Steiner Triple Systems Intersecting in Pairwise Disjoint Blocks

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Abstract

Two Steiner triple systems $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are said to intersect in m pairwise disjoint blocks if $|\mathcal{A} \cap \mathcal{B}| = m$ and all blocks in $\mathcal{A} \cap \mathcal{B}$ are pairwise disjoint. For each v, we completely determine the possible values of m such that there exist two Steiner triple systems of order v intersecting in m pairwise disjoint blocks.

1 Introduction

A *set system* is a pair $(X, \mathcal{A})$, where X is a finite set of *points*, and $\mathcal{A}$ is a set of subsets of X, called *blocks*. Let $\mathcal{K}$ be a set of positive integers. The set $\mathcal{K}$ is a *set of block sizes* for $(X, \mathcal{A})$ if $|A| \in \mathcal{K}$ for every $A \in \mathcal{A}$.

Let $(X, \mathcal{A})$ be a set system, and let $\mathcal{G} = \{G_1, \ldots, G_s\}$ be a partition of X into subsets, called *groups*. The triple $(X, \mathcal{G}, \mathcal{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group appears in exactly one block and $|A \cap G| \leq 1$ for all $A \in \mathcal{A}$ and $G \in \mathcal{G}$. We denote a GDD $(X, \mathcal{G}, \mathcal{A})$ by $\mathcal{K}$-GDD if $\mathcal{K}$ is a set of block sizes for $(X, \mathcal{A})$. The *group type*, or simply *type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $[|G| \mid G \in \mathcal{G}]$. When more convenient, we use the exponential notation to describe the type of a GDD: A GDD of type $g_1^{t_1} \cdots g_s^{t_s}$ is a GDD where there are exactly t_i groups of size g_i. If $\mathcal{G}$ is not specified in a GDD $(X, \mathcal{G}, \mathcal{A})$, it is taken that all groups are of size one: $\mathcal{G} = \{\{x\} \mid x \in X\}$. A $\{3\}$-GDD of type 1^v is a *Steiner triple system* of order v, and is denoted by STS(v).

Two GDDs $\mathcal{D}_1 = (X, \mathcal{G}_1, \mathcal{A}_1)$ and $\mathcal{D}_1 = (X, \mathcal{G}_2, \mathcal{A}_2)$ of the same type are said to *intersect in m blocks* if $|\mathcal{A}_1 \cap \mathcal{A}_2| = m$. If in addition, the blocks in $\mathcal{A}_1 \cap \mathcal{A}_2$ are pairwise disjoint (that is, for any $A, A' \in A_1 \cap A_2$, $A \neq A'$, we have $A \cap A' = \emptyset$), then $\mathcal{D}_1$ and $\mathcal{D}_2$

are said to *intersect in m pairwise disjoint blocks*. Define

$$\begin{aligned}\mathrm{Int}(T) &= \{m \mid \exists \text{ two } \{3\}\text{-GDDs of type } T \text{ intersecting in } m \text{ blocks}\}; \text{ and}\\ \mathrm{Int_d}(T) &= \{m \mid \exists \text{ two } \{3\}\text{-GDDs of type } T \text{ intersecting in } m \text{ pairwise disjoint blocks}\}.\end{aligned}$$

The *intersection problem* and *disjoint intersection problem* for $\{3\}$-GDDs of type T is to determine $\mathrm{Int}(T)$ and $\mathrm{Int_d}(T)$, respectively. A pair of GDDs is said to be *disjoint* if they intersect in no blocks.

The intersection problem for Steiner triple systems was completely solved by Lindner and Rosa [6]. The intersection problem for $\{3\}$-GDDs of type 2^t has also been solved by Hoffman and Lindner [5]. The case of $\{3\}$-GDDs of type g^3, which are equivalent to Latin squares of side g, has been settled by Fu [4]. Butler and Hoffman [2] finally put the intersection problem for $\{3\}$-GDDs of type g^t to rest with the following result.

Let $b(g^t) = g^2t(t-1)/6$ (the number of blocks in a $\{3\}$-GDD of type g^t), and denote by $\mathcal{I}(g^t) = \{0, 1 \ldots, b(g^t)\} \setminus \{b(g^t)-5, b(g^t)-3, b(g^t)-2, b(g^t)-1\}$.

Theorem 1.1 (Butler and Hoffman) *Let g and t be positive integers such that $t \geq 3$, $g^2\binom{t}{2} \equiv 0 \pmod 3$, and $g(t-1) \equiv 0 \pmod 2$. Then $\mathrm{Int}(g^t) = \mathcal{I}(g^t)$, except that*

(i) $\mathrm{Int}(1^9) = \mathcal{I}(1^9) \setminus \{5, 8\}$*;*

(ii) $\mathrm{Int}(2^4) = \mathcal{I}(2^4) \setminus \{1, 4\}$*;*

(iii) $\mathrm{Int}(3^3) = \mathcal{I}(3^3) \setminus \{1, 2, 5\}$*; and*

(iv) $\mathrm{Int}(4^3) = \mathcal{I}(4^3) \setminus \{5, 7, 10\}$.

While the intersection problem has received considerable attention, the disjoint intersection problem seems not to be well-studied. The purpose of this paper is to determine completely $\mathrm{Int_d}(1^v)$. This is the disjoint intersection problem for $\mathrm{STS}(v)$. Hence, we assume throughout this paper that $v \equiv 1, 3 \pmod 6$. Note that since $0, 1 \in \mathrm{Int}(1^v)$, we also have $0, 1 \in \mathrm{Int_d}(1^v)$.

Define

$$\mathcal{I}_\mathrm{d}(1^v) = \begin{cases}\{0, 1, \ldots, (v-1)/3\} & \text{if } v \equiv 1 \pmod 6,\\ \{0, 1, \ldots, v/3\} & \text{if } v \equiv 3 \pmod 6.\end{cases}$$

Since there can be at most $\lceil v/3 \rceil$ pairwise disjoint blocks in an $\mathrm{STS}(v)$, we have $\mathrm{Int_d}(1^v) \subseteq \mathcal{I}_\mathrm{d}(1^v)$. So in the remaining of this paper, we shall focus on showing membership of elements of $\mathcal{I}_\mathrm{d}(1^v)$ in $\mathrm{Int_d}(1^v)$, rather than the other way round.

2 Latin Squares Intersecting in a Transversal

A *Latin square* of side n is an $n \times n$ array with the property that every row and every column contains every element from $\{1, \ldots, n\}$ exactly once. A Latin square A of side n is *idempotent* if $A(i, i) = s_i$ for $1 \leq i \leq n$. A *transversal* in a Latin square of side n is a set of n cells, no two from the same row or from the same column, or contain the same entry. The *intersection* of two Latin squares A and B is the set of cells (i,j) such that $A(i,j) = B(i,j)$. In this section, we show that for $n \neq 2,3,6$, there exists a pair of Latin squares of side n intersecting in a transversal.

Two Latin squares A and B of side n are said to be *orthogonal* if the n^2 ordered pairs $(A(i,j), B(i,j))$ for $1 \leq i,j \leq n$ are all distinct. The following result is well-known (see, for example, [1]).

Theorem 2.1 *There exists a pair of orthogonal idempotent Latin squares of side n for all $n \neq 2,3,6$.*

Theorem 2.2 *Let $n \neq 2,3,6$. Then there exists a pair of Latin squares of side n intersecting in a transversal.*

Proof. Let A and B be a pair of orthogonal idempotent Latin squares of side n. We claim that A and B intersect in a transversal. Obviously, the transversal formed by the main diagonal of either A or B is in the intersection. It remains to show that for $1 \leq i,j \leq n$, $i \neq j$, we have $A(i,j) \neq B(i,j)$. Suppose on the contrary that $A(i,j) = B(i,j)$. Then we have $(A(i,j), B(i,j)) = (k,k) = (A(k,k), B(k,k))$ for some k. This contradicts the fact that A and B are orthogonal. $\Box$

A $\{3\}$-GDD of type g^3 can be obtained from a Latin square A of side g as follows. Let $R = \{r_1, \ldots, r_g\}$ and $C = \{c_1, \ldots, c_g\}$ be the set of row and column indices of A, respectively. Let $S = \{1, \ldots, n\}$ be the set of entries of A. Define

$$\begin{aligned} X &= R \cup C \cup S; \\ \mathcal{G} &= \{R, C, S\}; \\ \mathcal{A} &= \{\{r_i, c_j, A(i,j)\} \mid 1 \leq i,j \leq g\}. \end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type g^3. From this construction, it is easy to see that Theorem 2.2 implies the following.

Corollary 2.1 *For $g \neq 2,3,6$, we have $g \in \mathrm{Int}_{\mathrm{d}}(g^3)$.*

3 Wilson-Type Constructions

Wilson [7] established the following fundamental construction for GDDs which has significant impact on design theory.

Wilson's Fundamental Construction	
Input:	(master) GDD $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$;
	weight function $\omega : X \to \mathbf{Z}_{\geq 0}$;
	(ingredient) $\mathcal{K}$-GDD $\mathcal{D}_A = (X_A, \mathcal{G}_A, \mathcal{B}_A)$ of type $[\omega(a) \mid a \in A]$,
	for each block $A \in \mathcal{A}$, where
	$X_A = \cup_{a\in A}\{\{a\} \times \{1, \ldots, \omega(a)\}\}$ and
	$\mathcal{G}_A = \{\{a\} \times \{1 \ldots, \omega(a)\} \mid a \in A\}$.
Output:	$\mathcal{K}$-GDD $\mathcal{D}^* = (X^*, \mathcal{G}^*, \mathcal{A}^*)$ of type $[\sum_{x\in G} \omega(x) \mid G \in \mathcal{G}]$, where
	$X^* = \cup_{x\in X}(\{x\} \times \{1, \ldots, \omega(x)\})$,
	$G^* = \{\cup_{x\in G}(\{x\} \times \{1, \ldots, \omega(x)\}) \mid G \in \mathcal{G}\}$, and
	$\mathcal{A}^* = \cup_{A\in\mathcal{A}}\mathcal{B}_A$.
Notation:	$\mathcal{D}^* = \mathrm{WFC}(\mathcal{D}, \omega, \{D_A \mid A \in \mathcal{A}\})$.

We use Wilson's fundamental construction to produce pairs of disjoint GDDs.

Theorem 3.1 *Let $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$ be a GDD. Let $\omega : X \to \mathbf{Z}_{\geq 0}$ be a weight function. Suppose that for each block $A \in \mathcal{A}$, there exist a pair of disjoint $\{k\}$-GDDs of type $[\omega(a) \mid a \in A]$. Then there exists a pair of disjoint $\{k\}$-GDDs of type $[\sum_{x\in G} \omega(x) \mid G \in \mathcal{G}]$.*

Proof. For each block $A \in \mathcal{A}$, let $\mathcal{D}_A^1 = (X_A, \mathcal{G}_A, \mathcal{B}_A^1)$ and $\mathcal{D}_A^2 = (X_A, \mathcal{G}_A, \mathcal{B}_A^2)$ be a pair of disjoint $\{k\}$-GDDs of type $[\omega(a) \mid a \in A]$. Let $\mathcal{D}_i^* = \mathrm{WFC}(\mathcal{D}, \omega, \{\mathcal{D}_A^i \mid A \in \mathcal{A}\})$, $i = 1, 2$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\{k\}$-GDDs of type $[\sum_{x\in G} \omega(x) \mid G \in \mathcal{G}]$.

We claim that $\mathcal{D}_i^* = (X^*, \mathcal{G}^*, \mathcal{A}_i^*)$, $i = 1, 2$, is a pair of disjoint GDDs. To prove this, we have to establish the following:

$$\mathcal{B}_A^1 \cap \mathcal{B}_A^2 = \emptyset, \text{ for all } A \in \mathcal{A}\text{; and} \tag{1}$$

$$\mathcal{B}_A^1 \cap \mathcal{B}_B^2 = \emptyset, \text{ for all } A, B \in \mathcal{A},\ A \neq B. \tag{2}$$

(1) follows trivially from the fact that $(X_A, \mathcal{G}_A, \mathcal{B}_A^1)$ and $(X_A, \mathcal{G}_A, \mathcal{B}_A^2)$ are disjoint, for all $A \in \mathcal{A}$. To see that (2) holds, note that for $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, blocks in $\mathcal{B}_A^1$ have the form $\{y_1, \ldots, y_k\}$, where $y_i = (a_i, w)$ for some $w \in \{1, \ldots, \omega(a_i)\}$, and blocks in $\mathcal{B}_B^2$ have the form $\{z_1, \ldots, z_k\}$, where $z_i = (b_i, w)$ for some $w \in \{1, \ldots, \omega(z_i)\}$. Hence $y_i = z_j$ is possible only if $a_i = b_j$. It follows that $\{y_1, \ldots, y_k\} = \{z_1, \ldots, z_k\}$ only if $A = B$, establishing (2). $\square$

Theorem 3.2 (Colbourn, Hoffman, and Rees) *Let g, t, and u be positive integers. There exists a $\{3\}$-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(i) $t \geq 3$, *or* $t = 2$ *and* $u = g$;

(ii) $u \leq g(t-1)$;

(iii) $g(t-1) + u \equiv 0 \pmod 2$;

(iv) $gt \equiv 0 \pmod 2$;

(v) $g^2\binom{t}{2} + gtu \equiv 0 \pmod 3$.

Theorem 3.1 together with the existence of some $\{3\}$-GDDs provided by the above result of Colbourn, Hoffman and Rees [3] yields the following.

Corollary 3.1 *For each type listed in the table below, there exist a pair of disjoint $\{3\}$-GDDs.*

Type	*Type of Master $\{3\}$-GDD*	*Weight Function*
$3^4 9^1$	$1^4 3^1$	$\omega(\cdot) = 3$
$3^6 15^1$	$1^6 5^1$	$\omega(\cdot) = 3$
$9^t 15^1$, $t \equiv 0 \pmod 2$, $t \geq 4$	$3^t 5^1$	$\omega(\cdot) = 3$
$9^t 21^1$, $t \equiv 0 \pmod 2$, $t \geq 4$	$3^t 7^1$	$\omega(\cdot) = 3$
$9^t 27^1$, $t \equiv 0 \pmod 2$, $t \geq 4$	$3^t 9^1$	$\omega(\cdot) = 3$
$(6s)^t(6r)^1$, $t \geq 3$, $r \leq s(t-1)$, $s^2\binom{t}{2} + str \equiv 0 \pmod 3$	$(2s)^t(2r)^1$	$\omega(\cdot) = 3$

Proof. For each desired pair of disjoint $\{3\}$-GDDs, apply Theorem 3.1 with the listed master GDD, weight function, and use as ingredient GDDs a pair of disjoint $\{3\}$-GDDs of type 3^3, which exists by Theorem 1.1. The master GDDs all exist by Theorem 3.2. $\square$

Theorem 3.3 (Filling in Groups) *Suppose there exists a pair of disjoint $\mathcal{K}$-GDDs of type $[g_1, \ldots, g_s]$. Furthermore, for each g_i, $1 \leq i \leq s$, there exists a pair of $\mathcal{L}$-GDDs of type 1^{g_i} intersecting in m_i pairwise disjoint blocks. Then there exists a pair of $(\mathcal{K} \cup \mathcal{L})$-GDDs of type $1^{\sum_{i=1}^{s} g_i}$ intersecting in $\sum_{i=1}^{s} m_i$ pairwise disjoint blocks.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be a pair of disjoint $\mathcal{K}$-GDDs of type $[g_1, \ldots, g_s]$, where $\mathcal{G} = \{G_1, \ldots, G_s\}$ and $|G_i| = g_i$, $1 \leq i \leq s$. For each g_i, $1 \leq i \leq s$, let $(G_i, \mathcal{A}_i)$ and $(G_i, \mathcal{B}_i)$ be a pair of $\mathcal{L}$-GDDs intersecting in m_i pairwise disjoint blocks. Define

$$\begin{aligned} \mathcal{D}_1 &= (X, \mathcal{A} \cup (\cup_{i=1}^{s} \mathcal{A}_i)); \text{ and} \\ \mathcal{D}_2 &= (X, \mathcal{B} \cup (\cup_{i=1}^{s} \mathcal{B}_i)). \end{aligned}$$

$\mathcal{D}_1$ and $\mathcal{D}_2$ are clearly $(\mathcal{K} \cup \mathcal{L})$-GDDs of type $1^{\sum_{i=1}^{s} g_i}$. We claim that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect in $\sum_{i=1}^{s} m_i$ pairwise disjoint blocks. To see this, note that

(i) $\mathcal{A} \cap \mathcal{B} = \emptyset$, since $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ are disjoint GDDs;

(ii) for $1 \leq i \leq s$, we have $\mathcal{A} \cap \mathcal{B}_i = \emptyset$ and $\mathcal{B} \cap \mathcal{A}_i = \emptyset$, since any block in $\mathcal{A}$ contains at most one element from each group; and

(iii) for $1 \leq i, j \leq s$, $i \neq j$, we have $\mathcal{A}_i \cap \mathcal{B}_j = \emptyset$, since groups are pairwise disjoint.

Hence, the blocks that $\mathcal{D}_1$ and $\mathcal{D}_2$ have in common are precisely the blocks in $\cup_{i=1}^{s}(\mathcal{A}_i \cap \mathcal{B}_i)$. It follows that $\mathcal{D}_1$ and $\mathcal{D}_2$ are the desired pair of GDDs intersecting in $\sum_{i=1}^{s} m_i$ pairwise disjoint blocks. $\square$

For $A, B \subseteq \mathbf{Z}$, we use the notation $A+B$ to denote the set $\{a+b \mid a \in A \text{ and } b \in B\}$. For $A_i \subseteq \mathbf{Z}$, $1 \leq i \leq n$, we use the notation $\sum_{i=1}^{n} A_i$ to denote the set $\{a_1 + \cdots + a_n \mid a_i \in A_i, \text{ for } 1 \leq i \leq n\}$.

Corollary 3.2 *If there exists a pair of disjoint $\{3\}$-GDDs of type $g_1^{t_1} \cdots g_s^{t_s}$, then*

$$\mathrm{Int_d}(1^{\sum_{i=1}^{s} g_i t_i}) \supseteq \sum_{i=1}^{s} \sum_{j=1}^{t_i} \mathrm{Int_d}(1^{g_i}).$$

Proof. Apply Theorem 3.3 with GDDs intersecting in the appropriate number of pairwise disjoint blocks. $\square$

The above results are sufficient to settle the case $v \equiv 3 \pmod 6$.

4 Product Constructions

We use the following Singular Direct Product construction.

	Singular Direct Product Construction
Input:	positive integers u, w, t;
	for $i = 1, \ldots, t$, (master) $\mathcal{K}$-GDDs $\mathcal{D}_i = (X_i \cup U, \mathcal{G}_i, \mathcal{A}_i)$ of type $u^1 1^{w_i}$, where U is the group of size u in each GDD;
	(ingredient) $\mathcal{K}$-GDD $\mathcal{E} = (\cup_{i=1}^{t} X_i, \{X_i \mid 1 \leq i \leq t\}, \mathcal{B})$ of type $[w_i \mid 1 \leq i \leq t]$;
	(fill-in) $\mathcal{K}$-GDD $\mathcal{F} = (G, \mathcal{C})$ of type 1^u.
Output:	$\mathcal{K}$-GDD $\mathcal{D}^* = (X^*, \mathcal{A}^*)$ of type $1^{u+\sum_{i=1}^{t} w_i}$, where
	$X^* = U \cup (\cup_{i=1}^{t} X_i)$, and
	$A^* = (\cup_{i=1}^{t} \mathcal{A}_i) \cup \mathcal{B} \cup \mathcal{C}$.
Notation:	$\mathcal{D}^* = \mathrm{SDP}(\{D_i \mid 1 \leq i \leq t\}, \mathcal{E}, \mathcal{F})$.

Lemma 4.1 *Let $v \equiv 3 \pmod 6$ and $k \equiv 0, 1 \pmod 3$, $k \geq 3$. Then*

$$\mathrm{Int_d}(1^{k(v-1)+1}) \supseteq \mathrm{Int_d}(1^v) + \sum_{i=1}^{k-1} (\mathrm{Int_d}(1^v) \setminus \{v/3\}).$$

Proof. For $1 \leq i \leq k$, let $\mathcal{D}_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}_i)$ and $\mathcal{D}'_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}'_i)$ be a pair of $\{3\}$-GDDs of type 1^v intersecting in m_i pairwise disjoint blocks, where $m_1 \in \mathrm{Int_d}(1^v)$ and $m_i \in \mathrm{Int_d}(1^v) \setminus \{v/3\}$ for $2 \leq i \leq k$. We may assume that the point ∞ is not contained in any blocks in the intersection of $\mathcal{D}_i$ and $\mathcal{D}'_i$, $2 \leq i \leq k$.

Let $\mathcal{E} = (\cup_{i=1}^k X_i, \{X_i \mid 1 \le i \le k\}, \mathcal{B})$ and $\mathcal{E}' = (\cup_{i=1}^k X_i, \{X_i \mid 1 \le i \le k\}, \mathcal{B}')$ be a pair of disjoint $\{3\}$-GDDs of type $(v-1)^k$, which exists by Theorem 1.1. Let $\mathcal{F} = (\{\infty\}, \emptyset)$. Now apply the Singular Direct Product construction to obtain $\mathcal{D}_1^* = \mathrm{SDP}(\{\mathcal{D}_1, \ldots, D_k\}, \mathcal{E}, \mathcal{F})$ and $\mathcal{D}_2^* = \mathrm{SDP}(\{\mathcal{D}'_1, \ldots, \mathcal{D}'_k\}, \mathcal{E}', \mathcal{F})$. It is easy to see that the intersection of $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ is $\cup_{i=1}^k (A_i \cap A'_i)$, which contains $\sum_{i=1}^k m_i$ pairwise disjoint blocks. $\square$

Lemma 4.2 *Let s and t be positive integers, $t \ge 3$. Then*

$$\mathrm{Int_d}(1^{6st+1}) \supseteq \sum_{i=1}^t \mathrm{Int_d}(1^{6s+1}).$$

Proof. For $1 \le i \le t$, let $\mathcal{D}_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}_i)$ and $\mathcal{D}'_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}'_i)$ be a pair of $\{3\}$-GDDs of type 1^{6s+1} intersecting in m_i pairwise disjoint blocks, where $m_i \in \mathrm{Int_d}(1^{6s+1})$. We may assume that the point ∞ is not contained in any of the intersections.

Let $\mathcal{E} = (\cup_{i=1}^t X_i, \{X_i \mid 1 \le i \le t\}, \mathcal{B})$ and $\mathcal{E}' = (\cup_{i=1}^t X_i, \{X_i \mid 1 \le i \le t\}, \mathcal{B}')$ be a pair of disjoint $\{3\}$-GDD of type $(6s)^t$, which exists by Theorem 1.1. Let $\mathcal{F} = (\{\infty\}, \emptyset)$. Now apply the Singular Direct Product construction to obtain two $\{3\}$-GDDs of type 1^{6st+1} $\mathcal{D}_1^* = \mathrm{SDP}(\{\mathcal{D}_i \mid 1 \le i \le t\}, \mathcal{E}, \mathcal{F})$ and $\mathcal{D}_2^* = \mathrm{SDP}(\{\mathcal{D}'_i \mid 1 \le i \le t\}, \mathcal{E}', \mathcal{F})$. It is easy to see that the intersection of $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ is $\cup_{i=1}^t (\mathcal{A}_i \cap \mathcal{A}'_i)$, which contains $\sum_{i=1}^t m_i$ pairwise disjoint blocks. $\square$

Lemma 4.3 *Let r, s, t be positive integers such that $t \ge 3$, $r \le s(t-1)$, and $s^2\binom{t}{2} + str \equiv 0 \pmod 3$. Then*

$$\mathrm{Int_d}(1^{6st+6r+1}) \supseteq \mathrm{Int_d}(1^{6r+1}) + \sum_{i=1}^t \mathrm{Int_d}(1^{6s+1}).$$

Proof. For $1 \le i \le t$, let $\mathcal{D}_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}_i)$ and $\mathcal{D}'_i = (X_i \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}'_i)$ be a pair of $\{3\}$-GDDs of type 1^{6s+1} intersecting in m_i pairwise disjoint blocks, where $m_i \in \mathrm{Int_d}(1^{6s+1})$. We may assume that the point ∞ is not contained in any of the intersections. Let $\mathcal{D}_{t+1} = (X_{t+1} \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}_{t+1})$ and $\mathcal{D}'_{t+1} = (X_{t+1} \cup \{\infty\}, \mathcal{G}_i, \mathcal{A}'_{t+1})$ be a pair of $\{3\}$-GDDs of type 1^{6r+1} intersecting in m_{t+1} pairwise disjoint blocks, where $m_{t+1} \in \mathrm{Int_d}(1^{6r+1})$.

Let $\mathcal{E} = (\cup_{i=1}^{t+1} X_i, \{X_i \mid 1 \le i \le t+1\}, \mathcal{B})$ and $\mathcal{E}' = (\cup_{i=1}^{t+1} X_i, \{X_i \mid 1 \le i \le t+1\}, \mathcal{B}')$ be a pair of disjoint $\{3\}$-GDDs of type $(6s)^t(6r)^1$, which exists by Corollary 3.1. Let $\mathcal{F} = (\{\infty\}, \emptyset)$. Now apply the Singular Direct Product construction to obtain two $\{3\}$-GDDs of type $1^{6st+6r+1}$ $\mathcal{D}_1^* = \mathrm{SDP}(\{\mathcal{D}_i \mid 1 \le i \le t+1\}, \mathcal{E}, \mathcal{F})$ and $\mathcal{D}_2^* = \mathrm{SDP}(\{\mathcal{D}'_i \mid 1 \le i \le t+1\}, \mathcal{E}', \mathcal{F})$. It is easy to see that the intersection of $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ is $\cup_{i=1}^{t+1} (\mathcal{A}_i \cap \mathcal{A}'_i)$, which contains $\sum_{i=1}^{t+1} m_i$ pairwise disjoint blocks. $\square$

5 Direct Constructions

In this section, we determine $\mathrm{Int_d}(1^v)$ for some small values of v.

Lemma 5.1 *The following equalities hold:*

(i) $\mathrm{Int_d}(1^1) = \{0\}$;

(ii) $\mathrm{Int_d}(1^3) = \{1\}$;

(iii) $\mathrm{Int_d}(1^7) = \{0, 1\}$;

(iv) $\mathrm{Int_d}(1^9) = \{0, 1, 3\}$; *and*

(v) *for* $v \equiv 1, 3 \pmod 6$, $13 \le v \le 31$, $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$.

5.1 Proof of Lemma 5.1

(i) and (ii) hold trivially and recall that $0, 1 \in \mathrm{Int_d}(1^v)$ for $v \equiv 1, 3 \pmod 6$, $v \ge 7$. For $7 \le v \le 31$, we have the following table.

v	*Elements in* $\mathrm{Int_d}(1^v)$	*Authority*
7	$0, 1$	Trivial
9	$0, 1, 3$	Corollary 3.2: $\mathrm{Int_d}(1^9) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^3)$
13	$0, 1$	Trivial
15	$0, 1, 5$	Corollary 3.2: $\mathrm{Int_d}(1^{15}) \supseteq \sum_{i=1}^{5} \mathrm{Int_d}(1^3)$
19	$0, 1, 2, 3$	Lemma 4.2: $\mathrm{Int_d}(1^{19}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^7)$
21	$0, 1, 2, 3, 4, 5, 7$	Corollary 3.2: $\mathrm{Int_d}(1^{21}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^7)$; Corollary 3.2: $\mathrm{Int_d}(1^{21}) \supseteq \mathrm{Int_d}(1^9) + \sum_{i=1}^{4} \mathrm{Int_d}(1^3)$
25	$0, 1, 2, 3, 4, 5$	Lemma 4.1: $\mathrm{Int_d}(1^{25}) \supseteq \mathrm{Int_d}(1^9) + \sum_{i=1}^{2} (\mathrm{Int_d}(1^9) \setminus \{3\})$
27	$0, 1, 2, 3, 4, 5, 6, 7, 9$	Corollary 3.2: $\mathrm{Int_d}(1^{27}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^9)$
31	$0, 1, 2, 3, 4, 5, 6, 7$	Lemma 4.3: $\mathrm{Int_d}(1^{31}) \supseteq \mathrm{Int_d}(1^{13}) + \sum_{i=1}^{3} \mathrm{Int_d}(1^7)$

Let $\mathcal{S}_v$, $v \in \{13, 15, 19, 21, 25, 27, 31\}$ be the STS(v)'s given in Appendix A. For an STS(v) $\mathcal{S}_v = (X, \mathcal{B})$ and a permutation $\pi : X \to X$, π acts on $\mathcal{S}_v$ by canonical extension as follows: $\pi(\mathcal{S}_v) = \{\{\pi(a), \pi(b), \pi(c)\} \mid \{a, b, c\} \in \mathcal{B}\}$. The permutations π listed below is such that $\mathcal{S}_v$ and $\pi(\mathcal{S}_v)$ intersect in m pairwise disjoint blocks, where m is a number whose membership in $\mathrm{Int_d}(1^v)$ has not been established in the table above.

v	*Elements in* $\mathrm{Int_d}(1^v)$	π
13	2	`(0 7 3)(1 5 b 4 9 a 2 c)(6)(8)`
	3	`(0 c 3 4 a 8 2 6)(1 7 b 5)(9)`
	4	`(0 5)(1 c)(2 a 6)(3 b)(4 8)(7)(9)`
15	2	`(0 8 b e 9 4 3 d c 2 a 7 6)(1 5)`
	3	`(0 e 6 c 1 4 3 a 7 5 2 9 b d 8)`
	4	`(0 2 7 6 e a 3 b 1)(4 c 5 9 d 8)`
19	4	`(0 a 5)(1 d 8 f c 4 6 9 2 e)(3 7)(b g)(h)(i)`
	5	`(0 2 6 7 a 1 c 5 f h 8 b 4 i e 3)(9)(d g)`
25	6	`(0 o 6 3)(1 4 9 j 5 g k 2 l d)(7 h n e)(8 m)(a c i f b)`

Any two blocks of an STS(7) intersect in a point. So $2 \notin \mathrm{Int_d}(1^7)$. In an STS(9) $(X, \mathcal{B})$, if $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$, then $X \setminus (A \cup B)$ is a block in $\mathcal{B}$. So $2 \notin \mathrm{Int_d}(1^9)$. For each of the remaining possible disjoint intersection size $m \in \mathrm{Int_d}(1^v)$, we exhibit an STS($v$) $\mathcal{D}_v$, listed in Appendix B, whose intersection with $\mathcal{S}_v$ contains m pairwise disjoint blocks.

v	*Remaining values in* $\mathrm{Int_d}(1^v)$	$\mathcal{T}_v$
19	6	$\mathcal{T}_1$
21	6	$\mathcal{T}_2$
25	7	$\mathcal{T}_3$
	8	$\mathcal{T}_4$
27	8	$\mathcal{T}_5$
31	8	$\mathcal{T}_6$
	9	$\mathcal{T}_7$
	10	$\mathcal{T}_8$

6 Piecing Things Together

We first deal with the easier case of $v \equiv 3 \pmod 6$.

6.1 The Case $v \equiv 3 \mod 6$

6.1.1 $v \equiv 3 \pmod{18}$

By Theorem 1.1, there exists a pair of disjoint $\{3\}$-GDDs of type 13^3. Corollary 3.2 then gives $\mathrm{Int_d}(1^{39}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^{13}) = \sum_{i=1}^{3}\{0, \ldots, 4\} = \{0, \ldots, 12\}$. By Theorem 1.1, there also exists a pair of disjoint $\{3\}$-GDDs of type 3^{13}. Corollary 3.2 now gives $\mathrm{Int_d}(1^{39}) \supset \sum_{i=1}^{13} \mathrm{Int_d}(1^3) = \sum_{i=1}^{13}\{1\} = \{13\}$. Hence $\mathrm{Int_d}(1^{39}) = \mathcal{I}_\mathrm{d}(1^{39})$.

For $v \geq 57$, consider a pair of disjoint $\{3\}$-GDDs of type $9^t 21^1$, where $t = (v-21)/9$, existence of which is provided by Corollary 3.1. Now apply Corollary 3.2 to obtain

$$\begin{aligned}\mathrm{Int_d}(1^v) &\supseteq \mathrm{Int_d}(1^{21}) + \sum_{i=1}^{t} \mathrm{Int_d}(1^9) \\ &= \{0,\ldots,7\} + \sum_{i=1}^{t}\{0,1,3\} \\ &= \{0,\ldots,3t+7\} \\ &= \mathcal{I}_\mathrm{d}(1^v).\end{aligned}$$

This establishes the following.

Lemma 6.1 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 3 \pmod{18}$, $v \geq 39$.

6.1.2 $v \equiv 9 \pmod{18}$

By Theorem 1.1, there exists a pair of disjoint $\{3\}$-GDDs of type 15^3. Corollary 3.2 then gives $\mathrm{Int_d}(1^{45}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^{15}) = \sum_{i=1}^{3}\{0,\ldots,5\} = \{0,\ldots,15\} = \mathcal{I}_\mathrm{d}(1^{45})$.

For $v \geq 63$, consider a pair of disjoint $\{3\}$-GDDs of type $9^t 27^1$, where $t = (v-27)/9$, existence of which is provided by Corollary 3.1. Now apply Corollary 3.2 to obtain

$$\begin{aligned}\mathrm{Int_d}(1^v) &\supseteq \mathrm{Int_d}(1^{27}) + \sum_{i=1}^{t} \mathrm{Int_d}(1^9) \\ &= \{0,\ldots,9\} + \sum_{i=1}^{t}\{0,1,3\} \\ &= \{0,\ldots,3t+9\} \\ &= \mathcal{I}_\mathrm{d}(1^v).\end{aligned}$$

This establishes the following.

Lemma 6.2 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 9 \pmod{18}$, $v \geq 45$.

6.1.3 $v \equiv 15 \pmod{18}$

By Corollary 3.1, there exists a pair of disjoint $\{3\}$-GDDs of type $3^6 15^1$. Corollary 3.2 then gives $\mathrm{Int_d}(1^{33}) \supseteq \mathrm{Int_d}(1^{15}) + \sum_{i=1}^{6} \mathrm{Int_d}(1^3) = \{0,\ldots,5\} + \{6\} = \{6,\ldots,11\}$. Also, by Lemma 4.1, we have $\mathrm{Int_d}(1^{33}) \supseteq \mathrm{Int_d}(1^9) + \sum_{i=1}^{3}(\mathrm{Int_d}(1^9) \setminus \{3\}) = \{0,1,3\} + \sum_{i=1}^{3}\{0,1\} = \{0,1,2,3,4,5,6\}$. Hence we have $\mathrm{Int_d}(1^{33}) = \mathcal{I}_\mathrm{d}(1^{33})$.

For $v \geq 51$, consider a pair of disjoint $\{3\}$-GDDs of type $9^t 15^1$, where $t = (v-15)/9$, whose existence is provided by Corollary 3.1. Now apply Corollary 3.2 to obtain

$$\begin{aligned}\mathrm{Int_d}(1^v) &\supseteq \mathrm{Int_d}(1^{15}) + \sum_{i=1}^{t} \mathrm{Int_d}(1^9) \\ &= \{0,\ldots,5\} + \sum_{i=1}^{t} \{0,1,3\} \\ &= \{0,\ldots,3t+5\} \\ &= \mathcal{I}_\mathrm{d}(1^v).\end{aligned}$$

This establishes the following.

Lemma 6.3 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 15 \pmod{18}$, $v \geq 33$.

6.2 The Case $v \equiv 1 \pmod 6$

6.2.1 $v \equiv 1 \pmod{18}$

Let $s = 2$, $t = 3$. Then Lemma 4.2 gives $\mathrm{Int_d}(1^{37}) \supseteq \sum_{i=1}^{3} \mathrm{Int_d}(1^{13}) = \sum_{i=1}^{3} \{0,\ldots,4\} = \{0,\ldots,12\} = \mathcal{I}_\mathrm{d}(1^{37})$.

For $v \geq 55$, let $s = 3$, $t = (v-1)/18$. Now apply Lemma 4.2 to obtain

$$\begin{aligned}\mathrm{Int_d}(1^v) &\supseteq \sum_{i=1}^{t} \mathrm{Int_d}(1^{19}) \\ &= \sum_{i=1}^{t} \{0,\ldots,6\} \\ &= \{0,\ldots,6t\} \\ &= \mathcal{I}_\mathrm{d}(1^v).\end{aligned}$$

This establishes the following.

Lemma 6.4 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 1 \pmod{18}$, $v \geq 37$.

6.2.2 $v \equiv 7 \pmod{18}$

Let $v \geq 43$. Let $w = (v+2)/3$. Then $w \equiv 3 \pmod 6$ and $w \geq 15$. Now apply Corollary 4.1 to obtain

$$\begin{aligned}\mathrm{Int_d}(1^v) &\supseteq \mathrm{Int_d}(1^w) + (\mathrm{Int_d}(1^w) \setminus \{w/3\}) + (\mathrm{Int_d}(1^w) \setminus \}w/3\}) \\ &= \{0,\ldots,w/3\} + \{0,\ldots,w/3-1\} + \{0,\ldots,w/3-1\} \\ &= \{0,\ldots,w-2\} \\ &= \{0,\ldots,(v-1)/3-1\}.\end{aligned}$$

To show that $(v-1)/3 \in \mathrm{Int_d}(1^v)$, mimic the proof of Lemma 4.1 but with $m_1 = m_2 = m_3 = 0$ and replace the ingredient $\{3\}$-GDD of type $(w-1)^3$ with a pair of $\{3\}$-GDDs of type $(w-1)^3$ intersecting in $w-1$ disjoint blocks, existence of which is provided by Corollary 2.1. This establishes the following.

Lemma 6.5 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 7 \pmod{18}$, $v \geq 43$.

6.2.3 $v \equiv 13 \pmod{18}$

Let $s = 2$ and $t = 4$. Then Lemma 4.2 gives $\mathrm{Int_d}(1^{49}) \supseteq \sum_{i=1}^{4} \mathrm{Int_d}(1^{13}) = \sum_{i=1}^{4}\{0,\ldots,4\} = \{0,\ldots,16\} = \mathcal{I}_\mathrm{d}(1^{49})$.

For $v \geq 67$, let $t = (v-13)/18$. Then $t \geq 3$. Now apply Corollary 4.3 with $r = 2$ and $s = 3$ to obtain

$$\begin{aligned}
\mathrm{Int_d}(1^v) &= \mathrm{Int_d}(1^{18t+13}) \\
&\supseteq \mathrm{Int_d}(1^{13}) + \sum_{i=1}^{t} \mathrm{Int_d}(1^{19}) \\
&= \{0,\ldots,4\} + \sum_{i=1}^{t}\{0,\ldots,6\} \\
&= \{0,\ldots,6t+4\} \\
&= \mathcal{I}_\mathrm{d}(1^v).
\end{aligned}$$

This establishes the following.

Lemma 6.6 $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ *for* $v \equiv 13 \pmod{18}$, $v \geq 49$.

7 Conclusion

Lemmas 6.1 to 6.6 shows that $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$ for all $v \equiv 1, 3 \pmod 6$, $v \geq 33$. Lemma 5.1 gives $\mathrm{Int_d}(1^v)$ for all $v \equiv 1, 3 \pmod 6$, $v \leq 31$. This combines to give the main result of this paper below.

Theorem 7.1 *Let* $v \equiv 1, 3 \pmod 6$. *Then* $\mathrm{Int_d}(1^v) = \mathcal{I}_\mathrm{d}(1^v)$, *except that*

(i) $\mathrm{Int_d}(1^7) = \{0, 1\}$*; and*

(ii) $\mathrm{Int_d}(1^9) = \{0, 1, 3\}$.

If there exists a pair of STS(v) intersecting in t pairwise disjoint blocks, we can consider these t pairwise disjoint blocks as groups of a pair of disjoint $\{3\}$-GDDs of type 3^t1^{v-t}, and vice versa. Hence, Theorem 7.1 is equivalent to the following.

Theorem 7.2 *Let* $3t + r \equiv 1, 3 \pmod 6$, $3t + r \geq 13$. *Then there exists a pair of disjoint* $\{3\}$*-GDDs of type* 3^t1^r.

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A Some Small Steiner Triple Systems

For space efficiency, we list a block $\{a, b, c\}$ vertically as $\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}$.

A.1 An STS(13)

```
00000011111222223334445556
13579b3469a3467867868a7897
2468ac578bc95acbbacc9bbac9
```

A.2 An STS(15)

```
00000001111112222223333444455556666
13579bd3478bc34789c789a789a789a78ab
2468ace569ade65abedcdbeecdbbecdd9ce
```

A.3 An STS(19)

```
000000000111111112222222233333344444555556667778889999abcf
13456abce345678ac345678ad4689bg67acd678bd79fabe9beacdhedg
2789dgfihh9ibgdeffbeh9icg5acedieifghchagf8gidcffhgbhiiifh
```

A.4 An STS(21)

```
0000000000111111111222222222333333344444455555566667778888999aabbccdef
13456789af345679bdh345678bcf4679abc678adh678bde79dgabe9acgafieicedhgjg
2ejcbhidgk8egafcikjd9ajkehgi5fjghkicgbkfii9kfjh8ehkcdihdfjbjkfjjgekikh
```

A.5 An STS(25)

```
0000000000001111111111122222222222333333333444444445555555566666677777
1345689degin34567bcdefh345679abdfj4689abgik679ceghk679bcegj79abchabcef
27bafjchlkmoma8o9jniglkc8hkoeglmin5elhfdjnodlfimojnldknfimo8migjnkigjn
```

```
788888889999aaaabbbccdddeffhil
h9abcefgadgicejmcekdkfglhgjijm
mndhmkoibjlohnloofmelknoohmlkn
```

A.6 An STS(27)

```
0000000000000111111111111222222222222333333333344444444455555555566666
13456789abdeh3456789abdfg3456789acefj4678acdfkn6789abijp6789abcde79adl
2pgnkfqcmiljolhpecnjiqmokhdgimbpkqonl5be9ogijmqcnfkemloqjlihqomkf8gfhq
```

```
6677777788888899999aaabbbbbccccdeeefffggghhiikl
mn9abdghacdehkadeincdhcdefjdfiljghkgimjlmjkjoom
opqjkpoigjomplbflmopnlhgnlpeknoqiqphqpnpqmnkpqn
```

A.7 An STS(31)

```
0000000000000001111111111111122222222222222333333333333444444444445555
13456789acfhijn34567abcdghjms3456789bdefiln4679abcdefjn678bcdfghmn6789
2plbergdotkmqus89elfkoqnirptuoahjkcugqptmsr5mtgiurhklsqrqjeotiskpuiosm
```

```
5555555666666666777777778888888889999999aaaaaabbbbbbccccddddeeeeeeffgg
acdfglp79abcfghq9abcdijp9abdehikmachijkncefhnpcdfhjkdfgmfgikfghiorgjjn
rnjukqt8odpknusteglhmsnufuqrlntpobsqlrtpjqmltsisrtmnepluopulsmjntuhqto
```

```
ghhiijkkklllmor
qiojolmoqmptqps
rpukrorsunrusqt
```

B A Few More Small Steiner Triple Systems

B.1 An STS(19)

B.1.1 $\mathcal{T}_1$

```
000000000111111112222222233333344444555556666677788899bcf
1345678ag34578abd3459abde479bcf78acd68abd79ace9ce9abaehdg
2d9ebfhci6e9higcf867cfghi5ahegibfihggchfi8idfhdiggedbfieh
```

B.2 An STS(21)

B.2.1 $\mathcal{T}_2$

```
0000000000111111111222222222333333344444455555566666777788899aabbccdef
1345689abd345678cfg345789abc4789abd6789bi689cde7abcg9adh9efaeefhjdgihg
2ig7jkhcefeha9bdikj6kiejfdgh5fhkgcjecadfjkbgjhf8hdfijigkcgibikjikekkjh
```

B.3 Two STS(25)'s

B.3.1 $\mathcal{T}_3$

```
0000000000001111111111122222222222333333333444444445555555566666677777
13456789abcd345689adefj345689abefn4689abekl68abcdhl6789abeg7acdhm9abdf
2johemglkinfi7nbhgclkmo7icjdkmghlo5fcdhmgno9fjegknmijmfldok8olgknegcok
```

```
77888889999aabbbccccddeeffgggi
hl9abekachjdefhkdfhkhiijgiijmj
inoijnlbimnnfnloejomjmlmhonlok
```

B.3.2 $\mathcal{T}_4$

```
0000000000001111111111122222222222333333333444444445555555566666667777
13456789abfi34569cdegjk345789abcef467abdefh6789aehn89abcdeh7bdfghm9abd
2ndgckljehmo8b7afhniomlgi6mhdlonjk59cimokljkfjcglmonijfokml8ejnliolojh
```

```
778888889999aaabbbccccdefghijl
eg9abdefaegkdhkcdidfgiifgjkjlm
niockmgibhmnfnmlgnejkmlohnokon
```

B.4 An STS(27)

B.4.1 $\mathcal{T}_5$

```
0000000000000111111111111222222222222333333333344444444455555555566666
1345689abcdgk3456789abimn345679abdfgo46789abceh679bdeghm689abcdgk79bdg
2ml7fjehnqoipkcfjhlgedqopn8hcekjlimqp5lpqodgfijainkjfopqeoipjnmlq8hoqn
```

```
667777777888888889999aaaabbbccccddeeeeefffghiijl
ik9abcdfj9abdghmadflcfilcfhdhijhkghjkpgkljojnnm
pmmkqognlcnefkipbpjqgqmopimeklmnlmlonqhoppqkoqn
```

B.5 Three STS(31)'s

B.5.1 $\mathcal{T}_6$

```
000000000000000111111111111122222222222222333333333333344444444444445555
1345789abceghjp3456789acdegio345678bcdeghim478abdegjlqs6789abceijp6789
26fiklsmdoqtunrnrkbjsqfluhmpt9opqfaknstljur5cmhokfirptutmnhdgkulqsjnec
```

```
55555556666666666777777778888888899999999aaaaaabbbbbbcccccddddeeeefffff
abdfhmt79acdefkmn9abdghlr9bcdhikpadefgkmcijknpcehinqdfgimfhijiklogknpr
glrsqou8lohpgiusriesqoputtfjgroqubonjrpusqlrutujmtpretprqmlntmsrphloqu
```

```
ggghhhijjklllno
jkqiknjmommosqr
snusotkputnqtss
```

B.5.2 $\mathcal{T}_7$

```
000000000000000111111111111122222222222222333333333333344444444444445555
13456789abdehil345689abcdfgjn345679abdehmqs4678abfghlno69abceghikr6789
2qfunjmsctgokprs7qoklmeihuprtk8nclfgptjioru59redcimjuptldtjpnumqostajo
```

```
55555556666666666777777777888888888999999aaaaaabbbbbccccccddddeeeeffff
bcdefhk7abdeghkm9bcdefhin9abcdfgiqacegijmefhikldhknodfhjmtijmnghkrgkno
gsrimpl8qipfjursumgkpqotshrlnupsotbkqtrnpujnspofrsquelqorulsqolsmthtrs
```

```
ggghiijjjkllopp
ikoljmlmqnmqprs
nqrtkuptuunsqut
```

B.5.3 $\mathcal{T}_8$

```
000000000000000111111111111122222222222222333333333333344444444444445555
1345678abcdeijk3456789bcefgmn34567acdfijlnq4678abcdgioq679bcdefjpr6789
29hqgmtsunofplrkahipdoqlutjsre8gb9mhkpturos5plfthjsmnrumnitglkoqsurslp
```

```
55555556666666666777777778888888888999999aaaaaabbbbbbccccddddeeeefffgg
abdefik79acdefhnabcfghjk9abceghjkacdefhjlcdghkpefgjlmdfpshinpghjmginkl
octnjum8slojtqkuedquritomqnripsoubkgqruntifujnrpkirsoemtumrquolsrhlssq
```

```
ghhhiiijkkllmor
nnoqjmomlqmotps
tptrkqspptnuuqt
```