/4}3^{h(u-1)/2}$.*

5 Conclusion

Theorem 3.4 is a frame formulation of Theorem 3.3 from [7]. It provides a construction that can produce CURFs of index 1 from CURFs with higher index. We believe that this kind of construction is extremely potent and although it arose in the study of CURDs we anticipate that it will be a powerful tool for the construction of many more types of designs.

In this paper we have mainly constructed homogeneous CURFs, though many of the constructions given apply to the general case. The existence of non-homogeneous CURFs with constant group size is surprising, as are the weak combinatorial constraints that sometimes force homogeneity as in the proof of Theorem 2.2. The consideration of non-homogeneous CURFs is an open problem. In particular, non-homogeneity may provide avenues for new future constructions, even those that produce homogeneous objects.

Frame constructions often gain in power by the consideration of multiple group sizes. Frames with multiple group sizes are forced to have multiple partitions, at least one for each group size, thus strictly speaking these are not class-uniform structures. However, such designs may be very useful for finding class-uniform objects. The existence of multiple partitions is reminiscent of the fact that resolvable designs with holes have two types of resolution classes. The consideration of multiple partitions in a class-uniform context remains an interesting open problem.

A Appendix

We present some of the HCURFs required for the results in Section 4.1. Generally, these are cyclic designs which satisfy the conditions of Theorem 3.4. In some cases the triples are common to all classes and we then give them separately at the top. For other cases the triples are given with the class.

A.1 HCURFs of type 4^{9t+1}

We present the family of CURF_4 of type 1^{9t+1} with partition $2^{3t}3^t$ for $t = 1$ to 5. The cases $t = 1$ to 100 may be found in [5].

A CURF_4 of type 1^{10} with partition $2^3 3^1$ (t = 1)

Triples $\{1, 2, 4\}$

8 :	$\{0, 5\}$	$\{6, 7\}$	$\{9, 3\}$
9 :	$\{0, 5\}$	$\{6, 8\}$	$\{3, 7\}$
9 :	$\{3, 7\}$	$\{5, 8\}$	$\{6, 0\}$

A CURF_4 of type 1^{19} with partition $2^6 3^2$ (t = 2)

Triples $\{0, 1, 3\}$ $\{2, 6, 11\}$

10 :	$\{9, 17\}$	$\{12, 13\}$	$\{4, 15\}$	$\{18, 5\}$	$\{7, 14\}$	$\{8, 16\}$
17 :	$\{9, 4\}$	$\{16, 7\}$	$\{5, 13\}$	$\{14, 10\}$	$\{12, 18\}$	$\{8, 15\}$
18 :	$\{8, 5\}$	$\{4, 16\}$	$\{7, 13\}$	$\{9, 15\}$	$\{10, 17\}$	$\{12, 14\}$

A CURF_4 of type 1^{28} with partition $2^9 3^3$ (t = 3)

Triples $\{1, 2, 4\}$ $\{3, 7, 12\}$ $\{5, 11, 18\}$

25 :	{0, 14}	{8, 20}	{23, 27}	{10, 19}	{15, 26}	{13, 24}	{22, 6}	{9, 17}	{16, 21}
20 :	{0, 14}	{23, 25}	{13, 21}	{17, 6}	{9, 19}	{16, 26}	{22, 10}	{24, 27}	{8, 15}
24 :	{14, 20}	{27, 9}	{0, 13}	{6, 16}	{8, 19}	{21, 22}	{26, 10}	{17, 25}	{15, 23}

A CURF_4 of type 1^{37} with partition $2^{12} 3^4$ (t = 4)

Triples $\{0, 1, 3\}$ $\{2, 6, 11\}$ $\{4, 10, 17\}$ $\{5, 13, 23\}$

9 :	{7, 28}	{20, 29}	{32, 24}	{16, 34}	{8, 25}	{15, 31}	{18, 30}	{35, 14}
	{21, 33}	{22, 36}	{26, 19}	{12, 27}				
32 :	{12, 16}	{18, 34}	{15, 27}	{25, 26}	{24, 35}	{29, 9}	{19, 8}	{7, 20}
	{14, 28}	{36, 22}	{21, 31}	{30, 33}				
35 :	{29, 7}	{15, 32}	{8, 19}	{18, 33}	{34, 12}	{9, 20}	{28, 16}	{21, 26}
	{24, 30}	{22, 36}	{25, 27}	{31, 14}				

A CURF_4 of type 1^{46} with partition $2^{15} 3^5$ (t = 5)

Triples $\{1, 2, 4\}$ $\{3, 7, 12\}$ $\{5, 11, 18\}$ $\{6, 14, 24\}$ $\{8, 19, 33\}$

37 :	{0, 23}	{20, 22}	{31, 36}	{34, 44}	{35, 9}	{21, 43}	{27, 39}	{13, 41}
	{28, 42}	{17, 25}	{45, 29}	{40, 16}	{26, 38}	{10, 32}	{15, 30}	
44 :	{0, 23}	{13, 40}	{26, 10}	{27, 30}	{21, 37}	{29, 36}	{20, 42}	{17, 32}
	{22, 39}	{16, 31}	{38, 9}	{35, 41}	{34, 15}	{28, 43}	{25, 45}	
21 :	{28, 32}	{36, 27}	{22, 35}	{20, 41}	{37, 38}	{23, 40}	{9, 26}	{45, 10}
	{25, 44}	{39, 13}	{42, 16}	{29, 17}	{30, 0}	{31, 43}	{15, 34}	

A.2 HCURFs of type 5^{18t+1}

A CURF_5 of type 1^{19} with partition $2^3 3^4$ (t = 1) satisfying λ blowup. Point set $\mathbb{Z}_{19}$. Develop the following 3 partial classes, mod 19

{0, 1, 4}	{2, 3, 6}	{7, 8, 11}	{9, 10, 13}	{5, 15}	{12, 17}	{14, 18}
{0, 2, 9}	{1, 3, 10}	{4, 6, 13}	{5, 7, 14}	{12, 15}	{11, 17}	{8, 16}
{0, 5, 11}	{1, 6, 12}	{2, 7, 13}	{3, 8, 14}	{4, 16}	{15, 17}	{9, 10}

A CURF_5 of type 1^{37} with partition $2^6 3^8$ (t = 2) satisfying λ blowup. Point set $\mathbb{Z}_{37}$. Develop the following 3 partial classes, mod 37

{0, 1, 3}	{4, 5, 7}	{8, 9, 11}	{12, 13, 15}	{2, 6, 28}	{10, 14, 36}	{27, 31, 16}
{29, 33, 18}	{17, 20}	{21, 25}	{34, 35}	{19, 24}	{23, 30}	{22, 32}

{0, 5, 14}	{1, 6, 15}	{2, 7, 16}	{3, 8, 17}	{4, 10, 29}	{21, 27, 9}	{24, 30, 12}
{25, 31, 13}	{33, 11}	{22, 28}	{18, 32}	{23, 34}	{35, 26}	{20, 36}

{0, 7, 17}	{1, 8, 18}	{2, 9, 19}	{3, 10, 20}	{4, 12, 25}	{5, 13, 26}	{6, 14, 27}
{15, 23, 36}	{34, 16}	{11, 28}	{29, 31}	{24, 32}	{22, 35}	{21, 33}

A CURF_5 of type 1^{55} with partition $2^9 3^{12}$ (t = 3) satisfying λ blowup. Point set $\mathbb{Z}_{55}$. Develop the following 3 partial classes, mod 55

{0, 1, 3}	{4, 5, 7}	{8, 9, 11}	{12, 13, 15}	{10, 14, 19}	{16, 20, 25}	{17, 21, 26}
{18, 22, 27}	{23, 29, 39}	{24, 30, 40}	{28, 34, 44}	{31, 37, 47}	{53, 50}	{52, 38}
{6, 32}	{2, 36}	{33, 41}	{45, 46}	{42, 54}	{49, 51}	{35, 48}

{0, 7, 32}	{1, 8, 33}	{2, 9, 34}	{3, 10, 35}	{11, 19, 40}	{12, 20, 41}	{13, 21, 42}
{14, 22, 43}	{4, 15, 46}	{5, 16, 47}	{6, 17, 48}	{38, 49, 25}	{37, 44}	{24, 29}
{39, 30}	{18, 53}	{51, 26}	{27, 50}	{52, 28}	{36, 54}	{45, 23}

{0, 12, 27}	{1, 13, 28}	{2, 14, 29}	{3, 15, 30}	{4, 18, 40}	{5, 19, 41}	{6, 20, 42}
{7, 21, 43}	{8, 25, 45}	{9, 26, 46}	{16, 33, 53}	{17, 34, 54}	{50, 23}	{22, 11}
{36, 52}	{44, 38}	{10, 48}	{32, 51}	{35, 31}	{47, 37}	{24, 39}

A.3 HCURFs of type 5^{24t+1}

A CURF_5 of type 1^{25} with partition $2^9 3^2$ (t = 1)

Triples $\{0,1,3\}$ $\{2,6,11\}$

24 :	{12, 22}	{13, 23}	{8, 19}	{15, 10}	{9, 21}	{14, 18}	{5, 17}	{20, 7}	{4, 16}
24 :	{12, 23}	{8, 19}	{13, 20}	{9, 17}	{14, 21}	{18, 10}	{5, 16}	{4, 22}	{7, 15}
13 :	{21, 7}	{12, 15}	{17, 18}	{4, 23}	{24, 5}	{9, 19}	{14, 20}	{10, 8}	{16, 22}
22 :	{9, 19}	{12, 20}	{4, 13}	{18, 5}	{23, 8}	{21, 15}	{7, 14}	{10, 17}	{16, 24}

A CURF_5 of type 1^{49} with partition $2^{18} 3^4$ (t = 2)

Triples $\{0,1,3\}$ $\{2,6,11\}$ $\{4,10,17\}$ $\{5,13,23\}$

46 :	{18, 25}	{47, 12}	{33, 7}	{29, 44}	{35, 38}	{40, 15}	{16, 30}	{9, 21}
	{8, 28}	{32, 43}	{24, 41}	{14, 37}	{20, 39}	{22, 42}	{26, 34}	{27, 31}
	{45, 19}	{36, 48}						
44 :	{40, 15}	{7, 24}	{31, 41}	{14, 29}	{28, 42}	{36, 9}	{45, 20}	{35, 8}
	{27, 46}	{26, 43}	{30, 47}	{25, 39}	{16, 32}	{22, 33}	{18, 38}	{21, 19}
	{37, 12}	{34, 48}						
7 :	{21, 45}	{40, 12}	{14, 35}	{26, 37}	{42, 15}	{48, 27}	{28, 34}	{8, 20}
	{38, 29}	{31, 47}	{19, 39}	{18, 36}	{43, 44}	{22, 33}	{41, 25}	{30, 46}
	{24, 9}	{16, 32}						
32 :	{20, 40}	{14, 35}	{24, 43}	{16, 37}	{18, 30}	{28, 47}	{26, 41}	{7, 39}
	{38, 12}	{48, 22}	{8, 44}	{46, 19}	{9, 31}	{29, 34}	{21, 33}	{45, 15}
	{27, 42}	{25, 36}						

A.4 HCURFs of type 6^{2t+1}

We give an HCURF of type 6^{2t+1} for $t = 3, 4, 5, 6$, corresponding to $u = 7, 9, 11, 13$, $t = 2$ ($u = 5$) is given in the text. This is a generating set for $u \equiv 1 \bmod 2$ [1], with the 20 possible exceptions:

$$u \in \{43, 51, 59, 71, 75, 83, 87, 95, 107, 111, 115, 119, 131, 135, 139, 167, 179, 183, 191, 195\}.$$

A CURF of type 6^7 with partition $2^9 3^6$ (t = 3) Point set $\mathbb{Z}_6 \times \mathbb{Z}_7$, groups $\mathbb{Z}_6 \times \{i\}, i \in \mathbb{Z}_7$. Develop the following 4 classes, each missing group $\mathbb{Z}_6 \times \{0\}$, mod$(-, 7)$

{(2, 5), (0, 4), (1, 6)}	{(1, 1), (5, 3), (4, 6)}	{(3, 3), (2, 2), (0, 6)}	{(2, 1), (1, 3), (3, 6)}
{(5, 6), (3, 2), (2, 3)}	{(4, 2), (0, 1), (5, 4)}	{(4, 5), (4, 4)}	{(4, 3), (3, 4)}
{(0, 2), (4, 1)}	{(2, 4), (2, 6)}	{(0, 3), (0, 5)}	{(1, 2), (1, 4)}
{(1, 5), (5, 2)}	{(5, 1), (5, 5)}	{(3, 5), (3, 1)}	

{(3, 3), (1, 4), (5, 5)}	{(4, 4), (0, 2), (1, 1)}	{(1, 5), (5, 3), (4, 2)}	{(1, 6), (2, 2), (0, 3)}
{(3, 1), (5, 6), (1, 3)}	{(5, 1), (0, 6), (2, 3)}	{(2, 4), (1, 2)}	{(0, 4), (0, 5)}
{(5, 4), (5, 2)}	{(4, 3), (4, 6)}	{(2, 6), (3, 2)}	{(2, 5), (2, 1)}
{(0, 1), (4, 5)}	{(3, 5), (3, 6)}	{(3, 4), (4, 1)}	

{(4, 5), (5, 3), (2, 6)}	{(2, 5), (4, 2), (1, 1)}	{(5, 2), (3, 6), (0, 3)}	{(4, 6), (3, 1), (1, 4)}
{(1, 3), (0, 2), (2, 4)}	{(0, 6), (5, 4), (3, 5)}	{(5, 1), (1, 2)}	{(3, 2), (3, 4)}
{(0, 4), (2, 2)}	{(1, 6), (0, 1)}	{(5, 5), (2, 3)}	{(2, 1), (3, 3)}
{(1, 5), (4, 4)}	{(0, 5), (5, 6)}	{(4, 1), (4, 3)}	

{(3, 2), (1, 1), (0, 4)}	{(3, 3), (0, 1), (4, 6)}	{(2, 1), (5, 2), (4, 3)}	{(2, 4), (3, 1), (4, 2)}
{(4, 5), (5, 1), (2, 2)}	{(3, 6), (4, 1), (0, 5)}	{(0, 3), (0, 6)}	{(0, 2), (5, 6)}
{(1, 5), (1, 4)}	{(4, 4), (2, 3)}	{(5, 3), (5, 4)}	{(1, 3), (3, 5)}
{(3, 4), (5, 5)}	{(1, 6), (1, 2)}	{(2, 5), (2, 6)}	

A CURF of type 6^9 with partition $2^{12} 3^8$ (t = 4) Point set $\mathbb{Z}_6 \times \mathbb{Z}_9$, groups $\mathbb{Z}_6 \times \{i\}, i \in \mathbb{Z}_9$. Develop the following 4 classes, each missing group $\mathbb{Z}_6 \times \{0\}$, mod$(-, 9)$

{(5, 7), (2, 1), (4, 2)}	{(2, 5), (4, 3), (0, 1)}	{(1, 4), (0, 3), (4, 6)}	{(5, 5), (1, 1), (2, 7)}
{(0, 8), (3, 3), (4, 4)}	{(1, 3), (0, 4), (3, 7)}	{(3, 1), (5, 3), (2, 2)}	{(2, 4), (5, 8), (1, 5)}
{(4, 1), (3, 5)}	{(0, 2), (0, 5)}	{(1, 8), (0, 6)}	{(5, 6), (5, 1)}
{(4, 7), (1, 2)}	{(0, 7), (2, 8)}	{(3, 6), (3, 8)}	{(5, 4), (5, 2)}
{(3, 2), (4, 8)}	{(4, 5), (2, 3)}	{(1, 6), (1, 7)}	{(3, 4), (2, 6)}

{(4, 8), (1, 4), (2, 5)} {(0, 2), (2, 4), (4, 1)} {(0, 4), (1, 2)} {(2, 8), (2, 7)} {(3, 5), (0, 8)}	{(0, 1), (1, 7), (5, 4)} {(5, 6), (4, 3), (0, 5)} {(3, 1), (0, 3)} {(5, 7), (1, 6)} {(4, 7), (4, 5)}	{(2, 1), (3, 2), (4, 6)} {(0, 7), (3, 8), (5, 5)} {(3, 4), (2, 2)} {(2, 3), (2, 6)} {(1, 3), (4, 4)}	{(5, 8), (1, 1), (0, 6)} {(4, 2), (5, 1), (3, 3)} {(5, 3), (5, 2)} {(3, 7), (3, 6)} {(1, 5), (1, 8)}
{(1, 1), (0, 7), (4, 8)} {(3, 4), (5, 5), (1, 6)} {(0, 6), (5, 3)} {(2, 1), (2, 5)} {(1, 5), (1, 3)}	{(2, 2), (1, 7), (3, 6)} {(5, 7), (4, 3), (0, 8)} {(5, 6), (3, 1)} {(1, 8), (1, 4)} {(4, 6), (4, 5)}	{(3, 7), (5, 1), (0, 5)} {(3, 3), (0, 4), (2, 7)} {(0, 3), (2, 8)} {(4, 2), (5, 4)} {(5, 2), (4, 4)}	{(4, 7), (2, 3), (5, 8)} {(1, 2), (4, 1), (3, 8)} {(3, 5), (3, 2)} {(0, 2), (0, 1)} {(2, 6), (2, 4)}
{(4, 7), (2, 8), (0, 1)} {(1, 5), (3, 6), (2, 3)} {(1, 1), (4, 4)} {(0, 4), (0, 6)} {(0, 8), (0, 3)}	{(3, 8), (4, 6), (5, 3)} {(2, 1), (0, 2), (1, 7)} {(1, 8), (5, 1)} {(5, 4), (5, 7)} {(4, 3), (4, 8)}	{(3, 3), (5, 2), (0, 7)} {(1, 2), (3, 4), (2, 7)} {(2, 6), (5, 5)} {(0, 5), (2, 2)} {(3, 2), (1, 6)}	{(1, 4), (3, 7), (4, 1)} {(2, 5), (5, 8), (1, 3)} {(3, 1), (3, 5)} {(5, 6), (2, 4)} {(4, 5), (4, 2)}

A CURF of type 6^{11} with partition $2^{15}3^{10}$ (t = 5) Point set $\mathbb{Z}_6 \times \mathbb{Z}_{11}$, groups $\mathbb{Z}_6 \times \{i\}, i \in \mathbb{Z}_{11}$. Develop the following 4 classes, each missing group $\mathbb{Z}_6 \times \{0\}$, $\mathrm{mod}(-, 11)$

{(1, 8), (0, 1), (4, 4)} {(3, 10), (2, 5), (2, 4)} {(3, 1), (3, 8), (1, 6)} {(2, 6), (4, 9)} {(3, 6), (4, 5)} {(0, 4), (4, 3)} {(4, 1), (1, 2)}	{(1, 1), (3, 2), (0, 3)} {(5, 4), (4, 7), (0, 9)} {(5, 8), (2, 1), (1, 10)} {(4, 10), (5, 5)} {(0, 10), (4, 6)} {(0, 6), (3, 9)}	{(1, 4), (1, 9), (2, 10)} {(5, 1), (2, 8), (3, 7)} {(5, 3), (0, 7)} {(1, 3), (5, 9)} {(5, 2), (0, 5)} {(2, 7), (5, 10)}	{(5, 7), (4, 8), (5, 6)} {(0, 8), (3, 5), (3, 4)} {(3, 3), (0, 2)} {(2, 2), (1, 5)} {(2, 3), (2, 9)} {(4, 2), (1, 7)}
{(1, 4), (1, 7), (0, 10)} {(5, 3), (0, 1), (0, 5)} {(2, 7), (0, 4), (2, 9)} {(1, 5), (5, 2)} {(3, 3), (3, 5)} {(5, 1), (2, 4)} {(3, 9), (1, 1)}	{(1, 2), (0, 7), (1, 6)} {(5, 8), (4, 6), (2, 10)} {(4, 1), (4, 4), (3, 10)} {(2, 3), (3, 7)} {(4, 10), (0, 8)} {(1, 8), (2, 6)}	{(5, 7), (3, 4), (2, 2)} {(4, 2), (4, 9), (3, 6)} {(4, 3), (0, 2)} {(0, 9), (2, 5)} {(3, 8), (2, 1)} {(1, 3), (4, 5)}	{(1, 9), (4, 7), (3, 1)} {(5, 9), (5, 6), (1, 10)} {(5, 5), (5, 10)} {(5, 4), (3, 2)} {(0, 6), (2, 8)} {(4, 8), (0, 3)}
{(4, 2), (1, 8), (1, 9)} {(4, 5), (4, 4), (2, 3)} {(2, 1), (5, 7), (1, 5)} {(3, 2), (1, 6)} {(5, 9), (3, 1)} {(3, 5), (2, 8)} {(3, 8), (5, 3)}	{(5, 4), (5, 2), (4, 1)} {(0, 1), (0, 7), (2, 5)} {(1, 4), (2, 9), (4, 7)} {(0, 9), (1, 10)} {(2, 6), (3, 4)} {(3, 3), (0, 10)}	{(4, 10), (3, 9), (3, 6)} {(2, 4), (1, 1), (5, 6)} {(4, 6), (0, 2)} {(0, 6), (5, 10)} {(5, 1), (5, 5)} {(4, 8), (2, 2)}	{(0, 3), (0, 4), (1, 7)} {(5, 8), (3, 10), (2, 7)} {(4, 9), (2, 10)} {(0, 8), (0, 5)} {(1, 3), (3, 7)} {(1, 2), (4, 3)}
{(4, 8), (5, 2), (2, 3)} {(3, 4), (3, 9), (1, 10)} {(3, 8), (0, 2), (5, 7)} {(4, 10), (4, 4)} {(1, 1), (2, 5)} {(1, 3), (5, 6)} {(1, 4), (1, 6)}	{(5, 8), (0, 9), (3, 7)} {(4, 9), (5, 5), (3, 1)} {(0, 8), (5, 9), (3, 2)} {(2, 4), (2, 7)} {(0, 1), (5, 4)} {(1, 7), (3, 5)}	{(4, 5), (4, 7), (0, 10)} {(0, 7), (2, 8), (1, 9)} {(5, 3), (1, 2)} {(4, 6), (5, 10)} {(5, 1), (1, 8)} {(2, 1), (0, 6)}	{(0, 3), (2, 2), (0, 5)} {(2, 6), (4, 3), (2, 10)} {(1, 5), (4, 2)} {(4, 1), (3, 3)} {(2, 9), (3, 10)} {(0, 4), (3, 6)}

A CURF of type 6^{13} with partition $2^{18}3^{12}$ (t = 6) Point set $\mathbb{Z}_6 \times \mathbb{Z}_{13}$, groups $\mathbb{Z}_6 \times \{i\}, i \in \mathbb{Z}_{13}$. Develop the following 4 classes, each missing group $\mathbb{Z}_6 \times \{0\}$, $\mathrm{mod}(-, 13)$

{(1, 5), (4, 4), (5, 10)} {(4, 11), (2, 8), (3, 12)} {(2, 10), (5, 4), (0, 6)} {(1, 4), (3, 11)} {(0, 4), (2, 7)} {(3, 2), (0, 9)} {(0, 3), (5, 9)} {(4, 10), (1, 2)}	{(3, 5), (3, 9), (2, 11)} {(5, 2), (4, 6), (0, 5)} {(3, 3), (1, 11), (3, 10)} {(1, 3), (3, 1)} {(2, 3), (5, 5)} {(4, 8), (5, 12)} {(5, 11), (3, 6)} {(4, 3), (0, 12)}	{(4, 7), (0, 1), (0, 2)} {(5, 7), (1, 12), (1, 1)} {(2, 2), (1, 9), (2, 1)} {(0, 10), (4, 5)} {(4, 2), (5, 3)} {(5, 8), (2, 9)} {(3, 8), (1, 7)}	{(4, 9), (3, 4), (3, 7)} {(1, 8), (4, 1), (4, 12)} {(2, 5), (0, 7), (5, 6)} {(0, 8), (1, 10)} {(2, 4), (1, 6)} {(0, 11), (5, 1)} {(2, 6), (2, 12)}
{(0, 4), (4, 3), (1, 11)} {(2, 7), (2, 11), (3, 12)} {(3, 10), (4, 5), (4, 11)} {(0, 8), (3, 3)} {(0, 10), (2, 2)} {(3, 1), (4, 7)} {(2, 12), (3, 7)} {(1, 6), (2, 9)}	{(4, 4), (4, 12), (5, 9)} {(2, 1), (2, 4), (1, 5)} {(2, 10), (2, 8), (5, 5)} {(2, 6), (4, 2)} {(2, 5), (0, 3)} {(0, 12), (4, 1)} {(1, 4), (5, 7)} {(5, 8), (4, 9)}	{(1, 2), (1, 7), (3, 11)} {(5, 4), (5, 11), (5, 2)} {(1, 9), (1, 10), (0, 6)} {(1, 1), (2, 3)} {(5, 6), (4, 8)} {(4, 10), (1, 3)} {(3, 6), (0, 7)}	{(3, 2), (5, 1), (3, 4)} {(3, 5), (5, 3), (0, 2)} {(0, 1), (3, 8), (5, 10)} {(3, 9), (1, 12)} {(0, 5), (5, 12)} {(0, 11), (1, 8)} {(4, 6), (0, 9)}
{(5, 5), (0, 1), (2, 2)} {(1, 1), (0, 5), (0, 8)} {(2, 7), (4, 8), (0, 12)} {(4, 2), (2, 9)} {(0, 10), (3, 12)} {(2, 6), (0, 9)} {(4, 6), (5, 8)} {(5, 6), (4, 3)}	{(2, 12), (3, 8), (4, 4)} {(5, 4), (4, 9), (2, 11)} {(3, 10), (1, 4), (0, 6)} {(1, 11), (4, 1)} {(5, 3), (0, 11)} {(2, 5), (3, 4)} {(3, 7), (3, 6)} {(3, 9), (1, 7)}	{(5, 1), (1, 12), (1, 2)} {(2, 1), (0, 7), (3, 3)} {(5, 7), (5, 2), (1, 6)} {(1, 10), (2, 4)} {(0, 2), (2, 8)} {(3, 2), (4, 12)} {(5, 12), (4, 5)}	{(4, 7), (1, 9), (2, 10)} {(1, 8), (4, 10), (0, 3)} {(3, 1), (5, 9), (5, 10)} {(3, 11), (1, 3)} {(5, 11), (3, 5)} {(4, 11), (2, 3)} {(0, 4), (1, 5)}

$\{(2,8),(1,11),(4,7)\}$
$\{(0,4),(0,8),(3,5)\}$
$\{(2,10),(2,5),(5,1)\}$
$\{(1,7),(4,4)\}$
$\{(1,9),(5,3)\}$
$\{(3,2),(4,6)\}$
$\{(0,3),(5,5)\}$
$\{(5,12),(1,2)\}$

$\{(0,12),(0,6),(0,1)\}$
$\{(2,4),(3,10),(0,5)\}$
$\{(3,1),(4,12),(3,9)\}$
$\{(3,6),(2,3)\}$
$\{(5,9),(2,11)\}$
$\{(2,9),(4,3)\}$
$\{(1,5),(3,8)\}$
$\{(0,7),(4,10)\}$

$\{(2,1),(4,5),(3,11)\}$
$\{(5,8),(3,4),(5,11)\}$
$\{(4,1),(4,11),(4,2)\}$
$\{(5,6),(0,11)\}$
$\{(1,8),(4,9)\}$
$\{(1,3),(2,7)\}$
$\{(1,4),(2,12)\}$

$\{(4,8),(2,6),(0,10)\}$
$\{(1,1),(0,2),(1,10)\}$
$\{(5,10),(1,6),(1,12)\}$
$\{(5,7),(2,2)\}$
$\{(3,3),(5,4)\}$
$\{(3,12),(5,2)\}$
$\{(3,7),(0,9)\}$

A.5 HCURFs of type 7^{8t+1}

For $t \in \{2,3,4,5,6,7,8,11,12,13,14,15,16\}$ we give a CURF$_7$ of type 1^{8t+1} with partition $2^t 3^{2t}$ satisfying λ blow-up (Theorem 3.4), $t = 1, u = 9$ is given by Theorem 4.14. This corresponds to the generating set $u \in \{9,17,25,33,41,49,57,65,89,97,105,113,121,129\}$ from [2], leaving 66 possible exceptions. In each case we give 4 partial classes over the point set $\mathbb{Z}_{8t+1}$, each missing one point. The design is obtained by developing these 4 partial classes mod $8t+1$.

$t = 2, u = 17$

$\{0,6,4\}$ $\{2,7,10\}$ $\{5,8,13\}$ $\{11,1,12\}$ $\{9,15\}$ $\{14,16\}$

$\{0,8,1\}$ $\{3,9,5\}$ $\{6,10,7\}$ $\{12,14,2\}$ $\{13,4\}$ $\{11,15\}$

$\{0,6,4\}$ $\{2,7,10\}$ $\{5,8,13\}$ $\{11,1,12\}$ $\{9,14\}$ $\{15,16\}$

$\{0,8,1\}$ $\{3,9,5\}$ $\{6,10,7\}$ $\{12,14,2\}$ $\{13,16\}$ $\{11,4\}$

$t = 3, u = 25$

$\{0,1,3\}$ $\{2,6,13\}$ $\{4,9,17\}$ $\{5,11,20\}$ $\{7,8,10\}$ $\{12,16,23\}$ $\{14,18\}$
$\{19,22\}$ $\{15,21\}$

$\{0,1,3\}$ $\{2,6,13\}$ $\{4,9,17\}$ $\{5,11,20\}$ $\{10,15,23\}$ $\{16,22,7\}$ $\{24,19\}$
$\{12,14\}$ $\{18,8\}$

$\{0,1,3\}$ $\{2,6,13\}$ $\{4,9,17\}$ $\{5,11,20\}$ $\{7,8,10\}$ $\{12,16,23\}$ $\{19,18\}$
$\{14,22\}$ $\{24,15\}$

$\{0,1,3\}$ $\{2,6,13\}$ $\{4,9,17\}$ $\{5,11,20\}$ $\{10,15,23\}$ $\{16,22,7\}$ $\{12,24\}$
$\{14,21\}$ $\{8,19\}$

$t = 4, u = 33$

$\{1,2,4\}$ $\{5,6,8\}$ $\{9,10,12\}$ $\{3,7,13\}$ $\{14,18,24\}$ $\{15,23,32\}$ $\{16,20,26\}$
$\{0,11,22\}$ $\{27,17\}$ $\{28,25\}$ $\{21,29\}$ $\{19,30\}$

$\{0,5,18\}$ $\{1,6,19\}$ $\{2,7,20\}$ $\{3,10,22\}$ $\{4,11,23\}$ $\{8,15,27\}$ $\{9,17,26\}$
$\{13,21,30\}$ $\{28,12\}$ $\{14,32\}$ $\{16,25\}$ $\{24,31\}$

$\{1,2,4\}$ $\{5,6,8\}$ $\{9,10,12\}$ $\{3,7,13\}$ $\{14,18,24\}$ $\{15,23,32\}$ $\{16,20,26\}$
$\{0,11,22\}$ $\{27,28\}$ $\{30,25\}$ $\{19,21\}$ $\{17,31\}$

$\{0,5,18\}$ $\{1,6,19\}$ $\{2,7,20\}$ $\{3,10,22\}$ $\{4,11,23\}$ $\{8,15,27\}$ $\{9,17,26\}$
$\{13,21,30\}$ $\{16,29\}$ $\{32,28\}$ $\{12,24\}$ $\{25,31\}$

$t = 5, u = 41$

$\{0,15,19\}$ $\{1,3,17\}$ $\{4,11,21\}$ $\{2,8,20\}$ $\{7,18,10\}$ $\{5,23,22\}$ $\{9,29,24\}$
$\{13,25,32\}$ $\{26,35,39\}$ $\{37,6,12\}$ $\{16,34\}$ $\{30,14\}$ $\{28,40\}$ $\{27,38\}$
$\{31,33\}$

$\{0,11,8\}$ $\{1,18,20\}$ $\{2,10,22\}$ $\{4,9,13\}$ $\{3,16,17\}$ $\{12,15,30\}$ $\{5,19,21\}$
$\{14,23,24\}$ $\{27,40,6\}$ $\{26,31,37\}$ $\{39,32\}$ $\{28,36\}$ $\{25,35\}$ $\{7,33\}$
$\{29,38\}$

$\{0,15,19\}$ $\{1,3,17\}$ $\{4,11,21\}$ $\{2,8,20\}$ $\{7,18,10\}$ $\{5,23,22\}$ $\{9,29,24\}$
$\{13,25,32\}$ $\{26,35,39\}$ $\{37,6,12\}$ $\{30,33\}$ $\{34,28\}$ $\{38,16\}$ $\{40,27\}$
$\{31,14\}$

$\{0,11,8\}$ $\{1,18,20\}$ $\{2,10,22\}$ $\{4,9,13\}$ $\{3,16,17\}$ $\{12,15,30\}$ $\{5,19,21\}$
$\{14,23,24\}$ $\{27,40,6\}$ $\{26,31,37\}$ $\{28,7\}$ $\{34,35\}$ $\{32,36\}$ $\{39,25\}$
$\{33,38\}$

$t = 6, u = 49$

$\{0,1,3\}$ $\{4,5,7\}$ $\{8,9,11\}$ $\{6,10,15\}$ $\{12,16,21\}$ $\{13,17,22\}$ $\{14,20,31\}$
$\{18,24,35\}$ $\{19,26,42\}$ $\{23,30,46\}$ $\{27,33,44\}$ $\{25,32,48\}$ $\{47,38\}$ $\{34,41\}$
$\{29,37\}$ $\{36,2\}$ $\{39,43\}$ $\{28,40\}$

$\{0,8,30\}$ $\{1,9,31\}$ $\{2,10,32\}$ $\{3,13,34\}$ $\{4,14,35\}$ $\{5,15,36\}$ $\{6,20,40\}$
$\{7,19,43\}$ $\{11,25,45\}$ $\{12,24,48\}$ $\{29,41,16\}$ $\{33,47,18\}$ $\{44,17\}$ $\{21,27\}$
$\{28,38\}$ $\{26,46\}$ $\{39,22\}$ $\{23,37\}$

{0, 1, 3}	{4, 5, 7}	{8, 9, 11}	{6, 10, 15}	{12, 16, 21}	{13, 17, 22}	{14, 20, 31}
{18, 24, 35}	{19, 26, 42}	{23, 30, 46}	{27, 33, 44}	{25, 32, 48}	{36, 39}	{34, 47}
{41, 43}	{37, 38}	{29, 45}	{2, 28}			

{0, 8, 30}	{1, 9, 31}	{2, 10, 32}	{3, 13, 34}	{4, 14, 35}	{5, 15, 36}	{6, 20, 40}
{7, 19, 43}	{11, 25, 45}	{12, 24, 48}	{29, 41, 16}	{33, 47, 18}	{44, 26}	{22, 27}
{46, 21}	{42, 23}	{28, 39}	{17, 38}			

$t = 7, u = 57$

{1, 2, 4}	{5, 6, 8}	{9, 10, 12}	{7, 11, 16}	{13, 17, 22}	{14, 18, 23}	{15, 21, 28}
{20, 26, 33}	{24, 32, 50}	{29, 35, 42}	{44, 56, 27}	{48, 3, 31}	{41, 55, 25}	{0, 19, 38}
{30, 49}	{39, 54}	{45, 53}	{40, 46}	{51, 52}	{36, 43}	{34, 37}

{0, 8, 26}	{1, 9, 27}	{2, 12, 35}	{3, 13, 36}	{4, 14, 37}	{6, 17, 38}	{7, 18, 39}
{10, 21, 42}	{11, 23, 51}	{5, 20, 40}	{15, 30, 50}	{32, 46, 16}	{41, 55, 25}	{19, 34, 54}
{31, 53}	{43, 52}	{33, 44}	{48, 28}	{45, 22}	{47, 29}	{24, 49}

{1, 2, 4}	{5, 6, 8}	{9, 10, 12}	{7, 11, 16}	{13, 17, 22}	{14, 18, 23}	{15, 21, 28}
{20, 26, 33}	{24, 32, 50}	{29, 35, 42}	{44, 56, 27}	{48, 3, 31}	{41, 55, 25}	{0, 19, 38}
{40, 45}	{37, 49}	{34, 51}	{36, 52}	{43, 47}	{53, 39}	{30, 54}

{0, 8, 26}	{1, 9, 27}	{2, 12, 35}	{3, 13, 36}	{4, 14, 37}	{6, 17, 38}	{7, 18, 39}
{10, 21, 42}	{11, 23, 51}	{5, 20, 40}	{15, 30, 50}	{32, 46, 16}	{41, 55, 25}	{19, 34, 54}
{53, 22}	{29, 56}	{45, 47}	{28, 49}	{43, 33}	{44, 31}	{24, 52}

$t = 8, u = 65$

{0, 31, 23}	{1, 25, 11}	{2, 20, 13}	{3, 35, 9}	{4, 34, 6}	{5, 24, 27}	{14, 41, 29}
{10, 26, 30}	{15, 47, 32}	{12, 33, 40}	{18, 21, 22}	{19, 42, 50}	{28, 55, 44}	{37, 56, 61}
{16, 45, 36}	{39, 64, 51}	{53, 63}	{7, 48}	{52, 59}	{8, 43}	{17, 49}
{62, 46}	{38, 60}	{54, 57}				

{0, 2, 28}	{1, 15, 33}	{3, 23, 11}	{4, 20, 26}	{6, 27, 31}	{7, 17, 8}	{5, 10, 34}
{9, 22, 39}	{14, 16, 29}	{13, 35, 44}	{18, 48, 21}	{25, 32, 46}	{24, 53, 43}	{36, 62, 37}
{41, 47, 58}	{38, 56, 61}	{49, 54}	{64, 30}	{45, 51}	{63, 50}	{40, 59}
{19, 57}	{12, 52}	{42, 60}				

{0, 31, 23}	{1, 25, 11}	{2, 20, 13}	{3, 35, 9}	{4, 34, 6}	{5, 24, 27}	{14, 41, 29}
{10, 26, 30}	{15, 47, 32}	{12, 33, 40}	{18, 21, 22}	{19, 42, 50}	{28, 55, 44}	{37, 56, 61}
{16, 45, 36}	{39, 64, 51}	{63, 52}	{62, 48}	{49, 57}	{38, 59}	{54, 58}
{43, 60}	{53, 8}	{46, 7}				

{0, 2, 28}	{1, 15, 33}	{3, 23, 11}	{4, 20, 26}	{6, 27, 31}	{7, 17, 8}	{5, 10, 34}
{9, 22, 39}	{14, 16, 29}	{13, 35, 44}	{18, 48, 21}	{25, 32, 46}	{24, 53, 43}	{36, 62, 37}
{41, 47, 58}	{38, 56, 61}	{55, 57}	{42, 19}	{50, 51}	{45, 60}	{49, 12}
{59, 30}	{54, 63}	{40, 52}				

$t = 11, u = 89$

{0, 26, 32}	{1, 19, 42}	{2, 38, 22}	{3, 14, 33}	{4, 39, 35}	{5, 12, 20}	{7, 34, 51}
{6, 45, 37}	{8, 50, 43}	{9, 18, 29}	{16, 40, 41}	{11, 55, 21}	{24, 46, 48}	{17, 31, 59}
{25, 64, 52}	{13, 53, 56}	{36, 72, 49}	{30, 58, 70}	{15, 44, 47}	{23, 61, 28}	{62, 10, 78}
{63, 82, 67}	{68, 84}	{80, 57}	{74, 79}	{71, 86}	{69, 77}	{76, 27}
{85, 66}	{65, 87}	{81, 60}	{83, 73}	{54, 88}		

{0, 1, 25}	{2, 30, 20}	{3, 44, 39}	{4, 13, 34}	{6, 28, 40}	{5, 19, 21}	{9, 29, 42}
{7, 24, 50}	{8, 48, 52}	{12, 41, 49}	{15, 47, 53}	{11, 26, 36}	{16, 54, 45}	{31, 57, 62}
{33, 63, 76}	{43, 46, 65}	{18, 55, 32}	{37, 64, 71}	{69, 70, 22}	{17, 56, 38}	{51, 84, 86}
{60, 77, 66}	{82, 35}	{74, 85}	{73, 23}	{75, 27}	{58, 88}	{79, 72}
{80, 83}	{61, 81}	{67, 14}	{68, 10}	{59, 87}		

{0, 26, 32}	{1, 19, 42}	{2, 38, 22}	{3, 14, 33}	{4, 39, 35}	{5, 12, 20}	{7, 34, 51}
{6, 45, 37}	{8, 50, 43}	{9, 18, 29}	{16, 40, 41}	{11, 55, 21}	{24, 46, 48}	{17, 31, 59}
{25, 64, 52}	{13, 53, 56}	{36, 72, 49}	{30, 58, 70}	{15, 44, 47}	{23, 61, 28}	{62, 10, 78}
{63, 82, 67}	{81, 68}	{76, 77}	{88, 71}	{83, 65}	{66, 80}	{73, 69}
{79, 27}	{57, 86}	{54, 87}	{60, 85}	{84, 75}		

{0, 1, 25}	{2, 30, 20}	{3, 44, 39}	{4, 13, 34}	{6, 28, 40}	{5, 19, 21}	{9, 29, 42}
{7, 24, 50}	{8, 48, 52}	{12, 41, 49}	{15, 47, 53}	{11, 26, 36}	{16, 54, 45}	{31, 57, 62}
{33, 63, 76}	{43, 46, 65}	{18, 55, 32}	{37, 64, 71}	{69, 70, 22}	{17, 56, 38}	{51, 84, 86}
{60, 77, 66}	{58, 82}	{72, 78}	{74, 23}	{59, 85}	{88, 61}	{80, 35}
{75, 87}	{79, 81}	{73, 27}	{67, 10}	{68, 14}		

$t = 12, u = 97$

{27, 28, 75}	{29, 30, 77}	{31, 32, 79}	{10, 25, 45}	{11, 26, 46}	{18, 33, 53}	{0, 8, 39}
{1, 9, 40}	{4, 12, 43}	{34, 50, 74}	{36, 52, 76}	{38, 54, 78}	{7, 16, 35}	{13, 22, 41}
{14, 23, 42}	{37, 48, 66}	{44, 55, 73}	{51, 62, 80}	{56, 69, 86}	{57, 70, 87}	{58, 71, 88}
{5, 15, 49}	{72, 82, 19}	{47, 59, 92}	{61, 64}	{65, 85}	{89, 17}	{90, 96}
{67, 95}	{2, 81}	{93, 3}	{60, 83}	{21, 63}	{24, 68}	{84, 94}
{91, 6}						

{3, 13, 47}	{5, 17, 50}	{6, 18, 51}	{7, 21, 53}	{8, 22, 54}	{9, 23, 55}	{72, 69, 31}
{73, 70, 32}	{74, 71, 33}	{80, 75, 38}	{81, 76, 39}	{82, 77, 40}	{85, 78, 42}	{86, 79, 43}
{91, 84, 48}	{87, 89, 15}	{88, 90, 16}	{92, 94, 20}	{95, 2, 24}	{0, 4, 26}	{56, 62, 83}
{19, 25, 46}	{30, 36, 57}	{37, 41, 63}	{61, 1}	{68, 44}	{67, 96}	{59, 14}
{35, 65}	{66, 34}	{11, 45}	{93, 10}	{12, 52}	{27, 58}	{60, 64}
{28, 49}						

{27, 28, 75}	{29, 30, 77}	{31, 32, 79}	{10, 25, 45}	{11, 26, 46}	{18, 33, 53}	{0, 8, 39}
{1, 9, 40}	{4, 12, 43}	{34, 50, 74}	{36, 52, 76}	{38, 54, 78}	{7, 16, 35}	{13, 22, 41}
{14, 23, 42}	{37, 48, 66}	{44, 55, 73}	{51, 62, 80}	{56, 69, 86}	{57, 70, 87}	{58, 71, 88}
{5, 15, 49}	{72, 82, 19}	{47, 59, 92}	{95, 93}	{20, 3}	{17, 60}	{91, 64}
{68, 6}	{94, 81}	{90, 2}	{67, 83}	{65, 84}	{63, 89}	{96, 21}
{85, 24}						

{3, 13, 47}	{5, 17, 50}	{6, 18, 51}	{7, 21, 53}	{8, 22, 54}	{9, 23, 55}	{72, 69, 31}
{73, 70, 32}	{74, 71, 33}	{80, 75, 38}	{81, 76, 39}	{82, 77, 40}	{85, 78, 42}	{86, 79, 43}
{91, 84, 48}	{87, 89, 15}	{88, 90, 16}	{92, 94, 20}	{95, 2, 24}	{0, 4, 26}	{56, 62, 83}
{19, 25, 46}	{30, 36, 57}	{37, 41, 63}	{68, 27}	{96, 45}	{10, 58}	{52, 60}
{35, 93}	{12, 1}	{61, 66}	{49, 11}	{34, 67}	{44, 59}	{28, 29}
{14, 64}						

$t = 13, u = 105$

{2, 18, 36}	{3, 19, 37}	{4, 20, 38}	{1, 15, 34}	{7, 21, 40}	{8, 22, 41}	{11, 23, 43}
{12, 24, 44}	{13, 25, 45}	{16, 26, 47}	{17, 27, 48}	{29, 39, 60}	{42, 50, 72}	{46, 54, 76}
{49, 57, 79}	{52, 58, 81}	{53, 59, 82}	{55, 61, 84}	{5, 9, 33}	{28, 32, 56}	{62, 66, 90}
{65, 67, 92}	{69, 71, 96}	{73, 75, 100}	{68, 77, 10}	{0, 35, 70}	{97, 31}	{83, 86}
{78, 99}	{30, 101}	{85, 104}	{102, 14}	{51, 91}	{64, 87}	{88, 95}
{80, 93}	{98, 103}	{63, 94}	{74, 6}			

{0, 9, 47}	{1, 10, 48}	{4, 11, 50}	{5, 12, 51}	{6, 13, 52}	{14, 19, 59}	{15, 20, 60}
{16, 21, 61}	{22, 25, 66}	{23, 26, 67}	{24, 27, 68}	{2, 17, 53}	{3, 18, 54}	{28, 43, 79}
{31, 44, 81}	{32, 45, 82}	{33, 46, 83}	{41, 58, 84}	{55, 72, 98}	{56, 73, 99}	{38, 49, 91}
{86, 97, 34}	{89, 100, 37}	{29, 30, 78}	{39, 40, 88}	{63, 64, 7}	{95, 75}	{102, 57}
{36, 85}	{42, 101}	{69, 93}	{62, 77}	{87, 96}	{92, 80}	{74, 104}
{65, 94}	{8, 70}	{76, 35}	{71, 103}			

{2, 18, 36}	{3, 19, 37}	{4, 20, 38}	{1, 15, 34}	{7, 21, 40}	{8, 22, 41}	{11, 23, 43}
{12, 24, 44}	{13, 25, 45}	{16, 26, 47}	{17, 27, 48}	{29, 39, 60}	{42, 50, 72}	{46, 54, 76}
{49, 57, 79}	{52, 58, 81}	{53, 59, 82}	{55, 61, 84}	{5, 9, 33}	{28, 32, 56}	{62, 66, 90}
{65, 67, 92}	{69, 71, 96}	{73, 75, 100}	{68, 77, 10}	{0, 35, 70}	{74, 99}	{6, 63}
{80, 30}	{89, 91}	{104, 93}	{98, 97}	{64, 102}	{103, 85}	{88, 94}
{87, 95}	{86, 14}	{51, 78}	{31, 83}			

{0, 9, 47}	{1, 10, 48}	{4, 11, 50}	{5, 12, 51}	{6, 13, 52}	{14, 19, 59}	{15, 20, 60}
{16, 21, 61}	{22, 25, 66}	{23, 26, 67}	{24, 27, 68}	{2, 17, 53}	{3, 18, 54}	{28, 43, 79}
{31, 44, 81}	{32, 45, 82}	{33, 46, 83}	{41, 58, 84}	{55, 72, 98}	{56, 73, 99}	{38, 49, 91}
{86, 97, 34}	{89, 100, 37}	{29, 30, 78}	{39, 40, 88}	{63, 64, 7}	{101, 57}	{104, 62}
{76, 80}	{65, 87}	{94, 36}	{75, 103}	{77, 8}	{74, 90}	{96, 42}
{95, 69}	{71, 85}	{92, 102}	{35, 70}			

$t = 14, u = 113$

{0, 55, 25}	{2, 56, 19}	{1, 10, 43}	{3, 14, 52}	{6, 57, 7}	{5, 49, 53}	{4, 11, 51}
{9, 45, 15}	{8, 31, 46}	{12, 38, 65}	{16, 32, 18}	{17, 72, 69}	{13, 59, 24}	{22, 73, 54}
{26, 60, 82}	{27, 66, 76}	{21, 39, 63}	{20, 40, 48}	{35, 78, 47}	{34, 84, 77}	{33, 62, 81}
{37, 88, 41}	{30, 85, 86}	{58, 90, 98}	{28, 44, 67}	{29, 83, 70}	{71, 107, 80}	{61, 89, 92}
{99, 36}	{104, 94}	{111, 42}	{93, 106}	{91, 23}	{112, 75}	{105, 64}
{79, 110}	{87, 101}	{109, 74}	{108, 68}	{96, 100}	{97, 50}	{95, 102}

{0, 52, 17}	{1, 45, 12}	{2, 47, 26}	{3, 28, 40}	{4, 19, 24}	{5, 55, 23}	{7, 16, 29}
{13, 50, 30}	{8, 34, 9}	{6, 60, 36}	{14, 59, 63}	{15, 56, 68}	{18, 64, 20}	{10, 58, 48}
{21, 37, 73}	{27, 61, 67}	{25, 81, 39}	{33, 66, 38}	{41, 70, 49}	{31, 57, 62}	{65, 88, 75}
{35, 80, 53}	{42, 83, 44}	{71, 11, 77}	{69, 91, 98}	{72, 106, 87}	{43, 46, 89}	{76, 111, 90}
{82, 22}	{84, 101}	{96, 108}	{110, 85}	{93, 95}	{100, 51}	{112, 78}
{103, 92}	{102, 54}	{86, 104}	{99, 105}	{107, 79}	{94, 32}	{74, 97}

{0, 55, 25}	{2, 56, 19}	{1, 10, 43}	{3, 14, 52}	{6, 57, 7}	{5, 49, 53}	{4, 11, 51}
{9, 45, 15}	{8, 31, 46}	{12, 38, 65}	{16, 32, 18}	{17, 72, 69}	{13, 59, 24}	{22, 73, 54}
{26, 60, 82}	{27, 66, 76}	{21, 39, 63}	{20, 40, 48}	{35, 78, 47}	{34, 84, 77}	{33, 62, 81}
{37, 88, 41}	{30, 85, 86}	{58, 90, 98}	{28, 44, 67}	{29, 83, 70}	{71, 107, 80}	{61, 89, 92}
{110, 74}	{108, 50}	{96, 64}	{101, 102}	{106, 87}	{93, 109}	{100, 105}
{95, 36}	{23, 94}	{99, 42}	{111, 68}	{79, 103}	{112, 91}	{104, 75}

{0, 52, 17}	{1, 45, 12}	{2, 47, 26}	{3, 28, 40}	{4, 19, 24}	{5, 55, 23}	{7, 16, 29}
{13, 50, 30}	{8, 34, 9}	{6, 60, 36}	{14, 59, 63}	{15, 56, 68}	{18, 64, 20}	{10, 58, 48}
{21, 37, 73}	{27, 61, 67}	{25, 81, 39}	{33, 66, 38}	{41, 70, 49}	{31, 57, 62}	{65, 88, 75}
{35, 80, 53}	{42, 83, 44}	{71, 11, 77}	{69, 91, 98}	{72, 106, 87}	{43, 46, 89}	{76, 111, 90}
{105, 78}	{104, 84}	{99, 32}	{103, 51}	{109, 79}	{102, 22}	{86, 94}
{112, 74}	{95, 110}	{92, 101}	{93, 54}	{108, 82}	{85, 107}	{97, 100}

$t = 15, u = 121$

{35, 36, 95}	{37, 38, 97}	{39, 40, 99}	{12, 31, 56}	{13, 32, 57}	{14, 33, 58}	{0, 10, 49}
{1, 11, 50}	{5, 15, 54}	{41, 61, 91}	{42, 62, 92}	{43, 63, 93}	{9, 20, 44}	{16, 27, 51}
{17, 28, 52}	{34, 47, 70}	{46, 59, 82}	{53, 66, 89}	{8, 23, 45}	{64, 79, 101}	{65, 80, 102}
{60, 77, 98}	{67, 84, 105}	{68, 85, 106}	{18, 30, 73}	{69, 81, 3}	{88, 100, 22}	{72, 86, 7}
{90, 104, 25}	{94, 108, 29}	{115, 103}	{96, 21}	{116, 78}	{109, 114}	{83, 117}
{4, 55}	{87, 113}	{75, 107}	{74, 71}	{6, 112}	{26, 76}	{119, 19}
{48, 111}	{24, 118}	{110, 2}				

{7, 23, 64}	{8, 24, 65}	{9, 25, 66}	{10, 28, 68}	{11, 29, 69}	{12, 30, 70}	{89, 86, 38}
{90, 87, 39}	{91, 88, 40}	{97, 92, 45}	{98, 93, 46}	{99, 94, 47}	{102, 95, 49}	{103, 96, 50}
{107, 100, 54}	{110, 101, 56}	{113, 104, 59}	{114, 105, 60}	{109, 111, 19}	{116, 118, 26}	{117, 119, 27}
{120, 3, 31}	{0, 4, 32}	{73, 79, 106}	{75, 81, 108}	{15, 21, 48}	{33, 41, 67}	{2, 6, 34}
{43, 51, 77}	{44, 52, 78}	{1, 58}	{37, 17}	{20, 63}	{72, 83}	{18, 42}
{14, 80}	{71, 55}	{53, 13}	{112, 16}	{5, 74}	{57, 61}	{82, 22}
{76, 35}	{36, 85}	{115, 62}				

$\{35, 36, 95\}$	$\{37, 38, 97\}$	$\{39, 40, 99\}$	$\{12, 31, 56\}$	$\{13, 32, 57\}$	$\{14, 33, 58\}$	$\{0, 10, 49\}$
$\{1, 11, 50\}$	$\{5, 15, 54\}$	$\{41, 61, 91\}$	$\{42, 62, 92\}$	$\{43, 63, 93\}$	$\{9, 20, 44\}$	$\{16, 27, 51\}$
$\{17, 28, 52\}$	$\{34, 47, 70\}$	$\{46, 59, 82\}$	$\{53, 66, 89\}$	$\{8, 23, 45\}$	$\{64, 79, 101\}$	$\{65, 80, 102\}$
$\{60, 77, 98\}$	$\{67, 84, 105\}$	$\{68, 85, 106\}$	$\{18, 30, 73\}$	$\{69, 81, 3\}$	$\{88, 100, 22\}$	$\{72, 86, 7\}$
$\{90, 104, 25\}$	$\{94, 108, 29\}$	$\{6, 4\}$	$\{116, 26\}$	$\{2, 78\}$	$\{103, 110\}$	$\{120, 87\}$
$\{83, 21\}$	$\{75, 117\}$	$\{74, 118\}$	$\{111, 76\}$	$\{114, 115\}$	$\{109, 119\}$	$\{19, 55\}$
$\{96, 48\}$	$\{107, 113\}$	$\{71, 24\}$				

$\{7, 23, 64\}$	$\{8, 24, 65\}$	$\{9, 25, 66\}$	$\{10, 28, 68\}$	$\{11, 29, 69\}$	$\{12, 30, 70\}$	$\{89, 86, 38\}$
$\{90, 87, 39\}$	$\{91, 88, 40\}$	$\{97, 92, 45\}$	$\{98, 93, 46\}$	$\{99, 94, 47\}$	$\{102, 95, 49\}$	$\{103, 96, 50\}$
$\{107, 100, 54\}$	$\{110, 101, 56\}$	$\{113, 104, 59\}$	$\{114, 105, 60\}$	$\{109, 111, 19\}$	$\{116, 118, 26\}$	$\{117, 119, 27\}$
$\{120, 3, 31\}$	$\{0, 4, 32\}$	$\{73, 79, 106\}$	$\{75, 81, 108\}$	$\{15, 21, 48\}$	$\{33, 41, 67\}$	$\{2, 6, 34\}$
$\{43, 51, 77\}$	$\{44, 52, 78\}$	$\{115, 76\}$	$\{42, 14\}$	$\{63, 71\}$	$\{61, 84\}$	$\{5, 22\}$
$\{83, 112\}$	$\{57, 35\}$	$\{80, 13\}$	$\{85, 55\}$	$\{62, 53\}$	$\{1, 20\}$	$\{82, 17\}$
$\{18, 36\}$	$\{58, 72\}$	$\{37, 74\}$				

$t = 16, u = 129$

$\{2, 22, 44\}$	$\{3, 23, 45\}$	$\{4, 24, 46\}$	$\{1, 19, 42\}$	$\{7, 25, 48\}$	$\{8, 26, 49\}$	$\{11, 27, 51\}$
$\{12, 28, 52\}$	$\{13, 29, 53\}$	$\{16, 30, 55\}$	$\{17, 31, 56\}$	$\{18, 32, 57\}$	$\{9, 21, 47\}$	$\{38, 50, 76\}$
$\{54, 66, 92\}$	$\{58, 68, 95\}$	$\{59, 69, 96\}$	$\{60, 70, 97\}$	$\{63, 71, 99\}$	$\{64, 72, 100\}$	$\{65, 73, 101\}$
$\{74, 80, 109\}$	$\{75, 81, 110\}$	$\{77, 83, 112\}$	$\{6, 10, 40\}$	$\{33, 37, 67\}$	$\{84, 88, 118\}$	$\{87, 89, 120\}$
$\{91, 93, 124\}$	$\{111, 113, 15\}$	$\{105, 116, 34\}$	$\{0, 43, 86\}$	$\{85, 94\}$	$\{103, 127\}$	$\{61, 106\}$
$\{117, 36\}$	$\{108, 119\}$	$\{39, 5\}$	$\{20, 78\}$	$\{126, 107\}$	$\{41, 98\}$	$\{115, 14\}$
$\{82, 123\}$	$\{90, 104\}$	$\{79, 114\}$	$\{125, 102\}$	$\{35, 62\}$	$\{121, 122\}$	

$\{0, 11, 58\}$	$\{1, 12, 59\}$	$\{4, 13, 61\}$	$\{5, 14, 62\}$	$\{6, 15, 63\}$	$\{9, 16, 65\}$	$\{10, 17, 66\}$
$\{18, 25, 74\}$	$\{2, 7, 57\}$	$\{21, 26, 76\}$	$\{22, 27, 77\}$	$\{28, 31, 82\}$	$\{29, 32, 83\}$	$\{30, 33, 84\}$
$\{23, 42, 86\}$	$\{24, 43, 87\}$	$\{34, 53, 97\}$	$\{19, 36, 81\}$	$\{37, 54, 99\}$	$\{38, 55, 100\}$	$\{35, 50, 96\}$
$\{41, 56, 102\}$	$\{45, 60, 106\}$	$\{48, 69, 101\}$	$\{51, 72, 104\}$	$\{52, 73, 105\}$	$\{67, 80, 3\}$	$\{103, 116, 39\}$
$\{108, 121, 44\}$	$\{89, 90, 20\}$	$\{109, 110, 40\}$	$\{118, 119, 49\}$	$\{91, 93\}$	$\{68, 124\}$	$\{88, 126\}$
$\{70, 117\}$	$\{123, 71\}$	$\{122, 125\}$	$\{112, 75\}$	$\{113, 92\}$	$\{47, 107\}$	$\{115, 85\}$
$\{95, 64\}$	$\{128, 79\}$	$\{120, 46\}$	$\{78, 111\}$	$\{98, 8\}$	$\{94, 114\}$	

$\{2, 22, 44\}$	$\{3, 23, 45\}$	$\{4, 24, 46\}$	$\{1, 19, 42\}$	$\{7, 25, 48\}$	$\{8, 26, 49\}$	$\{11, 27, 51\}$
$\{12, 28, 52\}$	$\{13, 29, 53\}$	$\{16, 30, 55\}$	$\{17, 31, 56\}$	$\{18, 32, 57\}$	$\{9, 21, 47\}$	$\{38, 50, 76\}$
$\{54, 66, 92\}$	$\{58, 68, 95\}$	$\{59, 69, 96\}$	$\{60, 70, 97\}$	$\{63, 71, 99\}$	$\{64, 72, 100\}$	$\{65, 73, 101\}$
$\{74, 80, 109\}$	$\{75, 81, 110\}$	$\{77, 83, 112\}$	$\{6, 10, 40\}$	$\{33, 37, 67\}$	$\{84, 88, 118\}$	$\{87, 89, 120\}$
$\{91, 93, 124\}$	$\{111, 113, 15\}$	$\{105, 116, 34\}$	$\{0, 43, 86\}$	$\{108, 82\}$	$\{114, 122\}$	$\{41, 102\}$
$\{35, 78\}$	$\{104, 121\}$	$\{125, 79\}$	$\{115, 119\}$	$\{106, 62\}$	$\{94, 123\}$	$\{128, 39\}$
$\{36, 103\}$	$\{90, 14\}$	$\{127, 5\}$	$\{98, 85\}$	$\{126, 61\}$	$\{107, 20\}$	

$\{0, 11, 58\}$	$\{1, 12, 59\}$	$\{4, 13, 61\}$	$\{5, 14, 62\}$	$\{6, 15, 63\}$	$\{9, 16, 65\}$	$\{10, 17, 66\}$
$\{18, 25, 74\}$	$\{2, 7, 57\}$	$\{21, 26, 76\}$	$\{22, 27, 77\}$	$\{28, 31, 82\}$	$\{29, 32, 83\}$	$\{30, 33, 84\}$
$\{23, 42, 86\}$	$\{24, 43, 87\}$	$\{34, 53, 97\}$	$\{19, 36, 81\}$	$\{37, 54, 99\}$	$\{38, 55, 100\}$	$\{35, 50, 96\}$
$\{41, 56, 102\}$	$\{45, 60, 106\}$	$\{48, 69, 101\}$	$\{51, 72, 104\}$	$\{52, 73, 105\}$	$\{67, 80, 3\}$	$\{103, 116, 39\}$
$\{108, 121, 44\}$	$\{89, 90, 20\}$	$\{109, 110, 40\}$	$\{118, 119, 49\}$	$\{92, 124\}$	$\{120, 126\}$	$\{46, 112\}$
$\{95, 117\}$	$\{70, 85\}$	$\{79, 91\}$	$\{78, 8\}$	$\{75, 93\}$	$\{122, 68\}$	$\{47, 98\}$
$\{113, 88\}$	$\{125, 115\}$	$\{71, 107\}$	$\{123, 128\}$	$\{127, 111\}$	$\{64, 114\}$	

A.6 HCURFs of type 8^{21t+1}

An HCURF$_8$ of type 1^{22} with partition $2^9 3^1$ (t=1)

12 :	$\{0,1,3\}$	$\{2,13\}$	$\{4,8\}$	$\{5,9\}$	$\{6,10\}$
	$\{7,11\}$	$\{14,18\}$	$\{15,19\}$	$\{16,20\}$	$\{17,21\}$
2 :	$\{0,1,3\}$	$\{4,9\}$	$\{5,10\}$	$\{6,11\}$	$\{7,8\}$
	$\{12,17\}$	$\{13,18\}$	$\{14,19\}$	$\{15,20\}$	$\{16,21\}$
5 :	$\{0,1,3\}$	$\{2,18\}$	$\{4,20\}$	$\{6,12\}$	$\{7,13\}$
	$\{8,14\}$	$\{9,15\}$	$\{10,16\}$	$\{11,17\}$	$\{19,21\}$
19 :	$\{0,1,3\}$	$\{2,13\}$	$\{4,12\}$	$\{5,20\}$	$\{6,21\}$
	$\{7,14\}$	$\{8,15\}$	$\{9,16\}$	$\{10,17\}$	$\{11,18\}$
4 :	$\{0,1,3\}$	$\{2,5\}$	$\{6,14\}$	$\{7,15\}$	$\{8,16\}$
	$\{9,17\}$	$\{10,18\}$	$\{11,19\}$	$\{12,20\}$	$\{13,21\}$
4 :	$\{0,1,3\}$	$\{2,13\}$	$\{5,14\}$	$\{6,15\}$	$\{7,16\}$
	$\{8,17\}$	$\{9,18\}$	$\{10,19\}$	$\{11,20\}$	$\{12,21\}$
12 :	$\{0,1,3\}$	$\{2,13\}$	$\{4,14\}$	$\{5,15\}$	$\{6,16\}$
	$\{7,17\}$	$\{8,18\}$	$\{9,19\}$	$\{10,20\}$	$\{11,21\}$

An $HCURF_8$ of type 1^{43} with partition $2^{18}3^2$ (t=2)

24 :	{0, 1, 3}	{2, 6, 11}	{29, 39}	{42, 25}	{38, 7}
	{37, 22}	{10, 28}	{16, 36}	{12, 31}	{19, 27}
	{20, 30}	{33, 9}	{13, 34}	{15, 26}	{40, 8}
	{32, 4}	{14, 35}	{18, 23}	{21, 41}	{5, 17}
33 :	{0, 1, 3}	{2, 6, 11}	{42, 24}	{41, 16}	{25, 38}
	{10, 26}	{23, 35}	{27, 5}	{37, 18}	{12, 20}
	{31, 7}	{34, 8}	{32, 9}	{14, 22}	{4, 21}
	{29, 40}	{13, 28}	{39, 15}	{17, 30}	{19, 36}
21 :	{0, 1, 3}	{2, 6, 11}	{15, 22}	{35, 37}	{23, 24}
	{28, 41}	{39, 4}	{9, 27}	{36, 5}	{18, 31}
	{20, 40}	{38, 14}	{7, 10}	{16, 33}	{17, 32}
	{30, 42}	{34, 12}	{19, 29}	{8, 25}	{26, 13}
31 :	{0, 1, 3}	{2, 6, 11}	{41, 7}	{15, 27}	{12, 28}
	{9, 29}	{24, 39}	{8, 21}	{36, 30}	{25, 35}
	{23, 37}	{14, 22}	{4, 16}	{20, 38}	{10, 17}
	{34, 5}	{42, 19}	{26, 40}	{32, 13}	{18, 33}
38 :	{0, 1, 3}	{2, 6, 11}	{12, 23}	{27, 37}	{13, 19}
	{10, 31}	{41, 14}	{25, 32}	{4, 20}	{21, 35}
	{42, 9}	{22, 16}	{17, 24}	{40, 5}	{30, 36}
	{28, 34}	{8, 29}	{15, 26}	{39, 7}	{33, 18}
9 :	{0, 1, 3}	{2, 6, 11}	{40, 8}	{19, 27}	{25, 39}
	{5, 15}	{32, 16}	{17, 24}	{7, 20}	{29, 33}
	{26, 10}	{41, 18}	{34, 12}	{13, 23}	{37, 31}
	{21, 35}	{30, 4}	{42, 28}	{38, 14}	{36, 22}
18 :	{0, 1, 3}	{2, 6, 11}	{8, 33}	{31, 9}	{17, 30}
	{15, 38}	{20, 35}	{4, 10}	{7, 13}	{29, 36}
	{12, 19}	{14, 26}	{21, 37}	{40, 22}	{27, 34}
	{32, 24}	{25, 41}	{28, 39}	{5, 23}	{42, 16}

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