Degree powers in graphs with forbidden subgraphs

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Abstract

For every real $p > 0$ and simple graph G, set

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

and let $\phi(r, p, n)$ be the maximum of $f(p, G)$ taken over all K_{r+1}-free graphs G of order n. We prove that, if $0 < p < r$, then

$$\phi(r, p, n) = f(p, T_r(n)),$$

where $T_r(n)$ is the r-partite Turan graph of order n. For every $p \geq r + \lceil \sqrt{2r} \rceil$ and n large, we show that

$$\phi(p, n, r) > (1 + \varepsilon) f(p, T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$.

Our results settle two conjectures of Caro and Yuster.

1 Introduction

Our notation and terminology are standard (see, e.g. [1]).

Caro and Yuster [3] introduced and investigated the function

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

where $p \geq 1$ is integer and G is a graph. Writing $\phi(r, p, n)$ for the maximum value of $f(p, G)$ taken over all K_{r+1}-free graphs G of order n, Caro and Yuster stated that, for every $p \geq 1$,

$$\phi(r, p, n) = f(p, T_r(n)), \tag{1}$$

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where $T_r(n)$ is the r-partite Turán graph of order n. Although true for $p = 2$, $r \geq 2$, simple examples show that (1) fails for every fixed $r \geq 2$ and all sufficiently large p and n; this was observed by Schelp [4]. A natural problem arises: given $r \geq 2$, determine those real values $p > 0$, for which equality (1) holds. Furthermore, determine the asymptotic value of $\phi(r, p, n)$ for large n.

In this note we essentially answer these questions. In Section 2 we prove that (1) holds whenever $0 < p < r$ and n is large. Next, in Section 3, we describe the asymptotic structure of K_{r+1}-free graphs G of order n such that $f(p, G) = \phi(r, p, n)$. We deduce that, if $p \geq r + \lceil\sqrt{2r}\rceil$ and n is large, then

$$\phi(r, p, n) > (1 + \varepsilon) f(p, T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$. This disproves Conjecture 6.2 in [3]. In particular,

$$\frac{r}{pe} \geq \frac{\phi(r, p, n)}{n^{p+1}} \geq \frac{r-1}{(p+1)e}$$

holds for large n, and therefore, for any fixed $r \geq 2$,

$$\lim_{n\to\infty} \frac{\phi(r, p, n)}{f(p, T_r(n))}$$

grows exponentially in p.

The case $r = 2$ is considered in detail in Section 4; we show that, if $r = 2$, equality (1) holds for $0 < p \leq 3$, and is false for every $p > 3$ and n large.

In Section 5 we extend the above setup. For a fixed $(r + 1)$-chromatic graph H, $(r \geq 2)$, let $\phi(H, p, n)$ be the maximum value of $f(p, G)$ taken over all H-free graphs G of order n. It turns out that, for every r and p,

$$\phi(H, p, n) = \phi(r, p, n) + o\left(n^{p+1}\right). \tag{2}$$

This result completely settles, with the proper changes, Conjecture 6.1 of [3]. In fact, Pikhurko [5] proved this for $p \geq 1$, although he incorrectly assumed that (1) holds for all sufficiently large n.

2 The function $\phi(r, p, n)$ for $p < r$

In this section we shall prove the following theorem.

Theorem 1 *For every $r \geq 2$, $0 < p < r$, and sufficiently large n,*

$$\phi(r, p, n) = f(p, T_r(n)).$$

Proof Erdős [2] proved that, for every K_{r+1}-free graph G, there exists an r-partite graph H with $V(H) = V(G)$ such that $d_G(u) \leq d_H(u)$ for every $u \in V(G)$. As Caro and Yuster noticed, this implies that, for K_{r+1}-free graphs G of order n, if $f(p,G)$ attains a maximum then G is a complete r-partite graph. Every complete r-partite graph is defined uniquely by the size of its vertex classes, that is, by a vector $(n_i)_1^r$ of positive integers satisfying $n_1+...+n_r=n$; note that the Turán graph $T_r(n)$ is uniquely characterized by the condition $|n_i-n_j|\leq 1$ for every $i,j\in[r]$. Thus we have

$$\phi(r,p,n)=\max\left\{\sum_{i=1}^{r}n_i(n-n_i)^p \;:\; n_1+...+n_r=n,\ 1\leq n_1\leq ...\leq n_r\right\}. \qquad (3)$$

Let $(n_i)_1^r$ be a vector on which the value of $\phi(r,p,n)$ is attained. Routine calculations show that the function $x(n-x)^p$ increases for $0\leq x\leq \frac{n}{p+1}$, decreases for $\frac{n}{p+1}\leq x\leq n$, and is concave for $0\leq x\leq \frac{2n}{p+1}$. If $n_r\leq \left\lfloor \frac{2n}{p+1}\right\rfloor$, the concavity of $x(n-x)^p$ implies that $n_r-n_1\leq 1$, and the proof is completed, so we shall assume $n_r>\left\lfloor \frac{2n}{p+1}\right\rfloor$. Hence we deduce

$$n_1(r-1)+\left\lfloor\frac{2n}{p+1}\right\rfloor < n_1+...+n_r=n. \qquad (4)$$

We shall also assume

$$n_1\geq\left\lfloor\frac{n}{p+1}\right\rfloor, \qquad (5)$$

since otherwise, adding 1 to n_r and subtracting 1 from n_1, the value $\sum_{i=1}^{r}n_i(n-n_i)^p$ will increase, contradicting the choice of $(n_i)_1^r$. Notice that, as $n_1\leq n/r$, inequality (5) is enough to prove the assertion for $p\leq r-1$ and every n. From (4) and (5), we obtain that

$$(r-1)\left\lfloor\frac{n}{p+1}\right\rfloor+\left\lfloor\frac{2n}{p+1}\right\rfloor<n.$$

Letting $n\to\infty$, we see that $p\geq r$, contradicting the assumption and completing the proof. □

Maximizing independently each summand in (3), we see that, for every $r\geq 2$ and $p>0$,

$$\phi(r,p,n)\leq\frac{r}{p+1}\left(\frac{p}{p+1}\right)^p n^{p+1}. \qquad (6)$$

3 The asymptotics of $\phi(r,p,n)$

In this section we find the asymptotic structure of K_{r+1}-free graphs G of order n satisfying $f(p,G)=\phi(r,p,n)$, and deduce asymptotic bounds on $\phi(r,p,n)$.

Theorem 2 *For all $r\geq 2$ and $p>0$, there exists $c=c(p,r)$ such that the following assertion holds.*

If $f(p,G)=\phi(r,p,n)$ for some K_{r+1}-free graph G of order n, then G is a complete r-partite graph having $r-1$ vertex classes of size $cn+o(n)$.

Proof We already know that G is a complete r-partite graph; let $n_1 \leq ... \leq n_r$ be the sizes of its vertex classes and, for every $i \in [r]$, set $y_i = n_i/n$. It is easy to see that

$$\phi(r,p,n) = \psi(r,p)\, n^{p+1} + o\left(n^{p+1}\right),$$

where the function $\psi(r,p)$ is defined as

$$\psi(r,p) = \max\left\{\sum_{i=1}^{r} x_i (1-x_i)^p \;\;:\;\; x_1 + ... + x_r = 1,\; 0 \leq x_1 \leq ... \leq x_r\right\}$$

We shall show that if the above maximum is attained at $(x_i)_1^r$, then $x_1 = ... = x_{r-1}$. Indeed, the function $x(1-x)^p$ is concave for $0 \leq x \leq 2/(p+1)$, and convex for $2/(p+1) \leq x \leq 1$. Hence, there is at most one x_i in the interval $(2/(p+1) \leq x \leq 1]$, which can only be x_r. Thus $x_1, ..., x_{r-1}$ are all in the interval $[0, 2/(p+1)]$, and so, by the concavity of $x(1-x)^p$, they are equal. We conclude that, if

$$0 \leq x_1 \leq ... \leq x_r,\; x_1 + ... + x_r = 1,$$

and $x_j > x_i$ for some $1 \leq i < j \leq r-1$, then $\sum_{i=1}^{r} x_i (1-x_i)^p$ is below its maximum value. Applying this conclusion to the numbers $(y_i)_1^r$, we deduce the assertion of the theorem. □

Set

$$g(r,p,x) = (r-1)\, x (1-x)^p + (1-(r-1)x)(rx)^p.$$

From the previous theorem it follows that

$$\psi(r,p) = \max_{0 \leq x \leq 1/(r-1)} g(r,p,x).$$

Finding $\psi(r,p)$ is not easy when $p > r$. In fact, for some $p > r$, there exist $0 < x < y < 1$ such that

$$\psi(r,p) = g(r,p,x) = g(r,p,y).$$

In view of the original claim concerning (1), it is somewhat surprising, that for $p > 2r-1$, the point $x = 1/r$, corresponding to the Turán graph, not only fails to be a maximum of $g(r,p,x)$, but, in fact, is a local minimum.

Observe that

$$\frac{f(p, T_r(n))}{n^{p+1}} = \left(\frac{r-1}{r}\right)^p + o(1),$$

so, to find for which p the function $\phi(r,p,n)$ is significantly greater than $f(p, T_r(n))$, we shall compare $\psi(r,p)$ to $\left(\frac{r-1}{r}\right)^p$.

Theorem 3 *Let* $r \geq 2$, $p \geq r + \left\lceil \sqrt{2r} \right\rceil$. *Then*

$$\psi(r,p) > (1+\varepsilon)\left(\frac{r-1}{r}\right)^p$$

for some $\varepsilon = \varepsilon(r) > 0$.

Proof We have

$$\begin{aligned}\psi(r,p) &\geq g\left(r,p,\frac{1}{p}\right) = \frac{r-1}{p}\left(\frac{p-1}{p}\right)^p + \left(1-\frac{r-1}{p}\right)\left(\frac{r-1}{p}\right)^p \\ &> \frac{r-1}{p}\left(\frac{p-1}{p}\right)^p.\end{aligned}$$

To prove the theorem, it suffices to show that

$$\frac{r-1}{p}\left(\frac{(p-1)\,r}{p\,(r-1)}\right)^p > 1+\varepsilon \tag{7}$$

for some $\varepsilon = \varepsilon(r) > 0$. Routine calculations show that

$$\frac{r-1}{p}\left(1+\frac{p-r}{p\,(r-1)}\right)^p$$

increases with p. Thus, setting $q = \lceil\sqrt{2r}\rceil$, we find that

$$\begin{aligned}&\frac{r-1}{p}\left(1+\frac{p-r}{p\,(r-1)}\right)^p \\ &\geq \frac{r-1}{r+q}\left(1+\binom{r+q}{1}\frac{q}{(r+q)\,(r-1)}+\binom{r+q}{2}\frac{q^2}{(r+q)^2\,(r-1)^2}\right) \\ &= \frac{r-1}{r+q}+\frac{q}{r+q}+\frac{q^2\,(r+q-1)}{2\,(r+q)^2\,(r-1)} \geq 1-\frac{1}{r+q}+\frac{r\,(r+q-1)}{(r+q)^2\,(r-1)} \\ &= 1+\frac{r\,(r+q-1)-(r+q)\,(r-1)}{(r+q)^2\,(r-1)} = 1+\frac{q}{(r+q)^2\,(r-1)}.\end{aligned}$$

Hence, (7) holds with

$$\varepsilon = \frac{\lceil\sqrt{2r}\rceil}{\left(r+\lceil\sqrt{2r}\rceil\right)^2\,(r-1)},$$

completing the proof. □

We have, for n sufficiently large,

$$\begin{aligned}\frac{\phi(r,p,n)}{n^{p+1}} &= \psi(r,p)+o(1) \geq g\left(r,p,\frac{1}{p+1}\right)+o(1) \\ &= \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p+\left(1-\frac{r-1}{p+1}\right)\left(\frac{r-1}{p+1}\right)^p+o(1) \\ &> \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p.\end{aligned}$$

Hence, in view of (6), we find that, for n large,

$$\frac{r}{pe} \geq \frac{r}{p}\left(\frac{p}{p+1}\right)^{p+1} \geq \frac{\phi(r,p,n)}{n^{p+1}} \geq \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p \geq \frac{(r-1)}{(p+1)\,e}.$$

In particular, we deduce that, for any fixed $r \geq 2$,

$$\lim_{n\to\infty} \frac{\phi(r,p,n)}{f(p,T_r(n))}$$

grows exponentially in p.

4 Triangle-free graphs

For triangle-free graphs, i.e., $r = 2$, we are able to pinpoint the value of p for which (1) fails, as stated in the following theorem.

Theorem 4 *If $0 < p \leq 3$ then*

$$\phi(3,p,n) = f(p,T_2(n)). \tag{8}$$

For every $\varepsilon > 0$, there exists δ such that if $p > 3+\delta$ then

$$\phi(3,p,n) > (1+\varepsilon) f(p,T_2(n)) \tag{9}$$

for n sufficiently large.

Proof We start by proving (8). From the proof of Theorem 1 we know that

$$\phi(p,n,3) = \max_{k\in\lceil n/2\rceil} \{k(n-k)^p + (n-k)k^p\}.$$

Our goal is to prove that the above maximum is attained at $k = \lceil n/2\rceil$.

If $0 < p \leq 2$, the function $x(1-x)^p$ is concave, and (8) follows immediately.

Next, assume that $2 < p \leq 3$; we claim that the function

$$g(x) = (1+x)(1-x)^p + (1-x)(1+x)^p$$

is concave for $|x| \leq 1$. Indeed, we have

$$\begin{aligned} g(x) &= \left(1-x^2\right)\left((1-x)^{p-1} + (1+x)^{p-1}\right) = 2\left(1-x^2\right)\sum_{i=0}^{\infty}\binom{p-1}{2i}x^{2i} \\ &= 2 + 2\sum_{i=1}^{\infty}\left(\binom{p-1}{2i} - \binom{p-1}{2i-2}\right)x^{2i} \\ &= 2 + 2\sum_{i=1}^{\infty}\binom{p-1}{2i-2}\left(\frac{(p-2i-1)(p-2i-2)}{(2i-1)\,2i} - 1\right)x^{2i}. \end{aligned}$$

Since, for every i, the coefficient of x^{2i} is nonpositive, the function $g(x)$ is concave, as claimed.

Therefore, the function $h(x) = x(n-x)^p + (n-x)x^p$ is concave for $1 \leq x \leq n$. Hence, for every integer $k \in [n]$, we have

$$h\left(\left\lceil \frac{n}{2} \right\rceil\right) + h\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \geq h(k) + h(n-k) = 2h(k)$$
$$= 2\left(k(n-k)^p + (n-k)k^p\right),$$

proving (8).

Inequality (9) follows easily, since, in fact, for every $p > 3$, the function $g(x)$ has a local minimum at 0. $\square$

5 H-free graphs

In this section we are going to prove the following theorem.

Theorem 5 *For every $r \geq 2$, and $p > 0$,*

$$\phi(H, p, n) = \phi(r, p, n) + o\left(n^{p+1}\right).$$

A few words about this theorem seem in place. As already noted, Pikhurko [5] proved the assertion for $p \geq 1$; although he incorrectly assumed that (1) holds for all p and sufficiently large n, his proof is valid, since it is independent of the exact value of $\phi(r, p, n)$. Our proof is close to Pikhurko's, and is given only for the sake of completeness.

We shall need the following theorem (for a proof see, e.g., [1], Theorem 33, p. 132).

Theorem 6 *Suppose H is an $(r+1)$-chromatic graph. Every H-free graph G of sufficiently large order n can be made K_{r+1}-free by removing $o(n^2)$ edges.*

Proof of Theorem 5 Select a K_{r+1}-free graph G of order n such that $f(p, G) = \phi(r, p, n)$. Since G is r-partite, it is H-free, so we have $\phi(H, p, n) \geq \phi(r, p, n)$. Let now G be an H-free graph of order n such that

$$f(p, G) = \phi(H, p, n).$$

Theorem 6 implies that there exists a K_{r+1}-free graph F that may be obtained from G by removing at most $o(n^2)$ edges. Obviously, we have

$$e(G) = e(F) + o\left(n^2\right) \leq \frac{r-1}{2r}n^2 + o\left(n^2\right).$$

For $0 < p \leq 1$, by Jensen's inequality, we have

$$\left(\frac{1}{n}f(p, G)\right)^{1/p} \leq \frac{1}{n}f(1, G) = \frac{1}{n}2e(G) \leq \frac{r-1}{r}n + o(n).$$

Hence, we find that

$$f(p,G) \leq \left(\frac{r-1}{r}\right)^p n^{p+1} + o\left(n^{p+1}\right) = \phi(r,p,n) + o\left(n^{p+1}\right),$$

completing the proof.

Next, assume that $p > 1$. Since the function $xn^{p-1} - x^p$ is decreasing for $0 \leq x \leq n$, we find that

$$d_G^p(u) - d_F^p(u) \leq (d_G(u) - d_F(u))\, n^{p-1}$$

for every $u \in V(G)$. Summing this inequality for all $u \in V(G)$, we obtain

$$\begin{aligned} f(p,G) &\leq f(p,F) + (d_G(u) - d_F(u))\, n^{p-1} = f(p,F) + o\left(n^{p+1}\right) \\ &\leq \phi(r,p,n) + o\left(n^{p+1}\right), \end{aligned}$$

completing the proof. □

6 Concluding remarks

It seems interesting to find, for each $r \geq 3$, the minimum p for which the equality (1) is essentially false for n large. Computer calculations show that this value is roughly 4.9 for $r = 3$, and 6.2 for $r = 4$, suggesting that the answer might not be easy.

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