Multiple pattern avoidance with respect to fixed points and excedances

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Submitted: Oct 22, 2003; Accepted: Jul 25, 2004; Published: Jul 29, 2004
Mathematics Subject Classifications: 05A15, 05A05

Abstract

We study the distribution of the statistics 'number of fixed points' and 'number of excedances' in permutations avoiding subsets of patterns of length 3. We solve all the cases of simultaneous avoidance of more than one pattern, giving generating functions enumerating these two statistics. Some cases are generalized to patterns of arbitrary length. For avoidance of one single pattern we give partial results. We also describe the distribution of these statistics in involutions avoiding any subset of patterns of length 3.

The main technique is to use bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that the statistics in permutations that we study correspond to statistics on Dyck paths that are easy to enumerate.

1 Introduction

The problem of enumerating pattern-avoiding permutations, also known as restricted permutations, has generated a lot of research over the last few decades. One of the most referenced papers on this topic is [18], which contains a systematic enumeration of permutations avoiding any subset of patterns of length 3.

However, the study of statistics in pattern-avoiding permutations started developing very recently, and the interest in this topic is currently growing. Two of the most studied permutation statistics have been the number of fixed points and the number of excedances of a restricted permutation. For example, in [16, 6, 8] it is shown the surprising fact that the joint distribution of these two statistics is the same in 132- and in 321-avoiding permutations. In [5], involutions avoiding any single pattern of length 3 are studied with respect to the number of fixed points. Another paper [14] deals with the enumeration of permutations with a given number of fixed points avoiding simultaneously two or more patterns of length 3. Finally, [12] considers additional restrictions on 132-avoiding involutions.

Given two permutations $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{S}_m$, with $m \le n$, we say that π *contains* σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is in the same relative order as $\sigma_1\sigma_2\cdots\sigma_m$. If π does not contain σ, we say that π *avoids* σ, or that it is σ-*avoiding*. Denote by $\mathcal{S}_n(\sigma)$ the set of σ-avoiding permutations in $\mathcal{S}_n$. More generally, if $\Sigma \subseteq \bigcup_{k\ge1}\mathcal{S}_k$ is any set of permutations, denote by $\mathcal{S}_n(\Sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid simultaneously all the patterns in Σ (also called Σ-avoiding permutations).

We say that i is a *fixed point* of a permutation π if $\pi_i = i$, and that i is an *excedance* of π if $\pi_i > i$. Denote by $\mathrm{fp}(\pi)$ and $\mathrm{exc}(\pi)$ the number of fixed points and the number of excedances of π respectively. We are interested in enumerating pattern-avoiding permutations with respect to these two statistics. Most of the results will be expressed in terms of multivariate generating functions.

In Section 2 we introduce the definitions, the preliminaries, and the basic tools that we use in the paper. Section 3 is devoted to the study of permutations with a single restriction with respect to the statistics fp and exc (except in the case of the pattern 123, for which we can only give partial results regarding fp). In Section 4 we solve completely the case of permutations avoiding simultaneously any two patterns of length 3, giving generating functions counting the number of fixed points and the number of excedances. For some particular cases we can generalize the results, allowing one pattern of the pair to have arbitrary length. In Section 5 we give the analogous generating functions for permutations avoiding simultaneously any three patterns of length 3 or more. Finally, Section 6 is concerned with the study of the distribution of these statistics in involutions avoiding any subset of patterns of length 3. We end with a few final remarks about the results of the paper and possible extensions.

2 Bijections and statistics

We will represent a permutation $\pi \in \mathcal{S}_n$ by an $n \times n$ array with a cross in each one of the squares (i, π_i). Fixed points of π correspond to crosses on the main diagonal (from top-left to bottom-right corner) of the array, and excedances of π are represented by crosses to the right of this diagonal.

2.1 Trivial bijections in $\mathcal{S}_n$

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, define its complementation $\bar{\pi} = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$. The array of $\bar{\pi}$ is obtained from the array of π by a flip along a vertical axis, so fixed points (resp. excedances) of π correspond to crosses on (resp. to the left of) the antidiagonal (from top-right to bottom-left corner) of the array of $\bar{\pi}$. Similarly, define $\widehat{\pi}$ to be the permutation whose array is the one obtained from that of π by reflection along the antidiagonal. Note that reflecting the array of π along the main diagonal, we get the array of its inverse π^{-1}. For any set of permutations Σ, let $\bar{\Sigma}$ be the set obtained by reversing each of the elements of Σ. Define $\widehat{\Sigma}$ and Σ^{-1} analogously. The following trivial lemma will be used later on.

Lemma 2.1 *Let $\Sigma \subset \bigcup_{k\geq 1} \mathcal{S}_k$ be any set of permutations, and let $\pi \in \mathcal{S}_n$. We have that*

(1) $\pi \in \mathcal{S}_n(\Sigma) \iff \bar{\pi} \in \mathcal{S}_n(\bar{\Sigma}) \iff \widehat{\pi} \in \mathcal{S}_n(\widehat{\Sigma}) \iff \pi^{-1} \in \mathcal{S}_n(\Sigma^{-1})$,

(2) $\mathrm{fp}(\widehat{\pi}) = \mathrm{fp}(\pi)$, $\mathrm{exc}(\widehat{\pi}) = \mathrm{exc}(\pi)$,

(3) $\mathrm{fp}(\pi^{-1}) = \mathrm{fp}(\pi)$, $\mathrm{exc}(\pi^{-1}) = n - \mathrm{fp}(\pi) - \mathrm{exc}(\pi)$.

We are interested in the distribution of the statistics fp and exc among the permutations avoiding a certain set Σ of patterns. Given any such set Σ, we define the generating function F_Σ as

$$F_\Sigma(x, q, z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n.$$

If $\Sigma = \{\sigma\}$, we will write F_σ instead of $F_{\{\sigma\}}$. The following lemma restates the previous one in terms of generating functions.

Lemma 2.2 *Let Σ be any set of permutations. We have*

(1) $F_{\widehat{\Sigma}}(x, q, z) = F_\Sigma(x, q, z)$,

(2) $F_{\Sigma^{-1}}(x, q, z) = F_\Sigma(\frac{x}{q}, \frac{1}{q}, qz)$.

Proof. To prove (1), consider the bijection between $\mathcal{S}_n(\Sigma)$ and $\mathcal{S}_n(\widehat{\Sigma})$ that maps π to $\widehat{\pi}$. The equation follows from parts (1) and (2) of Lemma 2.1.

Equation (2) follows similarly from parts (1) and (3) of the previous lemma, noticing that

$$\sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma^{-1})} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi^{-1})} q^{\mathrm{exc}(\pi^{-1})} z^n$$
$$= \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{n-\mathrm{fp}(\pi)-\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} \left(\frac{x}{q}\right)^{\mathrm{fp}(\pi)} \left(\frac{1}{q}\right)^{\mathrm{exc}(\pi)} (qz)^n.$$

□

If for two sets of patterns Σ_1 and Σ_2 we have that $F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(x, q, z)$ (i.e., the joint distribution of fixed points and excedances is the same in Σ_1-avoiding as in Σ_2-avoiding permutations), as in part (1) of Lemma 2.2, we will write $\Sigma_1 \approx \Sigma_2$. If we are in the situation of part (2), that is, $F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(\frac{x}{q}, \frac{1}{q}, qz)$, we will write $\Sigma_1 \sim \Sigma_2$.

2.2 Bijection to Dyck paths

Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$ the class of all Dyck paths. One of the main ingredients we will need throughout the paper is a bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$, which we define in the next paragraph.

Any permutation $\pi \in \mathcal{S}_n$ can be represented as an $n \times n$ array with crosses in positions (i, π_i). From this array of crosses, we obtain the *diagram* of π as follows. For each cross, shade the cell containing it and the squares that are due south and due east of it. The diagram is defined as the region that is left unshaded. It is shown in [15] that this gives a bijection between $\mathcal{S}_n(132)$ and Young diagrams that fit in the shape $(n-1, n-2, \ldots, 1)$. Consider now the path determined by the border of the diagram of π, that is, the path with *up* and *right* steps that goes from the lower-left corner to the upper-right corner of the array, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. Define $\varphi(\pi)$ to be the Dyck path obtained from this path by reading an up-step every time it goes up and a down-step every time it goes right. Since the path in the array does not go below the diagonal, $\varphi(\pi)$ does not go below the x-axis. Figure 1 shows an example when $\pi = 67435281$.

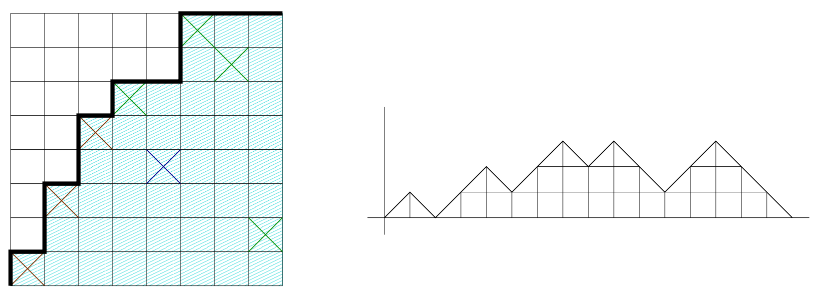

Figure 1: The bijection φ.

The bijection φ is essentially the same bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ given by Krattenthaler [13] (see also [11]), up to reflection of the path from a vertical line.

Next we define the inverse map $\varphi^{-1} : \mathcal{D}_n \longrightarrow \mathcal{S}_n(132)$. Given a Dyck path $D \in \mathcal{D}_n$, the first step needed to reverse the above procedure is to transform D into a path U from the lower-left corner to the upper-right corner of an $n \times n$ array, not going below the diagonal connecting these two corners. Then, the squares to the left of this path form a diagram, and we can shade all the remaining squares. From this diagram, the permutation $\pi \in \mathcal{S}_n(132)$ can be recovered as follows: row by row, put a cross in the leftmost shaded square such that there is exactly one cross in each column.

2.3 Statistics on Dyck paths

It is well-known that $|\mathcal{D}_n| = \mathbf{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. If $D \in \mathcal{D}_n$, we will write $|D| = n$ to indicate the *semilength* of D. The generating function that enumerates

Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} z^{|D|} = \sum_{n\geq 0} \mathbf{C}_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, which we denote by $\mathbf{C}(z)$. Sometimes it will be convenient to encode each up-step by a letter $\mathbf{u}$ and each down-step by $\mathbf{d}$, obtaining an encoding of the Dyck path as a *Dyck word*.

A *peak* of a Dyck path $D \in \mathcal{D}$ is an up-step followed by a down-step (i.e., an occurrence of $\mathbf{ud}$ in the associated Dyck word). A *hill* is a peak at height 1, where the height is the y-coordinate of the top of the peak. A *valley* of a Dyck path $D \in \mathcal{D}$ is a down-step followed by an up-step (i.e., an occurrence of $\mathbf{du}$ in the associated Dyck word). The *height* of D is the y-coordinate of the highest point of the path. Denote by $\mathcal{D}^{\leq k}$ the set of Dyck paths of height at most k. Define a *pyramid* to be a Dyck path that has only one peak, that is, a path of the form $\mathbf{u}^k\mathbf{d}^k$ with $k \geq 1$ (here the exponent indicates the number of times the letter is repeated).

As defined in [6], a *tunnel* of a Dyck path D is a horizontal segment between two lattice points of D that intersects D only in these two points, and stays always below D. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, where $B \in \mathcal{D}$ (no restrictions on A and C). In the decomposition, the tunnel is the segment that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$. If $D \in \mathcal{D}_n$, then D has exactly n tunnels, since such a decomposition can be given for each up-step $\mathbf{u}$ of D. The *length* of a tunnel is just its length as a segment, and the *height* is its y-coordinate. It will be useful to define the *depth* of a tunnel T as $\mathrm{depth}(T) := \frac{1}{2}\mathrm{length}(T) - \mathrm{height}(T) - 1$.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the x-coordinate of its midpoint is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that A and C have the same length. Denote by $\mathrm{ct}(D)$ the number of centered tunnels of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *right tunnel* if the x-coordinate of its midpoint is strictly greater than n, that is, the midpoint of the tunnel is to the right of the vertical line through the middle of D. Clearly, in terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that the length of A is strictly bigger than the length of C. Denote by $\mathrm{rt}(D)$ the number of right tunnels of D. In Figure 2, there is one centered tunnel drawn with a solid line, and four right tunnels drawn with dotted lines.

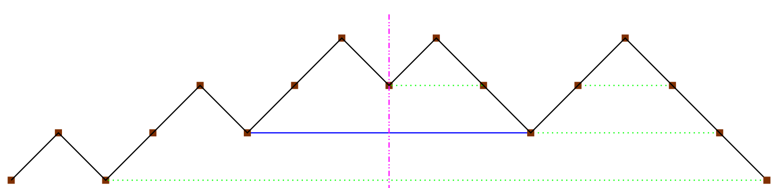

Figure 2: One centered and four right tunnels.

For a Dyck path $D \in \mathcal{D}_n$, denote by D^* the path obtained by reflection of D from a vertical line $x = n$. We say that D is *symmetric* if $D = D^*$. Denote by $\mathcal{D}s \subset \mathcal{D}$ the subclass of symmetric Dyck paths.

2.4 Properties of φ

In order to enumerate fixed points and excedances in permutations, we analyze what these statistics are mapped to by φ. Table 1 summarizes the correspondences of φ that we will use.

In the permutation π	In the array of π	In the Dyck path $\varphi(\pi)$
fixed points of π	crosses on the main diagonal	centered tunnels
excedances of π	crosses to the right of the main diagonal	right tunnels
fixed points of $\bar{\pi}$	crosses on the secondary diagonal	tunnels of depth 0
excedances of $\bar{\pi}$	crosses to the left of the secondary diagonal	tunnels of negative depth

Table 1: Behavior of φ on fixed points and excedances.

The correspondences between the first two columns have been discussed above. Now we show how φ maps the statistics on the second column to those on the third one. We repeat the reasoning in [6, 8].

For this purpose, instead of using $D = \varphi(\pi)$, it will be convenient to consider the path U from the lower-left corner to the upper-right corner of the array of π. We will talk about tunnels of U to refer to the corresponding tunnels of D under this trivial transformation.

Consider the arrangement of crosses of π as defined earlier. We now show how to associate a unique tunnel T of D to each cross X of this array. Observe that given a cross $X = (i, j)$, U has a vertical step in row i and a horizontal step in column j. In D, these two steps in U correspond to steps $\mathbf{u}$ and $\mathbf{d}$ respectively, so they determine a decomposition $D = A\mathbf{u}B\mathbf{d}C$ (see Figure 3), and therefore a tunnel T of D.

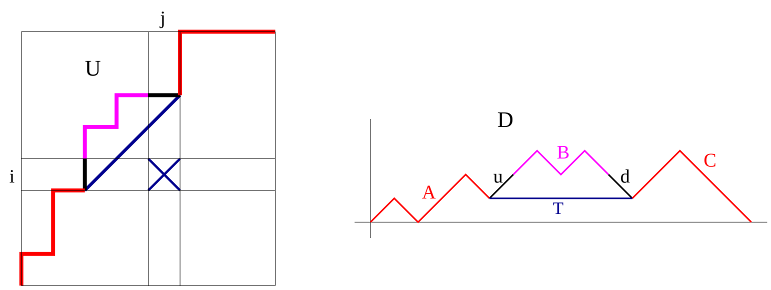

Figure 3: Each cross produces a tunnel.

The distance between these two steps determines the length of T, and the distance from these steps to the antidiagonal of the array determines the height of T. In order for the corresponding cross to lie on the antidiagonal, the relation between these two quantities must be $\frac{1}{2}\text{length}(T) = \text{height}(T) + 1$, which is equivalent to $\text{depth}(T) = 0$, by the definition of depth. The depth of T indicates how far from the antidiagonal X is. The cross lies to the left of the antidiagonal exactly when $\text{depth}(T) < 0$. This justifies the last two rows of the table. The first two correspondences are easier to see. Indeed, T is centered precisely when X is on the main diagonal, and T is a right tunnel when X lies to the right of the main diagonal.

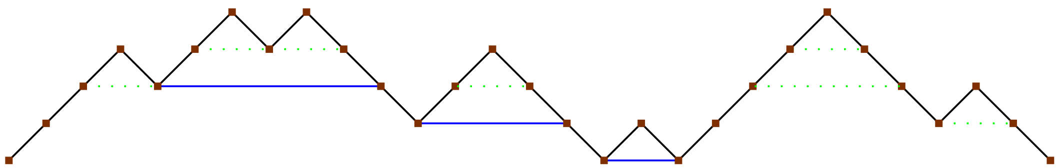

Figure 4: Three tunnels of depth 0 and seven tunnels of negative depth.

We define two new statistics on Dyck paths. For $D \in \mathcal{D}$, let $\mathrm{td}_0(D)$ be the number of tunnels of depth 0 of D, and let $\mathrm{td}_{<0}(D)$ be the number of tunnels of negative depth of D. In Figure 4, there are three tunnels of depth 0 drawn with a solid line, and seven tunnels of negative depth drawn with dotted lines. We have proved the following lemma.

Lemma 2.3 *Let $\pi \in \mathcal{S}_n(132)$, $\rho \in \mathcal{S}_n(312)$. We have*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$,

(3) $\mathrm{fp}(\rho) = \mathrm{td}_0(\varphi(\bar{\rho}))$,

(4) $\mathrm{exc}(\rho) = \mathrm{td}_{<0}(\varphi(\bar{\rho}))$.

2.5 Combinatorial classes and generating functions.

Here we direct the reader to [10] for a detailed account on combinatorial classes and the symbolic method. Let $\mathcal{A}$ be a class of unlabelled combinatorial objects and let $|\alpha|$ be the size of an object $\alpha \in \mathcal{A}$. If $\mathcal{A}_n$ denotes the objects in $\mathcal{A}$ of size n and $a_n = |\mathcal{A}_n|$, then the *ordinary generating function* of the class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} = \sum_{n \geq 0} a_n z^n.$$

Enumerations according to size and auxiliary parameters $\chi_1, \chi_2, \ldots, \chi_r$ are described by multivariate generating functions,

$$A(u_1, u_2, \ldots, u_r, z) = \sum_{\alpha \in \mathcal{A}} u_1^{\chi_1(\alpha)} u_2^{\chi_2(\alpha)} \cdots u_r^{\chi_r(\alpha)} z^{|\alpha|}.$$

Throughout the paper the variable z is reserved for marking the length of a permutation and the semilength of a Dyck path, x is used for marking the number of fixed points of a permutation and the number of centered tunnels or tunnels of depth 0 of a Dyck path, and q is the variable that marks the number of excedances of a permutation and the number of right tunnels or tunnels of negative depth of a Dyck path.

There is a direct correspondence between set theoretic operations (or "constructions") on combinatorial classes and algebraic operations on generating functions. Table 2 summarizes this correspondence for the operations that are used in the paper. There "union" means union of disjoint copies, "product" is the usual cartesian product, and "sequence" forms an ordered sequence in the usual sense. We assume that the parameters χ_i marked by the variables u_i are additive, that is, if $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ then $\chi_i((\alpha, \beta)) = \chi_i(\alpha) + \chi_i(\beta)$.

Construction		*Operation on generating functions*
Union	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$A(u_1, \ldots, u_r, z) = B(u_1, \ldots, u_r, z) + C(u_1, \ldots, u_r, z)$
Product	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(u_1, \ldots, u_r, z) = B(u_1, \ldots, u_r, z)C(u_1, \ldots, u_r, z)$
Sequence	$\mathcal{A} = \mathrm{Seq}(\mathcal{B})$	$A(u_1, \ldots, u_r, z) = \frac{1}{1-B(u_1,\ldots,u_r,z)}$

Table 2: The basic combinatorial constructions and their translation into generating functions.

3 Single restrictions

The most difficult case appears to be that of permutations avoiding one single pattern. It is well known that for any $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = \mathbf{C}_n$. By Lemma 2.2, we have that $132 \approx 213$, and that $231 \sim 312$. These are the only equivalences that follow from the trivial bijections. Recently it was shown that, surprisingly, $321 \approx 132$. The first proof of this result appears in [6] (see [8] for a bijective proof). The weaker version $321 \sim 132$ was proved earlier in [16].

So, we have the following equivalence classes of patterns of length 3 with respect to fixed points and excedances:

a) 123
b) $132 \approx 213 \approx 321$
c) $231 \quad \sim \quad$ **c')** 312

3.1 a) 123

For this case we have not been able to find a satisfactory expression for $F_{123}(x, q, z)$. We can nevertheless give summation formulas for the number of permutations in $\mathcal{S}_n(123)$ with a given number of fixed points. The first trivial observation is that if π avoids 123, then it can have at most two fixed points. If $\pi_i = i$, we say that i is a *big fixed point* of π if $i \geq \frac{n+1}{2}$, and that it is a *small fixed point* if $i < \frac{n+1}{2}$.

It is known that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Using the fact that π avoids 123 if and only if $\bar{\pi}$ avoids 321, we obtain a characterization of 123-avoiding permutations as those with the following property: the elements π_i such that $\pi_i < n+1-i$ form a decreasing subsequence, and so do the remaining elements. In particular, since no two fixed points can be in the same decreasing subsequence, this implies that π can have at most one big fixed point and one small fixed point.

In this subsection we use a bijection between $\mathcal{S}_n(123)$ and $\mathcal{D}_n$, which we denote by ψ. This bijection appears in an equivalent form in [13, 16], and it is closely related to the bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ given in [6, 11]. We can define ψ as follows. Let $\pi \in \mathcal{S}_n(123)$ be represented as an $n \times n$ array with a cross on each square (i, π_i). Consider the path with *down* and *left* steps that goes from the upper-right corner to the lower-left corner, leaving all the crosses to the left, and staying always as close to the diagonal connecting these two corners as possible. Then $\psi(\pi)$ is the Dyck path obtained from this path by reading an up-step every time the path goes down and a down-step every time it goes left. Figure 5 shows an example when $\pi = (9, 6, 10, 4, 8, 7, 3, 5, 2, 1)$.

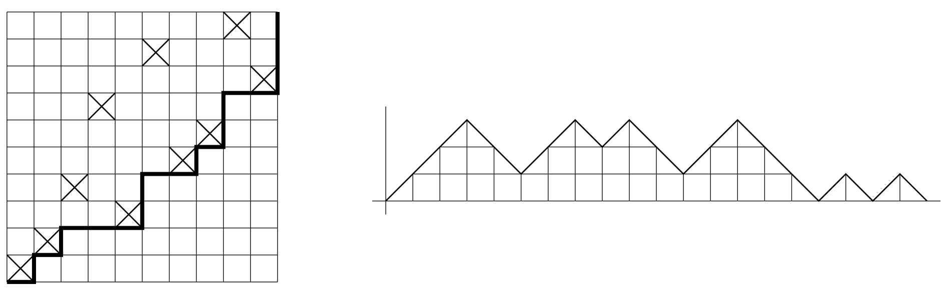

Figure 5: The bijection ψ.

Note that the peaks of the path are determined by the crosses of elements π_i such that $\pi_i \geq n+1-i$, which form a decreasing subsequence. Now it is easy to determine how many permutations have a big (resp. small) fixed point.

Proposition 3.1 *Let $n \geq 1$. We have*

(1) $|\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a big fixed point}\}| = \mathbf{C}_{n-1}$,

(2) $|\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a small fixed point}\}| = \begin{cases} \mathbf{C}_{n-1} & \text{if } n \text{ is even,} \\ \mathbf{C}_{n-1} - \mathbf{C}^2_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$

Proof. (1) It is clear from the definition of ψ that π has a big fixed point if and only if $\psi(\pi)$ has a peak in the middle. Now, we can easily define a bijection from the subset of elements of $\mathcal{D}_n$ with a peak in the middle and $\mathcal{D}_{n-1}$, by removing the two middle steps **ud**.

(2) Clearly, $\pi \in \mathcal{S}_n(123)$ if and only if $\widehat{\pi} \in \mathcal{S}_n(123)$. This involution switches big and small fixed points, except for the possible big fixed point in position $\frac{n+1}{2}$, which remains unchanged. Applying now ψ, a small fixed point of π is transformed into a peak in the

middle of $\psi(\widehat{\pi})$ of height at least two (indeed, a hill would correspond to the big fixed point $\frac{n+1}{2}$). Knowing that the number of paths in $\mathcal{D}_n$ with a peak in the middle is $\mathbf{C}_{n-1}$, we just have to subtract those where this middle peak has height 1. If n is even, paths in $\mathcal{D}_n$ cannot have a hill in the middle. If n is odd, such paths have the form $A\mathbf{ud}B$, where $A, B \in \mathcal{D}_{\frac{n-1}{2}}$, so the formula follows. $\square$

For $k \geq 0$, let $s_n^k(123) := |\{\pi \in \mathcal{S}_n(123) : \mathrm{fp}(\pi) = k\}|$. We have mentioned that $s_n^k(123) = 0$ for $k \geq 3$. The following corollary reduces the problem of studying the distribution of fixed points in $\mathcal{S}_n(123)$ to that of determining $s_n^2(123)$.

Corollary 3.2 *Let $n \geq 1$. We have*

$$(1)\ \ s_n^1(123) = \begin{cases} 2(\mathbf{C}_{n-1} - s_n^2(123)) & \textit{if } n \textit{ even,} \\ 2(\mathbf{C}_{n-1} - s_n^2(123)) - \mathbf{C}_{\frac{n-1}{2}}^2 & \textit{if } n \textit{ odd.} \end{cases}$$

$$(2)\ \ s_n^0(123) = \begin{cases} \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) & \textit{if } n \textit{ even,} \\ \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) + \mathbf{C}_{\frac{n-1}{2}}^2 & \textit{if } n \textit{ odd.} \end{cases}$$

Proof. (1) By inclusion-exclusion, $s_n^1(123) = |\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a big fixed point}\}| + |\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a small fixed point}\}| - 2s_n^2(123)$. Now we apply Proposition 3.1.

(2) Clearly, $s_n^0(123) = \mathbf{C}_n - s_n^1(123) - s_n^2(123)$. $\square$

The next theorem, together with the previous corollary, gives a formula for the distribution of fixed points in 123-avoiding permutations.

Theorem 3.3

$$s_n^2(123) = \sum_{i=1}^{n-1} \sum_{r,s=1}^{i} \left[\left(\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i} \right) \cdot \left(\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i} \right) \right.$$
$$\left. \cdot \sum_{\substack{h=1 \\ n-h \text{ even}}}^{n} \sum_{k=0}^{n-2i} f(k,r,h,n-2i+r) f(n-2i-k,s,h,n-2i+s) \right],$$

where

$$f(k,r,h,\ell) = \begin{cases} \binom{\frac{\ell+h-r}{2}-1}{k}\binom{\frac{\ell-h+r}{2}-1}{k-1} - \binom{\frac{\ell-h-r}{2}-1}{k}\binom{\frac{\ell+h+r}{2}-1}{k-1} & \textit{if } k \geq 1, \\ 1 & \textit{if } k = 0 \textit{ and } \ell = h - r, \\ 0 & \textit{otherwise,} \end{cases}$$

with the convention $\binom{a}{b} := 0$ if $a < 0$.

Proof. Recall that $s_n^2(123)$ counts permutations with both a big and a small fixed point. We have seen already that ψ maps a big fixed point of the permutation into a peak in the middle of the Dyck path. Now we look at how a small fixed point of the permutation is transformed by ψ. We claim that $\pi \in \mathcal{S}_n(123)$ has a small fixed point if and only if $D = \psi(\pi)$ satisfies the following condition (which we call condition C1): there exists i such that the i-th and $(i+1)$-st up-steps of D are consecutive, the i-th and $(i+1)$-st down-steps from the end are consecutive, and there are exactly $n+1-2i$ peaks of D between them. To see this, assume that i is a small fixed point of π (see Figure 6). Then, the path from the upper-right to the lower-left corner of the array of π, used to define $\psi(\pi)$, has two consecutive vertical steps in rows i and $i+1$, and two consecutive horizontal steps in columns i and $i+1$. Besides, there are $n+1-2i$ crosses below and to the right of cross (i,i), each one of which produces a peak in the Dyck path $\psi(\pi)$. Reciprocally, it can be checked that if $\psi(\pi)$ satisfies condition C1 then π has a small fixed point.

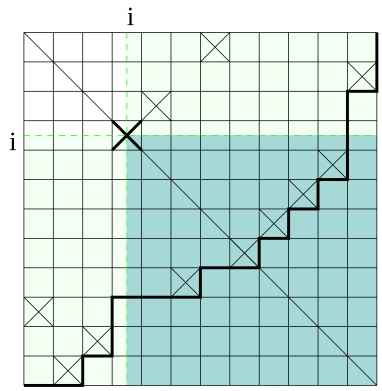

Figure 6: A small fixed point i has $n+1-2i$ crosses below and to the right.

All we have to do is count how many paths $D \in \mathcal{D}_n$ with a peak in the middle satisfy condition C1. For such a Dyck path D, define the following parameters: let i be the value such that condition C1 holds, let h be the height of D in the middle, r the height at which the i-th up-step ends, and s the height of the point between the i-th and $(i+1)$-st down-steps from the end. In the example of Figure 7, $n = 12$, $i = 4$, $h = 4$, $r = 3$, and $s = 1$.

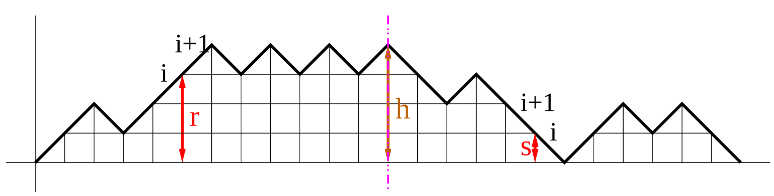

Figure 7: The parameters i, h, r and s in a Dyck path.

Fix n, i, h, r and s. We will count the number of Dyck paths D with these given parameters. We can write $D = A\mathbf{uu}B_1B_2\mathbf{dd}C$, where the distinguished $\mathbf{u}$'s are the i-th and $(i+1)$-st up-steps, similarly with the $\mathbf{d}$'s, and the middle of D is between B_1 and B_2. Then A is a path from $(0,0)$ to $(2i-r-1, r-1)$ not going below $x=0$. It is easy to see that there are $\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i}$ such paths A. By symmetry, there are $\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i}$ possibilities for C.

Now we count the possibilities for B_1 and B_2. It can be checked that $f(k,r,h,\ell)$ counts the number of paths from $(0,r)$ to (ℓ,h) having exactly k peaks, starting and ending with an up-step, and never going below $x=0$. The fragment $\mathbf{u}B_1$ is a path from $(2i-r,r)$ to (n,h) not going below $x=0$, and ending with an up-step (since D has a peak in the middle). If we fix k as the number of peaks of this fragment, then there are $f(k,r,h,n-2i+r)$ such paths $\mathbf{u}B_1$. Similarly, there are $f(n-2i-k, s, h, n-2i-s)$ possibilities for $B_2\mathbf{d}$ with $n-2i-k$ peaks.

Summing over all possible values of k, h, r, s and i we obtain the expression in the theorem. □

Using Corollary 3.2, we can prove that among the derangements (i.e., permutations without fixed points) of length n, the number of 123-avoiding ones is at least the number of 132-avoiding ones. This inequality was conjectured by Miklós Bóna and Olivier Guibert.

Theorem 3.4 ([2]) *For all $n \geq 4$, $s_n^0(132) < s_n^0(123)$.*

Proof. For $n \leq 12$ the result can be checked by exhaustive enumeration of all derangements. Let us assume that $n \geq 13$.

From part (2) of Corollary 3.2, we have that

$$s_n^0(123) \geq \mathbf{C}_n - 2\mathbf{C}_{n-1}.$$

It is known [16] that $s_n^0(132) = \mathbf{F}_n$, the n-th *Fine number*. Therefore, the theorem will be proved if we show that

$$\mathbf{F}_n < \mathbf{C}_n - 2\mathbf{C}_{n-1} \tag{1}$$

for $n \geq 13$. Using the identity $\mathbf{F}_n = \frac{1}{2}\sum_{i=0}^{n-2}(\frac{-1}{2})^i \mathbf{C}_{n-i}$, we get the inequality $\mathbf{F}_n < \frac{1}{2}\mathbf{C}_n - \frac{1}{4}\mathbf{C}_{n-1} + \frac{1}{8}\mathbf{C}_{n-2}$, which reduces (1) to showing that $\mathbf{C}_n > \frac{7}{2}\mathbf{C}_{n-1} + \frac{1}{4}\mathbf{C}_{n-2}$. This inequality certainly holds asymptotically, because $\mathbf{C}_n$ grows like $\frac{1}{\sqrt{\pi}} n^{-\frac{3}{2}} 4^n$ as n tends to infinity, and it is not hard to see that in fact it holds for all $n \geq 13$. □

3.2 b) $132 \approx 213 \approx 321$

This case has already been studied in [6]. The corresponding generating function, which appeared independently in [20] for the case of the pattern 321, is the following.

Theorem 3.5 ([6, 20])

$$F_{132}(x,q,z) = F_{213}(x,q,z) = F_{321}(x,q,z) =$$
$$= \frac{2}{1+(1+q-2x)z+\sqrt{1-2(1+q)z+(1-q)^2z^2}}.$$

In fact, permutations avoiding these patterns have been enumerated with respect to an additional statistic, the number of descents. Recall that $i \leq n-1$ is a *descent* of $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. Denote by $\mathrm{des}(\pi)$ the number of descents of π. Theorem 3.5 can be generalized as follows.

Theorem 3.6 ([6, 7]) *The generating function for 321-avoiding permutations with respect to fixed points, excedances, and descents is*

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)}z^n$$
$$= \frac{2}{1+(1+q-2x)z+\sqrt{1-2(1+q)z+((1+q)^2-4qp)z^2}}.$$

Similarly,

$$1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)+1}z^n = 1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(213)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)+1}z^n$$
$$= \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{f_1(q,z)}},$$

where $f_1(q,z) = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2-2q(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$.

3.3 c, c') $231 \sim 312$

Using the bijection

$$\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}), \end{array}$$

Lemma 2.3 implies that

$$F_{312}(x,q,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{td}_0(D)}q^{\mathrm{td}_{<0}(D)}z^{|D|}.$$

To enumerate tunnels of depth 0, we will separate them according to their height. For every $h \geq 0$, a tunnel at height h must have length $2(h+1)$ in order to have depth 0. It is important to notice that if a tunnel of depth 0 of D corresponds to a decomposition $D = A\mathbf{u}B\mathbf{d}C$, then D has no tunnels of depth 0 in the part given by B. In other words, the projections on the x-axis of all the tunnels of depth 0 of a given Dyck path are disjoint. This observation allows us to give a continued fraction expression for $F_{312}(x,1,z)$.

Theorem 3.7 $F_{312}(x,1,z)$ *is given by the following continued fraction:*

$$F_{312}(x,1,z) = \cfrac{1}{1-(x-1)z-\cfrac{z}{1-(x-1)z^2-\cfrac{z}{1-2(x-1)z^3-\cfrac{z}{1-5(x-1)z^4-\cfrac{z}{\ddots}}}}},$$

where at the n-th level, the coefficient of $(x-1)z^{n+1}$ is the Catalan number $\mathbf{C}_n$.

Proof. For every $h \geq 0$, let $\mathrm{td}_0^h(D)$ be the number of tunnels of D of height h and length $2(h+1)$. Note that $\mathrm{td}_0(D) = \sum_{h\geq 0} \mathrm{td}_0^h(D)$. We will show now that for every $h \geq 1$, the generating function for Dyck paths where x marks the statistic $\mathrm{td}_0^0(\cdot) + \cdots + \mathrm{td}_0^{h-1}(\cdot)$ is given by the continued fraction of the theorem truncated at level h, with the $(h+1)$-st level replaced with $\mathbf{C}(z)$.

A Dyck path D can be written uniquely as a sequence of elevated Dyck paths, that is, as $D = \mathbf{u}A_1\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each $A_i \in \mathcal{D}$. In terms of the generating function $\mathbf{C}(z) = \sum_{D\in\mathcal{D}} z^{|D|}$, this translates into the equation $\mathbf{C}(z) = \frac{1}{1-z\mathbf{C}(z)}$. A tunnel of height 0 and length 2 (i.e., a hill) appears in D for each empty A_i. Therefore, the generating function enumerating hills is

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)} z^{|D|} = \frac{1}{1-z[x-1+\mathbf{C}(z)]}, \tag{2}$$

since an empty A_i has to be accounted as x, not as 1.

Let us enumerate simultaneously hills (as above), and tunnels of height 1 and length 4. The generating function (2) can be written as

$$\frac{1}{1-z\left[x-1+\dfrac{1}{1-z\mathbf{C}(z)}\right]}.$$

Combinatorially, this corresponds to expressing each A_i as a sequence $\mathbf{u}B_1\mathbf{d}\cdots\mathbf{u}B_s\mathbf{d}$, where each $B_j \in \mathcal{D}$. Notice that since each $\mathbf{u}B_j\mathbf{d}$ starts at height 1, a tunnel of height 1 and length 4 is created by each $B_j = \mathbf{ud}$ in the decomposition. Thus, if we want x to mark also these tunnels, such a B_j has to be accounted as xz, not z. The corresponding generating function is

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)} z^{|D|} = \frac{1}{1-z\left[x-1+\dfrac{1}{1-z[(x-1)z+\mathbf{C}(z)]}\right]}.$$

Now it is clear how iterating this process indefinitely we obtain the continued fraction of the theorem. From the generating function where x marks $\mathrm{td}_0^0(\cdot) + \cdots + \mathrm{td}_0^{h-1}(\cdot)$, we can obtain the one where x marks $\mathrm{td}_0^0(\cdot) + \cdots + \mathrm{td}_0^h(\cdot)$ by replacing the $\mathbf{C}(z)$ at the lowest level with

$$\frac{1}{1-z[(x-1)\mathbf{C}_h z^h + \mathbf{C}(z)]},$$

to account for tunnels of height h and length $2(h+1)$, which in the decomposition correspond to elevated Dyck paths at height h. □

The same technique can be used to enumerate excedances in 312-avoiding permutations, which correspond to tunnels of negative depth in the Dyck path.

Theorem 3.8 *Let*

$$\mathbf{C}_{<i}(z) = \sum_{j=0}^{i-1} \mathbf{C}_j z^j$$

be the series for the Catalan numbers truncated at degree i. $F_{312}(x,q,z)$ is given by the following continued fraction:

$$F_{312}(x,q,z) = \cfrac{1}{1 - zK_0 + \cfrac{z}{1 - zK_1 + \cfrac{z}{1 - zK_2 + \cfrac{z}{1 - zK_3 + \cfrac{z}{\ddots}}}}},$$

where $K_n = (x-1)\mathbf{C}_n q^n z^n + (q-1)\mathbf{C}_{<n}(qz)$ for $n \geq 0$.

Note that the first values of K_n are

$$K_0 = x - 1, \qquad K_1 = (x-1)qz + q - 1,$$
$$K_2 = 2(x-1)q^2z^2 + (q-1)(1+qz), \quad K_3 = 5(x-1)q^3z^3 + (q-1)(1+qz+2q^2z^2).$$

Proof. We use the same decomposition as above, now keeping track of tunnels of negative depth as well. For every $h \geq 0$, let $\mathrm{td}^h_{<0}(D)$ be the number of tunnels of D of height h and length less than $2(h+1)$. Note that $\mathrm{td}_{<0}(D) = \sum_{h\geq 0} \mathrm{td}^h_{<0}(D)$. To follow the same structure as in the previous proof, counting tunnels height by height, it will be convenient that at the h-th step of the iteration, q marks not only tunnels of negative depth up to height h but also all the tunnels at higher levels. Denote by $\mathrm{alltun}^{>h}(D)$ the number of tunnels of D of height strictly greater than h.

We will show now that for every $h \geq 1$, the generating function for Dyck paths where x marks the statistic $\mathrm{td}^0_0(\cdot)+\cdots+\mathrm{td}^{h-1}_0(\cdot)$ and q marks $\mathrm{td}^0_{<0}(\cdot)+\cdots+\mathrm{td}^{h-1}_{<0}(\cdot)+\mathrm{alltun}^{>h-1}(\cdot)$ is given by the continued fraction of the theorem truncated at level h, with the $(h+1)$-st level replaced with $\mathbf{C}(qz)$.

The analogue of equation (2) is now

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}^0_0(D)} q^{\mathrm{td}^0_{<0}(D)+\mathrm{alltun}^{>0}(D)} z^{|D|} = \frac{1}{1 - z[x - 1 + \mathbf{C}(qz)]}. \tag{3}$$

Indeed, decomposing D as $\mathbf{u}A_1\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, q counts all the tunnels that appear in any A_i, and whenever an A_i is empty we must mark it as x.

Let us enumerate now tunnels of depth 0 and negative depth at both height 0 and height 1. Modifying (3) so that q no longer counts tunnels at height 1, we get

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}^0_0(D)} q^{\mathrm{td}^0_{<0}(D)+\mathrm{alltun}^{>1}(D)} z^{|D|} = \frac{1}{1 - z\left[x - 1 + \frac{1}{1 - z\mathbf{C}(qz)}\right]}, \tag{4}$$

which corresponds to writing each A_i as $A_i = \mathbf{u}B_1\mathbf{d}\cdots\mathbf{u}B_s\mathbf{d}$, and having q count all tunnels in each B_j. Now, in order for x to mark tunnels of depth 0 at height 1, each $B_j = \mathbf{ud}$, that in (4) is counted as qz, has to be now counted as xqz instead. Similarly, to have q mark tunnels of negative depth at height 1, we must count each empty B_j as q, not as 1. This gives us the following generating function:

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)} q^{\mathrm{td}_{<0}^0(D)+\mathrm{td}_{<0}^1(D)+\mathrm{alltun}^{>1}(D)} z^{|D|}$$
$$= \frac{1}{1-z\left[x-1+\dfrac{1}{1-z[(x-1)qz+q-1+\mathbf{C}(qz)]}\right]}.$$

Iterating this process level by level indefinitely we obtain the continued fraction of the theorem. At each step, from the generating function where x marks $\mathrm{td}_0^0(\cdot)+\cdots+\mathrm{td}_0^{h-1}(\cdot)$, and q marks $\mathrm{td}_{<0}^0(\cdot)+\cdots+\mathrm{td}_{<0}^{h-1}(\cdot)+\mathrm{alltun}^{>h-1}(\cdot)$, we can obtain the one where x marks $\mathrm{td}_0^0(\cdot)+\cdots+\mathrm{td}_0^h(\cdot)$ and q marks $\mathrm{td}_{<0}^0(\cdot)+\cdots+\mathrm{td}_{<0}^h(\cdot)+\mathrm{alltun}^{>h}(\cdot)$ by replacing the $\mathbf{C}(qz)$ at the lowest level with

$$\frac{1}{1-z[(x-1)\mathbf{C}_h q^h z^h+(q-1)\mathbf{C}_{<h}(qz)+\mathbf{C}(qz)]}. \tag{5}$$

This change makes x account for tunnels of depth 0 at height h, which in the decomposition correspond to the $\mathbf{C}_h$ possible elevated Dyck paths of length $2(h+1)$ when they occur at height h. It also makes q count tunnels of negative depth at height h, which in the decomposition correspond to elevated Dyck paths at height h of length less than $2(h+1)$. The generating function for these ones becomes $q\mathbf{C}_{<h}(qz)$ instead of $\mathbf{C}_{<h}(qz)$, since for every $j<h$, an elevated path $\mathbf{u}C\mathbf{d}$ with $C\in\mathcal{D}_j$ contributes to one extra tunnel of negative depth at height h, aside from the j tunnels of height more than h that it contains. □

For 231-avoiding permutations we get the following generating function.

Corollary 3.9 *Let $\mathbf{C}_{<i}(z)$ be as in Theorem 3.8. $F_{231}(x,q,z)$ is given by the following continued fraction:*

$$F_{231}(x,q,z) = \cfrac{1}{1-zK'_0+\cfrac{qz}{1-zK'_1+\cfrac{qz}{1-zK'_2+\cfrac{qz}{1-zK'_3+\cfrac{qz}{\ddots}}}}},$$

where $K'_n = (x-q)\mathbf{C}_n z^n + (1-q)\mathbf{C}_{<n}(z)$.

The first values of K'_n are

$$\begin{array}{ll} K'_0 = x-q, & K'_1 = (x-q)z+1-q, \\ K'_2 = 2(x-q)z^2+(1-q)(1+z), & K'_3 = 5(x-q)z^3+(1-q)(1+z+2z^2). \end{array}$$

Proof. By Lemma 2.2, we have that $F_{231}(x,q,z) = F_{312}(\frac{x}{q},\frac{1}{q},qz)$, so the expression follows from Theorem 3.8. □

4 Double restrictions

In this section we consider simultaneous avoidance of any two patterns of length 3. Using Lemma 2.2, the pairs of patterns fall into the following equivalence classes.

$$\begin{array}{c}
\textbf{a)}\ \{123,132\} \approx \{123,213\} \\
\textbf{b)}\ \{231,321\} \quad \sim \quad \textbf{b')}\ \{312,321\} \\
\textbf{c)}\ \{132,213\} \\
\textbf{d)}\ \{231,312\} \\
\textbf{e)}\ \{132,231\} \approx \{213,231\} \quad \sim \quad \textbf{e')}\ \{132,312\} \approx \{213,312\} \\
\textbf{f)}\ \{132,321\} \approx \{213,321\} \\
\textbf{g)}\ \{123,231\} \quad \sim \quad \textbf{g')}\ \{123,312\} \\
\textbf{h)}\ \{123,321\}
\end{array}$$

In [18] it is shown that the number of permutations in $\mathcal{S}_n$ avoiding any of the pairs in the classes **a)**, **b)**, **b')**, **c)**, **d)**, **e)**, and **e')** is 2^{n-1}, and that for the pairs in **f)**, **g)** and **g')**, the number of permutations avoiding any of them is $\binom{n}{2}+1$. The case **h)** is trivial because this pair is avoided only by permutations of length at most 4.

In terms of generating functions, this means that when we substitute $x = q = 1$ in $F_\Sigma(x, q, z)$, where Σ is any of the pairs in the classes from **a)** to **e')**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} 2^{n-1} z^n = \frac{1-z}{1-2z}.$$

If Σ is a pair from the classes **f)**, **g)**, **g')**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} (\binom{n}{2}+1) z^n = \frac{1-2z+2z^2}{(1-z)^3}.$$

4.1 a) $\{123,132\} \approx \{123,213\}$

Proposition 4.1

$$\begin{aligned}
F_{\{123,132\}}&(x,q,z) = F_{\{123,213\}}(x,q,z) \\
&= \frac{1 + xz + (x^2-4q)z^2 + (-3xq+q+q^2)z^3 + (xq+xq^2-3x^2q+3q^2)z^4}{(1-qz^2)(1-4qz^2)}.
\end{aligned}$$

Proof. Consider the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Subsection 2.2. It is shown in [13] that the height of the Dyck path $\varphi(\pi)$ is the length of the longest increasing subsequence of π. In particular, $\pi \in \mathcal{S}_n(12\cdots(k+1), 132)$ if and only if $\varphi(\pi)$ has height at most k. Thus, by Lemma 2.3, $F_{\{123,132\}}(x, q, z)$ can be written in terms of Dyck paths as

$$\sum_{D\in\mathcal{D}^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}. \tag{6}$$

Let us first find the univariate generating function for paths of height at most 2 (with no statistics). Clearly, the generating function for Dyck paths of height at most 1 is $\frac{1}{1-z}$, since such paths are just sequences of hills. A path D of height at most 2 can be written uniquely as $D = \mathbf{u}A_1\mathbf{d}\mathbf{u}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each A_i is a path of height at most 1. The generating function for each $\mathbf{u}A_i\mathbf{d}$ is $\frac{z}{1-z}$. Thus,

$$\sum_{D\in\mathcal{D}^{\leq 2}} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = \sum_{n\geq 0} 2^{n-1}z^n.$$

In the rest of this proof, we assume that all Dyck paths that appear have height at most 2 unless otherwise stated. To compute (6), we will separate paths according to their height at the middle. Consider first paths whose height at the middle is 0. Splitting such a path at its midpoint we obtain a pair of paths of the same length. Thus, the corresponding generating function is

$$\sum_{m\geq 0} 2^{m-1}z^m \cdot 2^{m-1}q^m z^m = \frac{1-3qz^2}{1-4qz^2}, \tag{7}$$

since the number of right tunnels of such a path is the semilength of its right half.

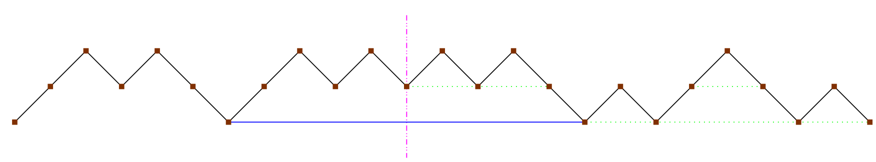

Figure 8: A path of height 2 with a centered tunnel.

Now we consider paths whose height at the middle is 1. It is easy to check that without the variables x and q, the generating function for such paths is

$$\frac{z}{1-4z^2}. \tag{8}$$

Let us first look at paths of this kind that have a centered tunnel. They must be of the form $D = A\mathbf{u}B\mathbf{d}C$ where $A, C \in \mathcal{D}^{\leq 2}$ have the same length, and B is a sequence of an even number of hills. Thus, their generating function is

$$xz \cdot \frac{1}{1-qz^2} \cdot \frac{1-3qz^2}{1-4qz^2}, \tag{9}$$

where x marks the centered tunnel, $\frac{1}{1-qz^2}$ corresponds to the sequence of hills B, half of which give right tunnels, and the last fraction comes from the pair AC, which is counted as in (7). From (8) and (9) it follows that the univariate generating function (with just variable z) for paths with height at the middle 1, not having a centered tunnel, is

$$\frac{z}{1-4z^2} - \frac{z(1-3z^2)}{(1-z^2)(1-4z^2)} = \frac{2z^3}{(1-z^2)(1-4z^2)}.$$

By symmetry, in half of these paths, the tunnel of height 0 that goes across the middle is a right tunnel. Thus, the multivariate generating function for all paths with height 1 at the middle is

$$\frac{xz(1-3qz^2)}{(1-qz^2)(1-4qz^2)}+\frac{(q+1)qz^3}{(1-qz^2)(1-4qz^2)}. \tag{10}$$

Here the right summand corresponds to paths with no centered tunnel: the term $(q+1)$ distinguishes whether the tunnel that goes across the middle is a right tunnel or not, and the other q's mark tunnels completely contained in the right half.

Paths with height 2 at the middle are easy to enumerate now. Indeed, they must have a peak **ud** in the middle, whose removal induces a bijection between these paths and paths with height 1 at the middle. This bijection preserves the number of right tunnels, and decreases the length and the number of centered tunnels by one. Thus, the generating function for paths with height 2 at the middle is xz times expression (10). Adding up this generating function, for paths with height 2 at the middle, to the expressions (7) and (10), for paths with height at the middle 0 and 1 respectively, we obtain the desired expression for $F_{\{123,132\}}(x,q,z)$. □

Let us see how the same technique used in this proof can be generalized to enumerate fixed points in $\mathcal{S}_n(132, 12\cdots(k+1))$ for an arbitrary $k \geq 0$.

Theorem 4.2 *For $k \geq 0$, let*

$$M_k(x,z) := F_{\{132,12\cdots(k+1)\}}(x,1,z) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132,12\cdots(k+1))} x^{\mathrm{fp}(\pi)}z^n.$$

Then the M_k's satisfy the recurrence:

$$M_k(x,z) = \sum_{\ell=0}^{k} R_{k,\ell}(z)(1+(x-1)zM_{\ell-1}(x,z)), \tag{11}$$

where $M_{-1}(x,z) := 0$, and $R_{k,\ell}(z)$ is defined as

$$R_{k,\ell}(z) := \sum_{n\geq 0} g^2_{k,\ell}(n)z^n, \text{ where } \sum_{n\geq 0} g_{k,\ell}(n)z^n = \frac{U_\ell(\frac{1}{2z})}{zU_{k+1}(\frac{1}{2z})},$$

where U_m are the Chebyshev polynomials of the second kind, defined by the recurrence

$$U_0(t)=1,\ U_1(t)=2t,\ \text{and } U_r(t)=2tU_{r-1}(t)-U_{r-2}(t).$$

Before proving this theorem, let us show how to apply it to obtain the generating functions M_k for the first few values of k. For $k=1$, we have $R_{1,0}(z)=\frac{z}{1-z^2}$, $R_{1,1}(z)=\frac{1}{1-z^2}$, so

$$M_1(x,z)=\frac{1+xz}{1-z^2}.$$

For $k = 2$, we get $R_{2,0}(z) = \frac{z^2}{1-4z^2}$, $R_{2,1}(z) = \frac{z}{1-4z^2}$, $R_{2,2}(z) = 1 + \frac{z^2}{1-4z^2}$, so

$$M_2(x,z) = \frac{1 + xz + (x^2-4)z^2 + (2-3x)z^3 + (3+2x-3x^2)z^4}{(1-z^2)(1-4z^2)},$$

which is the expression of Proposition 4.1 for $q = 1$.

For $k = 3$, we obtain $R_{3,0}(z) = \frac{z^3+z^5}{(1-z^2)(1-7z^2+z^4)}$, $R_{3,1}(z) = \frac{z^2+z^4}{(1-z^2)(1-7z^2+z^4)}$, $R_{3,2}(z) = \frac{z(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, $R_{3,2}(z) = 1 + \frac{z^2(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, so

$$\begin{aligned} M_3(x,z) = {} & [1+xz+(x^2-12)z^2+(x^3-11x+2)z^3+(-10x^2+4x+45)z^4+(-10x^3+4x^2+37x-10)z^5 \\ & +(25x^2-22x-52)z^6+(25x^3-22x^2-41x+16)z^7+(-12x^2+16x+16)z^8+(-12x^3+16x^2 \\ & +12x-8)z^9] \;/\; [(1-z^2)^2(1-4z^2)(1-7z^2+z^4)]. \end{aligned}$$

Proof. As shown at the beginning of the previous proof, φ induces a bijection between $\mathcal{S}_n(132, 12\cdots(k+1))$ and $\mathcal{D}^{\leq k}$, the set of Dyck paths of height at most k. Thus, by Lemma 2.3,

$$M_k(x,z) = \sum_{D \in \mathcal{D}^{\leq k}} x^{\mathrm{ct}(D)} z^{|D|}.$$

In order to find a recursion for this generating function, we are going to count pairs (D, S) where $D \in \mathcal{D}^{\leq k}$ and S is a subset of $\mathrm{CT}(D)$, the set of centered tunnels of D. In other words, we are considering Dyck paths where some centered tunnels (namely those in S) are marked. Each such pair is given weight $(x-1)^{|S|}z^{|D|}$, so that for a fixed D, the sum of weights of all pairs (D,S) is $\sum_{S\subseteq \mathrm{CT}(D)} (x-1)^{|S|}z^{|D|} = ((x-1)+1)^{|\mathrm{CT}(D)|}z^{|D|} = x^{\mathrm{ct}(D)}z^{|D|}$, which is precisely the weight that D has in $M_k(x,z)$.

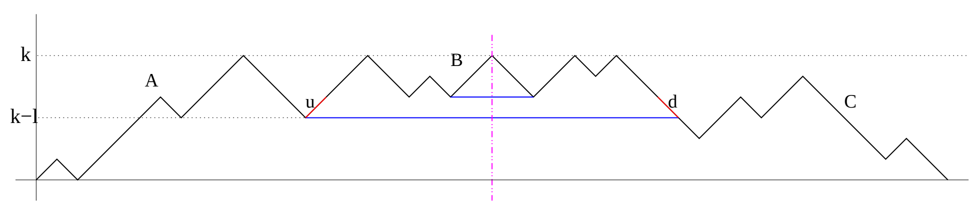

Figure 9: A path of height k with two marked centered tunnels.

If $D \in \mathcal{D}^{\leq k}$ has some marked centered tunnel, consider the decomposition $D = A\mathbf{u}B\mathbf{d}C$ given by the longest marked tunnel (i.e., all the other marked tunnels are inside the part B of the path). Let ℓ be the distance between this tunnel and the line $y = k$ (see Figure 9). Equivalently, A ends at height $k - \ell$, the same height where C begins. Then, B is an arbitrary Dyck path of height at most $\ell - 1$ with possibly some marked centered tunnels, so its corresponding generating function is $M_{\ell-1}(x,z)$ (with the convention $M_{-1}(x,z) := 0$, since for $\ell = 0$ there is no such B). Giving weight $(x-1)$ to the tunnel that determines our decomposition, we have that the part $\mathbf{u}B\mathbf{d}$ of the path contributes $(x-1)zM_{\ell-1}(x,z)$ to the generating function.

Now we look at the generating function for the part A of the path. Let $\widetilde{R}_{k,\ell}(w) := \sum_{n\geq 0} g_{k,\ell}(n)w^n$, where $g_{k,\ell}(n)$ is the number of paths from $(0,0)$ to $(n, k-\ell)$ staying always between $y = 0$ and $y = k$. It is known (see for example [13, Appendix]) that

$$\widetilde{R}_{k,\ell}(w) = \frac{U_\ell(\frac{1}{2w})}{wU_{k+1}(\frac{1}{2w})}.$$

The part C of the path D, flipped over a vertical line, can be regarded as a path with the same endpoints as A, since it must have the same length and end at the same height $k-\ell$. Thus, the generating function for pairs (A,C) of paths of the same length from height 0 to height $k-\ell$ and not going above $y = k$ is $\sum_{n\geq 0} g_{k,\ell}^2(n)z^n = R_{k,\ell}(z)$.

Hence, the generating function for paths $D \in \mathcal{D}^{\leq k}$ having the longest marked centered tunnel at height $k-\ell$ is $R_{k,\ell}(z)(x-1)zM_{\ell-1}(x,z)$.

If D has no marked tunnel, decompose it as $D = AC$ where A and C have the same length. Letting $k-\ell$ be again the height where A ends and C begins, the situation is the same as above but without any contribution coming from the central part of D. The parameter ℓ can take any value between 0 and k. Thus, summing over all possible decompositions of D, we get

$$M_k(x,z) = \sum_{\ell=0}^{k} R_{k,\ell}(z)(1 + (x-1)zM_{\ell-1}(x,z)).$$

□

4.2 b, b') $\{231, 321\} \sim \{312, 321\}$

Proposition 4.3

$$F_{\{312,321\}}(x,q,z) = \frac{1-qz}{1-(x+q)z+(x-1)qz^2}. \tag{12}$$

Proof. The length of the longest decreasing subsequence of π equals the height of the Dyck path $\varphi(\bar{\pi})$. In particular, we have a bijection

$$\begin{array}{ccc} \mathcal{S}_n(312,321) & \longleftrightarrow & \mathcal{D}_n^{\leq 2} \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$$

Thus, by Lemma 2.3,

$$F_{\{312,321\}}(x,q,z) = \sum_{D\in\mathcal{D}^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

But the only tunnels of depth 0 that a Dyck path of height at most 2 can have are hills, and the only tunnels of negative depth that it can have are peaks at height 2. A path $D \in \mathcal{D}^{\leq 2}$ can be written uniquely as $D = \mathbf{u}A_1\mathbf{du}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each A_i is a (possibly empty) sequence of hills. An empty A_i creates a tunnel of depth 0 in D, so it contributes as x. An A_i of length $2j > 0$ contributes as $q^j z^j$, since it creates j peaks at height 2 in D. Thus,

$$F_{\{312,321\}}(x,q,z) = \frac{1}{1 - z\left(x + \dfrac{qz}{1-qz}\right)},$$

which is equivalent to (12). $\square$

Corollary 4.4

$$F_{\{231,321\}}(x,q,z) = \frac{1-z}{1-(x+1)z+(x-q)z^2}.$$

Proof. It follows from Lemma 2.2 that $F_{\{231,321\}}(x,q,z) = F_{\{312,321\}}(\frac{x}{q}, \frac{1}{q}, qz)$. $\square$

As in the previous section, these results can be generalized to the case when instead of the pattern 321 we have a decreasing pattern $(k+1)k\cdots 21$ of arbitrary length. For $i, h \geq 0$, let $\mathbf{C}_i^{\leq h}$ be the number of Dyck paths of length $2i$ and height at most h. As mentioned before, it is known that

$$\sum_{i\geq 0} \mathbf{C}_i^{\leq h} z^i = \frac{U_h(\frac{1}{2\sqrt{z}})}{\sqrt{z}U_{h+1}(\frac{1}{2\sqrt{z}})},$$

where U_m are the Chebyshev polynomials of the second kind. Let

$$\mathbf{C}_{<i}^{\leq h}(z) := \sum_{j=0}^{i-1} \mathbf{C}_j^{\leq h} z^j.$$

The following theorem deals with fixed points and excedances in $\mathcal{S}_n(312, (k+1)k\cdots 1)$ for any $k \geq 0$.

Theorem 4.5 *Let $\mathbf{C}_i^{\leq h} = |\mathcal{D}_i^{\leq h}|$ and $\mathbf{C}_{<i}^{\leq h}(z)$ be defined as above. Then, for $k \geq 0$,*

$$F_{\{312,(k+1)k\cdots 1\}}(x,q,z) = A_0^k(x,q,z),$$

where A_i^k is recursively defined by

$$A_i^k(x,q,z) = \begin{cases} \dfrac{1}{1 - z[(x-1)\mathbf{C}_i^{\leq k-i-1} q^i z^i + (q-1)\mathbf{C}_{<i}^{\leq k-i-1}(qz) + A_{i+1}^k(x,q,z)]} & \text{if } i < k, \\ 1 & \text{if } i = k. \end{cases}$$

For example, for $k=2$ we obtain Proposition 4.3, and for $k=3$ we get

$$\begin{aligned} F_{\{312,4321\}}(x,q,z) &= \frac{1}{1 - z\left[x - 1 + \dfrac{1}{1 - z\left[(x-1)qz + q - 1 + \frac{1}{1-qz}\right]}\right]} \\ &= \frac{1 - 2qz + (q^2 - xq)z^2 + (xq^2 - q^2)z^3}{1 - (x+2q)z + (xq + q^2 - q)z^2 + (x^2q - xq)z^3 + (-x^2q^2 + 2xq^2 - q^2)z^4}. \end{aligned}$$

Proof. It is analogous to the proof of Theorem 3.8, with the only difference that here we consider only those paths that do not go above the line $y = k$. $\square$

Making the appropriate substitutions in the statement of Theorem 4.5, we obtain an expression for the generating function $F_{\{231,(k+1)k\cdots 1\}}(x,q,z) = F_{\{312,(k+1)k\cdots 1\}}(\frac{x}{q}, \frac{1}{q}, qz)$.

4.3 c) $\{132, 213\}$

Proposition 4.6

$$F_{\{132,213\}}(x,q,z) = \frac{1-(1+q)z-2qz^2+4q(1+q)z^3-(xq^2+xq+5q^2)z^4+2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}.$$

Proof. We use again the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$. From its description given in Subsection 2.2, it is not hard to see that a permutation $\pi \in \mathcal{S}_n(132)$ avoids 213 if and only if all the valleys of the corresponding Dyck path $\varphi(\pi)$ have their lowest point on the x-axis. A path with such property can be described equivalently as a *sequence of pyramids*. Denote by $\mathcal{P}yr_n \subseteq \mathcal{D}_n$ the set of sequences of pyramids of length $2n$, and let $\mathcal{P}yr := \bigcup_{n\geq 0} \mathcal{P}yr_n$. We have just seen that φ restricts to a bijection between $\mathcal{S}_n(132, 213)$ and $\mathcal{P}yr_n$. By Lemma 2.3, we can write $F_{\{132,213\}}(x,q,z)$ as

$$\sum_{D\in\mathcal{P}yr} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

Since for each $n \geq 1$ there is exactly one pyramid of length $2n$, the univariate generating function of sequences of pyramids is just $\sum_{D\in\mathcal{P}yr} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = 1 + \sum_{n\geq 1} 2^{n-1}z^n$.

Let us first consider elements of $\mathcal{P}yr$ that have height 0 in the middle (equivalently, the two central steps are **du**). Each one of their halves is a sequence of pyramids, both of the same length. They have no centered tunnels, and the number of right tunnels is given by the semilength of the right half. Thus, their multivariate generating function is

$$1 + \sum_{m\geq 1} 4^{m-1} q^m z^{2m} = 1 + \frac{qz^2}{1-4qz^2}. \tag{13}$$

Now we count elements of $\mathcal{P}yr$ whose two central steps are **ud**. They are obtained uniquely by inserting a pyramid of arbitrary length in the middle of a path with height 0 at the middle. The tunnels created by the inserted pyramid are all centered tunnels, so the corresponding generating function is

$$\frac{xz}{1-xz}\left(1 + \frac{qz^2}{1-4qz^2}\right). \tag{14}$$

It remains to count the elements of $\mathcal{P}yr$ that in the middle have neither a peak nor a valley. From a non-empty sequence of pyramids with height 0 in the middle, if we increase the size of the leftmost pyramid of the right half by an arbitrary number of

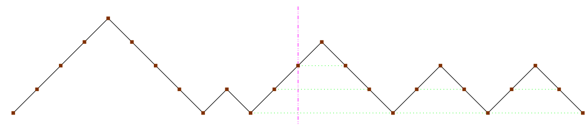

Figure 10: A sequence of pyramids.

steps, we obtain a sequence of pyramids whose two central steps are **uu**. Reciprocally, by this procedure every such sequence of pyramids can be obtained in a unique way from a sequence of pyramids with height 0 in the middle. Thus, the generating function for the elements of $\mathcal{P}yr$ whose two central steps are **uu** is

$$\frac{qz}{1-qz} \cdot \frac{qz^2}{1-4qz^2}. \tag{15}$$

By symmetry, the generating function for the elements of $\mathcal{P}yr$ whose two central steps are **dd** is

$$\frac{z}{1-z} \cdot \frac{qz^2}{1-4qz^2}, \tag{16}$$

where the difference with respect to 15 is that now the pyramid across the middle does not create right tunnels. Adding up (13), (14), (15) and (16) we get the desired generating function. □

4.4 d) $\{231, 312\}$

Proposition 4.7

$$F_{\{231,312\}}(x,q,z) = \frac{1-qz^2}{1-xz-2qz^2}.$$

Proof. We have shown in the proof of Proposition 4.6 that φ induces a bijection between $\mathcal{S}_n(132, 213)$ and $\mathcal{P}yr_n$, the set of sequences of pyramids of length $2n$. Composing it with the complementation operation, we get a bijection $\pi \mapsto \varphi(\bar{\pi})$ between $\mathcal{S}_n(231, 312)$ and $\mathcal{P}yr_n$. Together with Lemma 2.3, this allows us to express $F_{\{231,312\}}(x,q,z)$ as

$$\sum_{D\in\mathcal{P}yr} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

All that remains is to observe how many tunnels of zero and negative depth are created by a pyramid according to its size. A pyramid of odd semilength $2m+1$ creates one tunnel of depth 0 and m tunnels of negative depth. A pyramid of even semilength $2m$ creates only m tunnels of negative depth. Thus, we have that

$$F_{\{231,312\}}(x,q,z) = \frac{1}{1 - \frac{xz}{1-qz^2} - \frac{qz^2}{1-qz^2}},$$

which equals the expression above. □

4.5 e, e') $\{132, 231\} \approx \{213, 231\} \sim \{132, 312\} \approx \{213, 312\}$

Proposition 4.8

$$F_{\{132,231\}}(x, q, z) = F_{\{213,231\}}(x, q, z) = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}.$$

Proof. As usual, we use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$. Now we are interested in how the condition that π avoids 231 is reflected in the Dyck path $\varphi(\pi)$. It is easy to see from the description of φ and φ^{-1} in Subsection 2.2 that π is 231-avoiding if and only if $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step (equivalently, all the non-isolated up-steps occur at the beginning of the path). Let $\mathcal{E}_n \subseteq \mathcal{D}_n$ be the set of Dyck paths with this condition, and let $\mathcal{E} := \bigcup_{n\geq 0} \mathcal{E}_n$. Then, φ induces a bijection between $\mathcal{S}_n(132, 231)$ and $\mathcal{E}_n$. By Lemma 2.3, $F_{\{132,231\}}(x, q, z)$ can be written as

$$\sum_{D\in\mathcal{E}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

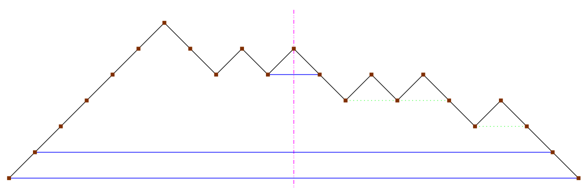

Figure 11: A path in $\mathcal{E}$ with a peak in the middle and two bottom tunnels.

If $D \in \mathcal{E}$, centered tunnels of D can appear only in the following two places. There can be a centered tunnel produced by a peak in the middle of D. All the other centered tunnels of D must have their endpoints in the initial ascending run and the final descending run of D (that is, in their corresponding decomposition $D = A\mathbf{u}B\mathbf{d}C$, A is a sequence of up-steps and C is a sequence of down-steps). For convenience we call this second kind of tunnels *bottom* tunnels. All the right tunnels of D come from peaks on the right half.

It is an exercise to check that the number of paths in $\mathcal{E}_n$ having a peak in the middle and r peaks on the right half is $\binom{n-r-1}{r}2^{r-1}$ if $r \geq 1$, and 1 if $r = 0$. Similarly, the number of paths in $\mathcal{E}_n$ with no peak in the middle and r peaks on the right half is $\binom{n-r}{r}2^{r-1}$ if $r \geq 1$, and 0 if $r = 0$. Let us ignore for the moment the bottom tunnels. For peaks in the middle and right tunnels we have the following generating function.

$$\begin{aligned}
P(x, q, z) &:= \sum_{D\in\mathcal{E}\setminus\mathcal{E}_0} x^{\#\{\text{peaks in the middle of D}\}} q^{\mathrm{rt}(D)} z^{|D|} \\
&= \sum_{n\geq 1} \left[\sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} 2^{r-1} q^r + x \left(1 + \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r} 2^{r-1} q^r \right) \right] z^n \\
&= \frac{xz + (q - x)z^2 - xqz^3}{(1 - z)(1 - z - 2qz^2)}. \qquad (17)
\end{aligned}$$

Now, to take into account all centered tunnels, we use that every $D \in \mathcal{E}$ can be written uniquely as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and $D' \in \mathcal{E}$ has no bottom tunnels. The generating function for elements of $\mathcal{E}$ that do have bottom tunnels, where x marks peaks in the middle, is $zx + zP(x,q,z)$ (the term zx is the contribution of the path $\mathbf{ud}$). Hence, the sought generating function where x marks all centered tunnels is

$$F_{\{132,231\}}(x,q,z) = \frac{1}{1-xz}[1 + P(x,q,z) - zx - zP(x,q,z)] = 1 + \frac{1-z}{1-xz}P(x,q,z),$$

which together with (17) implies the proposition. □

Corollary 4.9

$$F_{\{132,312\}}(x,q,z) = F_{\{213,312\}}(x,q,z) = \frac{1 - qz - qz^2 + xqz^3}{(1-xz)(1-qz-2qz^2)}.$$

Proof. It follows from Lemma 2.2 that $F_{\{132,312\}}(x,q,z) = F_{\{132,231\}}(\frac{x}{q}, \frac{1}{q}, qz)$. □

4.6 f) $\{132, 321\} \approx \{213, 321\}$

Proposition 4.10

$$F_{\{132,321\}}(x,q,z) = F_{\{213,321\}}(x,q,z) = \frac{1-(1+q)z+2qz^2}{(1-z)(1-xz)(1-qz)}.$$

Proof. From the definition of the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ and the description of its inverse given in Subsection 2.2, it follows that the number of peaks of the Dyck path $\varphi(\pi)$ equals the length of the longest decreasing subsequence of π. In particular, π is 321-avoiding if and only if $\varphi(\pi)$ has at most two peaks. By Lemma 2.3, $F_{\{132,321\}}(x,q,z) = \sum x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}$, where the sum is over Dyck paths D with at most two peaks. Clearly, such a path can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and D' is either empty or a pair of adjacent pyramids (see Figure 12). Therefore,

$$F_{\{132,321\}}(x,q,z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z}\cdot\frac{qz}{1-qz}\right),$$

since centered tunnels are produced by the steps outside D', and right tunnels are created by the right pyramid of D'. □

This case can be generalized to the situation when instead of 321 we have a decreasing pattern of arbitrary length. Observe that by Lemma 2.2, $F_{\{132,(k+1)k\cdots 21\}}(x,q,z) = F_{\{213,(k+1)k\cdots 21\}}(x,q,z)$ for all k.

Theorem 4.11

$$\sum_{k\geq 0} F_{\{132,(k+1)k\cdots 21\}}(x,q,z)\, p^k = \frac{2(1+xz(p-1))}{(1-p)[1+(1+q-2x)z - qz^2(p-1)^2 + \sqrt{f_1(q,z)}]},$$

where $f_1(q,z) = 1 - 2(1+q)z + [(1-q)^2 - 2q(p-1)(p+3)]z^2 - 2q(1+q)(p-1)^2z^3 + q^2(p-1)^4z^4$.

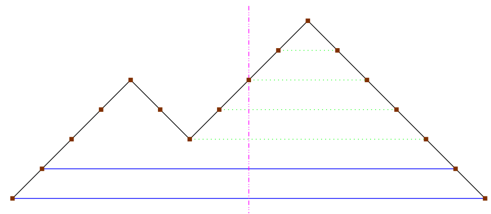

Figure 12: A path with two peaks.

Proof. As in the previous proof, we use the fact that the number of peaks of $\varphi(\pi)$ equals the length of the longest decreasing subsequence of π. Thus, φ induces a bijection between $\mathcal{S}_n(132,(k+1)k\cdots 21)$ and the subset of $\mathcal{D}_n$ of paths with at most k peaks. This implies that $\sum_{k\geq 0} F_{\{132,(k+1)k\cdots 21\}}(x,q,z)\,p^k$ can be expressed as

$$\frac{1}{1-p}\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}.$$

The result now follows from [7, Theorem 6.5] and the expression for the generating function $\sum x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}$ given in its proof. $\square$

4.7 g, g') $\{123,231\}\sim\{123,312\}$

Proposition 4.12

$$F_{\{123,312\}}(x,q,z) = \frac{1+xz+(x^2-2q)z^2+(-x^2q+xq^2+3q^2)z^4+3q^3z^5-q^3z^6-4q^4z^7-2xq^4z^8}{(1-qz^2)^3(1-q^2z^3)}.$$

Proof. We have seen in the proof of Proposition 4.10 that φ induces a bijection between $\mathcal{S}_n(132,321)$ and the set of paths in $\mathcal{D}_n$ with at most two peaks. Composing it with the complementation bijection, we get a bijection $\pi\mapsto\varphi(\bar{\pi})$ between $\mathcal{S}_n(123,312)$ and such set of Dyck paths. Using Lemma 2.3, we can write $F_{\{123,312\}}(x,q,z)=\sum x^{\mathrm{td}_0(D)}q^{\mathrm{td}_{<0}(D)}z^{|D|}$, where the sum is over Dyck paths D with at most two peaks. Again, such a D can be uniquely written as $D=\mathbf{u}^kD'\mathbf{d}^k$, where $k\geq 0$ and D' is either empty or a pair of adjacent pyramids, i.e., $D'=\mathbf{u}^i\mathbf{d}^i\mathbf{u}^j\mathbf{d}^j$ with $i,j\geq 1$. The idea is to consider cases depending on the relations among i, j and k.

To enumerate Dyck paths with at most two peaks with respect to $\mathrm{td}_0(\cdot)$ and $\mathrm{td}_{<0}(\cdot)$, it is important to look at where the tunnels of depth 0 and depth 1 occur. For convenience in this proof, we call such tunnels *frontier tunnels*, since they determine where tunnels of negative depth are: above them all tunnels have negative depth, and below them tunnels have positive depth. There are four possibilities according to where the frontier tunnels of D occur in the decomposition above:

(1) outside D',

(2) inside one of the pyramids of D',

(3) inside both pyramids of D',

(4) D has no frontier tunnel.

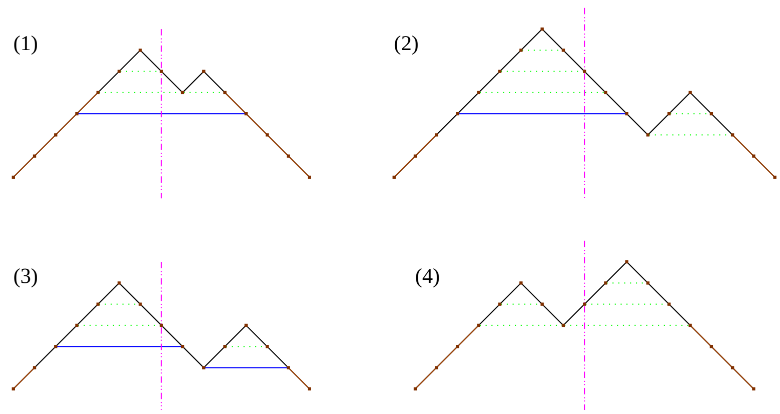

Figure 13: Four possible locations of the frontier tunnels.

Figure 13 shows an example of each of the four cases. The frontier tunnels (whose depth is 0 in this example) are drawn with a solid line, while the dotted lines are the tunnels of negative depth.

Note that in case (4) the tunnels of negative depth are exactly those in D'. We show as an example how to find the generating function in case (1). In this case, the frontier tunnel T gives a decomposition $D = A\mathbf{u}B\mathbf{d}C$ where $A = \mathbf{u}^m$, $C = \mathbf{d}^m$, $m \geq 0$, and B is a Dyck path with at most two peaks, of semilength $|B| = m$ if $\mathrm{depth}(T) = 0$, and $|B| = m+1$ if $\mathrm{depth}(T) = 1$. It follows from Proposition 4.10 that the generating function for Dyck paths with at most two peaks is $\frac{1-2z+2z^2}{(1-z)^3}$. In the situation where $\mathrm{depth}(T) = 0$, we have that $|D| = 2|B| + 1$ and $\mathrm{td}_{<0}(D) = |B|$. Thus, the corresponding generating function is

$$xz \cdot \frac{1 - 2qz^2 + 2q^2z^4}{(1 - qz^2)^3}.$$

Similarly, in the situation where $\mathrm{depth}(T) = 1$, we have that $|D| = 2|B|$ and $\mathrm{td}_{<0}(D) = |B|$, so the corresponding generating function is

$$\frac{1 - 2qz^2 + 2q^2z^4}{(1 - qz^2)^3}.$$

The other cases are similar. Adding up the generating functions obtained in each case, we get the desired expression for $F_{\{123,312\}}(x, q, z)$. □

Corollary 4.13

$$\begin{aligned} &F_{\{123,231\}}(x, q, z) \\ &= \frac{1 + xz + (x^2 - 2q)z^2 + (-x^2q + xq + 3q^2)z^4 + 3q^2z^5 - q^3z^6 - 4q^3z^7 - 2xq^3z^8}{(1 - qz^2)^3(1 - qz^3)}. \end{aligned}$$

Proof. By Lemma 2.2, we have that $F_{\{123,231\}}(x,q,z) = F_{\{123,312\}}(\frac{x}{q}, \frac{1}{q}, qz)$. $\square$

4.8 h) $\{123, 321\}$

Proposition 4.14

$$F_{\{123,321\}}(x,q,z) = 1 + xz + (x^2+q)z^2 + (2xq + q^2 + q)z^3 + 4q^2z^4.$$

Proof. By a well-known result of Erdős and Szekeres, any permutation of length at least 5 contains an occurrence of either 123 or 321. This reduces the problem to counting fixed points and excedances in permutations of length at most 4, which is trivial. $\square$

5 Triple restrictions

Here we consider simultaneous avoidance of any three patterns of length 3. Applying Lemma 2.2, the triplets of patterns fall into the following equivalence classes.

$$\begin{array}{rcl}
& \textbf{a)}\ \{123,132,213\} & \\
& \textbf{b)}\ \{231,312,321\} & \\
\textbf{c)}\ \{123,132,231\} \approx \{123,213,231\} & \sim & \textbf{c')}\ \{123,132,312\} \approx \{123,213,312\} \\
\textbf{d)}\ \{132,231,321\} \approx \{213,231,321\} & \sim & \textbf{d')}\ \{132,312,321\} \approx \{213,312,321\} \\
\textbf{e)}\ \{132,213,231\} & \sim & \textbf{e')}\ \{132,213,312\} \\
& \textbf{f)}\ \{132,231,312\} \approx \{213,231,312\} & \\
& \textbf{g)}\ \{123,231,312\} & \\
& \textbf{h)}\ \{132,213,321\} & \\
& \textbf{i)}\ \{123,132,321\} \approx \{123,213,321\} & \\
\textbf{j)}\ \{123,231,321\} & \sim & \textbf{j')}\ \{123,312,321\}
\end{array}$$

It is known [18] that the number of permutations in $\mathcal{S}_n$ avoiding the triplets in the classes **a)** and **b)** is the Fibonacci number F_{n+1}. The number of permutations avoiding any of the triplets in the classes **c)**, **c')**, **d)**, **d')**, **e)**, **e')**, **f)**, **g)** and **h)** is n. The cases of the triplets **i)**, **j)** and **j')** are trivial, because they are avoided only by permutations of length at most 4.

In terms of generating functions, when we substitute $x = q = 1$ in $F_\Sigma(x,q,z)$ where Σ is a triplet from one of the classes between **a)** and **g)**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} F_{n+1}z^n = \frac{1}{1-z-z^2}.$$

If Σ is any triplet from the classes between **c)** and **h)**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} nz^n = \frac{1-z+z^2}{(1-z)^2}.$$

The following theorem gives all the generating functions of permutations avoiding any triplet of patterns of length 3.

Theorem 5.1 a)

$$F_{\{123,132,213\}}(x,q,z) = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

b)

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}$$

c)

$$\begin{aligned} F_{\{123,132,231\}}(x,q,z) &= F_{\{123,213,231\}}(x,q,z) \\ &= \frac{1+xz+(x^2-q)z^2+qz^3+(-x^2q+xq+q^2)z^4}{(1-qz^2)^2} \end{aligned}$$

c')

$$\begin{aligned} F_{\{123,132,312\}}(x,q,z) &= F_{\{123,213,312\}}(x,q,z) \\ &= \frac{1+xz+(x^2-q)z^2+q^2z^3+(-x^2q+xq^2+q^2)z^4}{(1-qz^2)^2} \end{aligned}$$

d)

$$F_{\{132,231,321\}}(x,q,z) = F_{\{213,231,321\}}(x,q,z) = \frac{1-z+qz^2}{(1-z)(1-xz)}$$

d')

$$F_{\{132,312,321\}}(x,q,z) = F_{\{213,312,321\}}(x,q,z) = \frac{1-qz+qz^2}{(1-xz)(1-qz)}$$

e)

$$F_{\{132,213,231\}}(x,q,z) = \frac{1-z-qz^2+2qz^3+(-x^2q+q^2-xq)z^4+(x^2q-2q^2)z^5+xq^2z^6}{(1-z)(1-xz)(1-qz^2)^2}$$

e')

$$\begin{aligned} &F_{\{132,213,312\}}(x,q,z) \\ &= \frac{1-qz-qz^2+2q^2z^3+(-x^2q-xq^2+q^2)z^4+(x^2q^2-2q^3)z^5+xq^3z^6}{(1-xz)(1-qz)(1-qz^2)^2} \end{aligned}$$

f)

$$F_{\{132,231,312\}}(x,q,z) = F_{\{213,231,312\}}(x,q,z) = \frac{1+xqz^3}{(1-xz)(1-qz^2)}$$

g)

$$F_{\{123,231,312\}}(x,q,z) = \frac{1+xz+(x^2-q)z^2+xqz^3+q^2z^4}{(1-qz^2)^2}$$

h)

$$F_{\{132,213,321\}}(x,q,z) = \frac{1-(1+q)z+2qz^2-xqz^3}{(1-z)(1-xz)(1-qz)}$$

i)

$$F_{\{123,132,321\}}(x,q,z) = F_{\{123,213,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(xq+q^2+q)z^3+q^2z^4$$

j)

$$F_{\{123,231,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(2xq+q)z^3+q^2z^4$$

j')

$$F_{\{123,312,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(2xq+q^2)z^3+q^2z^4$$

Proof. Throughout this proof we will use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Subsection 2.2.

a) As in the proof of Proposition 4.1, we have that $\pi \in \mathcal{S}_n(132)$ avoids 123 if and only if the Dyck path $\varphi(\pi)$ has height at most 2. Similarly, from the proof of Proposition 4.6, π avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids. Thus, φ induces a bijection between $\mathcal{S}_n(123,132,213)$ and $\mathcal{P}yr_n^{\leq 2} := \mathcal{P}yr^{\leq 2} \cap \mathcal{D}_n$, where $\mathcal{P}yr^{\leq 2}$ denotes the set of sequences of pyramids of height at most 2. By Lemma 2.3,

$$F_{\{123,132,213\}}(x,q,z) = \sum_{D \in \mathcal{P}yr^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

To count centered and right tunnels, we distinguish cases according to which steps are the middle steps of D. A path in $\mathcal{P}yr^{\leq 2}$ of height 0 at the middle can be split in two elements of $\mathcal{P}yr^{\leq 2}$ of equal length, only the right one producing right tunnels. Since the number of $D \in \mathcal{P}yr_n^{\leq 2}$ is F_{n+1}, the generating function for paths of height 0 at the middle is

$$\sum_{n\geq 0} F_{m+1}^2 q^m z^{2m} = \frac{1-qz^2}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

Multiplying this expression by xz (resp. by x^2z^2) we obtain the generating function for paths in $\mathcal{P}yr^{\leq 2}$ having in the middle a centered pyramid of height 1 (resp. of height 2).

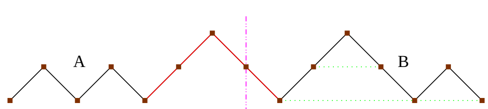

Figure 14: A sequence of pyramids of height at most 2.

Paths $D \in \mathcal{P}yr^{\leq 2}$ whose two middle steps are **dd** can be written as $D = A\mathbf{uudd}B$, where $A, B \in \mathcal{P}yr^{\leq 2}$ and $|B| = |A|+1$ (see Figure 14). Thus, the corresponding generating function is

$$\sum_{n\geq 1} F_m F_{m+1} q^m z^{2m+1} = \frac{qz^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

By symmetry, multiplying this expression by q we get the generating function for paths whose two middle steps are **uu**.

Adding up all the cases, we get the desired generating function

$$F_{\{123,132,213\}}(x,q,z) = \frac{(1+xz+x^2z^2)(1-qz^2)}{(1+qz^2)(1-3qz^2+q^2z^4)} + \frac{(q+q^2)z^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

b) Using the same reasoning as in **a)**, we have that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(231,312,321)$ and $\mathcal{P}yr_n^{\leq 2}$. Now, Lemma 2.3 implies that

$$F_{\{231,312,321\}}(x,q,z) = \sum_{D\in\mathcal{P}yr^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

Each pyramid of height 1 produces a tunnel of depth 0, and each pyramid of height 2 creates a tunnel of negative depth. Therefore,

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}.$$

c) We saw in the proof of Proposition 4.8 that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Therefore, φ induces a bijection between $\mathcal{S}_n(123,132,312)$, and paths in $\mathcal{D}_n$ with the above condition and height at most 2. Such paths (except the empty one) can be expressed uniquely as $D = \mathbf{u}A\mathbf{d}B$, where A and B are sequences of hills (i.e, they have the form $(\mathbf{ud})^k$ for some $k \geq 0$). Lemma 2.3 reduces the problem to enumerating centered tunnels and right tunnels on these paths.

If B is empty, $D = \mathbf{u}A\mathbf{d}$ has a centered tunnel at height 0. The contribution of paths of this kind to our generating function is $\frac{xz}{1-qz^2}$ for $|A|$ even, and $\frac{x^2z^2}{1-qz^2}$ for $|A|$ odd.

Assume now that $|A| < |B|$, so that A is within the left half of $D = \mathbf{u}A\mathbf{d}B$. If the middle of D is at height 0, then D is determined by the length of A and the number of hills in B to the left of the middle. Thus, the contribution of this subset to the generating function is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by xz gives the generating function for paths whose midpoint is on top of a hill of B.

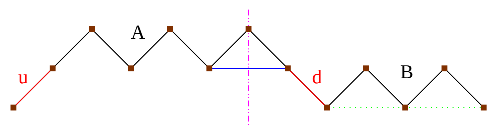

Figure 15: An example with $|A| = 3$ and $|B| = 2$.

It remains the case in which $|A| \geq |B| > 0$. If $|A| - |B|$ is even, the contribution of these paths to the generating function is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{1}{1-qz^2},$$

where the last factor counts how larger A is than B. If $|A|-|B|$ is odd, the corresponding generating function is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{xz}{1-qz^2},$$

since in this case there is a centered tunnel of height 1 inside A (see Figure 15).

Summing up all the cases, we get

$$F_{\{123,132,231\}}(x, q, z) = 1 + \frac{xz + x^2z^2}{1-qz^2} + \frac{(1+xz)qz^2}{(1-qz^2)^2} + \frac{qz^3(1+xz)}{(1-qz^2)^2}.$$

c') By Lemma 2.2, we have that $F_{\{123,132,312\}}(x, q, z) = F_{\{123,132,231\}}(\frac{x}{q}, \frac{1}{q}, qz)$, so the formula follows from part **c)**.

d) As in the proof of Proposition 4.8, we use that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Besides, as in Proposition 4.10, $\pi \in \mathcal{S}_n(132)$ avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. Thus, $\pi \in \mathcal{S}_n(132, 231, 321)$ if and only if $\varphi(\pi) \in \mathcal{D}_n$ has the form $\mathbf{u}^k B \mathbf{d}^k$, where B is either empty or is a pair of pyramids, the second of height 1. Fixed points and excedances of π are mapped to centered tunnels and right tunnels of $\varphi(\pi)$ respectively, by Lemma 2.3. Thus, $F_{\{123,132,312\}}(x, q, z)$ equals the generating function enumerating centered and right tunnels in these paths.

If B is not empty, the contribution of the first pyramid is $\frac{z}{1-z}$, and the second pyramid contributes qz. Centered tunnels come from the steps outside B. Hence,

$$F_{\{123,132,312\}}(x, q, z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z} \cdot qz\right).$$

d') It follows from part **d)** and Lemma 2.2.

e) Let $\pi \in \mathcal{S}_n(132)$. We have seen that the condition that π avoids 213 translates into $\varphi(\pi)$ being a sequence of pyramids. The additional restriction of π avoiding 231 implies that all but the first pyramid of the sequence $\varphi(\pi)$ must have height 1. Thus, by Lemma 2.3, $F_{\{132,213,231\}}(x, q, z)$ can be obtained enumerating centered and right tunnels in paths of the form $D = AB$, where A is any pyramid and B is a sequence of hills.

The contribution of such paths when B is empty is just $\frac{1}{1-xz}$. Assume now that B is not empty. If $|A| > |B|$, the corresponding contribution is

$$\frac{qz^2}{1-qz^2} \cdot \frac{z}{1-z},$$

where the second factor counts how larger A is than B. It remains the case $|A| \leq |B|$, in which A is within the left half of D. If the middle of D is at height 0, then D is determined by the length of A and the number of hills in B to the left of the middle. Thus, the contribution of this subset to the generating function is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by xz gives the generating function for paths whose midpoint is on top of a hill of B.

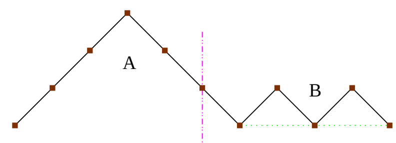

Figure 16: A pyramid followed by a sequence of hills.

Summing all this up, we get

$$F_{\{132,213,231\}}(x,q,z) = \frac{1}{1-xz} + \frac{qz^3}{(1-z)(1-qz^2)} + \frac{(1+xz)qz^2}{(1-qz^2)^2}.$$

e') It follows from part **e)** and Lemma 2.2.

f) Reasoning as in the proof of **e)**, we see that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(123, 231, 312)$ and the subset of paths in D_n consisting of a pyramid followed by a sequence of hills. By Lemma 2.3, it is enough to enumerate these paths according to the statistics $\mathrm{td}_0(\cdot)$ and $\mathrm{td}_{<0}(\cdot)$. If the path is not empty, the first pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size (since then it contains a tunnel of depth 0) and $\frac{qz^2}{1-qz^2}$ if it has even size. The sequence of hills contributes $\frac{1}{1-xz}$. Therefore,

$$F_{\{132,231,312\}}(x,q,z) = 1 + \frac{xz+qz^2}{1-qz^2} \cdot \frac{1}{1-xz}.$$

g) Let $\pi \in \mathcal{S}_n(132)$. We have seen that π avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids, and that π avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. In other words, φ induces a bijection between $\mathcal{S}_n(132, 213, 321)$ and the subset of paths in $\mathcal{D}_n$ that are a sequence of at most two pyramids. Composing with the complementation operation, we have that $\pi \in \mathcal{S}_n(123, 231, 312)$ if and only if $\varphi(\bar{\pi})$ is in that subset. Now, Lemma 2.3 implies that $F_{\{123,231,312\}}$ can be obtained enumerating sequences of at most 2 pyramids according to $\mathrm{td}_0(\cdot)$ and $\mathrm{td}_{<0}(\cdot)$. Each pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size and $\frac{qz^2}{1-qz^2}$ if it has even size. Thus,

$$F_{\{123,231,312\}}(x,q,z) = 1 + \frac{xz+qz^2}{1-qz^2} + \left(\frac{xz+qz^2}{1-qz^2}\right)^2.$$

h) We have shown in the proof of **g)** that $\pi \in \mathcal{S}_n(132, 213, 321)$ if and only if $\varphi(\pi)$ is a sequence of at most two pyramids. Using Lemma 2.3, it is enough to enumerate centered tunnels and right tunnels in such paths. The contribution of paths with exactly two pyramids is

$$\frac{z}{1-z} \cdot \frac{qz}{1-qz},$$

since only the one on the right gives right tunnels. Centered tunnels appear when there is only one pyramid. Thus we obtain

$$F_{\{132,213,321\}}(x, q, z) = \frac{1}{1-xz} + \frac{qz^2}{(1-z)(1-qz)}.$$

i, j, j') These cases are trivial because only permutations of length at most 4 can avoid 123 and 321 simultaneously. □

After having studied all the cases of double and triple restrictions, the next step is to consider restrictions of higher multiplicity. However, for $\Sigma \subseteq \mathcal{S}_3$, $|\Sigma| \geq 4$, the sets $\mathcal{S}_n(\Sigma)$ are very easy to describe (see for example [18]), and the distribution of fixed points and excedances is trivial. In particular, in these cases $|\mathcal{S}_n(\Sigma)| \in \{0, 1, 2\}$ for all n.

6 Pattern-avoiding involutions

Recall that a permutation π is an involution if $\pi = \pi^{-1}$. In terms of the array representation of π, this condition is equivalent to the array being symmetric with respect to the main diagonal. Denote by $\mathcal{I}_n$ the set of involutions of length n. In this section we consider the distribution of the statistics $\mathrm{fp}(\cdot)$ and $\mathrm{exc}(\cdot)$ in involutions avoiding any subset of patterns of length 3.

For any $\pi \in \mathcal{S}_n$, it is clear that $\mathrm{fp}(\pi) + \mathrm{exc}(\pi) + \mathrm{exc}(\pi^{-1}) = n$ (each cross in the array of π is either on, to the right of, or to the left of the main diagonal). Thus, if $\pi \in \mathcal{I}_n$, then $\mathrm{exc}(\pi) = \frac{1}{2}(n - \mathrm{fp}(\pi))$, so the number of excedances is determined by the number of fixed points. Therefore, it is enough here to consider only the statistic 'number of fixed points' in pattern-avoiding involutions.

For any set of patterns Σ, let $\mathcal{I}_n(\Sigma) := \mathcal{I}_n \cap \mathcal{S}_n(\Sigma)$, and let $i_n^k(\Sigma) := |\{\pi \in \mathcal{I}_n(\Sigma) : \mathrm{fp}(\pi) = k\}|$. Define

$$G_\Sigma(x, z) := \sum_{n \geq 0} \sum_{\pi \in \mathcal{I}_n(\Sigma)} x^{\mathrm{fp}(\pi)} z^n.$$

By the reasoning above, $\sum_{n\geq 0} \sum_{\pi \in \mathcal{I}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n = G_\Sigma(xq^{-1/2}, zq^{1/2})$.

Clearly, $\widehat{\pi}$ is an involution if and only if π is an involution. Therefore, from Lemma 2.1 we get the following.

Lemma 6.1 *Let Σ be any set of permutations. We have*

(1) $G_{\widehat{\Sigma}}(x, z) = G_\Sigma(x, z)$,

(2) $G_{\Sigma^{-1}}(x,z) = G_\Sigma(x,z)$.

The property stated in the following lemma is what allows us to apply our techniques for studying statistics in pattern-avoiding permutations to the case of involutions.

Lemma 6.2 *Let $\pi \in \mathcal{S}_n(132)$ and let $D = \varphi(\pi) \in \mathcal{D}_n$. Then,*

$$\pi \text{ is an involution} \Longleftrightarrow \varphi(\pi) \text{ is symmetric.}$$

Proof. The array of crosses representing π^{-1} is obtained from the one of π by reflection over the main diagonal. Therefore, from the description of the bijection φ given in Subsection 2.2, we have that $\varphi(\pi^{-1}) = D^*$. It follows that π is an involution if and only if $D = D^*$, which is equivalent to D being a symmetric Dyck path. □

6.1 Single restrictions

It is known [18] that for $\sigma \in \{123, 132, 213, 321\}$, $|\mathcal{I}_n(\sigma)| = \binom{n}{\lfloor n/2 \rfloor}$, and that for $\sigma \in \{231, 312\}$, $|\mathcal{I}_n(\sigma)| = 2^{n-1}$. From Lemma 6.1 it follows that for all $k \geq 0$, $i_n^k(132) = i_n^k(213)$ and $i_n^k(231) = i_n^k(312)$. It is shown in [5] (see also [8] for a bijective proof) that in fact $i_n^k(132) = i_n^k(321)$. So, for single restrictions there are three cases to consider.

Theorem 6.3 ([12, 5]) *Let $n \geq 1$, $k \geq 0$. We have*

(1) $i_n^0(123) = i_n^2(123) = \begin{cases} \binom{n-1}{\frac{n}{2}} & \textit{if } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd,} \end{cases}$

$i_n^1(123) = \begin{cases} \binom{n}{\frac{n-1}{2}} & \textit{if } n \textit{ is odd,} \\ 0 & \textit{if } n \textit{ is even,} \end{cases}$

$i_n^k(123) = 0$ *if* $k \geq 3$.

(2) $i_n^k(132) = i_n^k(213) = i_n^k(321) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \textit{if } n-k \textit{ is even,} \\ 0 & \textit{if } n-k \textit{ is odd.} \end{cases}$

(3) $G_{231}(x,z) = G_{312}(x,z) = \dfrac{1-z^2}{1-xz-2z^2}$.

Proof. (1) Clearly a 123-avoiding permutation cannot have more than two fixed points. On the other hand, if $\pi \in \mathcal{I}_n$, we have $\mathrm{fp}(\pi) = n - 2\mathrm{exc}(\pi)$, which explains that $i_n^k(123) = 0$ if $n-k$ is odd. This implies that for odd n, $\mathrm{fp}(\pi) = 1$ for all $\pi \in \mathcal{I}_n$, so $i_n^1(123) = |\mathcal{I}_n(123)| = \binom{n}{\frac{n-1}{2}}$. For even n, all we have to show is that $i_n^0(123) = i_n^2(123)$.

The bijection $\psi : \mathcal{S}_n(123) \longrightarrow \mathcal{D}_n$ described in Subsection 3.1 has the property that $\pi \in \mathcal{I}_n(123)$ if and only if $\psi(\pi)$ is a symmetric Dyck path. If n is even, involutions $\pi \in \mathcal{I}_n$ with $\mathrm{fp}(\pi) = 2$ are mapped to symmetric Dyck paths with a peak in the middle, and those with $\mathrm{fp}(\pi) = 0$ are mapped to symmetric Dyck paths with a valley in the middle. We can establish a bijection between these two sets of Dyck paths just by changing the middle peak **ud** into a middle valley **du** (this can always be done because the height at the middle of a Dyck path of even semilength is always even, so it cannot be 1). This proves that $i_n^0(123) = i_n^2(123)$, and in particular it equals $\frac{1}{2}|\mathcal{I}_n(123)| = \binom{n-1}{\frac{n}{2}}$.

(2) We use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, which by Lemma 6.2 restricts to a bijection between $\mathcal{I}_n(132)$ and $\mathcal{D}s$. Thus, by Lemma 2.3, $G_{123}(x,z)$ can be expressed as $\sum_{D\in\mathcal{D}s} x^{\mathrm{ct}(D)} z^{|D|}$, where the sum is over all symmetric Dyck paths. But the number of centered tunnels of a symmetric Dyck path is just its height at the middle. Therefore, taking only the first half of the path, $i_n^k(123)$ counts the number of paths from $(0,0)$ to (n,k) never going below the x-axis, which equals the ballot number given in the theorem.

(3) Consider the bijection $\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$. Then $\pi \in \mathcal{I}_n(312)$ if and only if $\varphi(\bar{\pi})$ is a sequence of pyramids. In fact, it turns out [18] that $\mathcal{I}_n(312) = \mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$. By Lemma 2.3, fixed points of π are mapped to tunnels of depth 0 of $\varphi(\bar{\pi})$, which are produced by pyramids of odd size. Thus, as in Proposition 4.7,

$$G_{312}(x,z) = \frac{1}{1 - \frac{xz+z^2}{1-z^2}}.$$

□

6.2 Multiple restrictions

Theorem 6.4 **a)**

$$G_{\{123,132\}}(x,z) = G_{\{123,213\}}(x,z) = \frac{1 + xz + (x^2-1)z^2}{1-2z^2}$$

b)

$$G_{\{231,321\}}(x,z) = G_{\{312,321\}}(x,z) = \frac{1}{1 - xz - z^2}$$

c)

$$G_{\{132,213\}}(x,z) = \frac{1-z^2}{(1-xz)(1-2z^2)}$$

d)

$$G_{\{231,312\}}(x,z) = \frac{1-z^2}{1-xz-2z^2}$$

e)

$$G_{\{132,231\}}(x,z) = G_{\{213,231\}}(x,z) = G_{\{132,312\}}(x,z) = G_{\{213,312\}}(x,z) = \frac{1+xz^3}{(1-xz)(1-z^2)}$$

f)

$$G_{\{132,321\}}(x,z) = G_{\{213,321\}}(x,z) = \frac{1}{(1-xz)(1-z^2)}$$

g)

$$G_{\{123,231\}}(x,z) = G_{\{123,312\}}(x,z) = \frac{1 + xz + (x^2-1)z^2 + xz^3 + z^4}{(1-z^2)^2}$$

h)

$$G_{\{123,321\}}(x,z) = 1 + xz + (x^2+1)z^2 + 2xz^3 + 2z^4$$

Proof. All the equalities between G_Σ for different Σ follow trivially from Lemma 6.1. To find expressions for these generating functions, the idea is to use again the same bijections as in Section 4, between permutations avoiding two patterns of length 3 and certain subclasses of Dyck paths. The main difference is that here we will have to deal only with symmetric Dyck paths, as a consequence of Lemma 6.2.

a) From the proof of Proposition 4.1 and Lemma 6.2, we have that φ restricts to a bijection between $\mathcal{I}_n(123,132)$ and symmetric Dyck paths $D \in \mathcal{D}_n$ of height at most 2. By Lemma 2.3, φ maps fixed points to centered tunnels, so all we have to do is count elements $D \in \mathcal{D}s$ of height at most 2 according to the number of centered tunnels. Such a D can be uniquely written as $D = ABC$, where $A = C^* \in \mathcal{D}^{\leq 2}$ and B is either empty or has the form $B = \mathbf{u}B_1\mathbf{d}$, where B_1 is a sequence of hills. If $|B_1|$ is even (resp. odd), then D has one (resp. two) centered tunnels, so the contribution of B is $1 + \frac{(1+xz)xz}{1-z^2}$. The contribution of A and C is $\frac{1-z^2}{1-2z^2}$. The product of these two quantities gives the expression for $G_{\{123,132\}}(x,z)$.

b) We have seen ([18]) that $\mathcal{I}_n(231) = \mathcal{S}_n(231,312)$. Therefore, $\mathcal{I}_n(231,321) = \mathcal{S}_n(231,312,321)$. This case was treated in Theorem 5.1 **b)**.

c) From the proof of Proposition 4.6 and Lemma 6.2, we have that φ gives a bijection between $\mathcal{I}_n(132,213)$ and symmetric sequences of pyramids $D \in \mathcal{P}yr_n$, and that it maps fixed points of the permutation to centered tunnels of the Dyck path. Such a D can be written uniquely as $D = ABC$, where $A = C^* \in \mathcal{P}yr$, and B is either empty or a pyramid. The contribution of B is $\frac{1}{1-xz}$, whereas A and C contribute $\frac{1-z^2}{1-2z^2}$. Multiplying these two expressions we get a formula for $G_{\{132,213\}}(x,z)$.

d) Again, $\mathcal{I}_n(231) = \mathcal{S}_n(231,312)$ implies that $\mathcal{I}_n(231,312) = \mathcal{S}_n(231,312)$, which has been considered in Proposition 4.7.

e) We have that $\mathcal{I}_n(132,231) = \mathcal{S}_n(132,231,312)$, so the fromula follows from Theorem 5.1 **f)**.

f) From the proof of Proposition 4.10 and Lemma 6.2, we have that φ gives a bijection between $\mathcal{I}_n(132,321)$ and symmetric paths $D \in \mathcal{D}_n$ with at most two peaks. Counting centered tunnels in such paths is very easy, since they have the form $D = \mathbf{u}^k B \mathbf{d}^k$, where $k \geq 0$ and B is either empty or a pair of identical pyramids. The contribution of B is $\frac{1}{1-z^2}$, whereas the rest contributes $\frac{1}{1-xz}$.

g) We have that $\mathcal{I}_n(123,231) = \mathcal{S}_n(123,231,312)$, so the formula follows from Theorem 5.1 **g)**.

h) It is trivial since $\mathcal{S}_n(123,321) = \emptyset$ for $n \geq 5$. □

The case of involutions avoiding simultaneously three or more patterns of length 3 is very easy and does not involve any new idea, so we omit it here.

7 Final remarks

Looking at the results of this paper, one can observe that the generating functions $F_\Sigma(x,q,z)$ that we have obtained for $\Sigma \subseteq \mathcal{S}_3$ are all rational functions when $|\Sigma| \geq 2$. This contrasts with the fact that they are not rational when $|\Sigma| = 1$, since in that case $F_\Sigma(1,1,z) = \frac{1-\sqrt{1-4z}}{2z} = \mathbf{C}(z)$. For the case of involutions, all the generating functions $G_\Sigma(x,z)$ for $\Sigma \subseteq \mathcal{S}_3$ are rational except when $\Sigma \in \{\{123\},\{132\},\{213\},\{321\}\}$.

Regarding possible extensions of this work, it would be interesting to find a generating function for fixed points and excedances in 123-avoiding permutations, the only case of patterns of length 3 that remains unsolved. In particular, just for the enumeration of fixed points in these permutations, we expect that an expression simpler than the one in Theorem 3.3 can be given.

Another further direction of research would consist in describing the cycle structure of pattern-avoiding permutations. Using the same bijective techniques from this paper, one can easily derive generating functions for the *cycle index* of permutations in $\mathcal{S}_n(231,312)$, in $\mathcal{S}_n(231,321)$ and in $\mathcal{S}_n(132,321)$. However, it is not clear whether for permutations avoiding other subsets of patterns of length 3, the distribution of the cycle type has a simple description.

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