A λ-ring Frobenius Characteristic for $G \wr S_n$

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Abstract

A λ-ring version of a Frobenius characteristic for groups of the form $G \wr S_n$ is given. Our methods provide natural analogs of classic results in the representation theory of the symmetric group. Included is a method decompose the Kronecker product of two irreducible representations of $G \wr S_n$ into its irreducible components along with generalizations of the Murnaghan-Nakayama rule, the Hall inner product, and the reproducing kernel for $G \wr S_n$.

1 Introduction

Let G be a finite group and let S_n be the symmetric group on n letters. In the early 1930's, Specht described the irreducible representations of the wreath product $G \wr S_n$ in his dissertation [16] but did not describe an analog of the Frobenius characteristic for the symmetric group.

Since then, there have been numerous accounts of the representation theory of $G \wr S_n$ [6, 7]. Most have not attempted to generalize the Frobenius map, although at least one has [10]. In [10], Macdonald gives a generalization of Schur's theory of polynomial functors before showing that a specialization of that theory naturally leads to Specht's results on the representations of $G \wr S_n$. Macdonald's version of the Frobenius map for $G \wr S_n$ is not the same as the Frobenius map in this paper, but it is shown to have some of the same properties. In particular, Macdonald verifies a sort of Frobenius reciprocity. These results are reproduced in [11]. Our presentation of the Frobenius map for $G \wr S_n$ can essentially be viewed as a detailed version of Macdonald's approach that exploits λ-ring notation. We explicitly give an analog of the Hall inner product which slightly differs from that in [11] and the reproducing kernel for $G \wr S_n$ which is not found in [11]. Moreover, our approach leads to a natural analog of the Murnaghan-Nakayama rule for $G \wr S_n$ and explicit formulas for the computation of Kronecker products for $G \wr S_n$.

Our version of the Murnaghan-Nakayama rule for computing the characters of $G \wr S_n$ yields an alternative but equivalent procedure to those found in [6, 7, 10, 11, 16]. In addition, a different proof of this rule has been given in [17]. Thus, our description cannot be viewed as new. However, our approach to decomposing the Kronecker product of representations of $G \wr S_n$ into irreducible components gives a more efficient algorithm than those which appear in the literature.

The approach we are taking has been developing for a number of years. In the late 1980's and early 1990's, Stembridge described a λ-ring version of the Frobenius characteristic for the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$ [17, 18]. This provided an account of the representation theory of the hyperoctahedral group through the manipulation of symmetric functions which paralleled the same ideas for the symmetric group [1]. The λ-ring Frobenius characteristic for $\mathbb{Z}_2 \wr S_n$ involved a class of symmetric functions over the hyperoctahedral group—in particular, Stembridge proved that the Frobenius characteristic of an irreducible character of $\mathbb{Z}_2 \wr S_n$ is a λ-ring symmetric function of the form $s_\lambda[X+Y]s_\mu[X-Y]$. These λ-ring versions of symmetric functions have similar relationships among themselves as the standard bases in the ring of symmetric functions over S_n [3]. These λ-ring symmetric functions have been used by Beck to give proofs of a variety of generating functions for permutation statistics for $\mathbb{Z}_2 \wr S_n$ [1, 2].

In 2000, Wagner described a natural extension of this λ-ring Frobenius characteristic for groups of the form $\mathbb{Z}_k \wr S_n$ [19]. A different generalization of Frobenius characteristic for $\mathbb{Z}_k \wr S_n$ was given by Poirier in [12].

Our Frobenius characteristic extends previously defined Frobenius characteristics for $\mathbb{Z}_k \wr S_n$ found in [1, 17, 18, 19]. A particularly nice aspect about our Frobenius characteristic is that is allows for a presentation of the representation theory of $G \wr S_n$ which mimics the presentation of the representation theory of the symmetric group found in [15].

The outline of this paper is as follows. The next section provides a very brief description of the group $G \wr S_n$. In Section 3, λ-ring notation is independently developed so that the Frobenius characteristic for $G \wr S_n$ may be defined in Section 4. Combinatorial proofs of classical λ-ring identities may be found there. In Section 4, a scalar product is defined are identified in the image of the Frobenius characteristic. Also in Section 4, an analog of the reproducing kernel for S_n is used to provide a criterion for determining dual bases. Characters of representations of G and S_n are induced up to the group $G \wr S_n$ in Section 5 which are then found to be the characters of the irreducible representations. The combinatorial interpretation of these irreducible characters is found in Section 6. Section 7 shows a way to compute the coefficients of the irreducible representations of $G \wr S_n$ in the Kronecker product of two irreducible representations of $G \wr S_n$. We end by giving an example of how the Kronecker product of two irreducible representations in the hyperoctahedral group may be decomposed.

2 The group $G \wr S_n$

In this section we record the results concerning wreath product groups which will be needed later. Specifically, we will identify the conjugacy classes and their sizes. The proofs of the assertions stated here may be found in [6, 11] (with different notation).

We define the group $G \wr S_n$ to be the set of $n \times n$ permutation matrices where each 1 in the matrix is replaced with an element of G. Group multiplication is defined to be matrix multiplication. Elements in $G \wr S_n$ may be written in matrix or cyclic notation. For example, if $g_1, \ldots, g_5$ are in G, an element in $G \wr S_n$ may be written as

$$\begin{pmatrix} 0 & g_1 & 0 & 0 & 0 \\ g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 \\ 0 & 0 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_5 \end{pmatrix}$$

or as $(g_1 1, g_2 2)(g_3 3, g_4 4)(g_5 5)$.

Throughout this paper, the c conjugacy classes of G will be denoted by $C_1, \ldots, C_{\mathsf{c}}$. If $g_1, \ldots, g_k \in G$, we define $(g_1 i_1, \ldots, g_k i_k)$ to be a C_j-cycle if $g_k g_{k-1} \cdots g_1 \in C_j$. For any partition $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, we write $\gamma \vdash n$ or $|\gamma| = n$ if $\gamma_1 + \cdots + \gamma_\ell = n$ and we let $\ell(\gamma)$ be the number of nonzero parts in the partition γ. Define $C_{(\gamma^1, \ldots, \gamma^{\mathsf{c}})}$ to be the set

$$\{\sigma \in G \wr S_n : \text{the } C_j\text{-cycles in } \sigma \text{ are of length } \gamma^j_1, \ldots, \gamma^j_{\ell(\gamma^j)} \text{ for } j = 1, \ldots, \mathsf{c}\};$$

that is, the set of $\sigma \in G \wr S_n$ where the C_j-cycles of σ induce the partition γ^j.

For convenience, we will write $(\gamma^1, \ldots, \gamma^{\mathsf{c}}) = \vec{\gamma}$ (where $\gamma^1, \ldots, \gamma^{\mathsf{c}}$ are partitions) and $\vec{\gamma} \vdash n$, alluding to the fact that $\sum_{i=1}^{\mathsf{c}} |\gamma^i| = n$.

Theorem 1. *A complete set of conjugacy classes for $G \wr S_n$ is $\{C_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$.*

Theorem 2. *The conjugacy class $C_{\vec{\gamma}}$ has size $n!|G|^n \prod_{i=1}^{\mathsf{c}} \frac{1}{z_{\gamma^i}} \left(\frac{|C_i|}{|G|}\right)^{\ell(\gamma^i)}$ where for any partition α with α_i parts of size i, $z_\alpha = 1^{\alpha_1} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!$.*

3 λ-Ring Notation

Since the Frobenius characteristic and the irreducible characters of $G \wr S_n$ will be written in λ-ring notation, this section independently develops λ-ring versions of symmetric functions. The idea of λ-rings have long been known to have a connection with the representation theory of the symmetric group [8]. Previous accounts of the theory have not included the fact that complex numbers may be factored out of the power symmetric functions p_n. Previously, it has been commonplace to only allow integer coefficients to have this property.

Let A be a set of formal commuting variables and A^* the set of words in A. The empty word will be identified with "1". Let $c \in \mathbb{C}$, $\gamma = (\gamma_1, \ldots, \gamma_\ell) \vdash n$, $x = a_1 a_2 \ldots a_i$ be any word in A^*, and $X, X_1, X_2, \ldots$ be any sequence of formal sums of the words in A^* with complex coefficients. Define λ-ring notation on the power symmetric functions by

$$\begin{aligned} p_r[0] &= 0, & p_r[1] &= 1, \\ p_r[x] &= x^r = a_1^r a_2^r \ldots a_i^r, & p_r[c\,X] &= c\, p_r[X], \\ p_r\left[\sum_i X_i\right] &= \sum_i p_r[X_i], & p_\gamma[X] &= p_{\gamma_1}[X] \cdots p_{\gamma_\ell}[X], \end{aligned}$$

where r is a nonnegative integer. These definitions imply that $p_r[XX_1] = p_r[X]p_r[X_1]$ and therefore $p_\gamma[XX_1] = p_\gamma[X]p_\gamma[X_1]$. These definitions also imply that for any complex number c and $\gamma \vdash n$, $p_\gamma[cX] = c^{\ell(\gamma)} p_\gamma[X]$.

When $X = x_1 + \cdots + x_N$, then our definitions ensure that

$$p_k[X] = \sum_{i=1}^{N} x_i^k,$$

which is the usual power symmetric function $p_k(x_1, \ldots, x_N)$. Furthermore, for any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$p_\lambda[X] = p_\lambda(x_1, \ldots, x_N).$$

The power symmetric functions are a basis for the ring of symmetric functions, so if Q is a symmetric function, then there are unique coefficients a_λ such that $Q = \sum_\lambda a_\lambda p_\lambda$. Define $Q[X] = \sum_\lambda a_\lambda p_\lambda[X]$. It follows that in the special case where $X = x_1 + \cdots + x_N$ is a sum of letters in A, $Q[X]$ is simply the symmetric function $Q(x_1, \ldots, x_N)$. We note that if $X = x_1 + x_2 + \cdots$ as an infinite sum of letters, the same reasoning will show that for any symmetric function Q, $Q[X] = Q$.

In particular, our definitions extend to the homogeneous, elementary, and Schur bases for the ring of symmetric functions, denoted by $\{h_\lambda : \lambda \vdash n\}$, $\{e_\lambda : \lambda \vdash n\}$, and $\{s_\lambda : \lambda \vdash n\}$, respectively. Using the transition matrices between these symmetric functions and the power basis, we define

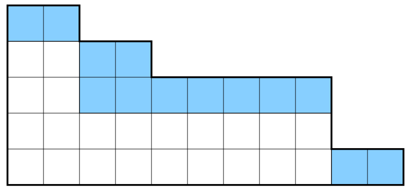
$$\begin{aligned} h_n[X] &= \sum_{\nu \vdash n} \frac{1}{z_\nu} p_\nu[X], & h_\lambda[X] &= h_{\lambda_1}[X] \cdots h_{\lambda_{\ell(\lambda)}}[X], \\ e_n[X] &= \sum_{\nu \vdash n} \frac{(-1)^{n-\ell(\nu)}}{z_\nu} p_\nu[X], & e_\lambda[X] &= e_{\lambda_1}[X] \cdots e_{\lambda_{\ell(\lambda)}}[X], \qquad \text{and} \end{aligned}$$

$$s_\lambda[X] = \sum_{\nu \vdash n} \frac{\chi_\nu^\lambda}{z_\nu} p_\nu[X]$$

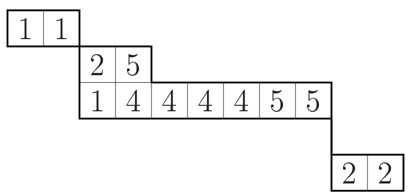
where χ_μ^λ is the irreducible character of S_n indexed by λ evaluated at the conjugacy class indexed by μ. Because $\|\frac{\chi_\nu^\lambda}{z_\nu}\|_{\lambda,\nu \vdash n}$ and $\|\chi_\lambda^\nu\|_{\lambda,\nu \vdash n}$ are inverses of each other,

$$p_\nu[X] = \sum_\lambda \chi_\nu^\lambda s_\lambda[X].$$

Given two partitions λ, μ, we write $\lambda \subseteq \mu$ provided the Ferrers diagram of λ fits inside the Ferrers diagram of μ. If $\lambda \subseteq \mu$, we let $|\mu/\lambda| = |\mu| - |\lambda|$ and we associate μ/λ with the cells in the Ferrers diagram of μ that are not in the Ferrers diagram of λ. The resultant cells are known as the skew shape μ/λ. Below, the skew shape (2,4,9,9,11)/(2,2,9,9) has been colored in teal.



A column strict tableau T of shape μ/λ is a filling of the skew shape μ/λ with positive integers such that the integers weakly increase when read from left to right and strictly increase when read from bottom to top. Let $CS(\mu/\lambda)$ be the set of all column strict tableaux of shape μ/λ. Given $T \in CS(\mu/\lambda)$, let $w_i(T)$ be the number of occurrences of i in T and let $w(T) = \prod_i x_i^{w_i(T)}$. Below we have provided an example of a column strict tableau T with $w(T) = x_1^3 x_2^3 x_4^4 x_5^3$.



Define the skew Schur function $s_{\mu/\lambda}$ by

$$s_{\mu/\lambda}(x_1, x_2, \dots) = \sum_{T \in CS(\mu/\lambda)} w(T).$$

When $\lambda = \varnothing$, this coincides with the definition of s_μ. Further, the decomposition of the skew Schur symmetric function $s_{\mu/\lambda}$ in terms of the Schur basis can be found via the well known Littlewood-Richardson coefficients. That is, if $c^\mu_{\lambda,\alpha}$ is the nonnegative integer coefficient of s_μ in $s_\lambda s_\alpha$, then

$$s_{\mu/\lambda} = \sum_\alpha c^\mu_{\lambda,\alpha} s_\alpha.$$

A rim hook in μ/λ is a sequence of cells along the northeast edge of the skew shape of μ/λ such that every pair of consecutive cells share an edge, there is not a 2 by 2 block of cells, and the removal of the cells from μ/λ leaves another skew shape. The sign of a rim hook ρ, $\mathrm{sgn}(\rho)$, is $(-1)^{r-1}$ where r is the number of rows in μ/λ which have a cell in ρ.

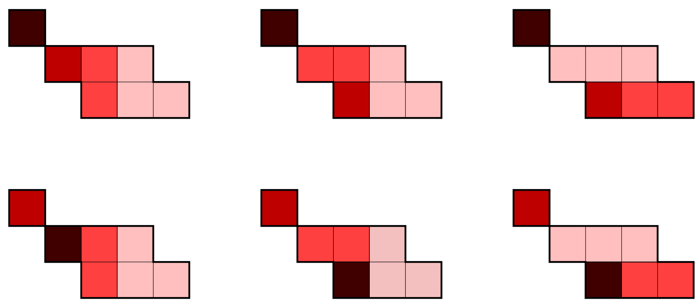
A rim hook tableau of shape μ/λ and type ν is a sequence of partitions $\lambda = \lambda^0, \dots, \lambda^j = \mu$ such that for each $1 \leq i \leq j$, λ^{i-1} is equal to λ^i with a rim hook of size ν_i removed. The sign of the rim hook tableau T, $\mathrm{sgn}(T)$, is the product of the signs of the rim hooks in T. If

$$\chi^{\mu/\lambda}_\nu = \sum \mathrm{sgn}(T)$$

where the sum runs over all rim hook tableaux T of shape μ/λ and type ν, then

$$s_{\mu/\lambda} = \sum_{\nu} \frac{\chi_{\nu}^{\mu/\lambda}}{z_{\nu}} p_{\nu} \tag{1}$$

[11]. The sum of signs of all rim hook tableaux is the same for any one order of the parts of ν. That is, the order that the parts of ν are placed in a rim hook tableau changes the appearance of the rim hook tableau but does not change the total sum of signs over all possible such objects. Unless otherwise specified, place rim hooks in a skew shape in order from smallest to largest. Below we have displayed all rim hook tableaux of shape $(1,4,5)/(1,2)$ and type $(1,1,2,3)$.



The rim hooks were placed in the above tableau according to darkness of color; that is, the darkest rim hook was placed first in the tableau and the lightest rim hook was placed last in the tableau.

If α, β are partitions of possibly different integers, let $\alpha + \beta$ be the partition created by combining the parts of the partitions α and β.

Lemma 3. *Suppose α, β are partitions such that $\alpha + \beta = \nu$. Then*

$$\chi_{\nu}^{\mu/\lambda} = \sum_{\lambda \subseteq \delta \subseteq \mu} \chi_{\alpha}^{\delta/\lambda} \chi_{\beta}^{\mu/\delta}.$$

Proof. This lemma is a result of placing the rim hooks in the skew shape μ/λ in two different ways. Instead of filling μ/λ with rim hooks in increasing order as usual, first fill μ/λ with the rim hooks with lengths found in α, then with the rim hooks with lengths found in β. Let δ be the partition formed by the rim hooks of α atop the partition λ. For different fillings of the rim hooks in α, different partitions δ may arise–each having the property that $\lambda \subseteq \delta \subseteq \mu$. For any fixed such δ, the sum over the weights of the fillings of δ/λ with rims hooks corresponding to the parts of α is $\chi_{\alpha}^{\delta/\lambda}$ and the sum over the weights of the fillings of μ/δ with rims hooks corresponding to the parts of β is $\chi_{\beta}^{\mu/\delta}$. Thus, the proof of the lemma is complete by summing over all possible δ. $\square$

Theorem 4. *For X, Y formal sums of words in A^* with complex coefficients,*

$$s_{\mu/\lambda}[X+Y] = \sum_{\lambda\subseteq\delta\subseteq\mu} s_{\mu/\delta}[X]s_{\delta/\lambda}[Y], \tag{2}$$

$$s_{\mu/\lambda}[-X] = (-1)^{|\mu'/\lambda'|}s_{\mu'/\lambda'}[X], \qquad \text{and} \tag{3}$$

$$s_{\mu}[XY] = \sum_{\lambda,\nu} K_{\mu,\lambda,\nu}s_{\lambda}[X]s_{\nu}[Y], \tag{4}$$

where λ' denotes the conjugate partition to λ and $K_{\mu,\lambda,\nu} = \sum_{\rho}\frac{1}{z_\rho}\chi^{\mu}_{\rho}\chi^{\lambda}_{\rho}\chi^{\nu}_{\rho}$.

Proof. Suppose $|\mu/\lambda| = n$. We have

$$\begin{aligned} s_{\mu/\lambda}[X+Y] &= \sum_{\nu\vdash n}\frac{\chi^{\mu/\lambda}_{\nu}}{z_\nu}p_\nu[X+Y] \\ &= \sum_{\nu=(1^{v_1},\ldots,n^{v_n})}\frac{\chi^{\mu/\lambda}_{\nu}}{z_\nu}\prod_{i=1}^{n}(p_i[X]+p_i[Y])^{v_i} \\ &= \sum_{\nu=(1^{v_1},\ldots,n^{v_n})}\frac{\chi^{\mu/\lambda}_{\nu}}{z_\nu}\prod_{i=1}^{n}\sum_{j_i=0}^{v_i}\binom{v_i}{j_i}p_i[X]^{v_i-j_i}p_i[Y]^{j_i}. \end{aligned}$$

By letting $\alpha = (1^{v_1-j_1},\ldots,n^{v_n-j_n})$, $\beta = (1^{j_1},\ldots,n^{j_n})$, and simplifying the binomial coefficients, the above string of equalities is equal to

$$\sum_{\nu\vdash n}\chi^{\mu/\lambda}_{\nu}\sum_{\alpha+\beta=\nu}\frac{1}{z_\alpha}p_\alpha[X]\frac{1}{z_\beta}p_\beta[Y].$$

Using Lemma 3, this expression may be written as

$$\begin{aligned} \sum_{\nu\vdash n}\sum_{\alpha+\beta=\nu}\sum_{\lambda\subseteq\delta\subseteq\mu}\frac{\chi^{\mu/\delta}_{\alpha}}{z_\alpha}p_\alpha[X]\frac{\chi^{\delta/\lambda}_{\beta}}{z_\beta}p_\beta[Y] &= \sum_{\lambda\subseteq\delta\subseteq\mu}\sum_{\nu\vdash n}\sum_{\alpha+\beta=\nu}\frac{\chi^{\mu/\delta}_{\alpha}}{z_\alpha}p_\alpha[X]\frac{\chi^{\delta/\lambda}_{\beta}}{z_\beta}p_\beta[Y] \\ &= \sum_{\lambda\subseteq\delta\subseteq\mu}\left(\sum_{\alpha\vdash|\mu/\delta|}\frac{\chi^{\mu/\delta}_{\alpha}}{z_\alpha}p_\alpha[X]\right)\left(\sum_{\beta\vdash|\delta/\lambda|}\frac{\chi^{\delta/\lambda}_{\beta}}{z_\beta}p_\beta[Y]\right) \\ &= \sum_{\lambda\subseteq\delta\subseteq\mu}s_{\mu/\delta}[X]s_{\delta/\lambda}[Y], \end{aligned}$$

which proves (2).

As for (3), we have

$$s_{\mu/\lambda}[-X] = \sum_{\nu}\frac{\chi^{\mu/\lambda}_{\nu}}{z_\nu}p_\nu[-X] = \sum_{\nu}(-1)^{\ell(\nu)}\frac{\chi^{\mu/\lambda}_{\nu}}{z_\nu}p_\nu[X]. \tag{5}$$

Every rim hook tableau of shape μ/λ and type ν is in one to one correspondence with a rim hook tableau of shape μ'/λ' of type ν via conjugation. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\nu)}$ are the rim hooks in a rim hook tableau of shape μ/λ and type ν. For every $i = 1, \ldots, \ell(\nu)$, $\mathrm{sgn}(\alpha_i') = (-1)^{|\alpha_i|-1}\mathrm{sgn}(\alpha_i)$. Therefore, the sign of a rim hook tableau of shape μ'/λ' and type ν is $(-1)^{|\mu/\lambda|-\ell(\nu)}$ times the sign of the corresponding rim hook tableau of shape μ/λ and type ν because

$$\prod_{i=1}^{\ell(\nu)} \mathrm{sgn}(\alpha_i') = \prod_{i=1}^{\ell(\nu)} (-1)^{|\alpha_i|-1}\mathrm{sgn}(\alpha) = (-1)^{|\mu/\lambda|-\ell(\nu)} \prod_{i=1}^{\ell(\nu)} \mathrm{sgn}(\alpha_i).$$

Using this in conjunction with (5) gives

$$\sum_{\nu\vdash n} (-1)^{\ell(\nu)}(-1)^{|\mu/\lambda|-\ell(\nu)} \frac{\chi_\nu^{\mu'/\lambda'}}{z_\nu} p_\nu[X] = \sum_{\nu\vdash n} (-1)^{|\mu'/\lambda'|} \frac{\chi_\nu^{\mu'/\lambda'}}{z_\nu} p_\nu[X] = (-1)^{|\mu'/\lambda'|} s_{\mu'/\lambda'}[X],$$

thereby proving (3).

Finally, we have

$$\begin{aligned} s_\mu[XY] &= \sum_{\rho\vdash n} \frac{\chi_\rho^\mu}{z_\rho} p_\rho[XY] \\ &= \sum_{\rho\vdash n} \frac{\chi_\rho^\mu}{z_\rho} p_\rho[X] p_\rho[Y] \\ &= \sum_{\rho\vdash n} \frac{\chi_\rho^\mu}{z_\rho} \left(\sum_{\lambda\vdash n} \chi_\rho^\lambda s_\lambda[X]\right) \left(\sum_{\nu\vdash n} \chi_\rho^\nu s_\nu[X]\right) \\ &= \sum_{\lambda,\nu\vdash n} K_{\mu,\lambda,\nu} s_\lambda[X] s_\nu[Y], \end{aligned}$$

which shows (4) and completes the proof. $\square$

A consequence of theorem 4 is corollary 5 below.

Corollary 5. *For X, Y formal sums of words in A^* with complex coefficients,*

$$h_r[X+Y] = \sum_{i=0}^{r} h_i[X] h_{r-i}[Y] \qquad \textit{and} \tag{6}$$

$$h_r[XY] = \sum_{\nu\vdash r} s_\nu[X] s_\nu[Y]. \tag{7}$$

Proof. This corollary follows from noting that $h_r = s_{(r)}$ and writing down the special cases of Theorem 4 which follow. For (6), we have

$$h_r[X+Y] = s_{(r)}[X+Y] = \sum_{\delta\subseteq(r)} s_\delta[X] s_{(r)/\delta}[Y] = \sum_{i=0}^{r} h_i[X] h_{r-i}[Y].$$

For (7) we have

$$h_r[XY] = s_{(r)}[XY] = \sum_{\lambda,\nu,\rho} \frac{1}{z_\rho} \chi_\rho^{(r)} \chi_\rho^\lambda \chi_\rho^\nu s_\lambda[X] s_\nu[Y] = \sum_{\lambda,\nu,\rho} \frac{1}{z_\rho} \chi_\rho^\lambda \chi_\rho^\nu s_\lambda[X] s_\nu[Y].$$

To complete the proof, notice that $\sum_\rho \frac{1}{z_\rho}\chi_\rho^\lambda \chi_\rho^\nu$ is 1 when $\lambda = \nu$ and 0 otherwise because $\|\frac{\chi_\nu^\lambda}{z_\nu}\|_{\lambda,\nu\vdash n}$ and $\|\chi_\lambda^\nu\|_{\lambda,\nu\vdash n}$ are inverses of each other. □

Finally, note that combining (2) and (3) in Theorem 4 gives Corollary 6 below.

Corollary 6. *For X, Y formal sums of words in A^* with complex coefficients,*

$$s_{\mu/\lambda}[X - Y] = \sum_{\lambda \subseteq \delta \subseteq \mu} (-1)^{|\delta/\lambda|} s_{\mu/\delta}[X] s_{\delta'/\lambda'}[Y].$$

4 The Frobenius Characteristic

In this section, a Frobenius characteristic for $G \wr S_n$ which preserves the inner product for functions constant on the conjugacy classes of $G \wr S_n$ (class functions) is defined. Dual bases in the space of λ-ring symmetric functions will be identified using an analog of the reproducing kernel.

For any group H, let $\mathcal{R}(H)$ be the center of the group algebra of H; that is, let $\mathcal{R}(H)$ be the set of functions mapping H into the complex numbers $\mathbb{C}$ which are constant on the conjugacy classes of H. Let $1_{\vec{\gamma}} \in \mathcal{R}(G \wr S_n)$ be the indicator function such that $1_{\vec{\gamma}}(\sigma) = 1$ provided $\sigma \in C_{\vec{\gamma}}$ and 0 otherwise. Then $\{1_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ is a basis for the center of the group algebra of $G \wr S_n$ because it is basis for the class functions. For $i = 1, \ldots, \mathsf{c}$ and variables $x_1^{(i)}, x_2^{(i)}, \ldots, x_N^{(i)}$, let $X^i = x_1^{(i)} + \cdots + x_N^{(i)}$. Define

$$\Lambda_{\mathsf{c},n} = \bigoplus_{n_1+n_2+\cdots+n_{\mathsf{c}}=n} \bigotimes_{i=1}^{\mathsf{c}} \Lambda_{n_i}(X^i)$$

where $\Lambda_{n_i}(X^i)$ is the space of homogeneous symmetric functions of degree n_i in the variables in X^i. Note that if $\{a_\lambda : \lambda \vdash n\}$ is a basis for $\Lambda_n(X^i)$, it follows that

$$\left\{ \prod_{i=1}^{\mathsf{c}} a_{\gamma^i}[X^i] : \vec{\gamma} = (\gamma^1, \ldots, \gamma^{\mathsf{c}}) \vdash n \right\}$$

is a basis for $\Lambda_{\mathsf{c},n}$.

Define the Frobenius characteristic F as a map from the center of the group algebra of $G \wr S_n$ to $\Lambda_{\mathsf{c},n}$ by

$$F(1_{\vec{\gamma}}) = \prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{z_{\gamma^i}}. \tag{8}$$

We may extend the map F by linearity to an isomorphism from $\mathcal{R}(G \wr S_n)$ onto $\Lambda_{\mathsf{c},n}$ because $\left\{\prod_{i=1}^{\mathsf{c}} p_{\gamma^i}[X^i]\right\}_{\vec{\gamma}\vdash n}$ is a basis for $\Lambda_{\mathsf{c},n}$.

Any group G has a natural scalar product on the center of the group algebra $\mathcal{R}(G)$ defined by

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \overline{g(\sigma)}$$

where $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$. A scalar product $\langle \cdot, \cdot \rangle_{\Lambda_{\mathsf{c},n}}$ may be defined so that the Frobenius map is an isometry with respect to this scalar product. The scalar product on indicator functions gives

$$\begin{aligned}
\langle 1_{\vec{\gamma}}, 1_{\vec{\delta}} \rangle_{G \wr S_n} &= \frac{1}{n!|G|^n} \sum_{\sigma \in G \wr S_n} 1_{\vec{\gamma}}(\sigma) \overline{1_{\vec{\delta}}(\sigma)} \\
&= \begin{cases} \frac{1}{n!|G|^n} |C_{\vec{\gamma}}| & \text{if } \vec{\gamma} = \vec{\delta} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \prod_{i=1}^{\mathsf{c}} \frac{1}{z_{\gamma^i}} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)} & \text{if } \vec{\gamma} = \vec{\delta} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This tells us that in order to force the Frobenius map to be an isometry, we should define the scalar product on the basis $\left\{ \prod_{i=1}^{\mathsf{c}} p_{\gamma^i}[X^i] \right\}_{\vec{\gamma} \vdash n}$ of $\Lambda_{\mathsf{c},n}$ by

$$\left\langle \prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{z_{\gamma^i}}, \prod_{i=1}^{\mathsf{c}} \frac{p_{\delta^i}[X^i]}{z_{\delta^i}} \right\rangle_{\Lambda_{\mathsf{c},n}} = \begin{cases} \prod_{i=1}^{\mathsf{c}} \frac{1}{z_{\gamma^i}} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)} & \text{if } \vec{\gamma} = \vec{\delta} \\ 0 & \text{otherwise.} \end{cases}$$

This definition of a scalar product immediately provides a self dual basis for $\Lambda_{\mathsf{c},n}$:

$$\left\{ \prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)}}} : \vec{\gamma} \vdash n \right\}.$$

Before we continue with our development of a criterion for dual bases in $\Lambda_{\mathsf{c},n}$ using an analog of the reproducing kernel in the space of symmetric functions, we digress to discuss the difference between our Frobenius map for $G \wr S_n$ and that of Macdonald [11]. His approach is slightly different than one presented in this paper, but the resulting Frobenius characteristic and inner product is simply a scalar multiple of ours. We will rejoin our approach with Lemma 7 on page 12.

Macdonald defines a graded $\mathbb{C}$-algebra $\mathcal{R}(G \wr S)$ by $\bigoplus_{n \geq 0} \mathcal{R}(G \wr S_n)$ where the multiplication on $\mathcal{R}(G \wr S)$ is defined as follows. Given $u \in \mathcal{R}(G \wr S_n)$ and $v \in \mathcal{R}(G \wr S_m)$, then $u \times v \in \mathcal{R}(G \wr S_n \times G \wr S_m)$. Since one can naturally embed $G \wr S_n \times G \wr S_m$ into $G \wr S_{n+m}$, one can define the induced representation

$$A \times B \uparrow_{G \wr S_n \times G \wr S_m}^{G \wr S_{n+m}}$$

for any representations A of $G \wr S_n$ and B of $G \wr S_m$. Thus we can define

$$ind_{G\wr S_n \times G\wr S_m}^{G\wr S_{n+m}}(\chi^A \times \chi^B) = \chi^{A\times B\uparrow_{G\wr S_n \times G\wr S_m}^{G\wr S_{n+m}}}. \tag{9}$$

Since all irreducible characters $G \wr S_n \times G \wr S_m$ are of the form $\chi^{A\times B}$ as A and B run over the irreducible representations of $G \wr S_n$ and $G \wr S_m$ respectively, we can define $ind_{G\wr S_n \times G\wr S_m}^{G\wr S_{n+m}}(u \times v)$ for any $u \times v \in \mathcal{R}(G \wr S_n \times G \wr S_m)$ by linearity. Then we define the product of u and v in $\mathcal{R}(G \wr S)$ by

$$uv = ind_{G\wr S_n \times G\wr S_m}^{G\wr S_{n+m}}(u \times v). \tag{10}$$

In addition, $\mathcal{R}(G \wr S)$ carries a scalar product defined by

$$\langle f, g\rangle_{G\wr S} = \sum_{n\geq 0} \langle f_n, g_n\rangle_{G\wr S_n}$$

where $f = \sum_{n\geq 0} f_n$ and $g = \sum_{n\geq 0} g_n$ for $f_n, g_n \in G \wr S_n$. For $r \geq 1, i = 1, \ldots, \mathsf{c}$, Macdonald lets $p_r(i)$ be independent indeterminates over $\mathbb{C}$ and defines $\Lambda(G \wr S)$ by

$$\Lambda(G \wr S) = \mathbb{C}[p_r(i) : r \geq 1, i = 1, \ldots, \mathsf{c}].$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, Macdonald defines $p_\lambda(i) = \prod_{j=1}^k p_{\lambda_j}(i)$ and for any sequence $\vec{\rho} = (\lambda^1, \ldots, \lambda^c)$ of partitions, he lets $P_{\vec{\rho}} = \prod_{i=1}^c p_{\lambda^i}(i)$. The set of all $P_{\vec{\rho}}$, as $\vec{\rho}$ varies over all sequences of partitions of length c, forms a basis for $\Lambda(G \wr S)$. Macdonald defines a scalar product on $\Lambda(G \wr S)$ by declaring that

$$\langle P_{\vec{\rho}}, P_{\vec{\gamma}}\rangle = \begin{cases} \prod_{i=1}^{\mathsf{c}} z_{\rho^i} \left(\frac{|G|}{|C_i|}\right)^{\ell(\rho^i)} & \text{if } \vec{\rho} = \vec{\gamma} \\ 0 & \text{otherwise.} \end{cases}$$

where $\vec{\rho} = (\rho^1, \ldots, \rho^{\mathsf{c}})$. Next, Macdonald defines a function $\Psi_n : G \wr S_n \to \Lambda(G \wr S)$ such that $\Psi_n(g) = P_{\vec{\rho}}$ if g is in the conjucacy class indexed by $\vec{\rho}$. He then defines a $\mathbb{C}$-linear mapping by defining for each $f \in R(G \wr S_n)$

$$\begin{aligned} ch(f) &= \langle f, \Psi_n\rangle_{G\wr S_n} \\ &= \frac{1}{|G \wr S_n|} \sum_{g\in G\wr S_n} f(g)\Psi_n(g) \\ &= \sum_{\vec{\rho}\vdash n} \prod_{i=1}^{\mathsf{c}} \frac{1}{z_{\rho^i}} \left(\frac{|C_i|}{|G|}\right)^{\ell(\rho^i)} f_{\vec{\rho}} P_{\vec{\rho}} \end{aligned} \tag{11}$$

where $f_{\vec{\rho}}$ is the value of f on the conjugacy class of $G \wr S_n$ indexed by ρ. Macdonald then shows that his characteristic map ch is an isometric isomorphism of graded $\mathbb{C}$-algebras. To see the connection with our Frobenius characteristic F for $G \wr S_n$, we can follow

Macdonal's suggestion and think of $p_r(i)$ as the power symmetric function $p_r[X^i]$. Thus, if $\vec{\rho} = (\rho^1, \ldots, \rho^{\mathsf{c}})$, then

$$P_{\vec{\rho}} = \prod_{i=1}^{\mathsf{c}} p_{\rho^i}[X^i].$$

It is not difficult to see from (11) that

$$ch(1_{\vec{\rho}}) = \prod_{i=1}^{\mathsf{c}} \frac{p_{\rho^i}[X^i]}{z_{\rho^i}} \left(\frac{|C_i|}{|G|}\right)^{\ell(\rho^i)} \tag{12}$$

Thus comparing (12) with (8), we see that the only difference between our definition of the Frobenius characteristic for $G \wr S_n$ and Macdonald's version is the extra factor of $\prod_{i=1}^{\mathsf{c}} \left(\frac{|C_i|}{|G|}\right)^{\ell(\rho^i)}$. This causes our scalar product to differ from Macdonald's scalar product by a constant. That is, under Macdonald's scalar product,

$$\left\langle \prod_{i=1}^{\mathsf{c}} p_{\rho^i}[X^i], \prod_{i=1}^{\mathsf{c}} p_{\delta^i}[X^i] \right\rangle_{\Lambda_{n,\mathsf{c}}} = \begin{cases} \prod_{i=1}^{\mathsf{c}} z_{\rho^i} \left(\frac{|G|}{|C_i|}\right)^{\ell(\rho^i)} & \text{if } \vec{\rho} = \vec{\delta} \\ 0 & \text{otherwise} \end{cases}$$

while for our scalar product on $\Lambda_{n,\mathsf{c}}$,

$$\left\langle \prod_{i=1}^{\mathsf{c}} p_{\rho^i}[X^i], \prod_{i=1}^{\mathsf{c}} p_{\delta^i}[X^i] \right\rangle_{\Lambda_{n,\mathsf{c}}} = \begin{cases} \prod_{i=1}^{c} z_{\rho^i} \left(\frac{|C_i|}{|G|}\right)^{\ell(\rho^i)} & \text{if } \vec{\rho} = \vec{\delta} \\ 0 & \text{otherwise.} \end{cases}$$

Next we shall develop a criterion for dual bases in $\Lambda_{\mathsf{c},n}$. We start by proving a technical lemma which will be crucial for our criterion.

Lemma 7. *For a conjugacy class C_i of G,*

$$\left. \prod_{j,k} (1 - x_j y_k)^{-\frac{|G|}{|C_i|}} \right|_{2n} = \sum_{\mu \vdash n} \frac{p_\mu[X] p_\mu[Y]}{z_\mu \left(\frac{|C_i|}{|G|}\right)^{\ell(\mu)}}$$

where $\cdot|_{2n}$ picks out the degree $2n$ terms from $\cdot$, $X = x_1 + x_2 + \cdots$, and $Y = y_1 + y_2 + \cdots$.

Proof. The proof of this lemma is a result of formal power series manipulations.

$$\begin{aligned} \left. \prod_{j,k} (1 - x_j y_k)^{-\frac{|G|}{|C_i|}} \right|_{2n} &= \left. \exp\left(\frac{|G|}{|C_i|} \sum_{j,k} \log\left(\frac{1}{1 - x_j y_k}\right)\right) \right|_{2n} \\ &= \left. \exp\left(\sum_{\ell \geq 1} \frac{|G|}{|C_i|} \frac{p_\ell[X] p_\ell[Y]}{\ell} \right) \right|_{2n} \\ &= \left. \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{\ell \geq 1} \frac{|G|}{|C_i|} \frac{p_\ell[X] p_\ell[Y]}{\ell} \right)^m \right|_{2n}. \end{aligned}$$

Since our concern is of terms of degree $2n$, the tail of this last sum may be chopped off. This gives

$$\sum_{m=1}^{n} \frac{1}{m!} \sum_{a_1+\cdots+a_n=n} \binom{m}{a_1, a_2, \ldots, a_n} \prod_{j=1}^{n} \left(\frac{|G|}{|C_i|} \frac{p_j[X]p_j[Y]}{j} \right)^{a_j}$$

$$= \sum_{m=1}^{n} \sum_{a_1+\cdots+a_n=n} \prod_{j=1}^{n} \left(\frac{|G|}{|C_i|} \right)^{a_j} \frac{p_j[X]^{a_j} p_j[Y]^{a_j}}{j^{a_j} a_j!}$$

$$= \sum_{\mu \vdash n} \left(\frac{|G|}{|C_i|} \right)^{\ell(\mu)} \frac{p_\mu[X]p_\mu[Y]}{z_\mu},$$

which completes the proof of this lemma. □

Just as we let $X^i = x_1^{(i)} + \cdots + x_N^{(i)}$ for $i = 1, \ldots, \mathsf{c}$, let $Y^i = y_1^{(i)} + \cdots + y_N^{(i)}$ be a sum of variables for $i = 1, \ldots, \mathsf{c}$. Let $\Omega^{2n} = \Omega^{2n}(X^1, \ldots, X^{\mathsf{c}}, Y^1, \ldots, Y^{\mathsf{c}})$ be the terms of degree $2n$ in the expression

$$\prod_{i=1}^{\mathsf{c}} \prod_{j,k} (1 - x_j^{(i)} y_k^{(i)})^{-\frac{|G|}{|C_i|}}.$$

We will show that Ω^{2n} is the reproducing kernel for $\Lambda_{\mathsf{c},n}$.

Theorem 8.

$$\Omega^{2n} = \sum_{\vec{\gamma} \vdash n} \left(\prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)}}} \right) \left(\prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[Y^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)}}} \right).$$

Proof. We have

$$\Omega^{2n} = \sum_{n_1+\cdots+n_{\mathsf{c}}=n} \prod_{i=1}^{\mathsf{c}} \left(\prod_{j,k} (1 - x_j^{(i)} y_k^{(i)})^{-\frac{|G|}{|C_i|}} \Bigg|_{2n_i} \right),$$

which by Lemma 7 is equal to

$$\sum_{n_1+\cdots+n_{\mathsf{c}}=n} \prod_{i=1}^{\mathsf{c}} \left(\sum_{\mu^i \vdash n_i} \left(\frac{|G|}{|C_i|} \right)^{\ell(\mu^i)} \frac{p_{\mu^i}[X^i]p_{\mu^i}[Y^i]}{z_{\mu^i}} \right)$$

$$= \sum_{n_1+\cdots+n_{\mathsf{c}}=n} \sum_{\mu^1 \vdash n_1} \cdots \sum_{\mu^{\mathsf{c}} \vdash n_{\mathsf{c}}} \prod_{i=1}^{\mathsf{c}} \left(\frac{|G|}{|C_i|} \right)^{\ell(\mu^i)} \frac{p_{\mu^i}[X^i]p_{\mu^i}[Y^i]}{z_{\mu^i}}$$

$$= \sum_{\vec{\gamma} \vdash n} \left(\prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)}}} \right) \left(\prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[Y^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|} \right)^{\ell(\gamma^i)}}} \right).$$

This completes the proof of the theorem. □

Theorem 9. *Two bases* $\{a_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ *and* $\{b_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ *of* $\Lambda_{\mathsf{c},n}$ *are dual if and only if*

$$\sum_{\vec{\gamma} \vdash n} a_{\vec{\gamma}}(X^1, \ldots, X^{\mathsf{c}}) b_{\vec{\gamma}}(Y^1, \ldots, Y^{\mathsf{c}}) = \Omega^{2n}.$$

Proof. Let $\vec{\gamma}_1, \ldots, \vec{\gamma}_m$ be some ordering of the set $\{\vec{\gamma} : \vec{\gamma} \vdash n\}$ and let us think of the bases $\{a_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$, $\{b_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$, and $\left\{ \prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|}\right)^{\ell(\gamma^i)}}} : \vec{\gamma} \vdash n \right\}$ as m-dimensional column vectors where the i^{th} entry is equal to the corresponding basis element indexed by $\vec{\gamma}_i$. Let $\vec{a}, \vec{b}$, and $\vec{p}$ denote these column vectors. Let A and B be the two matrices such that $\vec{a} = A\vec{p}$ and $\vec{b} = B\vec{p}$.

The proof of this theorem is entirely linear algebra and does not depend on Ω^{2n} itself. We will show that two bases are dual if and only if $AB^\top = I_m$ after which it will be shown that $AB^\top = I_m$ if and only if $\sum_{\vec{\gamma}} a_{\vec{\gamma}}(X^1, \ldots, X^{\mathsf{c}}) b_{\vec{\gamma}}(Y^1, \ldots, Y^{\mathsf{c}}) = \Omega^{2n}$.

Let $\vec{a} \odot \vec{b}^\top$ denote the $m \times m$ matrix

$$\left\| \langle a_{\vec{\gamma}}(X^1, \ldots, X^{\mathsf{c}}), b_{\vec{\delta}}(Y^1, \ldots, Y^{\mathsf{c}}) \rangle_{\Lambda_{\mathsf{c},n}} \right\|_{\vec{\gamma}, \vec{\delta} \vdash n}.$$

The bases $\{a_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ and $\{b_{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ are dual if and only if $\vec{a} \odot \vec{b}^\top = I_m$. We have that

$$\vec{a} \odot \vec{b}^\top = (A\vec{p}) \odot (B\vec{p})^\top = A\vec{p} \odot \vec{p}^\top B^\top = AB^\top$$

because $\vec{p} \odot \vec{p}^\top = I_m$ and because this product is associative. Therefore, the bases are dual if and only if $AB^\top = I_m = A^\top B$.

We have that

$$\sum_{\vec{\gamma} \vdash n} a_{\vec{\gamma}}(X^1, \ldots, X^{\mathsf{c}}) b_{\vec{\gamma}}(Y^1, \ldots, Y^{\mathsf{c}}) = \vec{a}^\top \vec{b} = (A\vec{p})^\top (B\vec{p}) = \vec{p}^\top A^\top B\vec{p}.$$

From Theorem 8, $\vec{p}^\top \vec{p}$ is equal to Ω^{2n}, which means that the equation in the statement of this theorem holds if and only if $\vec{p}^\top A^\top B\vec{p} = \vec{p}^\top \vec{p}$. Using the fact that a basis for $\Lambda_{\mathsf{c},n}$ is $\left\{ \prod_{i=1}^{\mathsf{c}} \frac{p_{\gamma^i}[X^i]}{\sqrt{z_{\gamma^i} \left(\frac{|C_i|}{|G|}\right)^{\ell(\gamma^i)}}} : \vec{\gamma} \vdash n \right\}$, it is easy to see that the equation in the statement of this theorem holds if and only if $A^\top B = I_m$. This completes the proof. □

Let $\{\chi^1, \cdots, \chi^{\mathsf{c}}\}$ be a complete set of characters of irreducible representations of G.

Theorem 10. *Let* χ^i_j *be the irreducible character of the representation indexed by* i *on the conjugacy class* C_j *of the group* G*. Then*

$$\left\{ \prod_{i=1}^{\mathsf{c}} s_{\gamma^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right] : \vec{\gamma} \vdash n \right\} \qquad \textit{and} \qquad \left\{ \prod_{i=1}^{\mathsf{c}} s_{\gamma^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \overline{\chi^i_j} X^j \right] : \vec{\gamma} \vdash n \right\}$$

are dual.

Proof. By Theorem 9, we need only show that

$$\sum_{\vec{\gamma}\vdash n}\left(\prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}}\chi^i_j X^j\right]\right)\left(\prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}}\overline{\chi^i_j} Y^j\right]\right) = \Omega^{2n}.$$

The left hand side of the above equality may be rewritten to look like

$$\sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}}\sum_{\gamma^i\vdash n_i} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}}\chi^i_j X^j\right] s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}}\overline{\chi^i_j} Y^j\right]$$

which by (7) in Corollary 5 is equal to

$$\sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}} h_{n_i}\left[\left(\textstyle\sum_{j=1}^{\mathsf{c}}\chi^i_j X^j\right)\left(\textstyle\sum_{j=1}^{\mathsf{c}}\overline{\chi^i_j} Y^j\right)\right] = \sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}} h_{n_i}\left[\sum_{j,k}\chi^i_j\overline{\chi^i_k}X^jY^k\right].$$

By an iterated application of (6) in Corollary 5, this may be changed to look like

$$h_n\left[\sum_{i=1}^{\mathsf{c}}\sum_{j,k}\chi^i_j\overline{\chi^i_k}X^jY^k\right] = h_n\left[\sum_{j,k}X^jY^k\left(\sum_{i=1}^{\mathsf{c}}\chi^i_j\overline{\chi^i_k}\right)\right].$$

The (column-wise) orthogonality of the irreducible characters of the group G tells us that

$$\left(\sum_{i=1}^{\mathsf{c}}\chi^i_j\overline{\chi^i_k}\right) = \begin{cases}\frac{|G|}{|C_j|} & \text{if } j=k\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, the left hand side of the first equality in this proof is equal to

$$\begin{aligned}
h_n\left[\sum_{i=1}^{\mathsf{c}}\frac{|G|}{|C_i|}X^iY^i\right] &= \sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}} h_{n_i}\left[\frac{|G|}{|C_i|}X^iY^i\right]\\
&= \sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}}\sum_{\gamma^i\vdash n_i}\frac{1}{z_{\gamma^i}}p_{\gamma^i}\left[\frac{|G|}{|C_i|}X^iY^i\right]\\
&= \sum_{n_1+\cdots+n_{\mathsf{c}}=n}\prod_{i=1}^{\mathsf{c}}\sum_{\gamma^i\vdash n_i}\frac{p_{\gamma^i}[X^i]p_{\gamma^i}[Y^i]}{z_{\gamma^i}\left(\frac{|C_i|}{|G|}\right)^{\ell(\gamma^i)}}.
\end{aligned}$$

The sum in the last equality is the right hand side of Theorem 8 after breaking the denominator into two square roots. Thus, our string of equalities is equal to Ω^{2n}. □

5 Induced and Irreducible Characters

In this section, representations of subgroups of $G \wr S_n$ are induced to construct the irreducible representations of $G \wr S_n$. Just as in the case of the symmetric group, the image of the irreducible characters involves the Schur basis.

Let A^λ be the irreducible representation of S_n corresponding to the partition λ and let χ^λ be the character of A^λ. Let ε denote the identity element in G and let us write $G = \{\varepsilon = \tau_1, \ldots, \tau_k\}$.

Define $\hat{A}^\lambda$ as the representation of $G \wr S_n$ with the property that $\hat{A}^\lambda((\varepsilon i, \varepsilon(i+1)))$ is equal to $A^\lambda((i, i+1))$ and that $\hat{A}^\lambda((\tau_j n))$ is equal to the identity matrix of the proper dimension for $i = 1, \ldots, n-1$. This uniquely defines $\hat{A}^\lambda$ because $(\varepsilon i, \varepsilon(i+1))$ and $(\tau_j n)$ may be shown to generate $G \wr S_n$. Let $\hat{\chi}^\lambda_{\vec{\gamma}}$ be the character of $\hat{A}^\lambda$ on the conjugacy class $C_{\vec{\gamma}}$. It follows that

$$\hat{\chi}^\lambda_{\vec{\gamma}} = \chi^\lambda_{\gamma^1 + \cdots + \gamma^{\mathsf{c}}}$$

where $\gamma^1 + \cdots + \gamma^{\mathsf{c}}$ is the partition of n formed by combining the parts of the partitions $\gamma^1, \ldots, \gamma^{\mathsf{c}}$.

The character χ^i of the group G may be extended to the group $G \wr S_n$ in a similar way. Let A^i be the representation of G corresponding to χ^i. Define $\hat{A}^i$ to be the representation of the group $G \wr S_n$ such that $\hat{A}^i((\varepsilon i, \varepsilon(i+1)))$ is the identity matrix of the proper dimension and $\hat{A}^i((\tau_j n)) = A^i(\tau_j)$. Let $\hat{\chi}^i$ be the character of $\hat{A}^i$. For $g \in C_j$, one C_j-cycle of length k is

$$\begin{aligned}(g i_1, \varepsilon i_2, \ldots, \varepsilon i_k) &= (\varepsilon i_1, \ldots, \varepsilon i_k)(g i_1) \\ &= (\varepsilon i_1, \ldots, \varepsilon i_k)(\varepsilon i_1, \varepsilon n)(g n)(\varepsilon i_1, \varepsilon n),\end{aligned}$$

so it may be seen that

$$\hat{\chi}^i_{\vec{\gamma}} = \prod_{j=1}^{\mathsf{c}} \left(\chi^i_j\right)^{\ell(\gamma^j)}$$

where $\hat{\chi}^i_{\vec{\gamma}}$ is the value of the character $\hat{\chi}^i$ on the conjugacy class $C_{\vec{\gamma}}$.

Let A^i be the representation of G corresponding to χ^i and A^λ be the irreducible representation of S_n corresponding to the partition λ. The Kronecker product $\hat{A}^i \otimes \hat{A}^\lambda$ of the representations $\hat{A}^i$ and $\hat{A}^\lambda$ is defined such that for any $\sigma \in G$, $\hat{A}^i \otimes \hat{A}^\lambda(\sigma) = \hat{A}^i(\sigma) \otimes \hat{A}^\lambda(\sigma)$. Here, for any matrices $A = ||a_{i,j}||$ and B, $A \otimes B$ is the block matrix $||a_{i,j}B||$. It is not difficult to see that the character of $\hat{A}^i \otimes \hat{A}^\lambda$ is $\hat{\chi}^i \hat{\chi}^\lambda$ where $\hat{\chi}^i \hat{\chi}^\lambda(\sigma) = \hat{\chi}^i(\sigma) \cdot \hat{\chi}^\lambda(\sigma)$ for any $\sigma \in G \wr S_n$.

Lemma 11. *For $\lambda \vdash n$ and $i = 1, \ldots, \mathsf{c}$,*

$$s_\lambda\left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j\right] = F(\hat{\chi}^i \hat{\chi}^\lambda).$$

Proof. An iterated application of (2) in Theorem 4 gives

$$\begin{aligned}
s_\lambda\left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j\right] &= \sum_{\nu^{\mathsf{c}-1} \subseteq \cdots \subseteq \nu^1 \subseteq \lambda} s_{\lambda/\nu^1}[\chi^i_1 X^1] \cdots s_{\nu^{\mathsf{c}-2}/\nu^{\mathsf{c}-1}}[\chi^i_{\mathsf{c}-1} X^{\mathsf{c}-1}] s_{\nu^{\mathsf{c}-1}}[\chi^i_{\mathsf{c}} X^{\mathsf{c}}] \\
&= \sum_{\nu^{\mathsf{c}-1} \subseteq \cdots \subseteq \lambda} \left(\sum_{\gamma^1 \vdash |\lambda/\nu^1|} \frac{\chi^{\lambda/\nu^1}_{\gamma^1}}{z_{\gamma^1}} p_{\gamma^1}[\chi^i_1 X^1] \right) \cdots \left(\sum_{\gamma^{\mathsf{c}} \vdash |\nu^{\mathsf{c}-1}|} \frac{\chi^{\nu^{\mathsf{c}-1}}_{\gamma^{\mathsf{c}}}}{z_{\gamma^{\mathsf{c}}}} p_{\gamma^{\mathsf{c}}}[\chi^i_{\mathsf{c}} X^{\mathsf{c}}] \right) \\
&= \sum_{n_1 + \cdots + n_{\mathsf{c}} = n} \sum_{\gamma^1 \vdash n_1} \cdots \sum_{\gamma^{\mathsf{c}} \vdash n_{\mathsf{c}}} \sum_{\nu^{\mathsf{c}-1} \subseteq \cdots \subseteq \lambda} \chi^{\lambda/\nu^1}_{\gamma^1} \cdots \chi^{\nu^{\mathsf{c}-1}}_{\gamma^{\mathsf{c}}} \prod_{j=1}^{\mathsf{c}} \frac{p_{\gamma^j}[\chi^i_j X^j]}{z_{\gamma^j}}.
\end{aligned}$$

An iterated application of Lemma 3 gives a simplification of the product of skew rim hook tableaux in the above equation. The above string of equalities is equal to

$$\begin{aligned}\sum_{n_1+\cdots+n_{\mathsf{c}}=n}\sum_{\gamma^1\vdash n_1}\cdots\sum_{\gamma^{\mathsf{c}}\vdash n_{\mathsf{c}}}\chi^{\lambda}_{\gamma^1+\gamma^2+\cdots+\gamma^{\mathsf{c}}}\prod_{j=1}^{\mathsf{c}}\frac{p_{\gamma^j}[\chi^i_j X^j]}{z_{\gamma^j}} &= \sum_{\vec{\gamma}\vdash n}\hat{\chi}^{\lambda}_{\vec{\gamma}}\prod_{j=1}^{\mathsf{c}}\left(\chi^i_j\right)^{\ell(\gamma^j)}\frac{p_{\gamma^j}[X^i]}{z_{\gamma^i}}\\ &= \sum_{\vec{\gamma}\vdash n}\hat{\chi}^{i}_{\vec{\gamma}}\hat{\chi}^{\lambda}_{\vec{\gamma}}\prod_{j=1}^{\mathsf{c}}\frac{p_{\gamma^j}[X^i]}{z_{\gamma^i}}\\ &= F(\hat{\chi}^i\hat{\chi}^{\lambda}). \qquad\square\end{aligned}$$

If A is any representation of H where H is a subgroup of a group G, let $A{\uparrow}_H^G$ denote the induced representation of A to G. If $\phi^{A\uparrow_H^G}$ is the character of $A{\uparrow}_H^G$, then for any $\tau\in G$,

$$\phi^{A\uparrow_H^G}(\tau)=\frac{1}{|H|}\sum_{g\in G}\phi^A(g^{-1}\tau g) \tag{13}$$

where ϕ^A is extended to the entire group by defining $\phi^A(\tau)=0$ for $\tau\notin H$. If B is any representation of G, let $B{\downarrow}_H^G$ be the restriction of B to the subgroup H.

Suppose $n_1,\ldots,n_{\mathsf{c}}\geq 0$ are such that $n_1+\cdots+n_{\mathsf{c}}=n$. We may think of

$$(G\wr S_{n_1})\times\cdots\times(G\wr S_{n_{\mathsf{c}}})$$

as a subgroup of $G\wr S_n$ where S_{n_j} permutes $\left\{1+\sum_{i<j}n_i, 2+\sum_{i<j}n_i,\ldots,n_j+\sum_{i<j}n_i\right\}$. Let $A^1,\ldots,A^{\mathsf{c}}$ be representations of $G\wr S_{n_1},\ldots,G\wr S_{n_{\mathsf{c}}}$, respectively, and $\phi^1,\ldots,\phi^{\mathsf{c}}$ the characters of these representations. We may form a representation $A^1\times\cdots\times A^{\mathsf{c}}$ such that if $(g_1,\ldots,g_{\mathsf{c}})\in(G\wr S_{n_1})\times\cdots\times(G\wr S_{n_{\mathsf{c}}})$, then

$$A^1\times\cdots\times A^{\mathsf{c}}(g_1,\ldots,g_{\mathsf{c}})=A^1(g_1)\otimes\cdots\otimes A^{\mathsf{c}}(g_{\mathsf{c}}).$$

We will denote the character of this representation induced to $G\wr S_n$ by $\phi^1\times\cdots\times\phi^{\mathsf{c}}\big\uparrow^{G\wr S_n}$.

Lemma 12.

$$F(\phi^1\times\cdots\times\phi^{\mathsf{c}}\big\uparrow^{G\wr S_n})=\prod_{i=1}^{\mathsf{c}}F(\phi^i).$$

Proof. We have that

$$\begin{aligned}F(\phi^1\times\cdots\times\phi^{\mathsf{c}}\big\uparrow^{G\wr S_n}) &= \sum_{\vec{\gamma}\vdash n}(\phi^1\times\cdots\times\phi^{\mathsf{c}}\big\uparrow^{G\wr S_n})_{\vec{\gamma}}\prod_{i=1}^{\mathsf{c}}\frac{p_{\gamma^i}[X^i]}{z_{\gamma^i}}\\ &= \frac{1}{n!|G|^n}\sum_{\vec{\gamma}\vdash n}|C_{\vec{\gamma}}|(\phi^1\times\cdots\times\phi^{\mathsf{c}}\big\uparrow^{G\wr S_n})_{\vec{\gamma}}\prod_{i=1}^{\mathsf{c}}p_{\gamma^i}[X^i]\left(\frac{|G|}{|C_i|}\right)^{\ell(\gamma^i)}.\end{aligned}$$

Let ψ_n be a function mapping $G \wr S_n$ into $\Lambda_{\mathsf{c},n}$, which is constant on conjugacy classes, such that if $\sigma \in C_{\vec{\gamma}}$, then $\psi_n(\sigma) = \prod_{i=1}^{\mathsf{c}} p_{\gamma^i}[X^i] \left(\frac{|G|}{|C_i|}\right)^{\ell(\gamma^i)}$. Then our above string of equalities is equal to

$$\begin{aligned}
&\frac{1}{n!|G|^n} \sum_{\sigma \in G\wr S_n} \phi^1 \times \cdots \times \phi^{\mathsf{c}} \Big\uparrow^{G\wr S_n} (\sigma)\psi_n(\sigma) \\
&\qquad = \frac{1}{n!|G|^n} \sum_{\sigma \in G\wr S_n} \frac{1}{n_1!\cdots n_{\mathsf{c}}!|G|^n} \sum_{\tau \in G\wr S_n} \phi^1 \times \cdots \times \phi^{\mathsf{c}}(\tau\sigma\tau^{-1})\psi_n(\sigma) \\
&\qquad = \frac{1}{n!n_1!\cdots n_{\mathsf{c}}!|G|^{2n}} \sum_{\xi,\tau \in G\wr S_n} \phi^1 \times \cdots \times \phi^{\mathsf{c}}(\xi)\psi_n(\tau^{-1}\xi\tau) \\
&\qquad = \frac{1}{n_1!\cdots n_{\mathsf{c}}!|G|^n} \sum_{\xi \in (G\wr S_{n_1})\times\cdots\times(G\wr S_{n_{\mathsf{c}}})} \phi^1 \times \cdots \times \phi^{\mathsf{c}}(\xi)\psi_n(\xi).
\end{aligned}$$

The last line of the above equality follows from the fact that ψ_n is constant on conjugacy classes and from (13). It may be noted that these last few equalities provide a sort of Frobenius reciprocity for this group. Continuing this string of equalities, we have

$$\begin{aligned}
\prod_{i=1}^{\mathsf{c}} \frac{1}{n_i!|G|^{n_i}} \sum_{\sigma_i \in G\wr S_{n_i}} \psi_{n_i}(\sigma_i)\phi^i(\sigma_i) &= \prod_{i=1}^{\mathsf{c}} \frac{1}{n_i!|G|^{n_i}} \sum_{\vec{\gamma}^i \vdash n_i} |C_{\vec{\gamma}^i}|(\phi^i)_{\vec{\gamma}^i} \prod_{j=1}^{\mathsf{c}} p_{\vec{\gamma}^{ij}}[X^j] \left(\frac{|G|}{|C_i|}\right)^{\ell(\vec{\gamma}^{ij})} \\
&= \prod_{i=1}^{\mathsf{c}} \sum_{\vec{\gamma}^i \vdash n_i} (\phi^i)_{\vec{\gamma}^i} \prod_{j=1}^{\mathsf{c}} \frac{p_{\vec{\gamma}^{ij}}[X^j]}{z_{\vec{\gamma}^{ij}}} \\
&= \prod_{i=1}^{\mathsf{c}} F(\phi^i). \qquad \square
\end{aligned}$$

Suppose A^i is an irreducible representation of the group G, A^{γ^i} an irreducible representation of the group S_{n_i}, and $\hat{A}^i$ and $\hat{A}^{\gamma^i}$ are their extensions to the group $G \wr S_{n_i}$. Then we shall show that the representation

$$A^{\vec{\gamma}} = \left(\hat{A}^1 \otimes \hat{A}^{\gamma^1}\right) \times \left(\hat{A}^2 \otimes \hat{A}^{\gamma^2}\right) \times \cdots \times \left(\hat{A}^{\mathsf{c}} \otimes \hat{A}^{\gamma^{\mathsf{c}}}\right) \Big\uparrow_{G\wr S_{n_1}\times\cdots\times G\wr S_{n_{\mathsf{c}}}}^{G\wr S_n} \tag{14}$$

is irreducible for every $\vec{\gamma} \vdash n$. The character of $A^{\vec{\gamma}}$ will be denoted by

$$\chi^{\vec{\gamma}} = \chi^{A^{\vec{\gamma}}} = \hat{\chi}^1\hat{\chi}^{\gamma^1} \times \cdots \times \hat{\chi}^{\mathsf{c}}\hat{\chi}^{\gamma^{\mathsf{c}}} \Big\uparrow_{G\wr S_{n_1}\times\cdots\times G\wr S_{n_{\mathsf{c}}}}^{G\wr S_n}. \tag{15}$$

If χ^i is an irreducible character of the group G, then so is $\overline{\chi}^i$. This means that $\overline{\chi}^i = \chi^k$ for some k. Thus,

$$\overline{\hat{\chi}}^1\hat{\chi}^{\gamma^1} \times \cdots \times \overline{\hat{\chi}}^{\mathsf{c}}\hat{\chi}^{\gamma^{\mathsf{c}}} \Big\uparrow_{G\wr S_{n_1}\times\cdots\times G\wr S_{n_{\mathsf{c}}}}^{G\wr S_n}$$

is equal to $\chi^{\vec{\delta}}$ for some $\vec{\delta}$. Let us denote this character by $\chi^{\overline{\vec{\gamma}}}$. Combining Lemma 11 and Lemma 12, we have the following corollary.

Corollary 13. *If* $\vec{\gamma} \vdash n$*,*

$$F(\chi^{\vec{\gamma}}) = \prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi_j^i X^j\right] \qquad \textit{and} \qquad F(\chi^{\overline{\vec{\gamma}}}) = \prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}} \overline{\chi_j^i} X^j\right].$$

Now we are ready to identify the irreducible characters of $G \wr S_n$.

Theorem 14. *The set* $\{\chi^{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ *is a complete set of irreducible characters of* $G \wr S_n$*.*

Proof. Corollary 13 and Theorem 10 show

$$\left\langle \chi^{\vec{\gamma}}, \chi^{\overline{\vec{\delta}}} \right\rangle_{G\wr S_n} = \left\langle \prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi_j^i X^j\right], \prod_{i=1}^{\mathsf{c}} s_{\delta^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}} \overline{\chi}_j^i X^j\right] \right\rangle_{\Lambda_{\mathsf{c},n}} = \begin{cases} 1 & \text{if } \vec{\gamma} = \vec{\delta} \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

As $\vec{\delta}$ varies, $\chi^{\overline{\vec{\delta}}}$ runs through all possible characters of the form $\chi^{\vec{\gamma}}$. Therefore, since $\langle \chi^{\vec{\gamma}}, \chi^{\vec{\gamma}} \rangle$ is greater than or equal to 1 for any $\vec{\gamma}$, equation (16) implies that $\langle \chi^{\vec{\gamma}}, \chi^{\vec{\gamma}} \rangle$ is actually equal to 1.

The scalar product of a character of a representation A with itself gives the sum of the squares of the multiplicities of the irreducible representations occurring in A. Thus, $\chi^{\vec{\gamma}}$ is irreducible for every $\vec{\gamma}$. Since we now have the same number of irreducible characters as the number of conjugacy classes, $\{\chi^{\vec{\gamma}} : \vec{\gamma} \vdash n\}$ must be a complete set. $\square$

Theorem 14 not only tells us the characters of the irreducible representations of $G \wr S_n$, but the actual irreducible representations as well. Suppose A^i is an irreducible representation of the group G, A^{γ^i} an irreducible representation of the group S_{n_i}, and $\hat{A}^i$ and $\hat{A}^{\gamma^i}$ are the representations of $G \wr S_{n_i}$ described in the beginning of this section. Then the representations

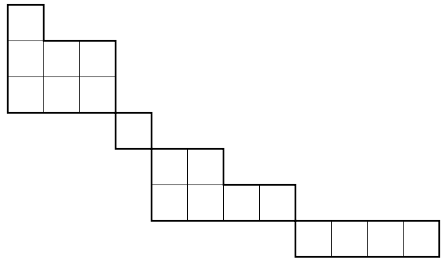
$$\left(\hat{A}^1 \otimes \hat{A}^{\gamma^1}\right) \times \cdots \times \left(\hat{A}^{\mathsf{c}} \otimes \hat{A}^{\gamma^{\mathsf{c}}}\right)\Big\uparrow_{G\wr S_{n_1} \times \cdots \times G\wr S_{n_{\mathsf{c}}}}^{G\wr S_n}$$

as $\vec{\gamma}$ runs over all $\vec{\gamma} \vdash n$ form a complete set of representatives of the irreducible representations of G (up to conjugation).

6 An Analog of the Murnaghan-Nakayama Rule

In this section we provide a combinatorial interpretation for the irreducible characters of $G \wr S_n$. Embedded within this interpretation is the Murnaghan-Nakayama rule for computing the irreducible characters of the symmetric group (consider $G = \{1\}$). This combinatorial interpretation may be used to find the value of the character indexed by $\vec{\gamma}$ on the conjugacy class $C_{\vec{\delta}}$. It also can provide algorithms to find the character table of $G \wr S_n$.

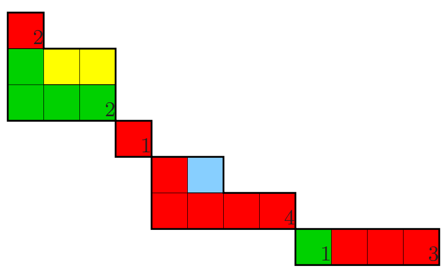
Given $\vec{\gamma}$, let $\gamma^1 \star \cdots \star \gamma^{\mathsf{c}}$ be the skew shape formed by placing the Ferrers diagrams of $\gamma^1, \ldots, \gamma^{\mathsf{c}}$ corner to corner. Let $\star\vec{\gamma}$ be shorthand notation for the resultant shape. For instance, below we have drawn the shape $(1, 3, 3) \star (1) \star (2, 4) \star (4)$.



The shape $\star\vec{\gamma}$ may be thought of as a skew shape μ/λ for some partitions μ, λ for which we have defined the notion of rim hook in Section 3. Note that our definitions ensure that if ρ is a rim hook of $\star\vec{\gamma}$, then ρ is a rim hook in γ^i for some $i = 1, \ldots, \mathsf{c}$. Suppose ρ is a rim hook of length k in γ^i in the shape $\star\vec{\gamma}$ where k is a part of the partition δ^j. We define the weight of ρ as $\mathrm{sgn}(\rho)\chi^i_j$ where $\mathrm{sgn}(\rho)$ is the usual sign of a rim hook; that is, $\mathrm{sgn}(\rho)$ is $(-1)^{r-1}$ where r is the number of rows occupied by ρ. As in the rest of this document, χ^i_j denotes the irreducible character indexed by i on the conjugacy class j of the group G.

The definition of rim hook tableaux may be extended by defining a $\star$-rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$ as a rim hook tableaux of shape $\star\vec{\gamma}$ where the lengths of the rim hooks are found in the parts of the partitions in $\vec{\delta}$. Define the weight of a $\star$-rim hook tableau T of shape $\vec{\gamma}$ and type $\vec{\delta}$ as the product of the weights of the rim hooks in T.

As an example, suppose G has 4 conjugacy classes. Then $\vec{\gamma} = ((1,3,3),(1),(2,4),(4))$ and $\vec{\delta} = ((1,4),(1,1,3,5),(2),(1))$ are both indices of conjugacy classes of $G \wr S_{18}$. The rim hooks in a $\star$-rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$ must have lengths found in $\vec{\delta}$. A $\star$-rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$ is found below.



The colors in the rim hook tabloid depicted above correspond to the colors in the parts of $\vec{\delta} = ((1,4),(1,1,3,5),(2),(1))$ and the numbers in some of the rim hooks indicate the order in which the rim hooks were placed into the tabloid among other rim hooks of the same color. The order of placement of the rim hooks in $\star\vec{\gamma}$ is done in the order of the partitions in $\vec{\delta}$. That is, we first place the rim hooks corresponding to the parts of δ^1, then we place the rim hooks corresponding to δ^2, etc. In this example, the weight of the $\star$-rim hook tableau is

$$\chi^1_2\chi^1_3(-\chi^1_1)\chi^2_2\chi^3_4(-\chi^3_2)\chi^4_1\chi^4_2.$$

Define $\star\chi^{\vec{\gamma}}_{\vec{\delta}}$ to be equal to $\sum \star w(T)$ where the sum runs over all $\star$-rim hook tableaux of shape $\vec{\gamma}$ and type $\vec{\delta}$. Throughout this document, every incarnation of the symbol "χ" has been used to denote the character of an irreducible representation (or a generalization thereof as in $\chi_\nu^{\mu/\lambda}$). At no time has the symbol "χ" been used for any other purpose. Our current definition is no exception as shown by the following theorem, the aforementioned analog to the Murnaghan-Nakayama rule.

Theorem 15. *The value of the irreducible character indexed by $\vec{\gamma}$ on the conjugacy class $C_{\vec{\delta}}$ is equal to $\star\chi_{\vec{\delta}}^{\vec{\gamma}}$. In other words, $\chi_{\vec{\delta}}^{\vec{\gamma}} = \star\chi_{\vec{\delta}}^{\vec{\gamma}}$.*

Proof. Let T be a $\star$-rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$. Let $\eta^{j,i}(T)$ be the partition formed by the lengths of the rim hooks of δ^j placed in the Ferrers diagram of γ^i in T. For example, in the $\star$-rim hook tableau T displayed on page 20, $\eta^{3,1}(T) = (2)$ and $\eta^{2,3}(T) = (5)$. Notice that

$$\star w(T) = \operatorname{sgn}(T) \prod_{i,j}^{\mathsf{c}} \left(\chi_j^i\right)^{\ell(\eta^{j,i}(T))}$$

where $\operatorname{sgn}(T)$ is the product of the signs of the rim hooks in T as usual. Suppose $\delta^j = (1^{d_{1,j}}, \ldots, n^{d_{n,j}})$ and $\eta^{j,i} = (1^{m_{1,j,i}}, \ldots, n^{m_{n,j,i}})$. Then we know that for every $k = 1, \ldots, n$,

$$d_{k,j} = m_{k,j,1} + \cdots + m_{k,j,\mathsf{c}}.$$

The number of $\star$-rim hook tableau T' of shape $\vec{\gamma}$ and type $\vec{\delta}$ with $\eta^{j,i}(T') = \eta^{j,i}(T)$ for all j, i is equal to

$$\begin{aligned}
\prod_{j=1}^{\mathsf{c}} \prod_{k=1}^{n} \binom{d_{k,j}}{m_{k,j,1}, \ldots, m_{k,j,\mathsf{c}}} &= \prod_{j=1}^{\mathsf{c}} \prod_{k=1}^{n} \frac{d_{k,j}!}{m_{k,j,1}! \cdots m_{k,j,\mathsf{c}}!} \\
&= \prod_{j=1}^{\mathsf{c}} \prod_{k=1}^{n} \frac{k^{d_{k,j}} d_{k,j}!}{k^{m_{k,j,1}} m_{k,j,1}! \cdots k^{m_{k,j,\mathsf{c}}} m_{k,j,\mathsf{c}}!} \\
&= \prod_{j=1}^{\mathsf{c}} \frac{z_{\delta^j}}{z_{\eta^{j,1}} \cdots z_{\eta^{j,\mathsf{c}}}}.
\end{aligned}$$

By definition, $\star\chi_{\vec{\delta}}^{\vec{\gamma}} = \sum \star w(T)$ where the sum runs over all possible $\star$-rim hook tableaux of shape $\vec{\gamma}$ and type $\vec{\delta}$. Counting this sum by partitions of the form $\eta^{j,i}$, we have that

$$\star\chi_{\vec{\delta}}^{\vec{\gamma}} = \sum_{\eta^{j,i}} \prod_{i=1}^{\mathsf{c}} \sum_{\substack{\varnothing = \beta^{0,i} \subseteq \cdots \subseteq \beta^{\mathsf{c},i} = \gamma^i \\ |\beta^{j,i}/\beta^{j-1,i}| = |\eta^{j,i}|}} \prod_{j=1}^{\mathsf{c}} \frac{z_{\delta^j}}{z_{\eta^{j,1}} \cdots z_{\eta^{j,\mathsf{c}}}} \chi_{\eta^{j,i}}^{\beta^{j,i}/\beta^{j-1,i}} \left(\chi_j^i\right)^{\ell(\eta^{j,i})} \tag{17}$$

where the first sum runs over all possible $\eta^{j,i}$ such that for all j, i, we have that both $|\eta^{1,i}| + \cdots + |\eta^{\mathsf{c},i}| = |\gamma^i|$ and $\eta^{j,1} + \cdots + \eta^{j,\mathsf{c}} = \delta^j$. The condition that $\eta^{j,1} + \cdots + \eta^{j,\mathsf{c}} = \delta^j$ may be eliminated from the sum as well as terms of the form z_{δ^j} if we multiply each term by $p_{\eta^{j,i}}[X^j]$ and then take the coefficient of $\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}$. That is, if we use the notation $\cdot\big|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}}$ to denote the coefficient of $\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}$ in $\cdot$, then (17) is equal to

$$
\begin{aligned}
&\sum_{\eta^{j,i}} \prod_{i=1}^{\mathsf{c}} \sum_{\substack{\varnothing=\beta^{0,i}\subseteq\cdots\subseteq\beta^{\mathsf{c},i}=\gamma^i \\ |\beta^{j,i}/\beta^{j-1,i}|=|\eta^{j,i}|}} \prod_{j=1}^{\mathsf{c}} \frac{\chi_{\eta^{j,i}}^{\beta^{j,i}/\beta^{j-1,i}}}{z_{\eta^{j,1}}\cdots z_{\eta^{j,\mathsf{c}}}} \left(\chi_j^i\right)^{\ell(\eta^{j,i})} p_{\eta^{j,i}}[X^j]\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}} \\
&\qquad = \sum_{\eta^{j,i}} \prod_{i=1}^{\mathsf{c}} \sum_{\substack{\varnothing=\beta^{0,i}\subseteq\cdots\subseteq\beta^{\mathsf{c},i}=\gamma^i \\ |\beta^{j,i}/\beta^{j-1,i}|=|\eta^{j,i}|}} \prod_{j=1}^{\mathsf{c}} \frac{\chi_{\eta^{j,i}}^{\beta^{j,i}/\beta^{j-1,i}}}{z_{\eta^{j,1}}\cdots z_{\eta^{j,\mathsf{c}}}} p_{\eta^{j,i}}[\chi_j^i X^j]\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}} \\
&\qquad = \prod_{i=1}^{\mathsf{c}} \sum_{\varnothing=\beta^{0,i}\subseteq\cdots\subseteq\beta^{\mathsf{c},i}=\gamma^i} \prod_{j=1}^{\mathsf{c}} \sum_{\eta^{j,i}\vdash|\beta^{j,i}/\beta^{j-1,i}|} \frac{\chi_{\eta^{j,i}}^{\beta^{j,i}/\beta^{j-1,i}}}{z_{\eta^{j,i}}} p_{\eta^{j,i}}[\chi_j^i X^j]\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}}.
\end{aligned}
$$

At this point, we may use (1) along with an iterated application of (2) in Theorem 4 to write the above expression in terms of Schur functions. In doing so, the above equalities are equal to

$$
\prod_{i=1}^{\mathsf{c}} \sum_{\varnothing=\beta^{0,i}\subseteq\cdots\subseteq\beta^{\mathsf{c},i}=\gamma^i} \prod_{j=1}^{\mathsf{c}} s_{\beta^{j,i}/\beta^{j-1,i}}\left[\chi_j^i X^j\right]\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}} = \prod_{i=1}^{\mathsf{c}} s_{\gamma^i}\left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi_j^i X^j\right]\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}}. \tag{18}
$$

Corollary 13 helps complete the proof of this theorem as the right hand side of (18) may be rewritten to look like

$$
F(\chi^{\vec{\gamma}})\big|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}} = \sum_{\vec{\delta}\vdash n} \chi_{\vec{\delta}}^{\vec{\gamma}} \prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}\Bigg|_{\prod_{j=1}^{\mathsf{c}} \frac{p_{\delta^j}[X^j]}{z_{\delta^j}}} = \chi_{\vec{\delta}}^{\vec{\gamma}}.
$$

We have shown that $\star\chi_{\vec{\delta}}^{\vec{\gamma}} = \chi_{\vec{\delta}}^{\vec{\gamma}}$, as desired. □

Theorem 15 is the analog to the Murnaghan-Nakayama rule as it provides a combinatorial interpretation for the characters of the irreducible representations of $G \wr S_n$. We may now use this combinatorial interpretation to produce the following corollary.

Let $h_{(i,j)}^{\star\vec{\gamma}}$ denote 1 plus the number of cells in $\star\vec{\gamma}$ to the right or above cell (i,j) in $\star\vec{\gamma}$. These are known as the hook numbers.

Corollary 16. *The degree of the irreducible representation of $G \wr S_n$ indexed by $\vec{\gamma}$ is equal to*

$$
\frac{n!}{\prod_{i,j} h_{(i,j)}^{\star\vec{\gamma}}} \prod_{k=1}^{\mathsf{c}} \left(f^k\right)^{\ell(\gamma^k)}
$$

where f^k is the dimension of the k^{th} irreducible representation of G.

Proof. We are concerned about the dimension of the representation corresponding to $\vec{\gamma}$. This number is given by the character of this representation on the conjugacy class $((1^n), \varnothing, \ldots, \varnothing)$ because this is the conjugacy class which contains the identity element in $G \wr S_n$. According to our combinatorial interpretation of the characters of $G \wr S_n$, we only need to count the number of ways to fill $\star\vec{\gamma}$ with rim hooks of length 1 and then multiply by the factor $\prod_{k=1}^{\mathsf{c}} \left(f^k\right)^{\ell(\gamma^k)}$ in order to find the dimension of the representation.

Frame, Robinson, and Thrall proved that $f^\lambda = \frac{n!}{\prod_{i,j} h^\lambda_{(i,j)}}$ where f^λ is the dimension of the irreducible character of S_n indexed by the partition λ [4]. Therefore, if $|\gamma^i| = n_i$, we have

$$\begin{aligned}
\chi^{\vec{\gamma}}_{((1^n),\varnothing,\ldots,\varnothing)} &= \binom{n}{n_1, \ldots, n_{\mathsf{c}}} f^{\gamma^1} \cdots f^{\gamma^{\mathsf{c}}} \prod_{k=1}^{\mathsf{c}} \left(f^k\right)^{\ell(\gamma^k)} \\
&= \frac{n!}{n_1! \cdots n_{\mathsf{c}}!} \prod_{l=1}^{\mathsf{c}} \frac{n_i!}{\prod_{i,j} h^{\gamma^l}_{(i,j)}} \prod_{k=1}^{\mathsf{c}} \left(f^k\right)^{\ell(\gamma^k)} \\
&= \frac{n!}{\prod_{i,j} h^{\star\vec{\gamma}}_{(i,j)}} \prod_{k=1}^{\mathsf{c}} \left(f^k\right)^{\ell(\gamma^k)}.
\end{aligned}$$

□

We conclude this section with an example of one of the character tables which can be easily computed using Theorem 15. Let us enumerate the conjugacy classes of the alternating group on five letters so that the character table reads like that below. For convenience, let $\mathrm{g} = \frac{1+\sqrt{5}}{2}$ and $\mathrm{g}' = \frac{1-\sqrt{5}}{2}$.

	C1	C2	C3	C4	C5
X1	1	1	1	1	1
X2	4	1	0	−1	−1
X3	5	−1	1	0	0
X4	3	0	−1	g	g′
X5	3	0	−1	g′	g

The character table for $\mathcal{A}_5 \wr S_2$ will be found. The vector partitions of 2 with 5 parts indexing the conjugacy classes of the group are listed along with the sizes of the conjugacy classes themselves. The conjugacy classes corresponding to the first two vector partitions have been flipped in order to list the dimension of the irreducible characters in the first column of the character table. Below is the character table of $\mathcal{A}_5 \wr S_2$:

	vector partition		size
X1	((2),∅,∅,∅,∅)	C2	60
X2	((1, 1),∅,∅,∅,∅)	C1	1
X3	((1),(1),∅,∅,∅)	C3	40
X4	(∅,(2),∅,∅,∅)	C4	1200
X5	(∅,(1, 1),∅,∅,∅)	C5	400
X6	((1),∅,(1),∅,∅)	C6	30
X7	(∅,(1),(1),∅,∅)	C7	600
X8	(∅,∅,(2),∅,∅)	C8	900
X9	(∅,∅,(1, 1),∅,∅)	C9	225
X10	((1),∅,∅,(1),∅)	C10	24

	vector partition		size
X11	(∅,(1),∅,(1),∅)	C11	480
X12	(∅,∅,(1),(1),∅)	C12	360
X13	(∅,∅,∅,(2),∅)	C13	720
X14	(∅,∅,∅,(1, 1),∅)	C14	144
X15	((1),∅,∅,∅,(1))	C15	24
X16	(∅,(1),∅,∅,(1))	C16	480
X17	(∅,∅,(1),∅,(1))	C17	360
X18	(∅,∅,∅,(1),(1))	C18	288
X19	(∅,∅,∅,∅,(2))	C19	720
X20	(∅,∅,∅,∅,(1, 1))	C20	144

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15	C16	C17	C18	C19	C20
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	−1	1	−1	1	1	1	−1	1	1	1	1	−1	1	1	1	1	1	−1	1
X3	8	0	5	0	2	4	1	0	0	3	0	−1	0	−2	3	0	−1	−2	0	−2
X4	16	4	4	1	1	0	0	0	0	−4	−1	0	−1	1	−4	−1	0	1	−1	1
X5	16	−4	4	−1	1	0	0	0	0	−4	−1	0	1	1	−4	−1	0	1	1	1
X6	10	0	4	0	−2	6	0	0	2	5	−1	1	0	0	5	−1	1	0	0	0
X7	40	0	1	0	−2	4	1	0	0	−5	1	−1	0	0	−5	1	−1	0	0	0
X8	25	5	−5	−1	1	5	−1	1	1	0	0	0	0	0	0	0	0	0	0	0
X9	25	−5	−5	1	1	5	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
X10	6	0	3	0	0	2	−1	0	−2	3+g	g	g−1	0	2g	3+g′	g′	g′−1	1	0	2g′
X11	24	0	3	0	0	−4	−1	0	0	4g−3	g	1	−g	g^2	4g′−3	g′	1	−1	−g′	g'^2
X12	30	0	−3	0	0	−2	1	0	−2	5g	−g	g	0	0	5g′	−g′	g′	0	0	0
X13	9	3	0	0	0	−3	0	−1	1	3g	0	−g	g	g^2	3g′	0	−g′	−1	g′	g'^2
X14	9	−3	0	0	0	−3	0	1	1	3g	0	−g	−g	g^2	3g′	0	−g′	−1	−g′	g'^2
X15	6	0	3	0	0	2	−1	0	−2	3+g′	g′	g′−1	0	2g′	3+g	g	g−1	1	0	2g
X16	24	0	3	0	0	−4	−1	0	0	4g′−3	g′	1	−g′	g'^2	4g−3	g	1	−1	−g	g^2
X17	30	0	−3	0	0	−2	1	0	−2	5g′	−g′	g′	0	0	5g	−g	g	0	0	0
X18	18	0	0	0	0	−6	0	0	2	3	0	−1	0	−2	3	0	−1	3	0	−2
X19	9	3	0	0	0	−3	0	1	1	3g′	0	−g′	g′	g'^2	3g	0	−g	−1	g	g^2
X20	9	−3	0	0	0	−3	0	1	1	3g′	0	−g′	−g′	g'^2	3g	0	−g	−1	−g	g^2

As an example, note that X19 corresponds to the vector partition $\vec{\gamma} = (\varnothing, \varnothing, \varnothing, \varnothing, (2))$ and C15 corresponds the vector partition $\vec{\delta} = ((1), \varnothing, \varnothing, \varnothing, (1))$. It is easy to check that there is exactly one rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$ and the weight of that rim hook tableau is $\chi_1^5 \chi_5^5 = 3g$. Thus the value of $X19$ at $C15$ is $3g$. The other entries in the table are computed in a similar manner.

7 Kronecker Products

In this section, we prove a generalization of a theorem of Littlewood which allows the Kronecker product of two irreducible representations of $G \wr S_n$ to be decomposed into irreducible components. In addition, we show that our analog of Littlewood's theorem can be used to give a natural extension of the algorithm developed by Garsia and Remmel to compute Kronecker products of representation of $G \wr S_n$ [5]. Specifically, let A^λ, A^μ be irreducible representations of S_n. Define $s_\lambda \otimes s_\mu$ to be the image of $\chi^{A^\lambda \otimes A^\mu}$ under the Frobenius characteristic for the symmetric group. Let $c^\nu_{\lambda,\mu}$ be the Littlewood-Richardson coefficients which give the occurrences of s_ν in $s_\lambda s_\mu$. Littlewood proved the following theorem in [9].

Theorem 17. *Let $\lambda \vdash n$, $\mu \vdash m$, and $\nu \vdash n+m$. Then*

$$s_\lambda s_\mu \otimes s_\nu = \sum_{\substack{\alpha \vdash m \\ \beta \vdash n}} c^\nu_{\alpha,\beta} (s_\lambda \otimes s_\alpha)(s_\mu \otimes s_\beta). \tag{19}$$

Later, Garsia and Remmel showed that Littlewood's theorem could be rewritten in the following way [5].

Theorem 18. *Let $\lambda \vdash n$, $\mu \vdash m$, and $\nu \vdash n+m$. Then*

$$s_\lambda s_\mu \otimes s_\nu = \sum_{\substack{\alpha \vdash m \\ \alpha \subseteq \nu}} (s_\lambda \otimes s_\alpha)(s_\mu \otimes s_{\nu/\alpha}). \tag{20}$$

We can use either (19) or (20) to compute Kronecker products for the symmetric group S_n as follows. Suppose one wants to compute $s_\lambda \otimes s_\nu$. First we can employ the Jacobi-Trudi identity to express s_λ as a signed sum of homogeneous symmetric functions h_μ so that we can reduce the problem of computing $s_\lambda \otimes s_\nu$ to the problem of computing $h_\mu \otimes s_\nu$. Then Theorem 17 may be iterated to prove the following.

Theorem 19. *Let $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ and $\nu \vdash n$. Then*

$$\begin{aligned} h_{\mu_1} \cdots h_{\mu_k} \otimes s_\nu &= \sum_{\alpha_i \vdash |\mu_i|} c^\nu_{\alpha_1,\ldots,\alpha_k} \prod_{i=1}^{k} h_{\mu_i} \otimes s_{\alpha_i} \\ &= \sum_{\alpha_i \vdash |\mu_i|} c^\nu_{\alpha_1,\ldots,\alpha_k} \prod_{i=1}^{k} s_{\alpha_i}. \end{aligned} \tag{21}$$

Here, $c^\nu_{\alpha_1,\ldots,\alpha_k} = \langle s_{\alpha_1} \cdots s_{\alpha_k}, s_\nu \rangle$. The second equality follows since $h_n \otimes s_\lambda = s_\lambda$ for all n and $\lambda \vdash n$. Similarly, one can iterate Theorem 18 to prove the following.

Theorem 20. *Let $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ and $\nu \vdash n$. Then*

$$h_{\mu_1} \cdots h_{\mu_k} \otimes s_\nu = \sum_{\substack{\emptyset = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k = \nu \\ |\alpha_i/\alpha_{i-1}| = |\mu_i|}} s_{\nu/\alpha_k} \cdots s_{\alpha_2/\alpha_1} s_{\alpha_1}. \tag{22}$$

From an algorithmic point of view, the advantage of (22) over (21) is that it usually has far fewer terms. Moreover, for (21), one often has to waste time computing Littlewood-Richardson coefficients $c^\nu_{\alpha_1,\ldots,\alpha_k}$ which turn out to be zero and thus make no contribution to the final sum. One might think that a disadvantage of (22) versus (21) is that we end up having to take a product of skew Schur functions rather the product of ordinary Schur functions. However, versions of the Littlewood-Richardson rule appearing in the literature imply that there is no difference in the complexity of computing the product of skew Schur functions versus computing the product of ordinary Schur functions; see, for example, the version of the Littlewood-Richardson rule due to Remmel and Whitney [14].

We shall show that a similar situation arises in $G \wr S_n$. For this section, let $s_{\vec{\gamma}}$ denote the analog the Schur function in the space $\Lambda_{\mathsf{c},n}$; in other words, $s_{\vec{\gamma}} = \prod_{i=1}^{\mathsf{c}} s_{\gamma^i} \left[\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right]$. In the same vein, let $\overline{s_{\vec{\gamma}}} = \prod_{i=1}^{\mathsf{c}} s_{\gamma^i} \left[\sum_{j=1}^{\mathsf{c}} \overline{\chi^i_j} X^j \right]$. Let $A^{\vec{\gamma}}$ and $A^{\vec{\delta}}$ be irreducible representations of $G \wr S_n$ and let us denote $F(\chi^{A^{\vec{\gamma}} \otimes A^{\vec{\delta}}})$ by $s_{\vec{\gamma}} \otimes s_{\vec{\delta}}$ and let $F(\chi^{\overline{A^{\vec{\nu}}}}) = \overline{s_{\vec{\nu}}}$ (the former will be known as the Kronecker product of Schur functions over $G \wr S_n$).

If $\vec{\xi} = (\xi^1, \ldots, \xi^{\mathsf{c}})$ and $\vec{\nu} = (\nu^1, \ldots, \nu^{\mathsf{c}})$, then we say $\vec{\xi} \subseteq \vec{\nu}$ if $\xi^i \subseteq \nu^i$ for $i = 1, \ldots, \mathsf{c}$. If $\vec{\xi} \subseteq \vec{\nu}$, then we let $\vec{\nu}/\vec{\xi} = (\nu^1/\xi^1, \ldots, \nu^{\mathsf{c}}/\xi^{\mathsf{c}})$. These are generalizations of the notions of $\xi \subseteq \nu$ and ν/ξ for partitions as defined in Section 3. Then we have the following generalization of Theorem 17.

Theorem 21. *Let $\vec{\gamma} \vdash n$, $\vec{\delta} \vdash m$ and $\vec{\nu} \vdash n+m$. Then*

$$s_{\vec{\gamma}} s_{\vec{\delta}} \otimes \overline{s_{\vec{\nu}}} = \sum_{\substack{\vec{\xi} \vdash n \\ \vec{\eta} \vdash m}} \prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i} \left(s_{\vec{\gamma}} \otimes s_{\vec{\xi}} \right) \left(s_{\vec{\delta}} \otimes s_{\vec{\eta}} \right)$$

where the sum runs over all $\vec{\xi} = (\xi^1, \ldots, \xi^c)$ and $\vec{\eta} = (\eta^1, \ldots, \eta^c)$ such that $|\xi^i| + |\eta^i| = |\nu^i|$ for all $i = 1, \ldots, c$.

Similarly, we have the following analog of Theorem 18.

Theorem 22. *Let $\vec{\gamma} \vdash n$, $\vec{\delta} \vdash m$ and $\vec{\nu} \vdash n+m$. Then*

$$s_{\vec{\gamma}} s_{\vec{\delta}} \otimes \overline{s_{\vec{\nu}}} = \sum_{\substack{\vec{\xi} \vdash n \\ \vec{\xi} \subseteq \vec{\nu}}} \left(s_{\vec{\gamma}} \otimes s_{\vec{\xi}} \right) \left(s_{\vec{\delta}} \otimes s_{\vec{\nu}/\vec{\xi}} \right).$$

Proof. A special case of Lemma 12 shows that

$$F\left(\chi^{A^{\vec{\gamma}} \otimes A^{\vec{\delta}} \uparrow^{G \wr S_{n+m}}_{G \wr S_n \times G \wr S_m}} \right) = F(\chi^{A^{\vec{\gamma}}}) F(\chi^{A^{\vec{\delta}}}).$$

According to Corollary 13, this is equal to $s_{\vec{\gamma}} s_{\vec{\delta}}$. Within the proof of Lemma 12, a class function ψ_{n+m} was defined on $G \wr S_{n+m}$ such that if $\sigma \in C_{\vec{\lambda}}$, then

$$\psi_{n+m}(\sigma) = \prod_{i=1}^{\mathsf{c}} p_{\lambda^i}[X^i] \left(\frac{|G|}{|C_i|} \right)^{\ell(\lambda^i)}.$$

With this definition, if $\alpha \in G \wr S_n$ and $\beta \in G \wr S_m$, then (α, β) is an element in $G \wr S_n \times G \wr S_m$ such that $\psi_{n+m}(\alpha, \beta) = \psi_n(\alpha)\psi_m(\beta)$. This fact follows from the multiplicative property of the power basis.

The Littlewood-Richardson coefficients for the analogs of Schur symmetric functions can be found:

$$\begin{aligned} s_{\vec{\gamma}} s_{\vec{\delta}} &= \prod_{i=1}^{\mathsf{c}} s_{\gamma^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right] s_{\delta^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right] \\ &= \prod_{i=1}^{\mathsf{c}} \sum_{\nu^i \vdash |\gamma^i| + |\delta^i|} c^{\nu^i}_{\gamma^i,\delta^i} s_{\nu^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right] \\ &= \sum_{\vec{\nu} \vdash n+m} \prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\gamma^i,\delta^i} s_{\nu^i} \left[\textstyle\sum_{j=1}^{\mathsf{c}} \chi^i_j X^j \right]. \end{aligned}$$

Due to Frobenius' reciprocity, we have that

$$\begin{aligned}
\left\langle \chi^{\overline{A^{\vec{\nu}}}\downarrow^{G\wr S_{n+m}}_{G\wr S_n\times G\wr S_m}}, \chi^{A^{\vec{\gamma}}\otimes A^{\vec{\delta}}}\right\rangle_{G\wr S_n\times G\wr S_m} &= \left\langle \chi^{\overline{A^{\vec{\nu}}}}, \chi^{A^{\vec{\gamma}}\otimes A^{\vec{\delta}}\uparrow^{G\wr S_{n+m}}_{G\wr S_n\times G\wr S_m}}\right\rangle_{G\wr S_{n+m}} \\
&= \left\langle \overline{s_{\vec{\nu}}}, s_{\vec{\gamma}}s_{\vec{\delta}}\right\rangle_{\Lambda_{\mathsf{c},n+m}} \\
&= \left\langle \overline{s_{\vec{\nu}}}, \sum_{\vec{\alpha}\vdash n}\prod_{i=1}^{\mathsf{c}} c^{\alpha^i}_{\gamma^i,\delta^i}s_{\alpha^i}\right\rangle_{\Lambda_{\mathsf{c},n+m}} \\
&= \prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\gamma^i,\delta^i}.
\end{aligned}$$

This shows that

$$\chi^{\overline{A^{\vec{\nu}}}\downarrow^{G\wr S_{n+m}}_{G\wr S_n\times G\wr S_m}} = \sum_{\substack{\vec{\gamma}\vdash n\\ \vec{\delta}\vdash m}}\prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\gamma^i,\delta^i}\chi^{A^{\vec{\gamma}}}\chi^{A^{\vec{\delta}}}. \tag{23}$$

Therefore,

$$\begin{aligned}
s_{\vec{\gamma}}s_{\vec{\delta}}\otimes\overline{s_{\vec{\nu}}} &= F\left(\chi^{A^{\vec{\gamma}}\otimes A^{\vec{\delta}}\uparrow^{G\wr S_{n+m}}_{G\wr S_n\times G\wr S_m}\otimes\overline{A^{\vec{\nu}}}}\right) \\
&= \frac{1}{|G\wr S_{n+m}|}\sum_{\sigma\in G\wr S_{n+m}}\chi^{A^{\vec{\gamma}}\otimes A^{\vec{\delta}}\uparrow^{G\wr S_{n+m}}_{G\wr S_n\times G\wr S_m}}(\sigma)\chi^{\overline{A^{\vec{\nu}}}}(\sigma)\psi_{n+m}(\sigma) \\
&= \frac{1}{|G\wr S_{n+m}|}\sum_{\sigma\in G\wr S_{n+m}}\frac{1}{|G\wr S_n||G\wr S_m|}\sum_{\tau\in G\wr S_{n+m}}\chi^{A^{\vec{\gamma}}\otimes A^{\vec{\delta}}}(\tau\sigma\tau^{-1})\chi^{\overline{A^{\vec{\nu}}}}(\sigma)\psi_{n+m}(\sigma) \\
&= \frac{1}{|G\wr S_n||G\wr S_m|}\sum_{\substack{\alpha\in G\wr S_n\\ \beta\in G\wr S_m}}\chi^{A^{\vec{\gamma}}}(\alpha)\chi^{A^{\vec{\delta}}}(\beta)\chi^{\overline{A^{\vec{\nu}}}}(\alpha,\beta)\psi_{n+m}(\alpha,\beta) \\
&= \frac{1}{|G\wr S_n||G\wr S_m|}\sum_{\substack{\alpha\in G\wr S_n\\ \beta\in G\wr S_m}}\chi^{A^{\vec{\gamma}}}(\alpha)\chi^{A^{\vec{\delta}}}(\beta)\sum_{\substack{\vec{\xi}\vdash n\\ \vec{\eta}\vdash m}}\prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i}\chi^{A^{\vec{\xi}}}(\alpha)\chi^{A^{\vec{\eta}}}(\beta)\psi_n(\alpha)\psi_m(\beta)
\end{aligned}$$

where this last equality comes from (23) and steps similar to those found in the proof of Lemma 12. Finally, we have that the above equation is equal to

$$\sum_{\substack{\vec{\xi}\vdash n\\ \vec{\eta}\vdash m}}\prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i}\left(\sum_{\alpha\in G\wr S_n}\frac{\chi^{A^{\vec{\gamma}}}(\alpha)\chi^{A^{\vec{\xi}}}(\alpha)\psi_n(\alpha)}{|G\wr S_n|}\right)\left(\sum_{\beta\in G\wr S_m}\frac{\chi^{A^{\vec{\delta}}}(\beta)\chi^{A^{\vec{\eta}}}(\beta)\psi_m(\beta)}{|G\wr S_m|}\right)$$

which may be written as

$$\sum_{\substack{\vec{\xi}\vdash n\\ \vec{\eta}\vdash m}}\prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i}\left(s_{\vec{\gamma}}\otimes s_{\vec{\xi}}\right)\left(s_{\vec{\delta}}\otimes s_{\vec{\eta}}\right). \tag{24}$$

This proves Theorem 21. To prove Theorem 22, one need only to observe that

$$\sum_{\substack{\vec{\xi}\vdash n\\ \vec{\eta}\vdash m}} \prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i} \left(s_{\vec{\gamma}} \otimes s_{\vec{\xi}}\right)\left(s_{\vec{\delta}} \otimes s_{\vec{\eta}}\right) = \sum_{\substack{\vec{\xi}\vdash n\\ \vec{\xi}\subseteq\vec{\nu}}} \left(s_{\vec{\gamma}} \otimes s_{\vec{\xi}}\right)\left(s_{\vec{\delta}} \otimes \sum_{\vec{\eta}\vdash m} \prod_{i=1}^{\mathsf{c}} c^{\nu^i}_{\xi^i,\eta^i} s_{\vec{\eta}}\right)$$
$$= \sum_{\substack{\vec{\xi}\vdash n\\ \vec{\xi}\subseteq\vec{\nu}}} \left(s_{\vec{\gamma}} \otimes s_{\vec{\xi}}\right)\left(s_{\vec{\delta}} \otimes s_{\vec{\nu}/\vec{\xi}}\right). \qquad \square$$

Retracing the proof of the above theorem and examining where the conjugation of $s_{\vec{\nu}}$ took effect, Theorem 22 may be restated to read that for $\vec{\gamma}\vdash n$, $\vec{\delta}\vdash m$ and $\vec{\nu}\vdash n+m$,

$$s_{\vec{\gamma}} s_{\vec{\delta}} \otimes s_{\vec{\nu}} = \sum_{\substack{\vec{\xi}\vdash n\\ \vec{\xi}\subseteq\vec{\nu}}} \left(s_{\vec{\gamma}} \otimes \overline{s_{\vec{\xi}}}\right)\left(s_{\vec{\delta}} \otimes \overline{s_{\vec{\nu}/\vec{\xi}}}\right). \tag{25}$$

The proof of the following corollary exploits the linearity of the Kronecker product and uses a simple iterated application of Theorem 22 and its above restatement.

Corollary 23. *Let $\vec{\gamma}^1,\ldots,\vec{\gamma}^k$ be vector partitions of $n_1,\ldots,n_k$, respectively and let $\vec{\nu}$ be a vector partition of $n_1+\cdots+n_k$. Then,*

$$s_{\vec{\gamma}^1}\cdots s_{\vec{\gamma}^k} \otimes s_{\vec{\nu}} = \sum \left(s_{\vec{\gamma}^k} \otimes \overline{s_{\vec{\nu}/\vec{\xi}^{k-1}}}\right)\left(s_{\vec{\gamma}^{k-1}} \otimes s_{\vec{\xi}^{k-1}/\vec{\xi}^{k-2}}\right)\cdots\left(s_{\vec{\gamma}^1} \otimes \overline{s_{\vec{\xi}^1}}\right)$$

where the sum runs over all vector partitions $\vec{\xi}^1\vdash n_1, \vec{\xi}^2\vdash n_1+n_2,\ldots,\vec{\xi}^k\vdash n_1+\cdots+n_k$ where $\vec{\xi}^1\subseteq\cdots\subseteq\vec{\xi}^k\subseteq\vec{\nu}$. Conjugation of the skew Schur functions alternates except for the final two terms which are either both conjugated or both not conjugated.

Corollary 23 above reduces the problem of taking the Kronecker product of a product of Schur functions to finding the Kronecker product of two Schur functions. By definition, we have that for any $\vec{\gamma}$, $s_{(\gamma^1,\ldots,\gamma^k)} = s_{(\gamma_1,\varnothing,\ldots,\varnothing)}\cdots s_{(\varnothing,\ldots,\varnothing,\gamma^k)}$. The Jacobi-Trudy identity may be applied to change functions of the form $s_{(\varnothing,\ldots,\gamma^i,\ldots,\varnothing)}$ into a sum of products of Schur functions of the form $s_{(\varnothing,\ldots,(l),\ldots,\varnothing)}$. Applying Corollary 23 over this sum of Schur functions, the problem of finding $s_{\vec{\gamma}}\otimes s_{\vec{\delta}}$ boils down to finding the Kronecker product of two Schur functions of the form $s_{(\varnothing,\ldots,(l),\ldots,\varnothing)}$. That is, the situation is reduced to finding

$$s_{(\varnothing,\ldots,(l),\ldots,\varnothing)} \otimes s_{(\varnothing,\ldots,(k),\ldots,\varnothing)} = F\left(\chi^{A^{(\varnothing,\ldots,(l),\ldots,\varnothing)}\otimes A^{(\varnothing,\ldots,(k),\ldots,\varnothing)}}\right).$$

The description of the irreducible characters given in Section 5 may be used to see that the representation $A^{(\varnothing,\ldots,(l),\ldots,\varnothing)}$ where (l) appears in the i^{th} place is equal to $\hat{A}^i\otimes\hat{A}^{(l)}$; however, $\hat{A}^{(l)}$ is the trivial representation, so $A^{(\varnothing,\ldots,(l),\ldots,\varnothing)}$ is actually the i^{th} irreducible representation of G. Therefore, the problem of breaking the Kronecker product of irreducible representations of $G\wr S_n$ into its irreducible components is reduced to finding the Kronecker product of irreducible representations in G.

For some groups, breaking the Kronecker products of irreducible representations into irreducible components is well known. For instance, all characters of finite abelian groups are linear, making Kronecker products simply point-wise multiplication. For these groups, the methods outlined above provide a way to break apart Kronecker products in $G \wr S_n$.

Let us turn our attention to groups of the form $\mathbb{Z}_k \wr S_n$ since the irreducible characters of this group are linear—making Kronecker products in this group easy to find. We will explicitly calculate an example of how to decompose Kronecker products for the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$. Recall that the character table of $\mathbb{Z}_k$ is $\|\varepsilon^{(i-1)(j-1)}\|_{i,j=1,\ldots,k}$ where $\varepsilon = e^{2\pi i/k}$ is a primitive k^{th} root of unity.

Theorem 24. *For the group $\mathbb{Z}_k \wr S_n$,*

$$\chi^{(\gamma^1,\ldots,\gamma^k)} = \chi^{(\varnothing,(n),\varnothing,\ldots,\varnothing)}\chi^{(\gamma^2,\ldots,\gamma^k,\gamma^1)}.$$

Proof. We give a bijective, combinatorial proof using the notion of $\star$-rim hook tableaux developed in Section 6. Let $\{\chi^1,\ldots,\chi^k\}$ be the irreducible characters of the group $\mathbb{Z}_k = \{1,\sigma,\ldots,\sigma^{k-1}\}$ such that $\chi^j(\sigma^s) = \varepsilon^{s(j-1)}$ where $\varepsilon = e^{2\pi i/k}$ is a primitive k^{th} root of unity. This way, since the conjucagy classes of $\mathbb{Z}_k$ are of the form $\{\sigma^i\}$, the value χ^i_j is equal to $\varepsilon^{(i-1)(j-1)}$ and thus the character table of $\mathbb{Z}_k$ is $\|\varepsilon^{(i-1)(j-1)}\|_{i,j=1,\ldots,k}$.

Let T be a $\star$-rim hook tableau of shape $\vec{\gamma}$ and type $\vec{\delta}$. Let us form a $\star$-rim hook tableau T' of shape $(\gamma^2,\ldots,\gamma^k,\gamma^1)$ by cyclically rotating the Ferrers diagrams of $\gamma^1,\ldots,\gamma^k$. This process is 1-1. We will track how the parts in δ^j affect the weight of T' with respect to the weight of T.

Say there are j_{i-1} parts of δ^j found in γ^i. This means $j_0+\cdots+j_{k-1} = \ell(\delta^j)$. The contributing factor of δ^j in $w(T)$ is

$$\left(\varepsilon^{0(j-1)}\right)^{j_0}\cdots\left(\varepsilon^{(k-2)(j-1)}\right)^{j_{k-2}}\left(\varepsilon^{(k-1)(j-1)}\right)^{j_{k-1}}$$

while the contributing factor of δ^j in $w(T')$ is

$$\left(\varepsilon^{0(j-1)}\right)^{j_1}\cdots\left(\varepsilon^{(k-2)(j-1)}\right)^{j_{k-1}}\left(\varepsilon^{(k-1)(j-1)}\right)^{j_0}.$$

Thus, by this cyclic rotation of T into T', a factor of

$$\left(\varepsilon^{0(j-1)}\right)^{j_0-j_1}\left(\varepsilon^{1(j-1)}\right)^{j_1-j_2}\cdots\left(\varepsilon^{(k-2)(j-1)}\right)^{j_{k-2}-j_{k-1}}\left(\varepsilon^{(k-1)(j-1)}\right)^{j_{k-1}-j_0} \tag{26}$$

arises. Using the fact that ε is a primitive k^{th} root of unity, the factor in (26) may be rewritten to look like $\varepsilon^{(j-1)(j_0+\cdots+j_{k-1})}$ which is equal to $\varepsilon^{(j-1)\ell(\delta^j)}$.

This means the change in weight when rotating T to create T' can be found. Since the rim hooks in T are not changing, the only possible difference in $w(T)$ and $w(T')$ can come from the weights of powers of ε from the parts found in $\delta^1,\ldots,\delta^k$. It follows that $w(T) = \varepsilon^{0\ell(\delta^1)}\varepsilon^{1\ell(\delta^2)}\cdots\varepsilon^{(k-1)\ell(\delta^k)}w(T')$. This coincides with the combinatorial interpretation of the character corresponding to the vector partition $(\varnothing,(n),\varnothing,\ldots,\varnothing)$ because

$$\chi^{(\varnothing,(n),\varnothing,\ldots,\varnothing)}_{\vec{\delta}} = \left(\varepsilon^0\right)^{\ell(\delta^1)}\left(\varepsilon^1\right)^{\ell(\delta^2)}\cdots\left(\varepsilon^{(k-1)}\right)^{\ell(\delta^k)}.$$

Therefore,

$$\begin{aligned}
\chi_{\vec{\delta}}^{(\gamma^1,\ldots,\gamma^{k-1},\gamma^k)} &= \sum_{\substack{T \text{ of shape } (\gamma^1,\ldots,\gamma^k) \\ \text{and type } \vec{\delta}}} w(T) \\
&= \sum_{\substack{T' \text{ of shape } (\gamma^2,\ldots,\gamma^k,\gamma^1) \\ \text{and type } \vec{\delta}}} \chi_{\vec{\delta}}^{(\varnothing,(n),\varnothing,\ldots,\varnothing)} w(T') \\
&= \chi_{\vec{\delta}}^{(\varnothing,(n),\varnothing,\ldots,\varnothing)} \chi_{\vec{\delta}}^{(\gamma^2,\ldots,\gamma^k,\gamma^1)}.
\end{aligned}$$

□

Corollary 25. *For the group $\mathbb{Z}_k \wr S_n$,*

$$\chi^{(\gamma^1,\ldots,\gamma^k)} = \chi^{(\varnothing,\ldots,(n),\ldots,\varnothing)} \chi^{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})}$$

where the part (n) in $\chi^{(\varnothing,\ldots,(n),\ldots,\varnothing)}$ appears in the i^{th} component and $(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})$ is a cyclic rotation of the partitions in $(\gamma^1,\ldots,\gamma^k)$.

Proof. According to Theorem 24,

$$\chi^{(\varnothing,(n),\varnothing,\ldots,\varnothing)} \chi^{(\varnothing,\ldots,(n)_j,\ldots,\varnothing)} = \chi^{(\varnothing,\ldots,(n)_{j+1},\ldots,\varnothing)} \tag{27}$$

where $(n)_j$ means that the partition (n) appears in the j^{th} component. Therefore,

$$\chi^{(\gamma^1,\ldots,\gamma^k)} = \chi^{(\varnothing,(n),\varnothing,\ldots,\varnothing)} \cdots \chi^{(\varnothing,(n),\varnothing,\ldots,\varnothing)} \chi^{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})} = \chi^{(\varnothing,\ldots,(n)_i,\ldots,\varnothing)} \chi^{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})}$$

via i applications of (27). □

Corollary 26. *For the group $\mathbb{Z}_k \wr S_n$,*

$$s_{(\varnothing,\ldots,(n),\ldots,\varnothing)} \otimes s_{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})} = s_{(\gamma^1,\ldots,\gamma^k)}$$

where the part (n) in the vector partition $(\varnothing,\ldots,(n),\ldots,\varnothing)$ appears in the i^{th} component and $(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})$ is a cyclic rotation of the partitions in $(\gamma^1,\ldots,\gamma^k)$.

Proof. Corollary 16 in Section 6 shows $A^{(\varnothing,\ldots,(n),\ldots,\varnothing)}$ is one dimensional where $A^{\vec{\gamma}}$ is the irreducible representation of $\mathbb{Z}_k \wr S_n$ associated with $\vec{\gamma}$. Corollary 25 gives that

$$\begin{aligned}
s_{(\varnothing,\ldots,(n),\ldots,\varnothing)} \otimes s_{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})} &= F\left(\chi^{A^{(\varnothing,\ldots,(n),\ldots,\varnothing)} \otimes A^{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})}}\right) \\
&= F\left(\chi^{(\varnothing,\ldots,(n),\ldots,\varnothing)} \chi^{(\gamma^i,\gamma^{i+1},\ldots,\gamma^{i-1})}\right) \\
&= F\left(\chi^{(\gamma^1,\ldots,\gamma^k)}\right) \\
&= s_{(\gamma^1,\ldots,\gamma^k)}.
\end{aligned}$$

□

It may be noted that Theorem 24, Corollary 25, and Corollary 26 all may be generalized to arbitrary finite abelian groups. After all, any finite abelian group is a product of groups of the form $\mathbb{Z}_k$.

We conclude this paper with an example of how to break Kronecker products into irreducible components in the case of the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$. The Schur analog in $\mathbb{Z}_2 \wr S_n$ is $\{s_{\gamma^1}[X^1 + X^2] s_{\gamma^2}[X^1 - X^2]\}$ where $|\gamma^1| + |\gamma^2| = n$. These characters are invariant under complex conjugation, providing the self-duality of this basis for $\Lambda_{n,2}$. Also due to this observation, the statement of Theorem 22 and (25) are the same. This means that there is no conjugation to keep track of when using this theorem or associated equation.

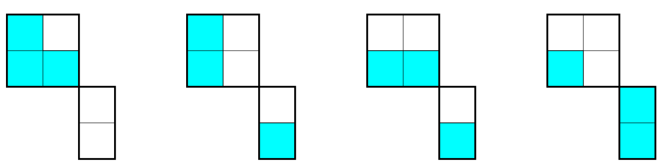
Suppose we are interested in decomposing $A^{((3),(3))} \otimes A^{((2^2),(1^2))}$ into irreducible components. Equation (25) may be employed to the Frobenius image of the character of this representation which is $s_{((3),(3))} \otimes s_{((2^2),(1^2))}$. The strategy is to break $s_{((3),(3))}$ into a product of the two Schur functions $s_{((3),\varnothing)}$ and $s_{(\varnothing,(3))}$ and then use Corollary 26. Throughout our calculations, the reader is assumed to know a method to find the coefficient of s_ν in the product $s_\alpha s_\beta$ (this is the Littlewood-Richardson coefficient $c^{\nu}_{\alpha,\beta}$).

We have according to Theorem 22 that

$$\begin{aligned}
s_{((3),(3))} \otimes s_{((2^2),(1^2))} &= s_{((3),\varnothing)} s_{(\varnothing,(3))} \otimes s_{((2^2),(1^2))} \\
&= \sum_{\substack{\vec{\xi} \vdash 3 \\ \vec{\xi} \subseteq ((2^2),(1^2))}} \left(s_{((3),\varnothing)} \otimes s_{\vec{\xi}} \right) \left(s_{(\varnothing,(3))} \otimes s_{((2^2),(1^2))/\vec{\xi}} \right) \\
&= \sum_{\substack{|\xi^1|+|\xi^2|=3 \\ \xi^1 \subseteq (2^2), \xi^2 \subseteq (1^2)}} s_{(\xi^1,\xi^2)} s_{((1^2)/\xi^2,(2^2)/\xi^1)}. \qquad (28)
\end{aligned}$$

The Kronecker products in the second line of the above equation were simplified using Corollary 26. In general, for partitions other than (n) or (1^n), the Jacobi-Trudy identity can be employed so that the resultant Kronecker products are in terms of partitions with corresponding linear characters.

There are only a few possibilities for partitions ξ^1 and ξ^2 which satisfy $|\xi^1| + |\xi^2| = 3$, $\xi^1 \subseteq (2^2)$, and $\xi^2 \subseteq (1^2)$. Below we give the $\star$-shape of $((2^2),(1^2))$ with the four possibilities for ξ^1 and ξ^2 shaded in teal.



Thus, we have four cases to consider:

	Case 1	Case 2	Case 3	Case 4
$\vec{\xi}$	$((1,2),\varnothing)$	$((1^2),(1))$	$((2),(1))$	$((1),(1^2))$
$((1^2)/\xi^2,(2^2)/\xi^1)$	$((1^2),(1))$	$((1),(1^2))$	$((1),(2))$	$(\varnothing,(1,2))$

In case 4, note that $s_{(2^2)/(1)} = s_{(1,2)}$. Equation (28) shows that we must multiply $s_{\vec{\xi}}$ together with $s_{((1^2)/\xi^2,(2^2)/\xi^1)}$ for each case and then sum the results to find $s_{((3),(3))} \otimes s_{((2^2),(1^2))}$. Since $s_{\vec{\xi}} = s_{(\xi^1,\varnothing)}s_{(\varnothing,\xi^2)}$, we now have reduced the situation to the relatively easy problem of multiplying Schur functions. Via methods in [13], it may be found that

$$s_{((1,2),\varnothing)}s_{((1^2),(1))} = s_{((1^3,2),(1))} + s_{((1,2^2),(1))} + s_{((1^2,3),(1))} + s_{((2,3),(1))} \tag{29}$$

$$s_{((1^2),(1))}s_{((1),(1^2))} = s_{((1^3),(1^3))} + s_{((1^3),(1,2))} + s_{((1,2),(1^3))} + s_{((1,2),(1,2))} \tag{30}$$

$$s_{((2),(1))}s_{((1),(2))} = s_{((1,2),(1,2))} + s_{((1,2),(3))} + s_{((3),(1,2))} + s_{((3),(3))} \tag{31}$$

$$s_{((1),(1^2))}s_{(\varnothing,(1,2))} = s_{((1),(1^3,2))} + s_{((1),(1,2^2))} + s_{((1),(1^2,3))} + s_{((1),(2,3))} \tag{32}$$

Cases 1-4 are represented in (29) through (32), respectively. Therefore, the sum on the right hand sides of (29) through (32) is the Schur function expansion of $s_{((3),(3))} \otimes s_{((2^2),(1^2))}$. Interpreting this in terms of irreducible representations of the hyperoctahedral group, if we let $A^{\vec{\xi}}$ be the irreducible representation indexed by the vector partition $\vec{\xi}$, we have found

$$\begin{aligned} A^{((3),(3))} \otimes A^{((2^2),(1^2))} = {} & A^{((1^3,2),(1))} \oplus A^{((1,2^2),(1))} \oplus A^{((1^2,3),(1))} \oplus A^{((2,3),(1))} \\ & \oplus A^{((1^3),(1^3))} \oplus A^{((1^3),(1,2))} \oplus A^{((1,2),(1^3))} \oplus A^{((1,2),(1,2))} \\ & \oplus A^{((1,2),(1,2))} \oplus A^{((1,2),(3))} \oplus A^{((3),(1,2))} \oplus A^{((3),(3))} \\ & \oplus A^{((1),(1^3,2))} \oplus A^{((1),(1,2^2))} \oplus A^{((1),(1^2,3))} \oplus A^{((1),(2,3))}. \end{aligned}$$

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