On the Associative Nijenhuis Relation

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Abstract

We give the construction of a free commutative unital associative Nijenhuis algebra on a commutative unital associative algebra based on an augmented modified quasi-shuffle product.

Keywords: quasi-shuffle, modified quasi-shuffle, associative Nijenhuis relation, Rota-Baxter relation, Loday-type algebras.

1 Introduction

The associative analog of the Nijenhuis relation [6] may be regarded as the homogeneous version of the Rota-Baxter relation [31, 29, 28, 30, 3, 4, 7, 9]. Some of its algebraic aspects especially

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with regard to the notion of quantum bihamiltonian systems were investigated by Cariñena *et al.* in [6]. The Lie algebraic version of the associative Nijenhuis relation is investigated in [12, 13] in the context of the classical Yang-Baxter equation, which is closely related to the Lie algebraic version of the Rota-Baxter relation (see especially [5]). Likewise in [18] the deformations of Lie brackets defined by Nijenhuis operators and in [17] the connections to the (modified) classical Yang-Baxter relation are studied. In Kreimer's work [19, 20] the connection of the Rota-Baxter relation to the Riemann-Hilbert problem in the realm of perturbative Quantum Field Theory is reviewed.

The algebraic properties of the associative notion of the Nijenhuis relation, respectively Nijenhuis algebras, provide interesting insights into associative analogs of Lie algebraic structures. Also, both the Rota-Baxter and the associative Nijenhuis relation are of interest with respect to the Hopf algebraic formulation of the theory of renormalization in perturbative Quantum Field Theory and especially to aspects of integrable systems.

After defining a commutative unital associative Nijenhuis algebra we use an augmented modified quasi-shuffle product to give explicitly the construction of the free commutative unital associative Nijenhuis algebra generated by a commutative unital associative $\mathbb{K}$-algebra. We follow thereby closely the inspiring work of Guo *et al.* [14, 15]. The main aspect of this construction of the free object is the use of a "homogenized" notion of Hoffman's quasi-shuffle product [16] giving an associative, unital, and commutative composition. This natural ansatz relies on the close resemblance between the Rota-Baxter and the associative Nijenhuis relation. The free construction of the former is essentially given by a modified quasi-shuffle product, called mixable shuffle in [14].

Let us remark here that in recent papers [9, 1, 2] it was shown that non-commutative[1] Rota-Baxter algebras always give Loday-type algebras, i.e. dendriform di- and trialgebra structures [22, 23, 24].

Also, we would like to point the reader to recent interesting work, which appeared after this article was prepared. In [10, 21], the associative Nijenhuis identity and its related shuffle relation were linked to Loday-type structures in a detailed manner, giving rise to new structures in this realm.

The paper is organized as follows. In Section 2 we give the definition of a commutative, unital and associative Nijenhuis $\mathbb{K}$-algebra and the associative Nijenhuis relation. We introduce the notion of Nijenhuis homomorphism and free Nijenhuis algebra. In Section 3 we define the modified and augmented modified quasi-shuffle product and establish their properties. In Section 4 the augmented modified quasi-shuffle product algebra is identified as the free commutative associative unital Nijenhuis algebra, thereby giving an explicit construction of the latter. This section closes with a remark concerning the connection to Loday-type algebras.

Throughout this paper, we will consider $\mathbb{K}$ to be a commutative field of characteristic zero. The term $\mathbb{K}$-algebra always means if not stated otherwise associative commutative unital $\mathbb{K}$-algebra.

[1] not necessarily commutative

2 The associative Nijenhuis Relation

Definition 2.1. *A Nijenhuis $\mathbb{K}$-algebra is a pair $(\mathcal{A}, N)$, where $\mathcal{A}$ is a $\mathbb{K}$-algebra and $N : \mathcal{A} \to \mathcal{A}$ is a linear operator satisfying the associative Nijenhuis relation:*

$$N(x)N(y) + N^2(xy) = N\big(N(x)y + xN(y)\big), \quad x, y \in \mathcal{A}. \tag{1}$$

The map $N : \mathcal{A} \to \mathcal{A}$ might be called associative Nijenhuis operator or just Nijenhuis map for short.

Remark 2.2. *(i) In this setting "associative" refers to the relation (1) to distinguish it clearly from its well-known Lie algebraic version [6, 12, 13, 18, 17]:*

$$[N(x), N(y)] + N^2([x, y]) = N\big([N(x), y] + [x, N(y)]\big). \tag{2}$$

(ii) A Nijenhuis map on a $\mathbb{K}$-algebra $\mathcal{A}$ gives also a Nijenhuis map for the associated Lie algebra $(\mathcal{A}, [\ ,\ ])$, where $[\ ,\ ]$ is the commutator.
(iii) With $N : \mathcal{A} \to \mathcal{A}$ being a Nijenhuis operator, also the operator $\widehat{N} := id_{\mathcal{A}} - N$ satisfies relation (1).

Equation (1) may be interpreted as the homogeneous version of the standard form of the Rota-Baxter relation of weight $\lambda \in \mathbb{K}$ [9]:

$$R(x)R(y) + \lambda R(xy) = R\big(R(x)y + xR(y)\big). \tag{3}$$

With respect to the Rota-Baxter relation, a simple transformation $R \to \lambda^{-1}R$ gives the so called standard form of (3), i.e., weight $\lambda = 1$. The homogeneity of relation (1) destroys this freedom to renormalize the operator N, so as to allow for either sign in front of the second term on the left-hand side of (1) for instance.

Let us give two examples of operators fulfilling the associative Nijenhuis relation.

Example 2.3. *(i) Left or right multiplication $L_a b := ab$, $R_a b := ba$, $a, b \in \mathcal{A}$ both satisfy relation (1).*
(ii) Another class of associative Nijenhuis operators comes from idempotent Rota-Baxter operators. Let $(\mathcal{A}, R)$ be a Rota-Baxter algebra with R being an idempotent ($R^2 = R$) Rota-Baxter operator satisfying relation (3) for $\lambda = 1$. The operator $\widehat{R} := id_{\mathcal{A}} - R$ being idempotent, too, also satisfies equation (3) (for $\lambda = 1$). Define the following operator:

$$N_\tau := \widehat{R} - \tau R = id_{\mathcal{A}} - (\tau + 1)R, \quad \tau \in \mathbb{K}. \tag{4}$$

The map N_τ is an associative Nijenhuis operator. The case $\tau = 1$ relates to the interesting modified Rota-Baxter relation:

$$N_1(x)N_1(y) + xy = N_1\big(N_1(x)y + xN_1(y)\big) \tag{5}$$

One finds similar examples and further algebraic aspects related to equation (1) in [6].

2.1 Nijenhuis Homomorphism and the Universal Property

We introduce now the notion of a Nijenhuis and Rota-Baxter algebra homomorphism.

Definition 2.4. *We call an algebra homomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ between Nijenhuis (respectively Rota-Baxter) $\mathbb{K}$-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ a Nijenhuis (resp. Rota-Baxter) homomorphism if and only if ϕ intertwines with the Nijenhuis (resp. Rota-Baxter) operators N_1, N_2 (resp. R_1, R_2):*

$$\phi \circ X_1 = X_2 \circ \phi, \ \ X_i = N_i \ (resp. \ R_i), \ i = 1, 2. \tag{6}$$

Definition 2.5. *A Nijenhuis (respectively Rota-Baxter) $\mathbb{K}$-algebra $F(\mathcal{A})$ generated by a $\mathbb{K}$-algebra $\mathcal{A}$ together with an associative algebra homomorphism $j_\mathcal{A} : \mathcal{A} \to F(\mathcal{A})$ is called a free Nijenhuis (resp. Rota-Baxter of weight $\lambda \in \mathbb{K}$) $\mathbb{K}$-algebra if the following universal property is satisfied: for any associative Nijenhuis (resp. Rota-Baxter) $\mathbb{K}$-algebra $\mathcal{X}$ and $\mathbb{K}$-algebra homomorphism $\phi : \mathcal{A} \to \mathcal{X}$ there exists a unique Nijenhuis (resp. Rota-Baxter) homomorphism $\tilde{\phi} : F(\mathcal{A}) \to \mathcal{X}$ such that the diagram:*

$$\begin{tikzcd} \mathcal{A} \arrow[d, "j_\mathcal{A}"'] \arrow[dr, "\phi"] & \\ F(\mathcal{A}) \arrow[r, "\tilde{\phi}"'] & \mathcal{X} \end{tikzcd}$$

commutes.

In the next section we introduce Hoffman's quasi-shuffle product and an augmented respectively augmented modified version of it to give the construction of the free Nijenhuis algebra.

3 Modified Quasi-Shuffle Product

Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. We denote the product on $\mathcal{A}$ by $[ab] \in \mathcal{A}$, $a, b \in \mathcal{A}$ and the algebra unit by $e \in \mathcal{A}$, $[ea] = a$, $a \in \mathcal{A}$.

Consider the tensor module of $\mathcal{A}$:

$$\mathcal{T}(\mathcal{A}) := \bigoplus_{n \geq 0} \mathcal{A}^{\otimes^n}, \tag{7}$$

where $\mathcal{A}^{\otimes^0} = \mathbb{K}$. We denote generators $a_1 \otimes \cdots \otimes a_n$ by concatenation i.e., words $a_1 \dots a_n$. Generally we use capitals $U \in \mathcal{A}^{\otimes^n}$, $n > 1$ for words and lower case letters $a \in \mathcal{A}$ for letters. The natural grading on $\mathcal{T}(\mathcal{A})$ is defined by the length $\mathfrak{l}$ of words, such that $\mathfrak{l}(UV) := \mathfrak{l}(U) + \mathfrak{l}(V) = n + m$, for $U \in \mathcal{A}^{\otimes^n}$ and $V \in \mathcal{A}^{\otimes^m}$, n, $m \geq 0$ i.e., for the empty word denoted by $1 \in \mathbb{K} = \mathcal{A}^{\otimes^0}$ we have $\mathfrak{l}(1) = 0$. We denote the associative commutative quasi-shuffle product [16] on $\mathcal{T}(\mathcal{A})$ by:

$$aU * bV := a(U * bV) + b(aU * V) - \lambda[ab](U * V) \tag{8}$$

$\lambda \in \mathbb{K}$ being an arbitrary parameter. The "merged" term $[ab] \in \mathcal{A}$ gives a letter. The empty word $1 \in \mathcal{T}(\mathcal{A})$ gives the product unit, i.e. $1 * U := U * 1 := U$, $U \in \mathcal{T}(\mathcal{A})$.

Remark 3.1. *(1) If $\lambda = 0$ we get the ordinary shuffle product of two words $U := u_1 \cdots u_n, V = v_1 \cdots v_m \in \mathcal{T}(\mathcal{A})$ [16], denoted in general by:*

$$U \sqcup\!\sqcup V := u_1(u_2 \cdots u_n \sqcup\!\sqcup v_1 \cdots v_m) + v_1(u_1 \cdots u_n \sqcup\!\sqcup v_2 \cdots v_m). \tag{9}$$

The shuffle sum in (9) consists of $\binom{m+n}{m}$ words.

(2) With $\lambda \in \mathbb{K}$ it is obvious that the quasi-shuffle product $*$ is not grade preserving with respect to the length $\mathfrak{l}$ of words, due to the last term on the right-hand side of (8).

Let us mention here, that Hoffman assumes the algebra $\mathcal{A}$ to be equipped with a grading $deg_{\mathcal{A}}$ which is preserved by the product $[\ ] \in \mathcal{A}$ i.e., $deg_{\mathcal{A}}([ab]) = deg_{\mathcal{A}}(a) + deg_{\mathcal{A}}(b)$. This grading is extended to $\mathcal{T}(\mathcal{A})$ by identifying the degree of a word $a_1 \dots a_n$ with the sum of the degrees of the letters $a_1, \cdots, a_n$. The above $*$-product then becomes grade preserving with respect to this second grading on $\mathcal{T}(\mathcal{A})$. We refer the reader to [16] and [11] for details, related to Hopf algebraic aspects related to the quasi-shuffle product construction.
Here we will work exclusively with the natural grading $\mathfrak{l}$ given by the length of words.

We define now the following two linear operators on $\mathcal{T}(\mathcal{A})$:

Definition 3.2. *Let $B_a^+,\ B^- : \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$, $a \in \mathcal{A}$ be the linear operators with:*

$$B_a^+(a_1 \dots a_n) \ := \ aa_1 \dots a_n \tag{10}$$
$$B^-(a_1 \dots a_n) \ := \ a_2 \dots a_n. \tag{11}$$

Proposition 3.3. *Denote the set of idempotent elements in $\mathcal{A}$ by $\mathcal{I} := \{x \in \mathcal{A} \mid [xx] = x\}$. The linear operator B_x^+ on $(\mathcal{T}(\mathcal{A}), *)$ satisfies the Rota-Baxter relation (3) of weight $\lambda \in \mathbb{K}$ when $x \in \mathcal{I}$.*

Proof. This follows by construction of the quasi-shuffle product i.e., relation (8). □

Remark 3.4. *For $\lambda = 0$ i.e., the ordinary shuffle relation, B_x^+ is a Rota-Baxter map of zero weight for general $x \in \mathcal{A}$.*

The augmented quasi-shuffle product on the augmented tensor module:

$$\bar{\mathcal{T}}(\mathcal{A}) := \bigoplus_{n>0} \mathcal{A}^{\otimes n} \tag{12}$$

is defined in the following way:

$$aU \boxdot bV := [ab](U * V) \tag{13}$$

whereby the unit of this product is given by the algebra unit $e \in \mathcal{A}$, $e \boxdot U = U \boxdot e = U$, $U \in \bar{\mathcal{T}}(\mathcal{A})$. The associativity and commutativity of $\boxdot$ for arbitrary $\lambda \in \mathbb{K}$ follows from the associativity and commutativity of the quasi-shuffle product and the product $[\,]$ in $\mathcal{A}$.

Remark 3.5. *The operator B_x^+ on $(\bar{\mathcal{T}}(\mathcal{A}), \boxdot)$ is Rota-Baxter of weight $\lambda \in \mathbb{K}$ only if x is the unit $e \in \mathcal{A}$.*

Definition 3.6. *We define now a modified quasi-shuffle product on $\mathcal{T}(\mathcal{A})$ using the algebra unit $e \in \mathcal{A}$:*

$$aU \circledast bV := a(U \circledast bV) + b(aU \circledast V) - \lambda\, e\, [ab](U \circledast V). \tag{14}$$

For the empty word we have $1 \circledast U = U \circledast 1 := U$, $U \in \mathcal{T}(\mathcal{A})$.

Remark 3.7. *Commutativity of $\circledast$ on $\mathcal{T}(\mathcal{A})$ follows from commutativity of the product $[\,]$ in $\mathcal{A}$.*

Lemma 3.8. *The bilinear product $\circledast$ on $\mathcal{T}(\mathcal{A})$ defined above is associative if and only if $\lambda = 1$ in (14).*

Proof. We prove the associativity property for this product by induction on the length of words. The linearity allows us to reduce the proof to generators in $\mathcal{T}(\mathcal{A})$. Choose $a, b, c \in \mathcal{T}(\mathcal{A})$, $\mathfrak{l}(a) = \mathfrak{l}(b) = \mathfrak{l}(c) = 1$:

$$
\begin{aligned}
a \circledast (b \circledast c) &= a \circledast \big(bc + cb - \lambda e[bc]\big) \\
&= abc + b\big(ac + ca - \lambda e[ac]\big) - \lambda e[ab]c \\
&\qquad + acb + c\big(ab + ba - \lambda e[ab]\big) - \lambda e[ac]b \\
&\qquad - \lambda\big(ae[bc] + e(a[bc] + [bc]a - \lambda e[a[bc]])\big) + \lambda^2(e[ae][bc]) \\
&= abc + bac + bca + acb + cab + cba \\
&\quad - \lambda\big(be[ac] + e[ab]c + ce[ab] + e[ac]b + ae[bc] + ea[bc] + e[bc]a\big) \\
&\qquad + \lambda^2\big(ee[a[bc]] + ea[bc]\big) \\
(a \circledast b) \circledast c &= (ab + ba - \lambda e[ab]) \circledast c \\
&= a\big(bc + cb - \lambda e[bc]\big) + cab - \lambda e[ac]b \\
&\qquad + b\big(ac + ca - \lambda e[ac]\big) + cba - \lambda e[bc]a \\
&\qquad - \lambda\big(e([ab]c + c[ab] - \lambda e[[ab]c]) + ce[ab]\big) + \lambda^2(e[ec][ab]) \\
&= abc + acb + cab + bac + bca + cba \\
&\quad - \lambda\big(ae[bc] + e[ac]b + be[ac] + e[bc]a + e[ab]c + ec[ab] + ce[ab]\big) \\
&\qquad + \lambda^2\big(ee[[ab]c] + ec[ab]\big)
\end{aligned}
$$

We see already here that this product is associative if and only if $\lambda = 1$, eliminating the two unwanted terms in the λ^2-part, i.e. $ec[ab]$ and $ea[bc]$! Now we assume that associativity is true for $\mathfrak{l}(U) + \mathfrak{l}(V) + \mathfrak{l}(W) = k - 1 > 3$ and choose $aX, bY, cZ \in \mathcal{T}(\mathcal{A})$, $\mathfrak{l}(aX) + \mathfrak{l}(bY) + \mathfrak{l}(cZ) = k$ (we now use $\lambda = 1$):

$$
\begin{aligned}
aX \circledast (bY \circledast cZ) &= aX \circledast \big(b(Y \circledast cZ) + c(bY \circledast Z) - e[bc](Y \circledast Z)\big) \\
&= aX \circledast b(Y \circledast cZ) + aX \circledast c(bY \circledast Z) - aX \circledast e[bc](Y \circledast Z) \\
&= a\big(X \circledast b(Y \circledast cZ)\big) + b\big(aX \circledast (Y \circledast cZ)\big) - e[ab]\big(X \circledast (Y \circledast cZ)\big) \\
&\quad + a\big(X \circledast c(bY \circledast Z)\big) + c\big(aX \circledast (bY \circledast Z)\big) - e[ac]\big(X \circledast (bY \circledast Z)\big) \\
&\quad - a\big(X \circledast e[bc](Y \circledast Z)\big) + e\big(aX \circledast [bc](Y \circledast Z)\big) + e[ae]\big(X \circledast [bc](Y \circledast Z)\big) \\
&= a\big((X \circledast bY) \circledast cZ\big) + b\big((aX \circledast Y) \circledast cZ\big) - c\big((aX \circledast bY) \circledast Z\big) \\
&\quad + ee[[ab]c]\big((X \circledast Y) \circledast Z\big) - e[ac]\big((X \circledast bY) \circledast Z\big) \\
&\quad - e[bc]\big((aX \circledast Y) \circledast Z\big) - e[ab]\big((X \circledast Y) \circledast cZ\big) \\
&= (aX \circledast bY) \circledast cZ
\end{aligned}
$$

□

The modified quasi-shuffle is homogeneous with respect to the length of the words in contrast to the ordinary quasi-shuffle product (8). Anticipating the result in the next section we mention here that this fact reflects the homogeneity difference between the Rota-Baxter relation (3) and the associative Nijenhuis relation (1).

Lemma 3.9. *For the algebra $(\mathcal{T}(\mathcal{A}), \circledast)$, the operator B_e^+ satisfies the associative Nijenhuis relation only if x is the unit $e \in \mathcal{A}$:*

$$B_e^+(X) \circledast B_e^+(Y) = B_e^+(X \circledast B_e^+(Y)) + B_e^+(B_e^+(X) \circledast Y) - B_e^+ B_e^+(X \circledast Y). \tag{15}$$

Proof. Again, this result follows from the construction of the modified quasi-shuffle product. □

Proposition 3.10. *Let the augmented modified quasi-shuffle product on the augmented tensor module $\bar{\mathcal{T}}(\mathcal{A})$ be defined as follows for $a, b \in \mathcal{A}$ and $U, V \in \bar{\mathcal{T}}(\mathcal{A})$:*

$$aU \boxtimes bV := [ab](U \circledast V). \tag{16}$$

Then $\boxtimes$ is associative and commutative, and we have $e \boxtimes V = V \boxtimes e = V$, for $V \in \bar{\mathcal{T}}(\mathcal{A})$ and the unit $e \in \mathcal{A}$.

In the following section we will show that the triple $(\bar{\mathcal{T}}(\mathcal{A}), \boxtimes, B_e^+)$ defines a Nijenhuis algebra; moreover, we will see that it fulfills the universal property.

4 Free Associative Nijenhuis Algebra

We will use the augmented modified quasi-shuffle product $\boxtimes$ to give the construction of the free Nijenhuis $\mathbb{K}$-algebra. The existence of such an object follows from general arguments in the theory of universal algebras [30, 8] since the category of Nijenhuis $\mathbb{K}$-algebras, defined through the identity (1) forms a variety (in the sense of universal algebras, see especially section 6 of [30] for a concise summary of the main ideas).

The following construction of the free associative Nijenhuis algebra $(\bar{\mathcal{T}}(\mathcal{A}), \boxtimes)$ was inspired by the mixable shuffle product of Guo *et al.* [14].

Let us remark here that the construction of the free Rota-Baxter algebra (of weight $\lambda \neq 0$) uses the augmented quasi-shuffle product $\boxdot$, defined in (13) and works analogously to what follows.

The quasi-shuffle product $*$ essentially embodies the structure of the Rota-Baxter relation (3). The case of weight $\lambda = 0$ gives the "trivial" Rota-Baxter algebra, i.e. relation (3) without the second term on the left-hand side. This construction was essentially given in [14] using a non-recursive notion of a so called mixable shuffle product. However the formulation using Hoffman's quasi-shuffle product [16] is new. For a detailed account of the link between the mixable and quasi-shuffle product, also addressing Hopf algebraic aspects, we refer the reader to the results in the recent work [11].

The essential point (for both constructions of the free objects) lies in the (augmented) quasi-shuffle product (or mixable shuffle in [14]) and its "merging" of letters $\lambda[ab]$, $a, b \in \mathcal{A}$ in the $*$-product on the right-hand side of (13), whereby $\lambda \in \mathbb{K}$ is arbitrary. It represents the second term on the left-hand side of (3). The modified quasi-shuffle product $\circledast$ (and hence the augmented modified quasi-shuffle product $\boxtimes$) reflects relation (1): in particular, the third term on the right-hand side of (14) is related to the second term on the left-hand side of the associative Nijenhuis relation (1). It has to be underlined that the associativity of $\boxtimes$ relies on the choice $\lambda = 1$ in equation (14). This is due to the fact [6] that the following bilinear map on a $\mathbb{K}$-algebra $\mathcal{A}$, N being an arbitrary $\mathbb{K}$-linear map:

$$\mu_N(a, b) := N(a)b + aN(b) - N(ab) \tag{17}$$

gives an associative product if and only if the following map, called μ-Nijenhuis torsion in [6]:

$$T_{\mu,N}(a,b) := N(\mu_N(a,b)) - N(a)N(b) \tag{18}$$

is a 2-Hochschild cocycle. Of course this holds if N is a Nijenhuis operator, since relation (1) implies that the right-hand side of (18) is zero.
In the sequel we will prove the following theorem.

Theorem 4.1. *For a commutative, associative, unital $\mathbb{K}$-algebra $\mathcal{A}$, $(\bar{\mathfrak{T}}(\mathcal{A}), \boxtimes, B_e^+)$ is the free (associative) Nijenhuis algebra generated by $\mathcal{A}$.*

Proof. The proof of the above Theorem (4.1) will be divided into three parts. In the following Proposition (4.2), we show that $(\bar{\mathfrak{T}}(\mathcal{A}), \boxtimes, B_e^+)$ is a Nijenhuis algebra i.e., the operator B_e^+ satisfies the associative Nijenhuis relation.

Proposition 4.2. *$\bar{\mathfrak{T}}(\mathcal{A})$ together with the augmented modified quasi-shuffle product $\boxtimes$ and the linear map B_e^+ is an associative Nijenhuis algebra.*

Proof.

$$\begin{aligned}
B_e^+(U) \boxtimes B_e^+(V) + (B_e^+)^2(U \boxtimes V) &= eU \boxtimes eV + ee(U \boxtimes V) \\
&= [ee](U \circledast V) + ee\big([u_1 v_1](B^-(U) \circledast B^-(V))\big) \\
&= e\big(u_1(B^-(U) \circledast V) + v_1(U \circledast B^-(V)) \\
&\qquad -e[u_1 v_1](B^-(U) \circledast B^-(V))\,\big) \\
&\qquad +ee[u_1 v_1]\big(B^-(U) \circledast B^-(V)\big) \\
&= B_e^+\big(U \boxtimes B_e^+(V) + B_e^+(U) \boxtimes V\big)
\end{aligned}$$

□

For the triple $(\bar{\mathfrak{T}}(\mathcal{A}), \boxtimes, B_e^+)$ to be the free Nijenhuis algebra over $\mathcal{A}$ we have to show that the universal property is satisfied. Let $\mathfrak{U}$ be an arbitrary Nijenhuis $\mathbb{K}$-algebra with Nijenhuis operator N and $\phi : \mathcal{A} \to \mathfrak{U}$ an $\mathbb{K}$-algebra homomorphism.

We have to extend the $\mathbb{K}$-algebra map ϕ to a Nijenhuis homomorphism (6) $\tilde{\phi} : \bar{\mathfrak{T}}(\mathcal{A}) \to \mathfrak{U}$ using the following important fact:

Lemma 4.3. *Every generator $a_1 \ldots a_n \in \bar{\mathfrak{T}}(\mathcal{A})$ can be written using the modified augmented quasi-shuffle product:*

$$a_1 \ldots a_n = a_1 \boxtimes B_e^+\big(a_2 \boxtimes B_e^+(a_3 \boxtimes B_e^+(\ldots B_e^+\big(a_{n-1} \boxtimes B_e^+(a_n)\big) \ldots))\big). \tag{19}$$

Since the extension $\tilde{\phi}$ is supposed to be a Nijenhuis algebra homomorphism we have the unique possible extension:

$$\tilde{\phi}(a_1 \ldots a_n) := \phi(a_1) N\big(\phi(a_2) N(\ldots N\big(\phi(a_{n-1}) N(\phi(a_n))\big) \ldots)\big). \tag{20}$$

Defining the map $N_w : \mathfrak{U} \to \mathfrak{U}$, $w \in \mathfrak{U}$, by $N_w(x) := N(wx)$, $x \in \mathfrak{U}$, we can write (20) in terms of iterated compositions (cf. §4 of [14]):

$$\tilde{\phi}(a_1 \ldots a_n) = \phi(a_1) \big\{ \circ_{i=2}^{n} N_{\phi(a_i)}(1_{\mathfrak{U}}) \big\}. \tag{21}$$

To complete the proof of Theorem (4.1) it remains to show the following:

Proposition 4.4. *$\tilde{\phi}$ is a Nijenhuis homomorphism.*

Proof. By construction $\tilde{\phi}$ is a well defined $\mathbb{K}$-linear map. The Nijenhuis property (6) follows as in [17, p.140]:

$$\begin{aligned}
\tilde{\phi}(B_e^+(a_1\dots a_n)) &= \tilde{\phi}(ea_1\dots a_n)\\
&\overset{(20)}{=} \circ_{i=1}^n N_{\phi(a_i)}(1_{\mathfrak{U}})\\
&= N\big(\phi(a_1)\circ_{i=2}^n N_{\phi(a_i)}(1_{\mathfrak{U}})\big)\\
&= N\big(\tilde{\phi}(a_1\dots a_n)\big).
\end{aligned}\tag{22}$$

Finally we would like to show that $\tilde{\phi}$ is a $\mathbb{K}$-algebra homomorphism, i.e. preserves multiplication. This we proof by induction on the length of words, $\mathfrak{l}(U)+\mathfrak{l}(V)=m+n=k$. For $k=2$ we have, $a,b\in\bar{\mathfrak{T}}(\mathcal{A})$:

$$\begin{aligned}
\tilde{\phi}(a\boxtimes b) &= \tilde{\phi}([ab])\\
&\overset{(20)}{=} \phi([ab])\\
&= \phi(a)\phi(b)=\tilde{\phi}(a)\tilde{\phi}(b).
\end{aligned}$$

Let $U:=u_1\dots u_n\in\mathcal{A}^{\otimes^n}$, $V:=v_1\dots v_m\in\mathcal{A}^{\otimes^m}$, with $m+n>2$:

$$\begin{aligned}
\tilde{\phi}(U\boxtimes V) &= \tilde{\phi}\big([u_1v_1]\,\{B^-(U)\circledast B^-(V)\}\,\big)\\
&= \tilde{\phi}\big([u_1v_1]\,\{B_e^+(B^-(U))\boxtimes B^-(V)+B^-(U)\boxtimes B_e^+(B^-(V))\\
&\qquad\qquad -B_e^+(B^-(U)\boxtimes B^-(V))\,\}\big)\\
&\overset{(20)}{=} \phi(u_1)\phi(v_1)\,N\big(\,\tilde{\phi}\{\,B_e^+(B^-(U))\boxtimes B^-(V)\\
&\quad +B^-(U)\boxtimes B_e^+(B^-(V))-B_e^+(B^-(U)\boxtimes B^-(V))\,\}\,\big)\\
&= \phi(u_1)\phi(v_1)\,N\big(\,\tilde{\phi}(B_e^+(B^-(U)))\,\tilde{\phi}(B^-(V))\\
&\quad +\tilde{\phi}(B^-(U))\,\tilde{\phi}(B_e^+(B^-(V)))-N\{\tilde{\phi}(B^-(U))\,\tilde{\phi}(B^-(V))\}\,\big)\\
&= \phi(u_1)\phi(v_1)\,N\{\,N\big(\tilde{\phi}(B^-(U))\big)\,\tilde{\phi}(B^-(V))\\
&\quad +\tilde{\phi}(B^-(U))\,N\big(\tilde{\phi}(B^-(V))\big)-N\big(\tilde{\phi}(B^-(U))\,\tilde{\phi}(B^-(V))\big)\,\}\\
&\overset{(\#)}{=} \phi(u_1)\phi(v_1)\,N\{\tilde{\phi}(B^-(U))\}\,N\{\tilde{\phi}(B^-(V))\}\\
&= \phi(u_1)N\{\tilde{\phi}(B^-(U))\}\,\phi(v_1)\,N\{\tilde{\phi}(B^-(V)\}\\
&= \tilde{\phi}(U)\,\tilde{\phi}(V)
\end{aligned}$$

□

In the third line we used the Nijenhuis property (22). Apparently by going from line (#) to the next we see that the use of the augmented modified quasi-shuffle product in the above construction of the free Nijenhuis algebra is limited to the commutative case.

This finishes the proof of Theorem (4.1), showing that $(\bar{\mathfrak{T}}(\mathcal{A}),\boxtimes,B_e^+)$ is the free Nijenhuis algebra generated by $\mathcal{A}$. □

Remark 4.5. *(1) The above construction may also be used to give the free Nijenhuis algebra on a set S by working with the polynomial algebra $\mathbb{K}[S]$.*

(2) Using the augmented quasi-shuffle product $\boxdot$, the proof of the universal property for the triple $(\bar{\mathcal{T}}(\mathcal{A}), \boxdot, \lambda, B_e^+)$ goes analogously, giving the free Rota-Baxter $\mathbb{K}$-algebra of weight $\lambda \in \mathbb{K}$ over $\mathcal{A}$.

As we already mentioned above, the algebras $(\mathcal{T}(\mathcal{A}), *, \lambda, B_e^+)$ and $(\mathcal{T}(\mathcal{A}), \circledast, B_e^+)$ define a Rota-Baxter algebra and a Nijenhuis algebra, respectively.

4.1 Dendriform Di- and Trialgebra Structures

We mentioned in the beginning the relation between Rota-Baxter algebras and Loday-type algebras [9, 1, 2], i.e. dendriform di- and trialgebra structures (see [22, 23] for definitions and further details).

In this final part we would like to relate the above given construction of the free Rota-Baxter algebra, i.e. the augmented quasi-shuffle products to these Loday-type structures.

Let us remark again, that recently in [10] and especially in [21] further results with respect to both the associative Nijenhuis and Rota-Baxter algebras were established.

A dendriform dialgebra (respectively trialgebra) is a $\mathbb{K}$-vector space V equipped with two (resp. three) binary bilinear operations $\succ$ and $\prec$ (resp. $\succ$, $\prec$, and $\bullet$), satisfying a certain set of axioms [22, 23], such that for $x, y \in V$ the binary composition

$$x * y := x \succ y + x \prec y$$

(resp. $x * y := x \succ y + x \prec y + x \bullet y$) gives an associative product, making $(V, *)$ an associative, but not necessarily commutative $\mathbb{K}$-algebra. The operations $\succ$ and $\prec$ (resp. $\succ$, $\prec$, and $\bullet$) "split" the associative product $*$ [24].

Rota-Baxter algebras $(\mathcal{A}, R)$ (resp. associative Nijenhuis algebras), not necessarily commutative, naturally fit into this picture, as they allow to define another product on the vector space underlying the Rota-Baxter algebra $\mathcal{A}$:

$$x * y := xR(y) + R(x)y - ab$$

(resp. $x * y := xN(y) + N(x)y - N(xy)$, see (17)), which is associative. It was shown in [1, 9] that this splitting defines dendrifrom di- and trialgebra structures on Rota-Baxter algebras. In [21] and later in [10] this notion was extended to associative Nijenhuis algebras.

In [26] Ronco defined a dendriform dialgebra structure on the tensor coalgebra $\bar{\mathcal{T}}(V)$ over a vector space V in connection to the ordinary shuffle product (9). The two compositions $\prec$, $\succ$ are defined using the fact that the shuffle sum (9) of two words $X = x_1 \dots x_n$ $Z = z_1 \dots z_m$ may be characterized by words either beginning with x_1 or z_1:

$$\begin{aligned} X \prec Z &:= x_1(B^-(X) \sqcup\!\sqcup Z) \\ X \succ Z &:= z_1(X \sqcup\!\sqcup B^-(Z)). \end{aligned}$$

In the context of $\bar{\mathcal{T}}(\mathcal{A})$ over the $\mathbb{K}$-algebra $\mathcal{A}$ we see how this relates to the augmented quasi-shuffle product $\boxdot$, with $\lambda = 0$ in the $*$-product on the right-hand side of equation (13), and therefore to the Rota-Baxter map B_e^+ of weight 0:

$$\begin{aligned} X \prec Z &:= X \boxdot B_e^+(Z) \\ X \succ Z &:= B_e^+(X) \boxdot Z, \end{aligned} \tag{23}$$

which gives the dendriform dialgebra structure related to the Rota-Baxter relation of weight zero found by Aguiar [1]. Assuming $\lambda \neq 0$, and using instead both Rota-Baxter maps i.e., B_e^+ and $\lambda id_{\bar{\mathfrak{T}}(\mathcal{A})} - B_e^+ =: \widehat{B_e^+}$, we have on the free Rota-Baxter algebra $(\bar{\mathfrak{T}}(\mathcal{A}), \boxdot)$ over $\mathcal{A}$ [9]:

Corollary 4.6. *Let $\bar{\mathfrak{T}}(\mathcal{A})$ be equipped with the augmented quasi-shuffle product $\boxdot$. The two compositions*

$$X \prec Z := X \boxdot B_e^+(Z) \qquad X \succ Z := -\widehat{B_e^+}(X) \boxdot Z, \quad X, Z \in \bar{\mathfrak{T}}(\mathcal{A})$$

defined in terms of the Rota-Baxter maps B_e^+ and $\lambda id_{\bar{\mathfrak{T}}(\mathcal{A})} - B_e^+ =: \widehat{B_e^+}$ satisfy the dendriform dialgebra relations.

Loday and Ronco related in [23] the dendriform trialgebra structure to Hoffman's quasi-shuffle product $*$. Here we see how this relates to the augmented quasi-shuffle product $\boxdot$ for weight $\lambda = 1$ in the $*$-product on the right-hand side of equation (13):

$$\begin{aligned} X \prec Z &:= x_1(B^-(X) * Z) = X \boxdot B_e^+(Z) && (24)\\ X \succ Z &:= z_1(X * B^-(Z)) = B_e^+(X) \boxdot Z && (25)\\ X \bullet Z &:= [x_1 z_1]\big(B^-(X) * B^-(Z)\big) = X \boxdot Z. && (26) \end{aligned}$$

This of course extends to general weight $\lambda \in \mathbb{K}$ i.e., the free Rota-Baxter algebra $(\bar{\mathfrak{T}}(\mathcal{A}), \boxdot)$ of weight $\lambda \neq 0$, and reduces to the case (23) for $\lambda = 0$.

Let us make here an observation with respect to the work [26] of Ronco. From the previous results it follows that Rota-Baxter algebras always give dendriform algebra structures. We used in the definition of the extension of an algebra homomorphism to a Rota-Baxter homomorphism the fact, analogous to (19), that the elements in $\bar{\mathfrak{T}}(\mathcal{A})$ may be written using the Rota-Baxter map B_e^+ ($\boxdot$ instead of $\boxtimes$):

$$\begin{aligned} a_1 \dots a_n &= a_1 \boxdot B_e^+\big(a_2 \boxdot B_e^+(a_3 \boxdot B_e^+(\dots B_e^+\big(a_{n-1} \boxdot B_e^+(a_n)\big) \dots))\big)\\ &= B_e^+\big(\dots B_e^+\big(B_e^+\big(B_e^+(a_n) \boxdot a_{n-1}\big) \boxdot a_{n-2}\big) \boxdot a_{n-3} \dots\big) \boxdot a_1. \end{aligned}$$

With respect to the dendriform (trialgebra) structures this may be written using either of the compositions $\prec$, $\succ$ and introducing the "identity" maps $\omega_{\prec,\succ}$ on $\bar{\mathfrak{T}}(\mathcal{A})$ ($a_i \in \mathcal{A}$ being letters):

$$\begin{aligned} \omega_{\prec}(a_1 \dots a_n) &:= a_1 \prec \big(a_2 \prec (a_3 \prec (\cdots \prec \big(a_{n-1} \prec a_n\big) \dots))\big) && (27)\\ &= \big((\dots((a_n \succ a_{n-1}) \succ a_{n-2}) \succ \cdots) \succ a_2\big) \succ a_1 && (28)\\ &=: \omega_{\succ}(a_n \dots a_1) && (29) \end{aligned}$$

These maps are of importance with respect to the primitive elements and brace-algebras [26, 27] in a dendriform Hopf algebra. The connection to Loday's recent work [24] has to be clarified in a future work concerning this link and the algebraic properties of the augmented modified quasi-shuffle product.

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