Minuscule Heaps Over Dynkin diagrams of type $\tilde{A}$

Manabu HAGIWARA *
Institute of Industrial Science, University of Tokyo
e-mail: manau@imailab.iis.u-tokyo.ac.jp

Submitted: Oct 18, 2002; Accepted: Dec 12, 2003; Published: Jan 2, 2004

Abstract

A minuscule heap is a partially ordered set, together with a labeling of its elements by the nodes of a Dynkin diagram, satisfying certain conditions derived by J. Stembridge. This paper classifies the minuscule heaps over the Dynkin diagram of type $\tilde{A}$.

1 Introduction

The aim of this paper is to classify the minuscule heaps over a Dynkin diagram of type $\tilde{A}$.

Let A be a symmetrizable generalized Cartan matrix, and let $\mathfrak{g}$ be a corresponding Kac-Moody Lie algebra. Let Γ be a Dynkin diagram which is an encoding of A. Minuscule heaps arose in connection with λ-minuscule elements of the Weyl group W of $\mathfrak{g}$. According to Proctor [9] and Stembridge [12] the notion of *λ-minuscule elements* of W was defined by Peterson in his unpublished work in the 1980's. For an integral weight λ, an element w of W is said to be a *λ-minuscule element* if it has a reduced decomposition $s_{i_1}s_{i_2}\dots s_{i_r}$ such that

$$s_{i_j}s_{i_{j+1}}\dots s_{i_r}\lambda = s_{i_{j+1}}\dots s_{i_r}\lambda - \alpha_{i_j} \ (1 \leq \forall j \leq r),$$

and it is called *minuscule* if w is λ-minuscule for some integral weight λ. Here α_i is the simple root corresponding to s_i. It is known that a minuscule element is *fully commutative*, namely any reduced decomposition can be converted into any other by exchanging adjacent commuting generators several times (see [9, §15], [10, Theorem A] and [11, Theorem 2.2], or [12, Proposition 2.1]). To a fully commutative element w, one can associate a Γ-labeled poset called its *heap*.

A Γ-*labeled poset* is a triple $(P, \leq, \phi)$ in which $(P, \leq)$ is a poset and $\phi : P \to N(\Gamma)$ is any map (called the *labeling map*). A linear extension of a Γ-labeled poset naturally

*The author's name is "HAGIWARA Manabu" by Japanese ordering.

determines a word in the generators of W. The heap of the fully commutative element w is a Γ-labeled poset whose linear extensions determine all reduced decompositions of w. A *minuscule heap* is the heap of a minuscule element of W.

Stembridge obtained certain structural conditions for a finite Γ-labeled poset to be a minuscule heap ([11, Proposition 3.1]). In this paper a minuscule heap is defined by a finite Γ-labeled poset which satisfies the conditions (H1-1), (H1-2) and (H2a) (see §2). In the following, we state a relation between minuscule elements and minuscule heaps.

Let $(P, \leq, \phi)$ be a minuscule heap. Put $r := \#P$. Let $\mu : P \to [1, r]$ be a *linear extension* of P, namely μ is a bijection and if $p \leq q$ then $\mu(p) \leq \mu(q)$. For μ, we associate a minuscule heap $(P, \leq, \phi)$ to $w \in W$ by the following expression

$$w := s_{\phi\circ\mu^{-1}(1)} s_{\phi\circ\mu^{-1}(2)} \dots s_{\phi\circ\mu^{-1}(r)}.$$

We note that an element obtained from a minuscule heap, by the relation above, is a minuscule element. Conversely for any minuscule element w there exists a unique minuscule heap which determines w.

In [11], there is an important condition which states that "the labels that occur in P index an acyclic subset of the Dynkin diagram". A nice consequence of this condition is that if it holds then P is a *ranked poset*. However Dynkin diagrams Γ of type $\tilde{A}$ are cyclic, and most of minuscule heaps over Γ are not ranked. In this paper, we introduce an analogy of slant lattices [6] (here called L_k) (§8) and use it to prove that a subset P of an extended slant lattice L_k is a convex subset if and only if P is a minuscule heap over Γ up to isomorphic. Slant lattices L were also used to classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams in [6].

It is known that the affine permutation group $\tilde{S}_{n+1}$ is isomorphic to an affine Weyl group $W(\tilde{A}_n)$. In [5], Green showed that the 321-avoiding permutations of affine permutations coincide with the fully commutative elements of $W(\tilde{A}_n)$. He showed also that the fully commutative elements of $W(\tilde{A}_n)$ form a union of Kazhdan-Lusztig cells. Here we show that the fully commutative elements of $W(\tilde{A}_n)$ coincide with its minuscule elements [Theorem 5.1].

The paper is organized as follows. In §2 we recall and provide some basic terminology. In §§3, 4 we collect some general facts on poset and on Γ-labeled poset with a general Dynkin diagram. In §5, we show that the fully commutative elements of $W(\tilde{A})$ coincide with the minuscule elements of $W(\tilde{A})$. From §4, we characterize the minuscule heaps over a Dynkin diagram of type $\tilde{A}$. In §6, we characterize the totally ordered minuscule heaps over a Dynkin diagram of type $\tilde{A}$. In §7, we determine the structure of a subposet which we call *v-interval*. In §8, we characterize the minuscule heaps over a Dynkin diagram of type $\tilde{A}$ up to isomorphism and introduce the notion of *extended slant lattices* L_k. In §9, we show that any minuscule heap over a Dynkin diagram of type $\tilde{A}$ is isomorphic to a convex subset of an extended slant lattice L_k.

2 Definitions

We start with the definition of general terms associated to a partially ordered set. We denote the number of elements of a set P by $\#P$.

Let $(P, \leq)$ be a poset (*partially ordered set*). For $p, q \in P$, we say that q *covers* p *(or p is covered by q)* if $p < q$ and $(p, q) = \emptyset$, and denote by $p \to q$. We say that p, q are a *covering pair* if $p \to q$ or $q \to p$. In this paper, we assume that P is completely determined by the covering relations, namely

(*) If $p, q \in P$ and $p \leq q$ then there exists a finite sequence of elements of P, say $p_0, p_1, \ldots, p_r$ such that $p_0 = p, p_r = q$ and p_i covers p_{i-1} for $1 \leq i \leq r$.

We call such a sequence $p_0, p_1, \ldots, p_r$ a *saturated chain from p to q*.

We denote ordering relations on posets as follows. Let P be a poset and let Q be its subset. For $x, y \in Q$, we write $[x, y]_P = \{z \in P | x \leq_P z \leq_P y\}$ and write $[x, y]_Q = \{z \in Q | x \leq_Q z \leq_Q y\}$. In general, a maximal connected subposet of P is called a *connected component* of P. A subset Q of P is said to be convex in P if whenever $p, q \in Q$ and $p \leq q$ we have $[p, q]_P \subset Q$.

Let Γ be a Dynkin diagram and let $N(\Gamma)$ be the node set of Γ. By an abuse of language, we sometimes identify $N(\Gamma)$ with Γ. We say that a triple $(P, \leq, \phi)$ (or simply P) is a Γ-*labeled* poset if $(P, \leq)$ is a partially ordered set, and ϕ is any map from P to $N(\Gamma)$. We call ϕ the *labeling map* and call $\phi(p)$ the *label* of p. We denote $\operatorname{Im} \phi$ by $\operatorname{supp} P$, and call it the *support* of P. For each $v \in N(\Gamma)$, we put

$$P_v := \{p \in P | \phi(p) = v\}.$$

For $v \in N(\Gamma)$ and $p, q \in P$ satisfying $p < q$, we say that $[p, q]$ is a *v-interval* if $p, q \in P_v$ and $(p, q) \cap P_v = \emptyset$.

Let $(P, \leq, \phi), (Q, \preceq, \psi)$ be Γ-labeled posets. We say that P and Q are *isomorphic* as Γ-labeled poset if there exists a poset isomorphism $\Phi : P \to Q$ such that $\phi(p) = \psi(\Phi(p)) (\forall p \in P)$.

Let Γ and Γ' be Dynkin diagrams, and let $(a_{i,j})_{i,j \in I}, (a'_{i,j})_{i,j \in I'}$ be the corresponding generalized Cartan matrices. Let P be a Γ-labeled poset and let Q be a Γ'-labeled poset. We say that P and Q are *abstractly isomorphic* (or isomorphic if no confusion arises) if there is a poset isomorphism $\alpha : P \to Q$ and an isomorphism of subdiagrams $\beta : \operatorname{supp} P \to \operatorname{supp} Q$ (namely a bijection $\operatorname{supp} P \to \operatorname{supp} Q$ such that $a_{i,j} = a'_{\beta(i),\beta(j)}$ for all $i, j \in \operatorname{supp} P$) such that β maps the label of p to the label of $\alpha(p)$ for every $p \in P$.

Let $D(3) := \{w, x, y, z\}$ be a set and let $\to$ be a binary relation on $D(3)$ with $w \to x, x \to z, w \to y, y \to z$. Let $\leq$ be an ordering on $D(3)$ which is the reflective, transitive closure of $\to$. Let Γ be the Dynkin diagram of type A_3 and let $N(\Gamma) = \{1, 2, 3\}$ be the node set of Γ. (Put $1, 2, 3$ on $N(\Gamma)$ from an edge node to another one.) Define a map $\phi : D(3) \to N(\Gamma)$ by putting $\phi(w) = \phi(z) := 2, \phi(x) := 1$ and $\phi(y) := 3$. We regard $D(3)$ as a Γ-labeled poset with ϕ. Let Γ' be a Dynkin diagram. We say that Γ'-labeled poset Q is a *diamond* if Q and $(D(3), \leq, \phi)$ are abstractly isomorphic.

Let $A = (a_{i,j})_{i,j\in N(\Gamma)}$ be a Cartan matrix corresponding to Γ. We say that a Γ-labeled poset $(P, \leq, \phi)$ is a *minuscule heap* if P is finite and satisfies the conditions (H1-1),(H1-2) and (H2a).

(H1-1) For $p, q \in P$, if $p \to q$, then $\phi(p)$ and $\phi(q)$ are either equal or adjacent in Γ.

(H1-2) For $p, q \in P$, if $\phi(p)$ and $\phi(q)$ are either equal or adjacent in Γ then p and q are comparable.

(H2a) For $p, q \in P$, if $\phi(p) = \phi(q)$ and $p \leq q$ then $\sum_{x\in[p,q]} a_{\phi(x),\phi(p)} = 2$.

In particular we regard an empty set as a minuscule heap.

Remark 2.1. In [11], Stembridge obtained two structural conditions, which he called (H1) and (H2), for a finite Γ-labeled poset to be a minuscule heap. In this paper we separate (H1) to two conditions (H1-1),(H1-2). And we use the condition (H2a) instead of (H2) which are equivalent. (Proposition 4.4.)

3 Basic Properties on Poset

In this section we provide some general facts on posets. We omit the proofs below since they are straightforward.

Proposition 3.1. *Let S be a set and let $\rightsquigarrow$ be a binary relation on S. Let $\preceq$ be the reflexive, transitive closure of $\rightsquigarrow$.*

Then $(S, \preceq)$ is a poset if and only if $s = s_0 \rightsquigarrow s_1 \rightsquigarrow \cdots \rightsquigarrow s_r = s$ implies $s_0 = s_1 = \cdots = s_r = s$ for some $r \geq 0$, where $s, s_0, s_1, \ldots, s_r \in S$.

Proposition 3.2. *Let $S, \rightsquigarrow, \preceq$ be the same as Proposition 3.1. Assume that $(S, \preceq)$ be a poset.*

Then $\rightsquigarrow$ is the covering relation on $(S, \preceq)$ if and only if $p \rightsquigarrow q$ and $p = p_0 \rightsquigarrow p_1 \rightsquigarrow \cdots \rightsquigarrow p_r = q$ implies $r = 1$.

Proposition 3.3. *Let $(P, \leq)$ be a poset and let G be a group which acts on P as a poset automorphism, namely $p \leq q$ if and only if $g(p) \leq g(q)$ for $g \in G$ and $p, q \in P$.*

Assume that G satisfies the following condition,

- *for $p \in P$ and $g \in G$, if p and $g(p)$ are comparable then $p = g(p)$.*

Put $P/G := \{\overline{p} | p \in P\}$, where $\overline{p} = \{g(p) | g \in G\}$, and put a relation $\preceq$ on P/G as following,

$$\overline{p} \preceq \overline{q} \text{ if and only if } p \leq g(q) \text{ for some } g \in G \ (\overline{p}, \overline{q} \in G/P).$$

Then it follows,

- *$\preceq$ is well-defined,*

- *$(P/G, \preceq)$ is a poset.*

Proposition 3.4. *Let P be a poset and let $p, q \in P$ satisfying $p \leq q$. Then $[p, q]$ is a convex subset of P.*

Proposition 3.5. *Let P be a poset and Q be a convex subset. If z is a minimal or maximal element of Q then $Q \setminus \{z\}$ is a convex subset of P.*

4 Basic Properties on Γ-labeled Posets

In this section we provide some general facts on Γ-labeled posets over a general Dynkin diagram. (See [7] or [8] for the definition of Dynkin diagrams.)

Proposition 4.1. *Let $(P, \leq, \phi)$ be a Γ-labeled poset. If $(P, \leq, \phi)$ satisfies (H1-2) then P_v is a totally ordered set for each $v \in N(\Gamma)$.*

Proof. By (H1-2), p and q are comparable, where $p, q \in P_v$. □

Proposition 4.2. *Let $(P, \leq, \phi)$ be a Γ-labeled poset which satisfies (H1-1) and (H1-2). Then P is connected if and only if* supp P *is connected.*

Proof. Assume that P is connected. For $u, v \in \mathrm{supp}\ P$, there exists $p \in P_v$ and $q \in P_u$. Now we can take a sequence $p = p_0, p_1, \ldots, p_r = q$ such that p_{i-1}, p_i are a covering pair. Then $\phi(p_0), \phi(p_1), \ldots, \phi(p_r)$ consists of a connected subdiagram of Γ by (H1-1). Hence supp P is connected.

Conversely assume that supp P is connected. Let $p, q \in P$ and put $v := \phi(p), u := \phi(q)$. Since Γ is connected, we can take a sequence $v = v_0, v_1, \ldots, v_r = u \in \mathrm{supp}\ P$ such that v_{i-1} and v_i are adjacent in Γ. Take some $p_i \in P_{v_i}$ $(1 \leq i \leq r-1)$ and put $p_0 = p, p_r = q$. Then p_{i-1}, p_i are comparable by (H1-2). So P is connected. □

Proposition 4.3. *Let $(P, \leq, \phi)$ be a Γ-labeled poset satisfying (H1-1) and (H1-2). Let $P_1, P_2, \ldots, P_r$ be the connected components of P. Then* $\mathrm{supp}\ P = \sqcup_{i=1}^{r} \mathrm{supp}\ P_i$. *In particular v and u are distinct and non-adjacent, where $v \in \mathrm{supp}\ P_i, u \in \mathrm{supp}\ P_j$ and $i \neq j$.*

Proof. If there exists $v \in \mathrm{supp}\ P_i \cap \mathrm{supp}\ P_j$ then we can take $p \in P_i \cap P_v$ and $q \in \mathrm{supp}\ P_j \cap P_v$. By (H1-2), p and q are comparable. This implies $i = j$ because P_i, P_j are connected components.

If there exists $v \in \mathrm{supp}\ P_i$ and $u \in \mathrm{supp}\ P_j$ such that v and u are adjacent in Γ then we obtain a contradiction by a similar argument. □

Proposition 4.4. *Let $(P, \leq, \phi)$ be a finite Γ-labeled poset. If $(P, \leq, \phi)$ satisfies (H1-1) and (H1-2) then each of the following each conditions are equivalent to (H2a).*

(H2) For any v-interval $[p, q]$, we have $\sum_{x \in [p,q]} a_{\phi(x), \phi(p)} = 2$.

(H2b) For any v-interval $[p, q]$, we have $\sum_{x \in (p,q)} a_{\phi(x), \phi(p)} = -2$.

Proof. By $a_{v,v} = 2$, it is obvious that (H2) and (H2b) are equivalent for any $v \in N(\Gamma)$. It is also obvious that (H2a) implies (H2).

We assume that (H2) holds. By (H1-2), P_v is a totally ordered set. Let p, q be elements of P_v which satisfy $p \leq q$. If we have $p = q$ then (H2a) trivially holds, since $a_{v,v} = 2$ for $v \in N(\Gamma)$. So we assume $p < q$. Let $p = p_0, p_1, \ldots, p_r = q$ be the elements of $P_v \cap [p, q]$ by an increasing ordering. For $x \in [p, q] \setminus (\cup_{1 \leq i \leq r} [p_{i-1}, p_i])$, x and p_j are incomparable for some $0 \leq j \leq r$. By (H1-2), $\phi(x)$ is different from v and not adjacent to v. Thus we have $a_{\phi(x),v} = 0$. This implies

$$\sum_{x \in [p,q]} a_{\phi(x),v} = \sum_{1 \leq i \leq r} \sum_{x \in [p_{i-1}, p_i)} a_{\phi(x),v} + a_{\phi(p_r),v} = 0 + 2 = 2.$$

□

Remark 4.5. Let Γ be a simply-laced Dynkin diagram and let $v \in N(\Gamma)$. Let $[p, q]$ be a v-interval. (H2b) requires that there exists just two elements of (p, q) whose labels are adjacent to v in Γ. This fact is very important since the Dynkin diagram of type $\tilde{A}_n$ $(n \geq 2)$ is simply-laced. However we investigate a minuscule heap over the Dynkin diagram of type $\tilde{A}_1$. In this case, (H2b) requires that there exists only one element of (p, q) whose labels are adjacent to v in Γ. These facts are used to prove Propositions 4.6, 6.1 and 6.2.

Proposition 4.6. *Let Γ be a simply-laced Dynkin diagram and let $(P, \leq, \phi)$ be a minuscule heap over Γ. Let $p, q \in P$ such that $\phi(p) = \phi(q)$. If there exists an element $x \in P$ such that $p \to x \to q$ then $[p, q]$ is a $\phi(p)$-interval. In particular $[p, q]$ is a diamond.*

Proof. We note that $\phi(x)$ and $\phi(p)$ are adjacent in Γ. If there exists $y \in (p, q)$ such that $\phi(y) = \phi(p)$ then x and y are comparable by (H1-2). This implies that we have either $p < x < y$ or $y < x < q$. It contradicts $p \to x \to q$. So we have $(p, q) \cap P_{\phi(p)} = \emptyset$. However there exists $y \in (p, q)$ such that $\phi(y)$ is adjacent to $\phi(p)$ by Remark 4.5. Let $p = p_0, p_1, \ldots, p_r = q$ be a saturated chain from p to q which contains y. We note that this saturated chain does not contain x. In fact a sequence p, x, q is a unique saturated chain which contains x.

For $p = p_0, p_1, \ldots, p_r = q$, y is the only element which can cover p, and y is the only element which can be covered by q. Thus this saturated chain consists of p, y, q. Hence we have $[p, q] = \{p, x, y, q\}$. By (H1-1) and (H2), $[p, q]$ is a diamond. □

Proposition 4.7. *Let Γ be a Dynkin diagram and let Φ be a graph automorphism on Γ. If a Γ-labeled poset $(P, \leq, \phi)$ is a minuscule heap then $(P, \leq, \Phi \circ \phi)$ is a minuscule heap. Furthermore these minuscule heaps are abstractly isomorphic.*

Proof. Since Φ is a graph automorphism, it is obvious that (H1-1) and (H1-2) hold. As $a_{v,u} = a_{\Phi(v),\Phi(u)}$, (H2a) holds. □

5 Relation Between Fully Commutative Elements and Minuscule Elements

First we show that the fully commutative elements of $W(\tilde{A})$ coincide with its minuscule elements.

Theorem 5.1. *Let Γ be a simply-laced Dynkin diagram with a finite node set. The fully commutative elements of $W(\Gamma)$ coincide with its minuscule elements if and only if Γ is of type A or $\tilde{A}$.*

Proof. It is well known that a minuscule element is fully commutative.

Assume Γ branches off. Then there exists a node $v \in N(\Gamma)$ such that the number of the adjacent nodes to v is larger than two. Let x, y, z be adjacent nodes to v. We can verify that $s_v s_x s_y s_z s_v$ is fully commutative, where s_u is a generator associated to $u \in N(\Gamma)$. But it violates (H2) and so it would not be minuscule. Hence Γ cannot have a junction. Thus Γ must be of type A or $\tilde{A}$.

If Γ is of type A, then it is well-known that a fully commutative element is minuscule. The remaining case is when Γ is of type $\tilde{A}$. Let w be a fully commutative element of $W(\tilde{A})$ and let $s_1 s_2 \ldots s_r$ be a reduced expression of w. By the commutativity of w, if s_i, s_j are consecutive occurrences of the generator s (meaning that $s_i = s_j = s$ for some generator s ($i <$j) and $s_i \neq s$ for $i < h < j$), then there are at least two generators s_{h_1}, s_{h_2} such that s_i and s_{h_1} (or s_{h_2}) are non-commutative $i < h_1, h_2 < j$. If there is no consecutive occurrences of any generators, then w is minuscule.

Remember that any heap of a fully commutative element satisfies (H1). For proving that w is minuscule, it is sufficient that there exists just two non-commutative generators s_{h_1}, s_{h_2}. If not, we can take three non-commutative generators from $s_{i+1}, \ldots, s_{j-1}$. Now, as Γ is of type $\tilde{A}$, each node has only two adjacent nodes. Thus we can take consecutive occurrences $s_{i'}, s_{j'}$ of s'. By the commutativity of w, we can take two generators $s_{h'_1}, s_{h'_2}$ from $s_{i'+1}, \ldots, s_{j'-1}$ which are non-commutative to s'. The nodes associated to $s_{i'+1}, s_{j'+1}$ are adjacent to the node associated to s' and they are different from s_i because $s_i \neq s_h$ for $i < h < j$. This implies $s_{i'+1} = s_{j'+1}$, in other words they are consecutive occurrences. By using a similar argument, the length of w must be infinite. It cannot happen. $\square$

6 Totally Ordered Minuscule Heaps over Dynkin Diagrams of Type $\tilde{A}$

In this section we determine the structure of totally ordered minuscule heaps over Dynkin diagrams of type $\tilde{A}$. From this section on we assume that Γ is a Dynkin diagram of type $\tilde{A}_n$ with the node set $N(\Gamma) := \{0, 1, \ldots, n\}$. (see Figure 1 for the definition of Dynkin diagram of type $\tilde{A}_n$.) We associate $i \in \mathbb{Z}$ to $j \in N(\Gamma) = \{0, 1, \ldots, n+1\}$ by the following rule $j = i \mod (n+1)$. We note that the Dynkin diagram of type $\tilde{A}_1$ and its Cartan matrix $A := A(\Gamma)$ are different from others of type $\tilde{A}_n (n \geq 2)$. First we classify the minuscule heaps over $\tilde{A}_1$. In fact a minuscule heap over $\Gamma(\tilde{A}_1)$ is a totally ordered set.

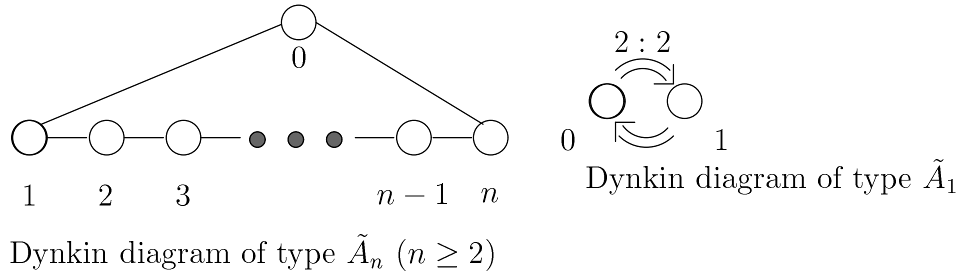

Figure 1: The Dynkin diagram of type $\tilde{A}_n$

The Cartan matrix $A = A(\Gamma(\tilde{A}_1)) = (a_{i,j})$ is

$$A := \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Proposition 6.1. *Let Γ be the Dynkin diagram of type $\tilde{A}_1$. A minuscule heap over Γ is a totally ordered set and is characterized by $r := \#P$ and the label v of its smallest element if $r \neq 0$. Namely, define a Γ-labeled poset $(P, \leq, \phi)$ by putting*

- $P = \{p_1, p_2, \ldots, p_r\}$,
- $p_{i-1} \to p_i$ *$(1 < i \leq r)$,*
- $\leq$ *is the transitive and reflective closure of $\to$,*
- $\phi(p_i) \equiv v + i - 1 \pmod 2$.

Then $(P, \leq, \phi)$ is a minuscule heap over Γ. Conversely a minuscule heap over Γ is abstractly isomorphic to a minuscule heap as defined above.

Proof. Define P as above. Then it is obvious that (H1-1) and (H1-2) hold on P. By the definition of P, a v-interval has the form $[p_i, p_{i+2}]$ for $v \in N(\Gamma)$. So we have

$$\sum_{x \in (p_i, p_{i+2})} a_{\phi(x),v} = \sum_{x \in \{p_{i+1}\}} a_{\phi(x),v} = -2.$$

This implies that P is a minuscule heap.

Conversely assume that $(P, \leq, \phi)$ is a minuscule heap. Then it is obvious that P is a totally ordered set by the shape of Γ and (H1-2). Hence we can write $P = \{p_1, p_2, \ldots, p_r\}$ with $p_{i-1} \to p_i$ $(1 < i \leq r)$. By (H2b), we have $\phi(p_{i-1}) \neq \phi(p_i)$. Since the labels of p_i are binary values ($\{0, 1\}$), two nodes always alternatively appear. Thus we have $\phi(p_i) \equiv v + i - 1 \pmod 2$. $\square$

By a similar argument above, we can determine the structure of a totally ordered minuscule heap over $\Gamma(\tilde{A}_n)$ $(n \geq 2)$.

Proposition 6.2. *Let $(P, \leq, \phi)$ be a totally ordered minuscule heap over Γ. Put $P = \{p_0, p_1, \dots, p_r\}$ with $p_{i-1} \to p_i (1 \leq i \leq r)$. Then the labels of the elements of P are either of type (T1) or (T2),*

(T1) $\phi(p_i) \equiv \phi(p_0) + i \pmod{n+1}$

(T2) $\phi(p_i) \equiv \phi(p_0) - i \pmod{n+1}$

Proof. By (H2b), we have $\phi(p_0) \neq \phi(p_1)$. By (H1-1), we have either $\phi(p_1) = \phi(p_0) + 1$ or $\phi(p_1) = \phi(p_0) - 1$.

Assume $\phi(p_1) = \phi(p_0) + 1$. Then we have either $\phi(p_2) = \phi(p_1) + 1$ or $\phi(p_2) = \phi(p_1) - 1$. If we have $\phi(p_2) = \phi(p_1) - 1$ then $\phi(p_2) = \phi(p_0)$. By Proposition 4.6, $[p_0, p_2]$ is a $\phi(p_0)$-interval. However we have

$$\sum_{x \in (p_0, p_2)} a_{\phi(x), \phi(p_0)} = \sum_{x \in \{p_1\}} a_{\phi(x), \phi(p_0)} = -1 \neq -2.$$

It contradicts (H2b). Thus we have $\phi(p_2) = \phi(p_1) + 1$. By repeating an argument above, the labels of the elements are of type (T1).

By a similar argument, we obtain the case (T2) from the assumption $\phi(p_1) = \phi(p_0) - 1$. □

We say that P is of type (T1) (resp. (T2)) if the labels of P are of type (T1) (resp. (T2)).

7 The Structure of v-intervals

In this section we investigate the structure of v-intervals for a minuscule heap over $\Gamma(\tilde{A}_n)$ with $n \geq 2$. To determine v-intervals for $v \in N(\Gamma)$ is useful to determine the structure of a minuscule heap.

By the symmetry of the shape of the Dynkin diagram of type $\tilde{A}$, to determine the structure of all of 1-intervals is equivalent to determine that of all of v-intervals for any $v \in N(\Gamma)$. Hence we investigate the structure of the 1-intervals.

Lemma 7.1. *Let P be a minuscule heap over $\Gamma(\tilde{A}_n)$ and let $[p_1, q_1] \subset P$ be a 1-interval. Let p_2, q_2 be elements of (p_1, q_1) whose labels are adjacent to 1 in Γ. Then we have $\phi(p_2) \neq \phi(q_2)$.*

Proof. We note that $\phi(p_2), \phi(q_2)$ must be 0 or 2. Our claim is that $\phi(p_2) = \phi(q_2)$ is impossible.

If we have $\phi(p_2) = \phi(q_2) = 2$ then p_2 and q_2 are comparable. Let us assume that $p_2 < q_2$ then $[p_2, q_2]$ is a 2-interval by (H2b). Hence there exist $p_3, q_3 \in (p_2, q_2)$ such that $p_2 \to p_3, q_3 \to q_2$ and $\phi(p_3), \phi(q_3)$ are adjacent to 2. Now $\phi(p_3), \phi(q_3)$ can be only equal to 1 or 3. However they cannot be equal to 1 since $[p_1, q_1]$ is a 1-interval. So we have $\phi(p_3) = \phi(q_3) = 3$. By repeating the arguments above, we can take a 0-interval $[p_{n+1}, q_{n+1}]$ from (p_1, q_1) and we know that $[p_{n+1}, q_{n+1}]$ must contain an element whose label is 1. It contradicts that $[p_1, q_1]$ is a 1-interval. □

For $i = 1, 2, 3$, we say that a 1-interval $[p, q]$ is *of type (Vi)* if $[p, q]$ satisfies the following:

(V1) $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $1, 2, 3, \dots, n-1, n, 0, 1$ respectively;

(V2) $[p, q]$ is a totally ordered set and consists of $n + 2$ elements; The labels of the elements in the increasing order are $1, 0, n, n-1, \dots, 3, 2, 1$ respectively.

(V3) $[p, q]$ is a diamond.

Proposition 7.2. *Any* 1*-interval* $[p, q]$ *is either of type (V1), (V2) or (V3).*

Proof. By Lemma 7.1, (p, q) contains a unique pair of elements x, y whose labels are $0, 2$ respectively. By (H1-1), only x or y can cover p and only x or y can be covered by q.

Assume that both x and y cover p then we claim that $[p, q]$ is of type (V3). By the assumption that x and y are incomparable, there is a saturated chain $x = p_1, p_2, \dots, p_r = q$ from x to q which does not contain y. We note that $\phi(p_{r-1})$ is either 0 or 2, namely p_{r-1} is either x or y. This implies $p_{r-1} = x$ and $r = 2$. Thus we have $x \to q$. By a similar argument, we have $y \to q$. So $[p, q] = \{p, x, y, q\}$ is of type (V3).

Next assume that only x covers p. We claim that $[p, q]$ is of type (V1). Let $p = p_0, x = p_1, p_2, \dots, p_r = q$ be a saturated chain from p to q. We note that $\phi(p_{i-1}) - \phi(p_i)$ is either 1 or -1 because $\phi(p_{i-1}), \phi(p_i)$ are adjacent. If these labels are all different then they are of type (V1) or (V2). If there are repeated labels then we can take a pair p_i, p_{i+2} such that $\phi(p_i) = \phi(p_{i+2})$. Let us choose such a minimal i. By Prop. 4.6, $[p_i, p_{i+2}]$ is a diamond. Thus there exists $p'_{i+1} \in [p_i, p_{i+2}]$ such that $\phi(p'_{i+1})$ is not equal to $\phi(p_i)$ and $\phi(p_{i+1})$. If we change p_{i+1} to p'_{i+1} then we take another saturated chain $p_0, p_1, \dots, p'_{i+1}, \dots, p_r$ from p to q such that $\phi(p_{i-1}) = \phi(p'_{i+1})$. By using the same argument, there exists a saturated chain $p = p_0, x = p_1, p_2, \dots, p_r = q$ with $\phi(p_1) = \phi(p_3)$. It contradicts that $[p, q]$ is a 1-interval. So each labels are different. There exists only one saturated chain from p to q is only $p_0, p_1, \dots, p_r$. Assume that there exists another saturated chain $p = q_0, x = q_1, \dots, q_r = q$. So, there exists q_i such that $q_i \neq p_i$ and $q_j = p_j$ $(0 \leq j < i)$. By the above argument, we have $\phi(p_i) = \phi(q_i)$. By (H1-2), p_i and q_i are comparable. If $p_i < q_i$ then we have $q_{i-1} < p_i < q_i$. If $q_i < p_i$ then we have $p_{i-1} < q_i < p_i$. These are contradictions. So $[p, q]$ is of type (V1).

By using a similar argument, if we assume that only y covers p then we obtain that $[p, q]$ is of type (V2). $\square$

For not only 1-intervals but also for a v-interval $[p, q]$, we say that $[p, q]$ is *of type (Vi)* if $[p, q]$ satisfies the following:

(V1) $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $v, v+1, v+2, \dots, v+n-1, v+n, v$ respectively;

(V2) $[p, q]$ is a totally ordered set and consists of $n + 2$ elements. The labels of the elements in the increasing order are $v, v+n, v+n-1, \dots, v+2, v+1, v$ respectively;

(V3) $[p, q]$ is a diamond.

Define a graph automorphism $\Phi : \Gamma \to \Gamma$ by $\Phi(i) := i+1 \pmod{n+1}$. If we apply Φ and Prop. 4.7 to Prop. 7.2 then we can characterize all the 2-intervals. Thus we characterize all the v-intervals for all $v \in N(\Gamma)$.

Proposition 7.3. *Any v-interval $[p, q]$ is either of type (V1), (V2) or (V3).*

Proposition 7.4. *If P contains a v-interval of type (V1) (resp. (V2)) for some $v \in N(\Gamma)$ then P is a totally ordered set and is of type (V1) (resp (V2)).*

Proof. Let $(P, \leq, \phi)$ be a minuscule heap over $\Gamma(\tilde{A}_n)$ which contains a v-interval $[p, q]$ of type (V1). It is sufficient to show that every element of P covers and is covered by at most one of its element. If we assume that there exists $x \in P$ which covers or is covered by two (or more) elements, then we get a contradiction as follows. By $\operatorname{supp}[p, q] = N(\Gamma)$, x is comparable to an element $p \in [p, q]$. Thus we have $p < x$ or $x < q$. Now we assume $p < x$ holds. (If we can consider the dual poset P^*, we can verify the case $x < q$.)

Assume $x \in P$ is covered by two (or more) elements x_1, x_2 of P. We can take a saturated chain $p = p_0 \to p_1 \to \cdots \to p_i = x$. By the shape of Γ, $\phi(x_1)$ or $\phi(x_2)$ is equal to $\phi(p_{i-1})$. Without loss of generality, we can assume that $\phi(x_1) = \phi(p_{i-1})$. By Prop. 4.6, $[p_{i-1}, x_1]$ is a diamond. Thus p_{i-1} must be covered by two (or more) elements. By repeating this argument, p is covered by two (or more) elements. We denote these elements by p_1, p_1'. Since $\phi(p_1')$ is next to $\phi(p)$, we have $\phi(p_1') = \phi(q)$. So p_1', q are comparable. Because p_1' covers p, we have $p_1' < q$. On the other hand, p_1' is not contained in $[p, q]$, a contradiction.

Assume $x \in P$ covers two (or more) elements x_1, x_2 of P. We can take two saturated chains $p = p_0 \to \cdots \to p_{i-1} = x_1 \to p_i = x$ and $p = q_0 \to \cdots \to q_{i-1} = x_2 \to q_i = x$. Thus there is an element $p_k = q_k$ in the saturated chain such that $p_0 = q_0, p_1 = q_1, \ldots, p_k = q_k$ and $p_{k+1} \neq q_{k+1}$. Hence p_k is covered by two elements, a contradiction.

The case for v-interval of type (V2) is proved in a similar way. □

The following corollary immediately follows from the above Proposition.

Corollary 7.5. *Let P be a minuscule heap over Γ. If P contains a v-interval of type (Vi) ($1 \leq i \leq 3$) for some $v \in N(\Gamma)$ then any u-interval is of type (Vi) for any $u \in N(\Gamma)$.*

We should make a remark about minuscule heaps $(P, \leq, \phi)$ with $\operatorname{supp} P \neq \Gamma$. Let $v \in N(\Gamma) \setminus \operatorname{supp} P$. If we choose a graph automorphism $\Phi : \Gamma \to \Gamma$ such that $\Phi(v) = 0$ then we can regard $(P, \leq, \Phi \circ \phi)$ as a minuscule heap over the Dynkin diagram Γ' of type A_n. And the minuscule heaps over Γ' are already classified. To summarize, we now know the following:,

- If P is a totally ordered set then P is either of type (T1) or (T2).
- If P is not a totally ordered set with $\operatorname{supp} P \neq \Gamma$ then P is a minuscule heap over a Dynkin diagram of type A.

The remaining case is that when P is not totally ordered and $\operatorname{supp} P = \Gamma$. We know that any v-interval of P is a diamond for any $v \in N(\Gamma)$. In §9 we study such minuscule heaps.

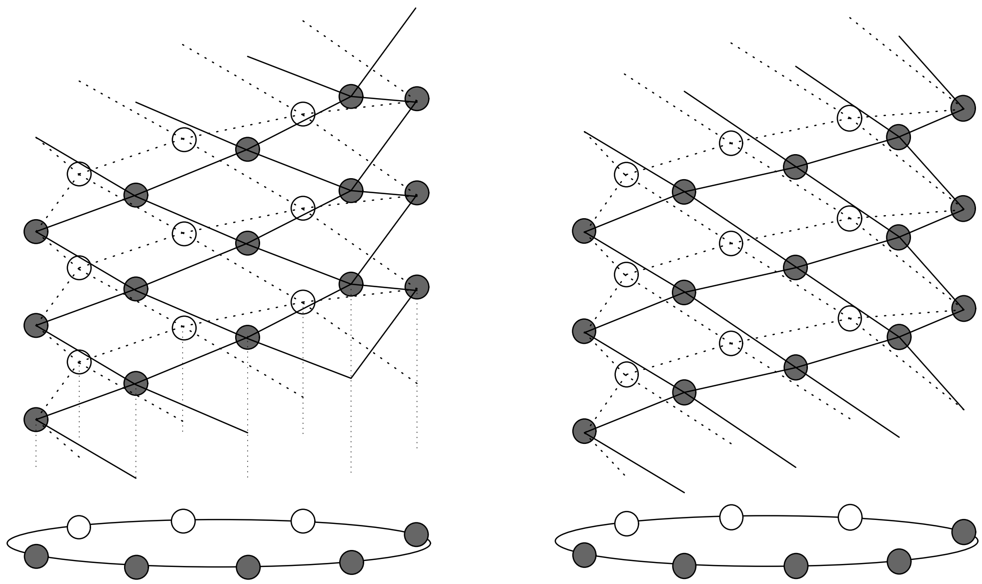

Figure 2: extended slant lattices L_3 (left figure) and L_4 (right figure) over $\Gamma(\tilde{A}_7)$

8 An Extended Slant Lattice

In this section we introduce the notion of extended slant lattices L_k $(1 \leq k \leq n)$ (see figure 2) which are Γ-labeled posets such that any minuscule heaps over $\Gamma(\tilde{A}_n)$ are isomorphic to those convex subsets.

We put $L_0 := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \pmod 2\}$ and define a relation $\rightarrow$ on L_0 by $(v, m) \rightarrow (v, m)+(1, -1)$ or $(v, m) \rightarrow (v, m)+(-1, -1)$. Let $\leq$ be the reflective, transitive closure of $\rightarrow$. We note that $(L_0, \leq)$ is a poset. Then the following lemma is obtained immediately.

Lemma 8.1. *Let Q be an order ideal generated by $(0, 0)$ of L_0. Then $Q = \{(a, b) | b \geq |a|\}$, where $||$ is the absolute value symbol.*

For $1 \leq k \leq n$, we define a map $\psi_k : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ by putting $\psi_k(v, m) := (v + n + 1, m - 2k + n + 1)$. Then ψ_k is a poset automorphism of $(\mathbb{Z} \times \mathbb{Z}, \leq)$. Let G_k be the group generated by ψ_k. Then G_k satisfies the conditions of Prop. 3.3. Hence we define a poset L_k by putting

$$L_k := (\mathbb{Z} \times \mathbb{Z}/G_k, \leq_{L_k}),$$

and call it *an extended slant lattice.* We note that the sum $+$, defined by putting $\overline{(v, m)} + \overline{(u, m')} := \overline{(v + u, m + m')}$ for $\overline{(v, m)}$ and $\overline{(u, m')}$, is well-defined on L_k. $\{(v, m) \in [0, n] \times \mathbb{Z} | v \equiv m \pmod 2\}$ is a complete representative system of L_k. Thus we can define a map $\phi : L_k \rightarrow [0, n]$ by putting $\phi(\overline{(v, m)}) := v$, where $0 \leq v \leq n$. Now we regard $L_k = (\mathbb{Z} \times \mathbb{Z}/G_k, \leq_{L_k}, \phi)$ as a Γ-labeled poset by this map.

Let us define a binary relation $\rightsquigarrow$ by $\overline{(v,m)} \rightsquigarrow \overline{(v+1,m-1)}, \overline{(v,m)} \rightsquigarrow \overline{(v-1,m-1)}$ for $\overline{(v,m)}, \overline{(v+1,m-1)}, \overline{(v-1,m-1)} \in L_k$ and put $\preceq$ the reflective and transitive closure of $\rightsquigarrow$.

Lemma 8.2. *$\preceq$ is equivalent to $\leq$.*

Proof. Let p, q be elements of L_k which satisfy $p \preceq q$. Then we can take a sequence $p = p_0 \rightsquigarrow p_1 \rightsquigarrow \cdots \rightsquigarrow p_r = q$. Now we have $p_i = p_{i-1} + \overline{(\pm 1, -1)}$. If we put $p_{i-1} = \overline{(v,m)}$ then we can write $p_i = \overline{(v \pm 1, m-1)}$. By the definition of $\leq$, we have $(v,m) \leq_{L_0} (v \pm 1, m-1)$. Thus we have $\overline{(v,m)} \leq_{L_k} \overline{(v \pm 1, m-1)}$. This implies $p \leq_{L_k} q$.

Conversely assume that $\overline{(v,m)} \leq_{L_k} \overline{(u,l)}$. Then we have $(v,m) \leq_{L_0} (u,l) + j(n+1, n+1-2k)$ for some $j \in \mathbb{Z}$. Let $p_0, p_1, \ldots, p_r$ be a saturated chain from (v,m) to $(u + j(n+1), l + j(n+1-2k))$. Then we have $\overline{p_0} \rightsquigarrow \overline{p_1} \rightsquigarrow \cdots \rightsquigarrow \overline{p_r}$. Hence $\overline{(v,m)} \preceq \overline{(u,l)}$. □

Although the following proposition maybe seems to be obvious, it does not hold on L_1, L_n.

Proposition 8.3. *For $2 \leq k \leq n-1$, $\overline{(v,m)} \in L_k$ covers $\overline{(v \pm 1, m+1)}$. $\overline{(v,m)}$ is covered by $\overline{(v \pm 1, m-1)}$.*

Proof. It is sufficient to show that $\rightsquigarrow$ is a cover relation on L_k. To do so, we show that $\overline{(a,b)} = s_0 \rightsquigarrow s_1 \rightsquigarrow \cdots \rightsquigarrow s_r = \overline{(a \pm 1, b-1)}$ implies $r = 1$ by Lemma 3.2.

First we prove in the case $s_r = \overline{(a+1, b-1)}$. Put $\alpha := \#\{1 \leq i \leq r | s_i = s_{i-1} + \overline{(1,-1)}\}$ and put $\beta := \#\{1 \leq i \leq r | s_i = s_{i-1} + \overline{(-1,-1)}\}$. (We note that $\alpha + \beta = r$ and $\alpha, \beta \geq 0$.) Then we have $s_r = \overline{(a + \alpha - \beta, b - \alpha - \beta)} = \overline{(a+1, b-1)}$. This implies that

$$\begin{aligned} a + \alpha - \beta &= a + 1 + \gamma(n+1), \\ b - \alpha - \beta &= b - 1 + \gamma(n+1-2k), \end{aligned}$$

for some $\gamma \in \mathbb{Z}$. The solution of above equations is $\alpha = 1 + k\gamma$, $\beta = \gamma(k-n-1)$. By the assumption, we have $2 \leq k \leq n-1$ and $\alpha \geq 0$. Thus, $\gamma \geq 0$. On the other hand, $k - n - 1 < 0$ and $\beta \geq 0$. So, $\gamma \leq 0$. This implies $\gamma = 0$. Then we have $\alpha = 1$ and $\beta = 0$. So we conclude that $r = \alpha + \beta = 1 + 0 = 1$ holds.

By using a similar argument, we have $\alpha = 0$ and $\beta = 1$ from the case $s_r = \overline{(a-1, b-1)}$. □

Remark 8.4. In the proof above, there exists a solution $\alpha = 0, \beta = n$ for $s_r = \overline{(a+1, b-1)}$, $k = 1$ and $\gamma = -1$. Hence $\rightsquigarrow$ is not a covering relation on L_1.

Proposition 8.5. *L_1, L_n are totally ordered sets.*

Proof. We give only proof for L_1. (We can similarly prove it for L_n.)

We know that $\{\overline{(v,m)} | 0 \leq v \leq n, m \equiv v \pmod 2\}$ is a complete representative system. For $\overline{(0, 2m)}$, we have $\overline{(0,2m)} = \overline{(m(n+1), m(n+1))}$. So we can write $\overline{(i, i+2m)} = \overline{(i + m(n+1), i + m(n+1))}$ for $0 \leq i \leq n$. Thus L_1 is isomorphic to $\mathbb{Z}$ as poset. (We note that $-j$ is an element on $\mathbb{Z}$ corresponding to $\overline{(j,j)}$. For example, we have $\overline{(0,0)} > \overline{(1,1)}$ in the ordering $\leq$ on L_1.) □

Corollary 8.6. *For $1 \le k \le n$, we have that (H1-1),(H1-2) and (H2a) hold on L_k, and their finite convex subsets are minuscule. Moreover their v-intervals are diamonds when $2 \le k \le n-1$.*

Proof. By Propositions 8.3, 8.5, (H1-1) holds on L_k. By the definition of $\preceq$ and by the fact that (H1-2) holds on L_0, (H1-2) holds on L_k.

Next we show that (H2a) holds. If $k = 1$ or $k = n$ then P has the structure in Prop. 6.2. So (H2a) holds. By the definition of L_k, a v-interval is a diamond when $2 \le k \le n-1$. So (H2a) holds.

Thus their finite convex subsets are minuscule. $\square$

For a finite subset Q of L_k and $v \in N(\Gamma)$, we can define a unique maximal (resp. minimal) element t_v (resp. b_v) of Q_v since (H1-2) holds on L_k

Lemma 8.7. *Let $y_0 \in \mathbb{Z}$ and put $Q := \{p \in L_k | p \le \overline{(k, y_0)}\}$. Then we have*

$$Q = \{\overline{(a,b)} | 0 \le a \le k, b \ge y_0 + k - a\} \cup \{\overline{(a,b)} | k < a \le n, b \ge y_0 + a - k\}.$$

Let $x_0 \in \mathbb{Z}$ and put $Q' := \{p \in L_k | p \ge \overline{(n+1-k, x_0)}\}$. Then we have

$$\begin{aligned} Q' = \{\overline{(a,b)} | 0 \le a \le n+1-k, b \le x_0 + k + a\} \\ \cup \{\overline{(a,b)} | n+1-k < a \le n, b \le x_0 + 2n + 2 - k - a\}. \end{aligned}$$

Proof. We can take a complete representative system $\{\overline{(v,m)} | 0 \le v \le n, m \equiv v \pmod 2\}$. For $0 \le a \le n$, we have

$$\begin{aligned} & \overline{(a,b)} \le \overline{(k, y_0)} \\ \iff & (a,b) + j(n+1, n+1-2k) \le (k, y_0) \ \exists j \in \mathbb{Z} \\ \iff & (a-k+j(n+1), b-y_0+j(n+1-2k)) \le (0,0) \ \exists j \in \mathbb{Z} \\ & \text{(By Lemma 8.1)} \\ \iff & b - y_0 + j(n+1-2k) \ge |a-k+j(n+1)| \ \exists j \in \mathbb{Z}. \end{aligned}$$

Assume $0 \le a \le k$. Then we have $a - k + j(n+1) \le 0 \iff j \le 0$ or equivalently we have $a - k + j(n+1) \ge 0 \iff j \ge 1$. For $j \le 0$, we have

$$\begin{aligned} b & \ge y_0 - j(n+1-2k) - a + k - j(n+1) \\ & = y_0 + k - a - 2j(n+1-k). \end{aligned}$$

Hence the necessary and sufficient condition for $(a-k+j(n+1), b-y_0+k+j(n+1-2k)) \le (0,0)$ (for some $j \le 0$) is

$$b \ge y_0 + k - a.$$

For $j \ge 1$, we have

$$\begin{aligned} b & \ge y_0 - j(n+1-2k) + a - k + j(n+1) \\ & = y_0 + a - k + 2jk. \end{aligned}$$

Hence the necessary and sufficient condition for $(a-k+j(n+1), b-y_0+j(n+1-2k)) \leq (0,0)$ (for some $j \geq 1$) is

$$b \geq y_0 + a.$$

Thus we have $b \geq y_0 + k - a$. Assume $k < a \leq n$. We have $a - k + j(n+1) \geq 0 \iff j \geq 0$ or equivalently $a - k + j(n+1) \leq 0 \iff j \leq -1$. For $j \geq 0$ we have

$$b \ \geq \ y_0 + a - k.$$

For $j \leq -1$ we have

$$b \ \geq \ y_0 - a + 2n + 2 - k.$$

On the other hand, we have $y_0 - a + 2n + 2 - k \geq y_0 + a - k$. Thus we have $b \geq y_0 + a - k$.

We can prove for the case for Q' in a similar way. $\square$

Let $x_0, y_0 \in \mathbb{Z}$ satisfying $x_0 \geq y_0$. For $v \in N(\Gamma)(= [0,n])$, we define y_v by putting $y_v := y_0 + k - v$ if $0 \leq v \leq k$ and $y_v := y_0 + a - k$ if $k < v \leq n$. We define x_v by putting $x_v := x_0 + k + v$ if $0 \leq v \leq n+1-k$ and $x_v := x_0 + 2n + 2 - k - v$ if $n+1-k < v \leq n$. Then we have the following,

Proposition 8.8.

$$[\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}] = \{(v,m) \in L_k | y_v \leq m \leq x_v\}$$

Proof. It is obvious if we take $Q \cap Q'$ in Lemma 8.7. $\square$

Proposition 8.9. *Let $2 \leq k \leq n-1$ and let Q be a finite subposet of L_k satisfying* $\operatorname{supp} Q = \Gamma$. *For $v \in N(\Gamma)$, put $b_v := \overline{(v, x_v)}$ and $t_v := \overline{(v, y_v)}$, where t_v (resp b_v) is a maximal (resp. minimal) element of P_v.*

Q is a convex subset if and only if Q satisfies the followings,

- $Q_v = \{\overline{(v,m)} \in L_k | y_v \leq m \leq x_v\}$,
- *t_{v-1}, t_v are a covering pair,*
- *b_{v-1}, b_v are a covering pair,*

for each $1 \leq v \leq n+1$.

Proof. Let Q be a convex subset. Since (H1-2) holds on L_k, we have $Q_v = \{\overline{(v,m)} \in L_k | y_v \leq m \leq x_v\}$.

Assume that $\overline{(v-1,m)}, \overline{(v,l)} \in Q$ for some $1 \leq v \leq n$. If we have $m < l$ then there exists a saturated chain $\overline{(v,l)} \to \overline{(v-1,l-1)} \to \overline{(v,l-2)} \to \cdots \to \overline{(v,m-1)} \to \overline{(v-1,m)}$ on L_k. Since Q is a convex subset, this saturated chain is contained in Q. This implies that we have $-1 \leq y_{v-1} - y_v$. Next we assume $m > l$. Then we have $y_{v-1} - y_v \leq 1$ by a similar argument. So we have $-1 \leq y_{v-1} - y_v \leq 1$. This implies that t_{v-1} and t_v are a covering pair. By a similar argument, b_{v-1} and b_v are a covering pair.

$\overline{(0,m)} = \overline{(n+1, m-2k+n+1)}$ implies that we have $|y_n - y_0 + 2k - n - 1| = |x_n - x_0 + 2k - n - 1| = 1$. This implies that t_0 and t_n (or b_0 and b_n) are a covering pair.

Conversely let Q be a subset satisfying the condition. We claim that Q is obtained from $[\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}]$ by deleting its maximal or minimal elements. Thus it follows that P is a convex subset by Propositions 3.4, 3.5.

For $1 \leq v \leq n$ if we have $t_{v-1} \leftarrow t_v$ and $t_v \rightarrow t_{v+1}$ then we construct another poset $Q' = Q_v \cup \{\overline{(v, y_v - 2)}\}$. We note that Q' satisfies the condition. We regard Q as $Q' \setminus \{\overline{(c, y_v - 2)}\}$. We also note that $Q_0 = Q'_0$. By repeating this operation, we obtain a poset Q' such that $t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_k$ and $t_k \leftarrow t_{k+1} \leftarrow \cdots \leftarrow t_n \leftarrow t_{n+1} = t_0$. For $1 \leq v \leq n$ if we have $b_{v-1} \rightarrow b_v$ and $b_v \leftarrow b_{v+1}$ then we obtain another poset $Q' \cup \{\overline{(v, x_v + 2)}\}$. By repeating this operation, we obtain a poset Q' such that $b_0 \leftarrow b_1 \leftarrow \cdots \leftarrow b_{n+1-k}$ and $b_{n+1-k} \rightarrow \cdots \rightarrow b_n \rightarrow b_{n+1} = b_0$. By Prop. 8.8, we have $Q' = [\overline{(n+1-k, x_0+n+1-k)}, \overline{(k, y_0-k)}]$. In particular Q' is a convex subset. Thus Q is a convex subset. $\square$

9 Classification of Minuscule Heaps over $\Gamma(\tilde{A})$

In this section, we classify the minuscule heaps over $\Gamma = \Gamma(\tilde{A}_n)$ (Theorem 9.8). In previous sections, we classified some minuscule heaps. The remaining case is that of minuscule heap P over Γ such that $n \geq 2$, supp $P = \Gamma$, and its v-intervals are of type (V3).

Let P be a minuscule heap over Γ which is not totally ordered. First we observe the structure of $P_i \cup P_{i+1}$, where $P_i = \{p \in P | \phi(p) = i\}$. By (H1-2), $P_i \cup P_{i+1}$ is a totally ordered set. We denote the number of element of P_i (resp. P_{i+1}) by r_i (resp. r_{i+1}). Therefore there are $(r_i - 1)$ i-intervals in $P_i \cup P_{i+1}$. We note that each i-interval contains one element of P_{i+1}. Thus we have $r_{i+1} \geq r_i - 1$. By a similar argument, we have $r_i \geq r_{i+1} - 1$. Hence we have $r_i + 1 \geq r_{i+1} \geq r_i - 1$.

Let us assume $r_{i+1} = r_i - 1$. Now there are $r_i - 1 (= r_{i+1})$ i-intervals. Each i-interval contains one element of P_{i+1}. Since an i-interval is a diamond, if we take an element of P_i and an element of P_{i+1} which is contained in the interval then they are a covering pair. Let $p_1, p_2, \ldots, p_{r_i}$ be the elements of P_i in the increasing order and let $q_1, q_2, \ldots, q_{r_{i+1}}$ be the elements of P_{i+1} in the increasing order. By the argument above, we have

$$p_1 \rightarrow q_1 \rightarrow p_2 \rightarrow q_2 \rightarrow \cdots \rightarrow q_{r_{i+1}} \rightarrow p_{r_i}.$$

By a similar argument if we assume $r_{i+1} = r_i + 1$ then we have

$$q_1 \rightarrow p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \cdots \rightarrow p_{r_i} \rightarrow q_{r_{i+1}}.$$

Next let us assume $r_i = r_{i+1}$. By a similar argument, we have either

$$q_1 \rightarrow p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow \cdots \rightarrow q_{r_{i+1}} < p_{r_i}$$

or

$$p_1 < q_1 \rightarrow p_2 \rightarrow q_2 \rightarrow \cdots \rightarrow q_{r_i} \rightarrow p_{r_i} \rightarrow q_{r_{i+1}}.$$

In the former case, $[p_{r_i-1}, p_{r_i}]$ is an i-interval. Thus we have $q_{r_{i+1}} \to p_{r_i}$. In the latter case, we have $p_1 \to q_1$. At this point, we have completed to determine the structure of $P_i \cup P_{i+1}$. We record these results in the following corollaries 9.1, 9.2,

Corollary 9.1. *Let P be a minuscule heap with* $\operatorname{supp} P = \Gamma$ *such that P is not a totally ordered set. Then t_i, t_{i+1} (b_i, b_{i+1}) are a covering pair for each $i \in N(\Gamma)$.*

Corollary 9.2. *For $v \in N(\Gamma)$, put $r_v := \#P_v$. Then we have,*

- *"$r_{i+1} = r_i$" is equivalent to either "$t_i \to t_{i+1}$ and $b_i \to b_{i+1}$" or "$t_i \leftarrow t_{i+1}$ and $b_i \leftarrow b_{i+1}$",*
- *$r_{i+1} =$ "$r_i + 1$" is equivalent to "$t_i \to t_{i+1}$ and $b_i \leftarrow b_{i+1}$",*
- *$r_{i+1} =$ "$r_i - 1$" is equivalent to "$t_i \leftarrow t_{i+1}$ and $b_i \to b_{i+1}$".*

We call the number of covering pairs satisfying $t_v \to t_{v+1}$ $(1 \le v \le n+1)$ the *gradient* of P.

Proposition 9.3. *Let P be a minuscule heap over Γ which is not totally ordered and* $\operatorname{supp} = \Gamma$. *Let k be a gradient of k. Then we have $2 \le k \le n-1$.*

Proof. If $k = 1$ then there is a unique covering pair t_{i-1}, t_i such that $t_{i-1} \to t_i$. On the other hand, we have $t_{i-1} \to t_{i-2} \to \ldots t_{i+1} \to t_i$. It contradicts that t_{i-1}, t_i are a covering pair.

We can prove, for the case $k = n$, by using a similar argument. □

Let P be a not totally ordered minuscule heap with $\operatorname{supp} P = \Gamma$ and let k be the gradient of P. Define a map $\nu : P \to L_k$ by putting,

$$\nu(t_0) := \overline{(0,0)},$$

$$\nu(t_i) := \begin{cases} \nu(t_{i-1}) + \overline{(1,-1)} & \text{if } t_{i-1} \to t_i, \\ \nu(t_{i-1}) + \overline{(1,1)} & \text{if } t_{i-1} \leftarrow t_i, \end{cases}$$

and if p is j-th largest element in P_v then

$$\nu(p) := \nu(t_v) + \overline{(0, 2j-2)}.$$

We note that ν is an injection.

Proposition 9.4. *Let t_n (resp. t_0) be the unique maximal element of P_n (resp. P_0). If $t_n \to t_0$ then we have $\nu(t_n) \to \nu(t_0)$. If $t_n \leftarrow t_0$ then we have $\nu(t_n) \leftarrow \nu(t_0)$.*

Proof. Assume $t_n \to t_0$. Now we have $\#\{1 \le i \le n | t_{i-1} \to t_i\} = k-1$. Thus,

$$\begin{aligned} \nu(t_n) &= \nu(t_0) + (k-1)\overline{(1,-1)} + (n-k+1)\overline{(1,1)} \\ &= \nu(t_0) + \overline{(n, n-2k+2)} \\ &= \nu(t_0) + \overline{(-1,1)} \\ &\to \nu(t_0). \end{aligned}$$

Assume $t_n \leftarrow t_0$. Now we have $\#\{1 \leq i \leq n | t_{i-1} \rightarrow t_i\} = k$. Thus we have

$$\begin{aligned} \nu(t_n) &= \nu(t_0) + k\overline{(1,-1)} + (n-k)\overline{(1,1)} \\ &= \nu(t_0) + \overline{(n, n-2k)} \\ &= \nu(t_0) + \overline{(-1,-1)} \\ &\leftarrow \nu(t_0). \end{aligned}$$

$\square$

We note that ν preserves the cover-relations on $t_0, t_1, \ldots, t_n$.

Proposition 9.5. *Let P be a minuscule heap over Γ which is not totally ordered and let k be the gradient of P. Let ν be a map defined as above. Then* $\operatorname{Im} \nu$ *is a convex subset in L_k.*

Proof. Put $Q := \operatorname{Im} \nu$. For $0 \leq v \leq n$, Q_v satisfies the condition in Prop. 8.9 by the definition of ν. Let t'_v be a maximal element of Q_v.

Then t'_v and t'_{v+1} are a covering pair since ν preserves the cover-relations on $t_0, t_1, \ldots, t_n$. Finally we show that b'_v and b'_{v+1} are a covering pair, where b'_v is a minimal element of Q_v. Assume $t_v \rightarrow t_{v+1}$. By Corollary 9.2, $b_v \rightarrow b_{v+1}$ implies $r_v = r_{v+1}$. Thus we have $b'_{v+1} = \nu(b_{v+1}) = \nu(t_{v+1}) + \overline{(0, 2(r_{v+1}-1))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r_v - 1))} = \nu(b_v) + \overline{(1,-1)} \leftarrow \nu(b_v) = b'_v$. $b_v \leftarrow b_{v+1}$ implies $r_{v+1} = r_v + 1$. Now we have $b'_{v+1} = \nu(b_{v+1}) = \nu(t_v) + \overline{(0, 2(r_{v+1}-1))} = \nu(t_{v+1}) + \overline{(0, 2r_v)} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2r_v)} = \nu(t_v) + \overline{(1,1)} + \overline{(0, 2(r_v - 1))} = \nu(b_v) + \overline{(1,1)} \rightarrow \nu(b_v) = b'_v$.

We can prove in the case $t_v \leftarrow t_{v+1}$ by a similar argument. $\square$

Proposition 9.6. *Let P be a minuscule heap over Γ which is not totally ordered and let k be the gradient of P. Then P is isomorphic to* $\operatorname{Im} \nu$ *as a Γ-labeled poset.*

Proof. Let p, q be elements of P which satisfy $p \rightarrow q$. Then we show that $\nu(p) < \nu(q)$. Assume $p \in P_v$ and $q \in P_{v+1}$. (We can similarly argue in the case $p \in P_{v+1}, q \in P_v$.) And we assume $t_v \rightarrow t_{v+1}$. If p is the r-th element in P_v by an increasing order then q is the r-th element in P_{v+1} by an argument in the preceding Corollary 9.1. Thus we have $\nu(q) = \nu(t_{v+1}) + \overline{(0, 2(r-1))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} \leftarrow \nu(p)$. Hence we have $\nu(q) \geq \nu(p)$. Assume $t_v \leftarrow t_{v+1}$. If p is the r-th element in P_v then q is the $(r-1)$-th element in P_{v+1}. Thus we have $\nu(q) = \nu(t_{v+1}) + \overline{(0, 2(r-2))} = \nu(t_v) + \overline{(1,1)} + \overline{(0, 2(r-2))} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(p) + \overline{(1,-1)} \leftarrow \nu(p)$. So we have $\nu(q) \geq \nu(p)$.

The arguments above show that $p \leq q$ implies $\nu(p) \leq \nu(q)$.

Conversely we assume $\nu(p) \rightarrow \nu(q)$. Since $\operatorname{Im} \nu$ is a convex subset, we have either $\nu(q) = \nu(p) + \overline{(1,-1)}$ or $\nu(q) = \nu(p) + \overline{(-1,-1)}$. Assume that $p \in P_v, q \in P_{v+1}$. (We can prove in the case $p \in P_{v+1}, q \in P_v$ by a similar argument.) If $t_v \rightarrow t_{v+1}$ then we have $\nu(q) = \nu(p) + \overline{(1,-1)} = \nu(t_v) + \overline{(0, 2(r-1))} + \overline{(1,-1)} = \nu(t_v) + \overline{(1,-1)} + \overline{(0, 2(r-1))} = \nu(t_{v+1}) + \overline{(0, 2(r-1))}$. Thus q is the r-th element in P_{v+1}. By an argument in the preceding Corollary 9.1, we have $p \rightarrow q$. If $t_v \leftarrow t_{v+1}$ then q is the $(r-1)$-th element in P_{v+1} by a similar argument. Thus we have $p \to q$. Now Im ν is a convex subset. Hence if $\nu(p) \le \nu(q)$ then we can take a saturated chain from $\nu(p)$ to $\nu(q)$. Thus we have $p \le q$. □

By Propositions 9.5, 9.6, it follows that

Corollary 9.7. *Let P be a minuscule heap with gradient k ($2 \le k \le n-1$) which is not totally ordered. Then P is isomorphic to a convex subset of L_k.*

Our main theorem gives the converse claim (Theorem 9.8).

In fact, a totally ordered minuscule heap is also isomorphic to a convex subset of L_k for $k = 1, n$. Let Q be a totally ordered minuscule heap of type (T1) (resp. (T2)). Define a map $\nu : Q \to L_n$ by the following. If the label of a maximal element of Q is i then $\nu(q) = \overline{(i-j+1, i-j+1)}$ (resp. $\overline{(i+j-1, i-j+1)}$ where q is the j-th element in Q by an increasing ordering. It is obvious that Q is isomorphic to Im ν as a Γ-labeled poset. Now Im Q is a convex subset of L_n (resp. L_1).

We note that a minuscule heap with supp $P \neq \Gamma$ can be identified with a minuscule heap over a Dynkin diagram of type A. Then it is isomorphic to a convex subset of L_k for some $1 \le k \le n$.

Theorem 9.8. *Let $n \ge 2$. Let Γ be a Dynkin diagram of type $\tilde{A}_n$. A minuscule heap P over Γ is isomorphic to a convex subset of L_k, where $1 \le k \le n$ is determined by P. Conversely a finite convex subset of L_k is a minuscule heap.*

Remark 9.9. If a minuscule heap P satisfies supp $P \neq \Gamma$ then k in Theorem 9.8 is not unique.

Acknowledgments. The author would like to thank Professors I. Terada and K. Koike for helpful discussions.

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