Classification of Six-Point Metrics

Bernd Sturmfels and Josephine Yu
Department of Mathematics
University of California
Berkeley, CA 94720
[bernd,jyu]@math.berkeley.edu

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Abstract

There are 339 combinatorial types of generic metrics on six points. They correspond to the 339 regular triangulations of the second hypersimplex $\Delta(6,2)$, which also has 14 non-regular triangulations.

1 The Metric Fan

We consider the cone of all metrics on the finite set $\{1,2,\ldots,n\}$:

$$C_n \quad = \quad \Big\{ d \in \mathbb{R}^{\binom{n}{2}} \;:\; d_{ij} \geq 0 \text{ and } d_{ij} + d_{jk} \geq d_{ik} \text{ for all } \quad 1 \leq i,j,k \leq n \Big\}.$$

This is a closed convex pointed polyhedral cone. Its extreme rays have been studied in combinatorial optimization [4, 5]. Among the extreme rays are the *splits*. The splits are the metrics $\sum_{i\in A}\sum_{j\notin A} e_{ij} \in \mathbb{R}^{\binom{n}{2}}$ as A ranges over nonempty subsets of $\{1,2,\ldots,n\}$. There is an extensive body of knowledge (see [5, 9]) also on the facets of the subcone of C_n generated by the splits.

Our object of study is a canonical subdivision of the metric cone C_n. It is called the *metric fan* and denoted MF_n. A quick way to define the metric fan MF_n is to say that it is the *secondary fan of the second hypersimplex*

$$\Delta(n,2) \quad = \quad \mathrm{conv}\big\{ e_i + e_j \;:\; 1 \leq i < j \leq n \big\} \quad \subset \quad \mathbb{R}^n.$$

Every metric d defines a regular polyhedral subdivision Δ_d of $\Delta(n,2)$ as follows. The vertices of $\Delta(n,2)$ are identified with the edges of the complete graph K_n, and subpolytopes of $\Delta(n,2)$ correspond to arbitrary subgraphs of K_n. A subgraph G is a cell of Δ_d if there exists an $x \in \mathbb{R}^n$ satisfying

$$x_i + x_j = d_{ij} \text{ if } \{i,j\} \in G \qquad \text{and} \qquad x_i + x_j > d_{ij} \text{ if } \{i,j\} \notin G.$$

Two metrics d and d' lie in the same cone of the metric fan MF_n if they induce the same subdivision $\Delta_d = \Delta_{d'}$ of the second hypersimplex $\Delta(n, 2)$. We say that the metric d is *generic* if d lies in an open cone of MF_n. This is equivalent to saying that Δ_d is a *regular triangulation* of $\Delta(n, 2)$.

These triangulation of $\Delta(n, 2)$ and the resulting metric fan MF_n were studied by De Loera, Sturmfels and Thomas [3], who had been unaware of an earlier appearance of the same objects in phylogenetic combinatorics [1, 6]. In [6], Dress considered the polyhedron dual to the triangulation Δ_d,

$$P_d \quad = \quad \left\{ x \in \mathbb{R}^n_{\geq 0} \ : \ x_i + x_j \ \geq \ d_{ij} \ \text{ for } 1 \leq i < j \leq n \right\},$$

and he showed that its *complex of bounded faces*, denoted T_d, is a natural object which generalizes the phylogenetic trees derived from the metric d. Both [3] and [6] contain the description of the metric fans MF_n for $n \leq 5$:

- The octahedron $\Delta(4, 2)$ has three regular triangulations Δ_d. They are equivalent up to symmetry. The corresponding *tight span* T_d is a quadrangle with an edge attached to each of its four vertices. The three walls of the fan MF_4 correspond to the trees on $\{1, 2, 3, 4\}$.

- The fan MF_5 has 102 maximal cones which come in three symmetry classes. The tight spans T_d of these three metrics are depicted in [6, Figure A3], and the corresponding triangulations Δ_d appear (in reverse order) in [3, page 414]. For instance, the *thrackle triangulation* of [3, §2] corresponds to the planar diagram in [6]. All three tight spans T_d have five two-cells. (The type T_{X,D_3} is slightly misdrawn in [6]: the two lower left quadrangles should form a flat pentagon).

The aim of this article is to present the analogous classification for $n = 6$. The following result was obtained with the help of Rambau's software TOPCOM [13] for enumerating triangulations of arbitrary convex polytopes.

Theorem 1 *There are* $194, 160$ *generic metrics on six points. These correspond to the maximal cones in* MF_6 *and to the regular triangulations of* $\Delta(6, 2)$*. They come in* 339 *symmetry classes. The hypersimplex* $\Delta(6, 2)$ *has also* $3, 840$ *non-regular triangulations which come in* 14 *symmetry classes.*

This paper is organized as follows. In Section 2 we describe all 12 generic metrics whose tight span T_d is two-dimensional, and in Section 3 we describe all 327 generic metrics whose tight span has a three-dimensional cell. Similarly, in Section 4, we describe the 14 non-regular triangulations of $\Delta(6, 2)$. In each case a suitable system of combinatorial invariants will be introduced. In Section 5 we study the geometry of the metric fan MF_6. The rays of MF_6 are precisely the *prime metrics* in [12]. We determine the maximal cones incident to each prime metric, and we discuss the corresponding minimal subdivisions of $\Delta(6, 2)$. In Section 6 we present a software tool for visualizing the tight span T_d of any finite metric d. This tool was written written in POLYMAKE [10] with the help of

Michael Joswig and Julian Pfeifle. We also explain how its output differs from the output of SPLITSTREE [8].

A complete list of all six-point metrics has been made available at

`bio.math.berkeley.edu/SixPointMetrics`

For each of the 339+14 types in Theorem 1, the regular triangulation, Stanley-Reisner ideal, and numerical invariants are listed. The notation is consistent with that used in the paper. In addition, the webpage contains interactive pictures in JAVAVIEW [11] of the tight span of each metric.

2 The 12 Two-Dimensional Generic Metrics

We identify each generic metric d with its tight span T_d, where the exterior segments have been contracted[1] so that every maximal cell has dimension ≥ 2. With this convention, generic four-point metrics are quadrangles and five-point metrics are glued from five polygons (cf. [6, Figure A3]). The generic six-point metrics, on the other hand, fall naturally into two groups.

Lemma 2 *Each generic metric on six points is either a three-dimensional cell complex with* 26 *vertices,* 42 *edges,* 18 *polygons and one* 3*-cell, or it is a two-dimensional cell complex with* 25 *vertices,* 39 *edges and* 15 *polygons. There are* 327 *three-dimensional metrics and* 12 *two-dimensional metrics.*

We first list the twelve types of two-dimensional metrics. In each case the tight span consists of 15 polygons which are either triangles, quadrangles or pentagons. Our first invariant is the vector $B = (b_3, b_4, b_5)$ where b_i is the number of polygons with i sides. The next two invariants are the order of the symmetry group and the number of cubic generators in the Stanley-Reisner ideal of the triangulation Δ_d. The last item is a representative metric $d = (d_{12}, d_{13}, d_{14}, d_{15}, d_{16}, d_{23}, d_{24}, d_{25}, d_{26}, d_{34}, d_{35}, d_{36}, d_{45}, d_{46}, d_{56})$:

Type 1: $(1, 10, 4), 1, 2, (9, 9, 10, 13, 18, 18, 17, 6, 11, 17, 14, 9, 11, 8, 17)$
Type 2: $(1, 10, 4), 1, 3, (8, 8, 8, 14, 15, 16, 14, 6, 9, 12, 12, 7, 8, 7, 13)$
Type 3: $(1, 10, 4), 1, 5, (5, 6, 7, 8, 12, 11, 10, 5, 7, 11, 6, 6, 7, 5, 10)$
Type 4: $(1, 10, 4), 2, 3, (7, 5, 7, 12, 12, 12, 12, 5, 7, 10, 9, 7, 7, 5, 10)$
Type 5: $(1, 10, 4), 2, 4, (6, 7, 8, 10, 14, 13, 12, 6, 8, 13, 9, 7, 6, 6, 10)$
Type 6: $(1, 10, 4), 2, 5, (7, 7, 7, 11, 14, 12, 12, 6, 7, 14, 10, 7, 6, 7, 11)$
Type 7: $(1, 10, 4), 8, 6, (5, 5, 5, 8, 10, 10, 8, 5, 5, 8, 5, 5, 5, 5, 8)$
Type 8: $(2, 8, 5), 1, 3, (5, 5, 7, 10, 11, 10, 10, 5, 8, 10, 7, 6, 5, 4, 7)$
Type 9: $(2, 8, 5), 2, 4, (7, 7, 8, 10, 14, 14, 13, 5, 9, 13, 9, 7, 10, 6, 14)$
Type 10: $(2, 8, 5), 2, 4, (5, 4, 5, 8, 9, 7, 8, 3, 6, 9, 6, 5, 5, 4, 7)$

[1] Note that the exterior segments do appear in Figures 1–5 of this paper and in the diagrams on our webpage. They are drawn in green for extra clarity.

Type 11: $(2,8,5), 2, 4, (4,5,5,8,9,9,7,4,7,8,5,4,5,4,7)$
Type 12: $(3,6,6), 12, 3, (3,3,5,6,6,6,6,3,5,6,5,3,3,3,6)$

The three metrics of types 9, 10 and 11 cannot be distinguished by the given invariants. In Section 5 we explain how to distinguish these three types.

The metric with the largest symmetry group is Type 12. Its symmetry group has order 12. This combinatorial type of this metric is given by the Stanley-Reisner ideal of the corresponding regular triangulation of $\Delta(6,2)$:

$$\begin{gathered}\langle x_{36}x_{14}, x_{25}x_{34}, x_{35}x_{46}, x_{16}x_{45}, x_{35}x_{12}, x_{26}x_{35}, x_{36}x_{45}, x_{15}x_{36}, x_{26}x_{45}, x_{12}x_{46},\\ x_{12}x_{56}, x_{25}x_{36}, x_{45}x_{23}, x_{24}x_{13}, x_{45}x_{12}, x_{34}x_{12}, x_{25}x_{46}, x_{23}x_{46}, x_{16}x_{25}, x_{13}x_{46},\\ x_{24}x_{36}, x_{35}x_{14}, x_{13}x_{56}, x_{26}x_{14}, x_{26}x_{13}, x_{15}x_{46}, x_{36}x_{12}, x_{45}x_{13}, x_{25}x_{14}, x_{25}x_{13},\\ x_{15}x_{26}x_{34},\ x_{23}x_{56}x_{14},\ x_{16}x_{24}x_{35}\,\rangle.\end{gathered}$$

The number of quadratic generators is 30, and this number is independent of the choice of generic metric. The number of cubic generators of this particular ideal is three (the last three generators), which is the third invariant listed under "Type 12". These cubic generators correspond to "empty triangles" in the triangulation Δ_d. For instance, the cubic $x_{15}x_{26}x_{34}$ means that $\operatorname{conv}\{e_1+e_5, e_2+e_6, e_3+e_4\}$ is not a triangle in Δ_d but each of its three edges is an edge in Δ_d. In the tight span T_d this can be seen as follows:

$$\begin{aligned}\{\text{geodesics between 1 and 5}\} \,\cap\, \{\text{geodesics between 2 and 6}\}&\\ \cap \quad \{\text{geodesics between 3 and 4}\} \quad &= \quad \emptyset,\end{aligned}$$

but any two of these sets of geodesics have a common intersection. This can be seen in the picture of the tight span of the type 12 metric in Figure 1.

The twelve generic metrics listed above demonstrate the subtle nature of the notion of *combinatorial dimension* introduced in [6]. Namely, the combinatorial dimension of a generic metric d can be less than that of a generic split-decomposable metric [1]. This implies that the space of all n-point metrics of combinatorial dimension ≤ 2 is a polyhedral fan whose dimension exceeds the expected number $4n-10$ (cf. [7, Theorem 1.1 (d)]).

For six-point metrics, this discrepancy can be understood by looking at the centroid $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ of the hypersimplex $\Delta(6,2)$. There are 25 simplices in $\Delta(6,2)$ which contain the centroid: the 15 triangles given by the perfect matchings of the graph K_6 and the 10 five-dimensional simplices corresponding to two disjoint triangles in K_6. In any given triangulation Δ_d, the centroid can lie in either one or the other. In the former case, the tight span T_d has a 3-dimensional cell dual to the perfect matching triangle in Δ_d. The combinatorial possibilities of these 3-cells will be explored in Section 3. In the latter case, the tight span T_d has a distinguished vertex dual to the two-disjoint-triangles simplex in Δ_d. This vertex lies in nine polygons of T_d which form a link of type $K_{3,3}$. But there is no 3-cell in T_d. The distinguished vertex is the one in the center in Figure 1.

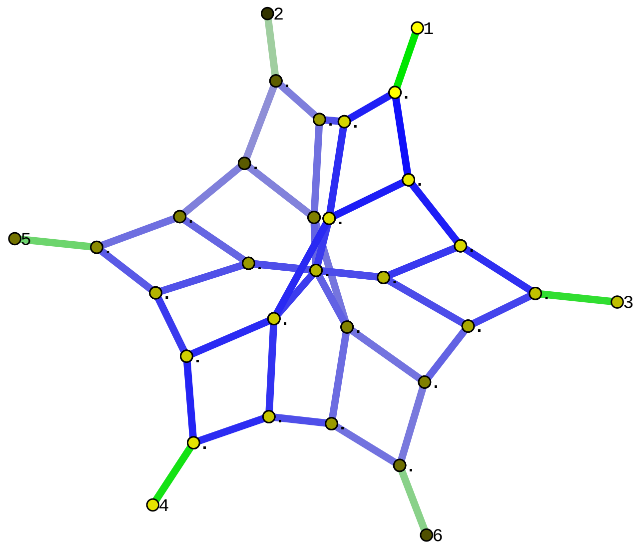

Figure 1: The tight span of the metric # 12

3 The 327 Three-Dimensional Generic Metrics

We next classify the 327 three-dimensional metrics. It turns out that in each case, the unique 3-dimensional cell is a simple polytope, so its numbers v of vertices and e of edges are determined by its number f of faces:

$$v = 2f - 4 \quad \text{and} \quad e = v + f - 2.$$

We consider the two vectors $R = (r_3, r_4, r_5, r_6)$ and $B = (b_3, b_4, b_5, b_6)$ where r_i is the number of polygons with i edges on the 3-cell and b_i is the number of polygons with i edges not on the 3-cell. It turns out that no type has a polygon with 7 or more sides. Hence the number of facets of the 3-cell is $f = r_3 + r_4 + r_5 + r_6$. Our third invariant is the pair $S = (s_2, s_3)$ where s_2 (resp. s_3) is the number of 2/4-splits (resp. 3/3-splits) lying on the cone of the metric fan containing d. The fourth invariant is the pair $C = (c_5, c_6)$ where c_i is the number of cubic generators of the Stanley-Reisner ideal which involve i of the points. And finally we list (g, t) where g is the order of the symmetry group and t is the number of types which share these invariants. These invariants divides the 327 three-dimensional metrics into 251 equivalence classes. We order the classes lexicographically according to the vector $(f, R, B, S, C, (g, t))$. The 251 invariants are given in the following long list of strings $R\ B\ S\ C\ gt$:

4000 0a22 60 02 41	4000 0941 50 11 11	4000 0941 50 21 21	4000 1751 40 10 11
4000 1751 40 20 11	4000 1832 50 11 11	4000 1832 50 21 11	4000 2642 40 20 11
4000 2642 40 20 21	4000 2642 40 30 21	4000 2642 40 40 21	4000 2723 50 21 21
4000 3533 40 40 11	4000 4424 40 60 81	2300 0a21 51 21 21	2300 0850 40 10 11
2300 0850 50 21 11	2300 0931 40 20 11	2300 0931 50 11 11	2300 0940 51 01 22

2300 0940 51 11 13	2300 0940 51 21 22	2300 0940 61 01 21	2300 1660 40 20 11
2300 1660 40 40 21	2300 1741 40 20 12	2300 1741 40 30 11	2300 1741 50 21 11
2300 1750 51 11 13	2300 1750 51 21 12	2300 1822 40 40 21	2300 1831 51 21 11
2300 2551 40 40 12	2300 2560 51 21 22	2300 2632 40 40 11	2300 2641 51 21 21
2300 3442 40 60 21	0600 0a11 41 10 11	0600 0a20 42 00 22	0600 0a20 52 00 12
0600 0a20 62 00 41	$\underline{0600\ 0c00\ 63\ 00\ c1}$	0600 0840 41 00 11	0600 0840 41 00 21
0600 0840 41 10 11	0600 0840 41 20 21	0600 0840 42 00 22	0600 0840 51 00 21
0600 0921 41 00 21	0600 0930 41 00 11	0600 0930 41 10 11	0600 0930 51 00 12
0600 0930 51 10 11	0600 1650 40 30 21	0600 1650 40 40 21	0600 1650 41 10 11
0600 1650 41 20 12	0600 1650 42 20 11	0600 1731 41 20 11	0600 1740 41 10 11
0600 1740 41 20 11	0600 1740 51 10 11	0600 1821 41 10 11	0600 1821 41 30 11
0600 1830 42 20 11	0600 1830 52 20 11	0600 2460 40 40 11	0600 2460 41 20 21
0600 2460 42 40 41	0600 2541 41 40 21	0600 2631 41 30 11	0600 2640 42 40 41
0600 3270 40 60 41	2220 0840 40 20 11	2220 0840 50 21 21	2220 0921 40 20 21
2220 0930 51 11 13	2220 0930 51 21 13	2220 1650 40 30 21	2220 1650 40 40 11
2220 1650 40 40 21	2220 1731 40 40 11	2220 1740 51 21 13	2220 2460 40 40 11
2220 2541 40 60 21	2220 3270 40 60 41	0520 0a01 31 00 21	0520 0a10 42 00 12
0520 0a10 52 00 11	0520 0b00 53 00 21	0520 0830 41 00 11	0520 0830 41 10 11
0520 0920 31 00 11	0520 0920 41 00 11	0520 0920 41 00 21	0520 0920 41 10 11
$\underline{0520\ 0920\ 42\ 00\ 14}$	0520 0920 51 00 21	0520 0920 52 00 21	0520 1640 41 10 11
0520 1640 41 20 12	0520 1640 42 20 11	0520 1730 31 20 11	0520 1730 41 10 11
0520 1730 41 20 11	0520 1730 41 30 11	0520 1730 42 20 12	0520 1811 31 20 11
0520 1820 42 20 12	0520 1820 52 20 11	0520 2450 41 40 11	0520 2450 42 40 21
0520 2540 41 30 11	0520 2621 31 40 21	0520 2630 42 40 21	1330 0830 41 10 11
1330 0830 41 20 11	1330 0920 41 01 11	1330 0920 41 10 12	1330 0920 41 11 11
1330 0920 51 10 11	1330 1640 40 40 11	1330 1640 41 20 13	1330 1640 42 20 11
1330 1730 41 11 11	1330 1730 41 20 11	1330 1730 41 30 12	1330 1820 42 20 11
1330 1820 52 20 11	1330 2450 40 60 21	1330 2450 41 40 11	1330 2450 42 40 21
1330 2540 41 30 11	1330 2630 42 40 21	2221 0920 51 21 22	2302 0920 51 21 21
3031 1640 40 40 11	3031 2450 40 60 21	0440 0a00 42 00 21	0440 0a00 43 00 21
0440 0910 32 00 12	0440 0910 41 00 11	0440 0910 42 00 11	0440 1720 31 10 12
0440 1720 31 20 14	0440 1720 32 10 14	0440 1720 32 20 11	0440 1720 41 10 12
0440 1720 42 20 11	0440 1810 32 10 14	0440 1810 42 10 12	0440 1810 42 10 22
0440 1810 42 20 11	0440 1900 43 10 11	0440 2530 31 20 12	0440 2530 31 40 11
0440 2530 32 30 12	0440 2620 32 30 12	0602 0a00 42 00 21	0602 0a00 43 00 41
0602 0820 42 00 41	0602 1720 42 20 11	0602 2440 42 40 41	0602 2620 42 40 41
1331 0820 41 10 11	1331 0910 41 10 11	1331 1720 31 20 12	1331 1720 41 30 11
1331 1720 42 20 12	1331 1810 42 20 11	1331 2440 41 40 11	1331 2440 42 40 21
1331 2530 31 40 11	1331 2620 42 40 21	1412 0910 41 11 11	2060 2440 42 40 41
2141 0820 41 20 21	2141 2620 42 40 41	2222 1720 41 30 11	2222 2440 40 60 41
2222 2440 41 40 21	4004 2440 40 60 81	0360 0900 32 00 21	0360 0900 33 00 61
0360 1710 31 10 11	0360 1710 32 20 11	0360 2610 22 20 13	0360 2610 32 20 13
0360 2700 33 20 11	0360 2700 33 20 21	0360 3420 22 40 23	0441 0900 31 00 21
0441 1710 31 10 11	0441 1710 32 10 11	0441 1800 32 10 12	0441 1800 33 10 11
0441 2520 31 20 11	0441 2520 32 30 11	0441 2610 22 20 11	0441 2610 32 20 11
0441 2610 32 30 11	0441 2700 33 20 21	0441 3420 22 40 21	0522 1710 32 10 11

0522 1800 32 10 12	0522 1800 33 10 11	0522 2520 31 20 21	0522 2520 32 30 11
0522 2610 32 30 11	0603 0900 31 01 61	1251 1710 31 20 11	1251 2520 32 30 11
1332 1710 31 20 11	1332 1710 32 20 11	1332 2520 31 40 11	1332 2520 32 30 11
1332 2610 32 30 12	2304 2520 31 40 21	0280 2600 22 20 11	0280 2600 23 20 21
0280 3410 22 30 21	0280 3410 22 40 21	0361 2600 22 20 11	0361 2600 23 20 11
0361 3410 22 30 11	0361 3410 22 40 22	0361 3500 23 30 11	0442 2600 22 20 13
0442 2600 22 20 22	0442 2600 23 20 22	0442 3410 22 30 11	0442 3410 22 30 23
0442 3500 23 30 13	0523 2600 22 20 21	0523 2600 23 20 21	0523 3410 22 40 21
0604 2600 22 20 41	1252 3410 22 40 22	1333 3410 22 40 21	1414 3410 22 40 21
0281 4300 13 40 12	0281 4300 13 40 21	0362 4300 13 40 11	0362 4300 13 40 21
0443 4300 13 40 13	0443 4300 13 40 21	0524 4300 13 40 21	0282 6000 04 60 41
0363 6000 04 60 21	0363 6000 04 60 61	0444 6000 04 60 83	

The letters a, b and c appearing in this list represent the integers $10, 11$ and 12. We label the 327 types of three-dimensional metrics as Type 13, Type 14, ..., Type 339, in the order in which they appear in this list. Whenever the string does not end in a 1 then that string refers to more than one type.

For instance, the first underlined string refers to Type 32 and Type 33. For both of these types, the three-dimensional cell in T_d is a triangular prism, hence $R = (2, 3, 0, 0)$. The next invariant $B = (0, 9, 4, 0)$ says that the two-dimensional part of T_d consists of nine quadrangles and four pentagons. Types 34 through 38 share these characteristics. What distinguishes Types 32-33 from Types 34-38 is the number of cubic generators in the Stanley-Reisner ideal. The relevant vectors $C = (c_5, c_6)$ in the table entries are 01, 11 and 21. The tight spans of Type 32 and 33 are depicted in Figure 2. The location of the four pentagons relative to the six exterior segments of the figure shows that these two types are non-isomorphic.

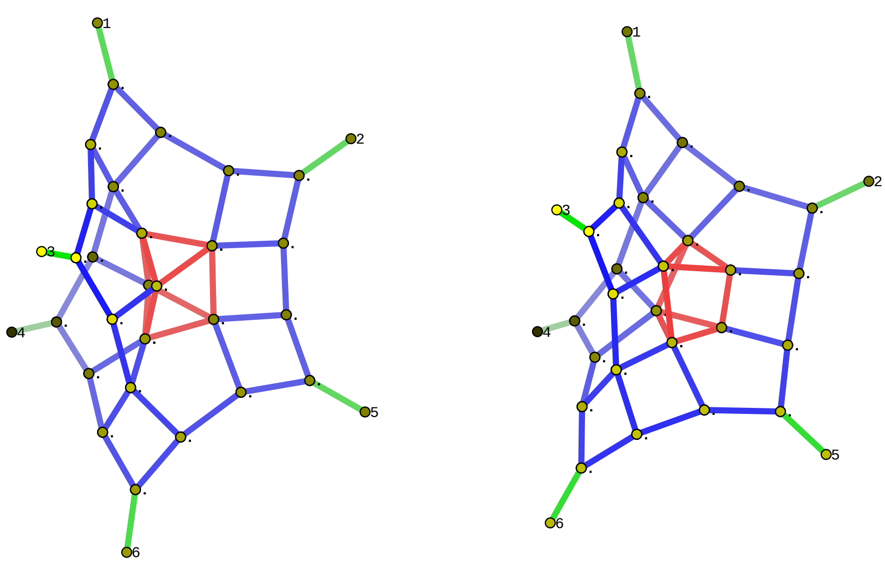

Figure 2: The tight spans of the metrics # 32 and # 33

The second underlined string in the long list is Type 66. Its 3-cell is a cube (hence

$R = 0600$), and its 2-dimensional part consists of twelve quadrangles (hence $B = 0c00$). This is the unique generic metric which is *split-decomposable* (hence $S = 63$) in the sense of [1]. It corresponds to the quadratic Gröbner bases (hence $C = 00$) and the *thrackle triangulation* described in [3, §2]. It has the symmetry group of a regular hexagon (hence $g = 12$) and it is uniquely characterized by R and B (hence $t = 1$). Its tight span is the logo for the conference on *Phylogenetic Combinatorics* which was held in Uppsala, Sweden, in July 2004, `http://www.lcb.uu.se/pca04/`.

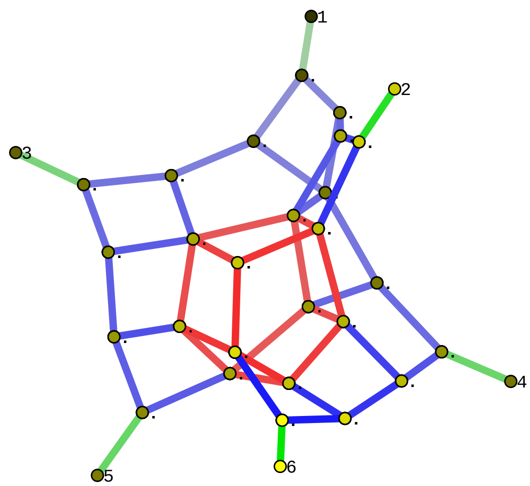

Figure 3: The tight span of the metric # 131

The third underlined string represents a class of four types, namely, Types 131, 132, 133 and 134. In each of these four cases, the 3-dimensional cell is a pentagonal prism (hence $R = 0520$), the two-dimensional part consists of nine quadrangles and two pentagons (hence $B = 0920$), and the Gröbner basis is quadratic (hence $C = 00$). Figure 3 shows one of these tight spans.

4 The 14 Non-regular Triangulations

Theorem 4.2 in [3] states that the hypersimplex $\Delta(n, 2)$ has non-regular triangulations for $n \geq 9$. In this section we strengthen this result as follows.

Theorem 3 *The second hypersimplex* $\Delta(n, 2)$ *admits non-regular triangulations if and only if* $n \geq 6$. *There are precisely* 14 *symmetry classes of non-regular triangulations of* $\Delta(6, 2)$.

It can be shown by explicit computations that all triangulations of $\Delta(4, 2)$ and $\Delta(5, 2)$ are regular. In what follows we list the 14 non-regular triangulations of $\Delta(6, 2)$. Each of them can be lifted to a non-regular triangulation of $\Delta(n, 2)$ for $n \geq 7$ using the technique described at the end of [3, §4].

Each non-regular triangulation Δ has a dual polyhedral cell complex T. This complex T shares all the combinatorial properties of a tight span T_d, but it cannot be realized as the complex of bounded faces of a polyhedron P_d. We call T the *abstract tight span* dual to Δ.

We use the labels Type 340, Type 341, ..., Type 353 to denote the 14 non-regular triangulations Δ of $\Delta(6,2)$. In each case we describe the abstract tight span T. The first four of the 14 abstract tight spans are two-dimensional. They can be characterized by means of the invariants in Section 2:

Type 340: $(0,12,3)$, 1, 1
Type 341: $(0,12,3)$, 1, 2
Type 342: $(0,12,3)$, 6, 0
Type 343: $(1,10,4)$, 4, 1

The remaining ten abstract tight spans have a unique three-dimensional cell. We characterize them using the invariants (R,B,S,C,g) of Section 3:

Type 344: $(4,0,0,0)$, $(0,8,6,0)$, $(4,0)$, $(0,0)$, 4
Type 345: $(0,4,4,0)$, $(0,8,2,0)$, $(4,0)$, $(0,0)$, 4
Type 346: $(0,4,4,0)$, $(2,4,4,0)$, $(4,0)$, $(4,0)$, 2
Type 347: $(0,4,4,0)$, $(2,4,4,0)$, $(4,0)$, $(2,0)$, 4
Type 348: $(0,4,4,0)$, $(2,4,4,0)$, $(4,0)$, $(2,0)$, 4
Type 349: $(0,4,4,0)$, $(2,4,4,0)$, $(4,0)$, $(6,0)$, 8
Type 350: $(0,3,6,0)$, $(2,5,2,0)$, $(3,0)$, $(2,0)$, 2
Type 351: $(0,3,6,0)$, $(2,5,2,0)$, $(3,0)$, $(4,0)$, 2
Type 352: $(0,2,8,0)$, $(2,6,0,0)$, $(2,2)$, $(2,0)$, 4
Type 353: $(0,0,12,0)$, $(6,0,0,0)$, $(0,4)$, $(6,0)$, 24.

Type # 353 is the most symmetric one among triangulations Δ of $\Delta(6,2)$. The corresponding abstract tight span T is a beautiful object, namely, it is a dodecahedron with six triangles and six edges attached, as shown in Figure 4. The Stanley-Reisner ideal corresponding to the dodecahedral type # 353 is generated by 30 quadrics and 6 cubics. The quadrics in this ideal are

$$
\begin{gathered}
x_{12}x_{35}, x_{12}x_{36}, x_{12}x_{45}, x_{12}x_{46}, x_{12}x_{56}, x_{13}x_{24}, x_{13}x_{26}, x_{13}x_{45},\\
x_{13}x_{46}, x_{13}x_{56}, x_{14}x_{23}, x_{14}x_{26}, x_{14}x_{35}, x_{14}x_{36}, x_{14}x_{56}, x_{15}x_{23},\\
x_{15}x_{24}, x_{15}x_{26}, x_{15}x_{36}, x_{15}x_{46}, x_{23}x_{45}, x_{23}x_{46}, x_{23}x_{56}, x_{24}x_{35},\\
x_{24}x_{36}, x_{24}x_{56}, x_{26}x_{35}, x_{26}x_{45}, x_{35}x_{46}, x_{36}x_{45}.
\end{gathered}
$$

The twelve variables x_{ij} appearing in this list can be identified with the edges of an octahedron. The 30 quadrics are precisely the pairs of disjoint edges of the octahedron. We note that these quadratic generators (and hence the global structure of the tight span) are also shared by the last six types (334, 335, 336, 337, 338, 339) in the table of Section 3. The simplicial complex represented by these 30 quadrics is (essentially) the boundary of the truncated octahedron (with the six square faces regarded as tetrahedra).

Now, each of the Types 334, 335, 336, 337, 338, 339 and 353 has six cubic generators in its ideal. The choice of these cubic generators amounts to subdividing each of the six square faces of the truncated octahedron with one of its two diagonals. Type 353 arises from the most symmetric choice of these diagonals. The six cubics in the ideal for Type 353 are

$$\underline{x_{34}}x_{12}x_{26},\ \underline{x_{34}}x_{15}x_{56},\ \ \underline{x_{16}}x_{23}x_{35},\ \underline{x_{16}}x_{24}x_{45},\ \ \underline{x_{25}}x_{13}x_{14},\ \underline{x_{25}}x_{36}x_{46}.$$

The underlined variables are the non-edges of the octahedron.

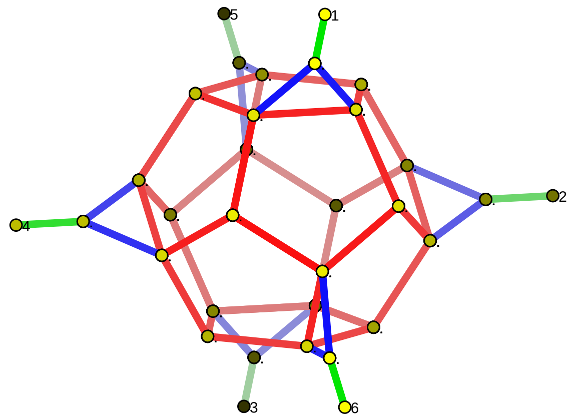

Figure 4: The dodecahedral (abstract) tight span # 353

The abstract tight span T has the following geometric description. The polytope dual to the truncated octahedron is gotten from the 3-cube by subdividing each of the six facets with a new vertex (thus creating $6 \times 4 = 24$ edges) and then erasing the 12 edges of the cube. Consider the six 4-valent vertices we just introduced. Each of them can be replaced by two trivalent vertices with a new edge in-between. If this replacement is done in the most symmetric manner then the result is a dodecahedron. Finally, we glue a triangle and an edge on each of the six new edges. The result is Figure 4.

5 Prime Metrics and Minimal Subdivisions

Koolen, Moulton and Tönges [12] classified all the *prime metrics* for $n = 6$. These are the rays in the metric fan MF_6. There are 14 symmetry classes:

1/5 Split: $d = (0,0,0,0,1,0,0,0,1,0,0,1,0,1,1)$
2/4 Split: $d = (1,1,0,1,1,0,1,0,0,1,0,0,1,1,0)$
3/3 Split: $d = (0,0,1,1,1,0,1,1,1,1,1,1,0,0,0)$
Prime P_1 : $d = (1,1,1,1,2,2,2,2,1,2,2,1,2,1,1)$
Prime P_2 : $d = (1,1,1,2,2,2,2,1,1,2,1,1,1,1,2)$
Prime P_3 : $d = (1,1,1,2,2,2,2,1,1,2,1,1,3,1,2)$
Prime P_4 : $d = (1,1,1,1,2,1,1,1,1,1,1,1,2,1,1)$
Prime P_5 : $d = (1,2,2,2,4,3,3,3,3,2,2,2,4,2,2)$
Prime P_6 : $d = (1,1,2,3,3,2,3,2,2,3,2,2,1,1,2)$

Prime P_7 : $d = (1,1,1,2,2,2,2,1,1,2,1,1,2,1,2)$
Prime P_8 : $d = (1,2,2,4,4,3,3,3,3,4,2,2,2,2,4)$
Prime P_9 : $d = (1,1,1,2,3,2,2,1,2,2,1,2,3,2,1)$
Prime P_{10} : $d = (0,1,1,1,2,1,1,1,2,2,2,1,2,1,1)$
Prime P_{11} : $d = (0,1,1,2,2,1,1,2,2,2,1,1,1,1,2)$

Our computations provide an independent verification of the correctness and completeness of the results in [12]. Namely, we computed the cone in the metric fan corresponding to each of the 339 metrics. For each cone we computed (using POLYMAKE [10]) the facets and the extreme rays of the cone. And it turned out that the extreme rays we found are precisely the 14 types listed above. All the facets and extreme rays of the 339 types of maximal cones in the metric fan MF_6 are posted at

`bio.math.berkeley.edu/SixPointMetrics`

Our computations lead to the following result.

Proposition 4 *If f_i is the number of types of maximal cones in the metric fan MF_6 with i facets and e_j is the number with j extreme rays then*

$$\begin{aligned}(f_{15}, f_{16}, \ldots, f_{21}) &= (197, 42, 63, 18, 8, 10, 1)\\ (e_{15}, e_{16}, \ldots, e_{24}) &= (197, 60, 28, 19, 20, 2, 5, 2, 1, 5).\end{aligned}$$

In particular, the metric fan MF_6 has 197 types of simplicial cones.

Proposition 4 shows that the largest number of facets of any cone is 21, attained by only one type, and the largest number of extreme rays is 24, attained by five types. The next two examples concern the extremal cases.

Example 5 *Type 12 is the last metric listed in the table of Section 2. Its cone in the metric fan is described by the following 21 linear inequalities:*

$$\begin{gathered}
d_{12}+d_{25} \geq d_{15}, \quad d_{13}+d_{36} \geq d_{16}, \quad d_{45}+d_{46} \geq d_{56},\\
d_{25}+d_{45} \geq d_{24}, \quad d_{36}+d_{46} \geq d_{34}, \quad d_{12}+d_{13} \geq d_{23},\\
d_{26}+d_{34} \geq d_{24}+d_{36},\ d_{15}+d_{26} \geq d_{16}+d_{25},\ d_{14}+d_{23} \geq d_{13}+d_{24},\\
d_{14}+d_{56} \geq d_{16}+d_{45},\ d_{16}+d_{35} \geq d_{15}+d_{36},\ d_{24}+d_{35} \geq d_{25}+d_{34},\\
d_{14}+d_{23} \geq d_{12}+d_{34},\ d_{14}+d_{56} \geq d_{15}+d_{46},\ d_{26}+d_{34} \geq d_{23}+d_{46},\\
d_{24}+d_{35} \geq d_{23}+d_{45},\ d_{15}+d_{26} \geq d_{12}+d_{56},\ d_{16}+d_{35} \geq d_{13}+d_{56},\\
d_{15}+d_{23}+d_{34}+d_{56} \geq d_{16}+d_{24}+2d_{35},\\
d_{16}+d_{23}+d_{24}+d_{56} \geq d_{15}+2d_{26}+d_{34},\\
d_{15}+d_{16}+d_{24}+d_{34} \geq 2d_{14}+d_{23}+d_{56}.
\end{gathered}$$

None of these 21 inequalities is redundant. This cone has 19 extreme rays: the six 1/5 splits, six 2/4 splits, three rays of type P_6, three rays of type P_7, and a unique ray of type P_2, namely, $(1,1,1,2,2,2,2,1,1,2,1,1,1,1,2)$.

□

The six 1/6-splits are among the extreme rays of every cone in the metric fan MF_6. In the next example, we only list the other extreme rays.

Example 6 *The five types of cones in MF_6 with 24 extreme rays are # 7, # 26, # 337, # 338, and # 339. Only # 7 corresponds to a two-dimensional tight span. The following 18 vectors are extreme rays of this cone:*

$$\begin{array}{rr}
2/4\,\mathrm{split}: (101111000111000) & 2/4\,\mathrm{split}: (100011110001011) \\
2/4\,\mathrm{split}: (011111111000000) & 2/4\,\mathrm{split}: (010011001110011) \\
P_2: (111222211211112) & P_6: (121333222311222) \\
P_6: (122233312221213) & P_6: (211333311222222) \\
P_6: (212233221312213) & P_7: (111222211211212) \\
P_8: (221444322322334) & P_8: (222344412432323) \\
P_8: (222344432412323) & P_8: (223444322322314) \\
P_{10}: (111222011211112) & P_{10}: (111222211011112) \\
P_{10}: (111022211211112) & P_{10}: (111222211211110)
\end{array}$$

The other four types with 24 extreme rays are three-dimensional, and each of them has three-dimensional prime metrics among its extreme rays. For instance, the following 18 vectors are extreme rays of the cone # 338:

$$\begin{array}{rr}
3/3\,\mathrm{split}: (110010110110011) & 3/3\,\mathrm{split}: (101011010101101) \\
3/3\,\mathrm{split}: (010111011100110) & 3/3\,\mathrm{split}: (001110111111000) \\
P_1: (111122221221211) & P_4: (111121111211111) \\
P_5: (322243331422222) & P_5: (222142232432323) \\
P_5: (122243333422222) & P_5: (222342232432321) \\
P_9: (221232121321121) & P_9: (121131222311222) \\
P_9: (112132122322112) & P_9: (212231221312211) \\
P_{11}: (111122221221011) & P_{11}: (111122221201211) \\
P_{11}: (111120221221211) & P_{11}: (111122021221211)
\end{array}$$

The corresponding lists for all 339 types appear on our website.

□

It is instructive to draw the tight spans of the eleven prime metrics. Four of them are actually three-dimensional. For instance, the metric P_4 has the structure of an octahedron. Please compare Figure 5 with [12, Figure 1].

For each of the 11 prime metrics, we list the f-vector of their tight span:

	# vertices	# edges	# polygons	# 3-cells
P_1 :	6	9	4	0
P_2 :	7	15	9	0
P_3 :	6	10	6	1
P_4 :	10	20	12	1
P_5 :	11	20	11	1
P_6 :	7	11	5	0
P_7 :	11	20	10	0
P_8 :	11	19	9	0
P_9 :	7	12	7	1
P_{10} :	5	7	3	0
P_{11} :	5	7	3	0

The metrics $P_1, \ldots, P_9$ each have six *trivial vertices* in their tight span. The list of non-trivial vertices given in [12, Table 1] is consistent with our list above. The prime metrics P_{10} and P_{11} are improper in the sense that two points have distance zero.

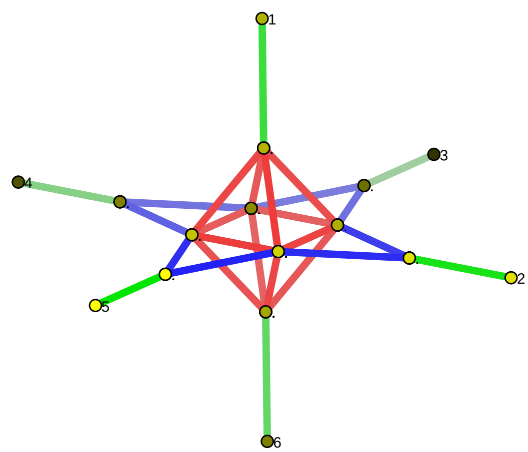

Figure 5: The tight span of the metric $P_4 + (1, 1, \ldots, 1)$

Remark 7 *The tight spans of the prime metrics P_3, P_4, P_5, P_9 have a 3-dimensional cell, while the tight spans of the other seven are 2-dimensional.*

The metric fan MF_6 defines an incidence relation between the prime metrics and the generic metrics. This leads to a finer invariant for distinguishing among the 339 types of generic types. This invariant is the vector $(S, P) = (s_2, s_3,\, p_1, p_2, \ldots, p_{11})$ where p_i is the number of extreme rays of type P_i which lie on the corresponding cone. This invariant resolves about half of the clusters which had been left unresolved by the earlier invariants.

Example 8 *The three two-dimensional types # 9, # 10 and # 11, all have the same invariants* $R = (2, 8, 5)$, $g = 2$ *and* $c = 4$ *in the list of Section 2. They are distinguished by the new invariant* $(S, P) = (s_2, s_3,\, p_1, p_2, \ldots, p_{11})$

Type 9 has $(S, P) = (4, 0, 0, 1, 0, 0, 0, 2, 1, 2, 0, 2, 0)$.

Type 10 has $(S, P) = (5, 0, 0, 1, 0, 0, 0, 1, 2, 2, 0, 2, 0)$.

Type 11 has $(S, P) = (5, 0, 0, 1, 0, 0, 0, 2, 2, 2, 0, 2, 0)$.

Example 9 *The last three three-dimensional types # 337, # 338 and # 339 all have the same invariants* (f, R, B, S, C) *in the big table of Section 3. They are distinguished by the new invariant* $(S, P) = (s_2, s_3,\, p_1, p_2, \ldots, p_{11})$

Type 337 has $(S, P) = (0, 4, 0, 0, 1, 1, 4, 0, 0, 0, 4, 0, 4)$.

Type 338 has $(S, P) = (0, 4, 1, 0, 0, 1, 4, 0, 0, 0, 4, 0, 4)$.

Type 339 has $(S, P) = (0, 4, 1, 0, 1, 0, 4, 0, 0, 0, 4, 0, 4)$.

We close this section with a remark aimed at experts in polytope theory. The metric fan is the secondary fan of the hypersimplex $\Delta(6, 2)$, hence it is the normal fan of the *secondary polytope* $\Sigma\big(\Delta(6, 2)\big)$. Following [2], the vertices of the secondary polytope correspond to the regular triangulations of $\Delta(6, 2)$, and the facets of the secondary polytope correspond to *minimal regular subdivisions* of $\Delta(6, 2)$. For instance, Figure 5 is the dual picture to a regular subdivision of $\Delta(6, 2)$ into 10 five-dimensional polytopes.

The results described in this section provide the vertex-facet incidence matrix of the secondary polytope $\Sigma\big(\Delta(6, 2)\big)$. In particular, the classification result of Koolen, Moulton and Tönges [12] can be restated as follows.

Corollary 10 *Up to the action of the symmetric group on* $\{1, 2, 3, 4, 5, 6\}$*, the secondary polytope* $\Sigma\big(\Delta(6, 2)\big)$ *has precisely* 14 *facets.*

6 Visualization of Tight Spans in POLYMAKE

POLYMAKE is a software package developed by Ewgenij Gawrilow and Michael Joswig for studying polytopes and polyhedra [10]. It allows us to define a polyhedron by a set of either inequalities or vertices and computes numerous properties of the polyhedron. We implemented a client program to POLYMAKE for visualizing the tight span T_d of a given metric d. In short, our program does the following to produce the figures in this paper:

1. Compute all faces of the polyhedron P_d from the given metric d.
2. Build the tight span T_d by extracting the bounded faces of P_d.
3. Spring-embed the 2-skeleton of T_d in 3-space using POLYMAKE's spring embedder and display it using JAVAVIEW.
4. Label the points corresponding to the finite set (the "taxa") on which the metric is defined. In our case, the set of labels is $\{1, 2, 3, 4, 5, 6\}$.

The program was developed by the second author with the assistance of Michael Joswig and Julian Pfeifle. The code can be downloaded at

`www.math.berkeley.edu/~jyu/`.

The figures above only show the combinatorial properties of the tight span and, because of the projection and the spring embedder, the edge lengths seen here do not represent the actual edge lengths in the tight span.

The pictures produced by our software are different from the output produced by the software SPLITSTREE [8]. SPLITSTREE is a program that can, among other things, compute and visualize the split-decompositions of metrics (see [1, 8]). It decomposes an input metric into a sum of splits plus a *split-prime* metric that cannot be further decomposed into splits. SPLITSTREE outputs a planar graph representing the split-decomposable part of the input metric, where sets of parallel edges represent splits.

When a metric is split-decomposable, then the tight span is a cubical complex, and the output from SPLITSTREE agrees with our visualization with a few edges removed to make it planar. The edges output by SPLITSTREE are scaled to be metrically accurate. By contrast, our implementation does not preserve the edge lengths. Among the 339 generic metrics on six points, only one type (namely, Type # 66) is split-decomposable. For all other 338 types, the picture produced by our program contains more refined combinatorial information than the output of SPLITSTREE.

The split-decomposition theory of Bandelt and Dress [1] has the following interpretation in terms of the metric fan. The 31 splits are among the extreme rays of the metric fan. Consider the *induced subfan* of MF_6 on the 31 splits. A key result in [1] states that this subcomplex is simplicial, i.e., all cones in this subfan are spanned by linearly independent vectors. Now consider any metric d, for instance, one of our 339 generic metrics, and let C be the cone of the metric fan MF_6 containing d in its relative interior. The intersection of C with the induced subfan of split-decomposable metrics is a simplicial face F of C. It follows that d can be written uniquely as a sum of a vector d' in F and a positive combination of extreme rays of C not in F. The output of SPLITSTREE is the edge graph of the tight span of d'.

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