BOTTOM SCHUR FUNCTIONS

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Abstract

We give a basis for the space spanned by the sum $\hat{s}_\lambda$ of the lowest degree terms in the expansion of the Schur symmetric functions s_λ in terms of the power sum symmetric functions p_μ, where $\deg(p_i) = 1$. These lowest degree terms correspond to minimal border strip tableaux of λ. The dimension of the space spanned by $\hat{s}_\lambda$, where λ is a partition of n, is equal to the number of partitions of n into parts differing by at least 2. Applying the Rogers-Ramanujan identity, the generating function also counts the number of partitions of n into parts $5k+1$ and $5k-1$.

We also show that a symmetric function closely related to $\hat{s}_\lambda$ has the same coefficients when expanded in terms of power sums or augmented monomial symmetric functions.

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of the integer n, i.e., $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and $\sum \lambda_i = n$. The *length* $\ell(\lambda)$ of a partition λ is the number of nonzero parts of λ. The (Durfee or Frobenius) *rank* of λ, denoted $\operatorname{rank}(\lambda)$, is the length of the main diagonal of the diagram of λ, or equivalently, the largest integer i for which $\lambda_i \geqslant i$. The rank of λ is the least integer r such that λ is a disjoint union of r border strips (defined below).

Nazarov and Tarasov [1, Sect. 1], in connection with tensor products of Yangian modules, defined a generalization of rank to skew partitions (or skew diagrams) λ/μ. The paper [3, Proposition 2.2] gives several simple equivalent definitions of $\operatorname{rank}(\lambda/\mu)$. One of the definitions is that $\operatorname{rank}(\lambda/\mu)$ is the least integer r such that λ/μ is a disjoint union of r border strips. It develops a general theory of minimal border strip tableaux of skew shapes, introducing the concepts of the snake sequence and the interval set of a skew shape λ/μ. These tools are used to count the number of minimal border strip decompositions and minimal border strip tableaux of λ/μ. In particular, the paper [3] gives an explicit

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combinatorial formula for the coefficients of the p_ν, where $\ell(\nu) = \mathrm{rank}(\lambda/\mu)$, which appear in the expansion of $s_{\lambda/\mu}$.

The paper [3] considered a degree operator $\deg(p_\nu) = \ell(\nu)$ and defined the *bottom Schur functions* to be the sum of the terms of lowest degree which appear in the expansion of $s_{\lambda/\mu}$ as a linear combination of the p_ν. We study the bottom Schur functions in detail when $\mu = \emptyset$. In particular, in Section 4 we give a basis for the vector space they span.

In Section 7 we show that when we substitute ip_i for p_i in the expansion of a bottom Schur function in terms of power sums, then the resulting symmetric function has the same coefficients when expanded in terms of power sums or augmented monomial symmetric functions.

2 Definitions

In general we follow [2, Ch. 7] for notation and terminology involving symmetric functions. Let λ be a partition of n with Frobenius rank k. Recall that k is the length of the main diagonal of the diagram of λ, or equivalently, the largest integer i for which $\lambda_i \geqslant i$. Let $m_i(\lambda) = \#\{j : \lambda_j = i\}$, the number of parts of λ equal to i. Define $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. A *border strip* (or *rim hook* or *ribbon*) is a connected skew shape with no 2×2 square. An example is 75443/4332 whose diagram is illustrated in Figure 1. Define the *height* $\mathrm{ht}(B)$ of a border strip B to be one less than its number of rows.

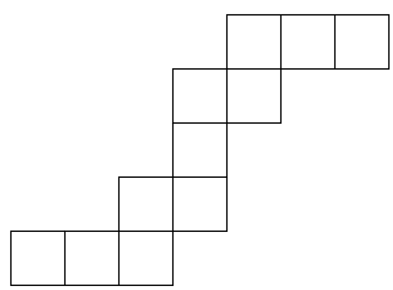


Figure 1: The border strip 75443/4332

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n, i.e., $\alpha_i \geqslant 0$ and $\sum \alpha_i = n$. Define a *border strip tableau* of shape λ and type α to be an assignment of positive integers to the squares of λ such that:

(a) every row and column is weakly increasing,

(b) the integer i appears α_i times, and

(c) the set of squares occupied by i forms a border strip.

Equivalently, one may think of a border-strip tableau as a sequence $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^r \subseteq \lambda$ of partitions such that each skew shape λ^i/λ^{i+1} is a border-strip of size α_i. For

1	1	5	5	5
1	2	5		
3	5	5		
3	5			
3				

Figure 2: A border strip tableau of 53321 of type $(3, 1, 3, 0, 7)$

instance, Figure 2 shows a border strip tableau of 53321 of type $(3, 1, 3, 0, 7)$. It is easy to see (in this nonskew case) that the smallest number of strips in a border-strip tableau is $\operatorname{rank}(\lambda)$. Define the *height* $\operatorname{ht}(T)$ of a border-strip tableau T to be

$$\operatorname{ht}(T) = \operatorname{ht}(B_1) + \operatorname{ht}(B_2) + \cdots + \operatorname{ht}(B_k)$$

where $B_1, \ldots, B_k$ are the (nonempty) border strips appearing in T. In the example we have $\operatorname{ht}(T) = 1 + 0 + 2 + 3 = 6$. Now we can define

$$\chi^\lambda(\nu) = \sum_T (-1)^{\operatorname{ht}(T)},$$

summed over all border-strip tableaux of shape λ and type ν. Since there are at least $\operatorname{rank}(\lambda)$ strips in every tableau, we have that $\chi^\lambda(\nu) = 0$ if $\ell(\nu) < \operatorname{rank}(\lambda)$. The numbers $\chi^\lambda(\nu)$ for $\lambda, \nu \vdash n$ are the values of the irreducible characters χ^λ of the symmetric group S_n.

Finally we can express the Schur function s_λ in terms of power sums p_ν, viz.,

$$s_\lambda = \sum_\nu \chi^\lambda(\nu) \frac{p_\nu}{z_\nu}. \tag{2.1}$$

Define $\deg(p_i) = 1$, so $\deg(p_\nu) = \ell(\nu)$. The *bottom Schur function* $\hat{s}_\lambda$ is defined to be the lowest degree part of s_λ, so

$$\hat{s}_\lambda = \sum_{\nu : \ell(\nu) = \operatorname{rank}(\lambda)} \chi^\lambda(\nu) \frac{p_\nu}{z_\nu}.$$

Also write $\tilde{p}_i = \frac{p_i}{i}$. For instance,

$$s_{321} = \frac{1}{45} p_1^6 - \frac{1}{9} p_3 p_1^3 + \frac{1}{5} p_1 p_5 - \frac{1}{9} p_3^2.$$

Hence

$$\begin{aligned} \hat{s}_{321} &= \frac{1}{5} p_1 p_5 - \frac{1}{9} p_3^2 \\ &= \tilde{p}_1 \tilde{p}_5 - \tilde{p}_3^2. \end{aligned}$$

We identify a partition λ with its *diagram*

$$\lambda = \{(i,j) \,:\, 1 \leqslant j \leqslant \lambda_i\}.$$

Let e be an edge of the lower envelope of λ, i.e., no square of λ has e as its upper or left-hand edge. We will define a certain subset S_e of squares of λ, called a *snake*. If e is horizontal and (i,j) is the square of λ having e as its lower edge, define

$$\begin{aligned} S_e \;\; &= \;\; (\lambda) \cap \{(i,j),(i-1,j),(i-1,j-1), \\ &\qquad (i-2,j-1),(i-2,j-2),\ldots\}. \end{aligned} \tag{2.2}$$

If e is vertical and (i,j) is the square of λ having e as its right-hand edge, define

$$\begin{aligned} S_e \;\; &= \;\; (\lambda) \cap \{(i,j),(i,j-1),(i-1,j-1), \\ &\qquad (i-1,j-2),(i-2,j-2),\ldots\}. \end{aligned} \tag{2.3}$$

In Figure 3 the nonempty snakes of the shape 533322 are shown with dashed paths through their squares, with a single bullet in the two snakes with just one square. The *length* $\ell(S)$ of a snake S is one fewer than its number of squares; a snake of length $i-1$ (so with i squares) is call an *i-snake*. Call a snake of **even** length a *left snake* if e is horizontal and a *right snake* if e is vertical. It is clear that the snakes are linearly ordered from lower left to upper right. In this linear ordering, replace a left snake with the symbol L, a right snake with R, and a snake of odd length with O. The resulting sequence (which does not determine λ) is called the *snake sequence* of λ, denoted $\mathrm{SS}(\lambda)$. For instance, from Figure 3 we see that

$$\mathrm{SS}(533322) = LLOOLORROOR.$$

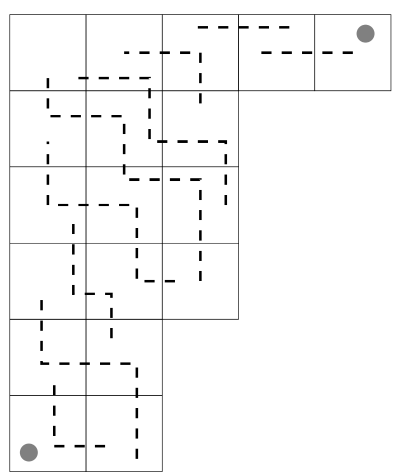


Figure 3: Snakes for the shape 533322

Lemma 2.1. *The L's in the snake sequence correspond exactly to horizontal edges of the lower envelope of λ which are below the line $x + y = 0$. The R's correspond exactly to vertical edges of the lower envelope of λ which are above the line $x+y=0$. All other edges of the lower envelope of λ are labelled by O's.*

Clearly we could have defined the snake sequence this way; however, the definitions above also hold for skew shapes. Lemma 2.1 only holds when λ is a straight (i.e., nonskew) shape.

Proof. Let e be an edge of the lower envelope of λ below the line $x+y=0$. Let (i,j) be the square of λ having e as its lower edge. The last square in the snake is some square in the first column of λ. So if e is horizontal then the last square is $(i-j+1,1)$, the snake has an odd number of squares and so has even length, and is labelled by L. If e is vertical then the last square is $(i-j,1)$, the snake has an even number of squares, so has odd length, and is labelled by R. The case when e is above $x+y=0$ is proved similarly. $\square$

Corollary 2.2. *In the snake sequence of λ, the L's occur strictly to the left of the R's.*

The number of horizontal edges of the lower envelope of λ which are below the line $x+y=0$ equals the length of the main diagonal of the diagram of λ, which is the rank of λ. Similarly the number of vertical edges of the lower envelope of λ which are above the line $x+y=0$ also equals the rank of λ. Henceforth we fix $k=\operatorname{rank}(\lambda)$.

Let $\mathrm{SS}(\lambda)=q_1q_2\cdots q_m$, and define an *interval set* of λ to be a collection $\mathcal{I}$ of k ordered pairs,

$$\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\},$$

satisfying the following conditions:

(a) the u_i's and v_i's are all distinct integers,

(b) $1\leqslant u_i<v_i\leqslant m$,

(c) $q_{u_i}=L$ and $q_{v_i}=R$.

Figure 4 illustrates the interval set $\{(1,11),(2,7),(5,8)\}$ of the shape 533322.

Figure 4: An interval set of the shape 533322

Given an interval set $\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}$, define the *crossing number* $c(\mathcal{I})$ to be the number of *crossings* of $\mathcal{I}$, i.e. the number of pairs (i,j) for which $u_i<u_j<v_i<v_j$.

Let $\boldsymbol{T}$ be a border strip tableau of shape λ. Recall that

$$\operatorname{ht}(\boldsymbol{T})=\sum_B \operatorname{ht}(B),$$

where B ranges over all border strips in $\boldsymbol{T}$ and $\operatorname{ht}(B)$ is one less than the number of rows of B. Define $z(\lambda)$ to be the height $\operatorname{ht}(\boldsymbol{T})$ of a "greedy border strip tableau" $\boldsymbol{T}$ of shape λ obtained by starting with λ and successively removing the largest possible border strip.

(Although $\boldsymbol{T}$ may not be unique, the set of border strips appearing in $\boldsymbol{T}$ is unique, so $\mathrm{ht}(\boldsymbol{T})$ is well-defined.)

The connection between bottom Schur functions and interval sets was given by Stanley [3, Theorem 5.2]:

$$\hat{s}_\nu = (-1)^{z(\nu)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i-u_i},$$

where $\mathcal{I}$ ranges over all interval sets of ν.

For example the shape 321 has snake sequence $LOLROR$. There are two interval sets, $\{(1,4),(3,6)\}$ with crossing number 1, and $\{(1,6),(3,4)\}$ with crossing number 0. So as we saw before

$$\hat{s}_{321} = \tilde{p}_1\tilde{p}_5 - \tilde{p}_3^2.$$

3 Bottom Schur Functions of straight shapes

Lemma 3.1. *The lexicographic order on shapes ν whose length $\ell(\nu)$ equals their rank k is equal to the reverse lexicographical order (with respect to the ordering L<R<O) on their snake sequences.*

Proof. Since $\ell(\nu)=k$, the snake sequence begins with k L's. If the length of the ith row of ν is $k+j$, then there are j O's to the left of the $(k-i+1)$st R. $\square$

Denote the *complete homogeneous symmetric functions* by h_λ. Recall that the Jacobi-Trudi identity expresses the s_λ's in terms of the h_μ's:

$$s_\lambda = \det(h_{\lambda_i-i+j})_{i,j=1}^{n},$$

where we define $h_i = 0$ for $i<0$. For example

$$s_{554421} = \det \begin{bmatrix} h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} \\ h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 & 1 & h_1 \end{bmatrix}.$$

Since $h_n = \sum_{\lambda\vdash n} \frac{p_\lambda}{z_\lambda}$, the term of lowest degree (in p) in the expansion of a given h_n in terms of the p_j is just $\frac{p_n}{n} = \tilde{p}_n$. For a product $h_{n_1}h_{n_2}\cdots h_{n_j}$ the term of lowest degree in the expansion in terms of the p_j is just $\tilde{p}_{n_1}\tilde{p}_{n_2}\cdots\tilde{p}_{n_j}$. So we have that $\hat{s}_\lambda$ = terms of lowest order in $\det(\tilde{p}_{\lambda_i-i+j})_{i,j=1}^{n}$ (since the p_λ are algebraically independent, and since

$\det(h_{\lambda_i-i+j}) = s_\lambda \neq 0$, this determinant will not vanish). For example

$$\hat{s}_{554421} = \text{ terms of lowest order in det} \begin{bmatrix} \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 & \tilde{p}_8 & \tilde{p}_9 & \tilde{p}_{10} \\ \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 & \tilde{p}_8 & \tilde{p}_9 \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 \\ \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 \\ 0 & 0 & 1 & \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 \\ 0 & 0 & 0 & 0 & 1 & \tilde{p}_1 \end{bmatrix}.$$

Since $p_0 = 1$, the terms of lowest order are those which contain the most number of 1's.

Row i of the matrix will have a 1 in position (i, j) if $\lambda_i - i + j = 0$, i.e. if $\lambda_i < i$ (this shows that the number of rows of JT_λ which do not contain a 1 is another definition of rank(λ) [3, Prop. 2.2]).

Let JT_p^* be the matrix obtained from the original Jacobi-Trudi matrix by removing every row and column which contains a 1 and replacing the h_i with $\tilde{p}_i$. We show below that this matrix is not singular and so we have

$$\hat{s}_\lambda = \det\ JT_p^*.$$

For example

$$\hat{s}_{554421} = \det \begin{bmatrix} \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_8 & \tilde{p}_{10} \\ \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_7 & \tilde{p}_9 \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_5 & \tilde{p}_7 \\ \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_4 & \tilde{p}_6 \end{bmatrix}.$$

Any minor of the Jacobi-Trudi matrix for a shape λ is the Jacobi-Trudi matrix for some skew shape μ/σ. For let JT^* be some minor of size m of some Jacobi-Trudi matrix JT. If the entry in position (i, j) is h_x put $jt^*_{i,j} = x$. Now we can set

$$\sigma_i = jt^*_{1,m} - jt^*_{1,i} - m + i,$$

and

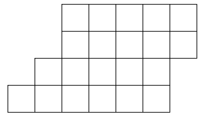
$$\mu_i = jt^*_{i,i} + \sigma_i.$$

Again note that since the p_λ are algebraically independent and $\det JT^* = s_{\mu/\sigma} \neq 0$, we have $\det JT_p^* \neq 0$.

In our running example, we have $\sigma_1 = 10 - 5 - 4 + 1 = 2, \sigma_2 = 10 - 6 - 4 + 2 = 2, \sigma_3 = 10 - 8 - 4 + 3 = 1$ and $\sigma_4 = 10 - 10 - 4 + 4 = 0$. Hence $\sigma = (2210)$. Also $\mu_1 = 5 + 2, \mu_2 = 5 + 2, \mu_3 = 5 + 1$ and $\mu_4 = 6 + 0$. Thus $\mu = (7766)$. Therefore we have that $\hat{s}_{554421}$ equals the determinant of the Jacobi-Trudi matrix of 7766/2210 with the h's replaced by $\tilde{p}$'s.

Lemma 3.2. *If the skew shape μ/σ has the Jacobi-Trudi matrix JT^* obtained by removing all rows and columns with a 1 from a Jacobi-Trudi matrix JT of a shape λ with rank k, then μ/σ contains a square of size k.*

The rank of 554421 is 4, and the diagram of 7766/2210 does indeed contain a square of size 4:



Proof. We give a proof due to Christine Bessenrodt, greatly improving our original proof. Define $\mu'_i = \ell(\lambda) - k + \lambda_i$ $(i = 1, \ldots, k)$ and $\sigma'_i = \#\{s|\lambda_s \leqslant k - i\}$ $(i = 1, \ldots, k)$. We give a diagrammatic definition of μ' and σ' which also illustrates that the skew diagram μ'/σ' contains a square of size k. Consider λ as a $k \times k$ square with two partitions α and β glued to it, i.e. $\lambda = (k + \beta_1, \ldots, k + \beta_k, \alpha_1, \ldots, \alpha_{\ell(\lambda)-k})$. Flip α over its anti-diagonal and then glue the bottom right corner of the result to the bottom left corner of the square. The final diagram is the skew diagram of μ'/σ'. We show that $\mu = \mu'$ and $\sigma = \sigma'$.

The k rows of JT^* are contained in the first k rows of JT, so $\mu_i = \lambda_i + c$ for some constant c. The last column of JT does not have a 1 in it, so it will not be removed, and its first k entries will be the last column of JT^*. Hence $jt^*_{1,k} = jt_{1,\ell(\lambda)} = \lambda_1 + \ell(\lambda) - 1$. Since $jt^*_{1,k} = \mu_1 - \sigma_k + k - 1$, we have $\mu_i = \ell(\lambda) - k + \lambda_i = \mu'_i$.

The first k entries of the last column of JT are retained. Then we remove the next $\#\{s|\lambda_s = 1\}$ columns to its left, do not remove the next column, remove the next $\#\{s|\lambda_s = 2\}$ columns to the left, and so on. Formally we have $jt^*_{1,k-j} = jt^*_{1,k-j+1} - 1 - \#\{s|\lambda_s = j\}$. Combining this with $\sigma_i = jt^*_{1,k} - jt^*_{1,i} - k + i$ gives us $\sigma_i = \#\{s|\lambda_s \leqslant k - i\} = \sigma'_i$. $\square$

4 The space spanned by the bottom Schur functions

Before we use the above results to give a basis for the space spanned by the bottom Schur functions, we must first recall some classical tableaux theory.

If λ/μ is a skew shape, then a *standard Young tableau* (SYT) of shape λ/μ is a labelling of the squares of λ/μ with the numbers $1, 2, \ldots, n$, each number appearing once, so that every row and column is increasing. A *semistandard Young tableau* (SSYT) of shape λ/μ is a labelling of the squares of λ/μ with positive integers that is weakly increasing in every row and strictly increasing in every column. We say that T has *type* $\alpha = (\alpha_1, \alpha_2, \ldots)$ if T has α_i parts equal to i.

1	3	5	6
2	7	8	9
4			

SYT

1	1	3	3
2	4	4	4
5			

SSYT

Now we define an operation (of Schützenberger) on standard Young tableaux called a *jeu de taquin slide*. Given a skew shape λ/μ, consider the squares b_0 that can be added to

λ/μ, so that b_0 shares at least one edge with λ/μ, and $\{b_0\} \cup \lambda/\mu$ is a valid skew shape. Suppose that b_0 shares a lower or right edge with λ/μ (the other situation is completely analogous). There is at least one square b_1 in λ/μ that is adjacent to b_0; if there are two such squares, then let b_1 be the one with a smaller entry. Move the entry occupying b_1 into b_0. Then repeat this procedure, starting at b_1. The resulting tableau will be a standard Young tableau. Analogously if b_0 shares an upper or left edge, the operation is the same except we let b_1 be the square with the bigger entry from two possibilities. For example we illustrate both situations in Figure 5; the tableau on the right results from playing jeu de taquin beginning at the square marked by a bullet on the tableau on the left (and vice versa).

$$\begin{array}{|c|c|c|c|} \bullet & 1 & 5 & 6 \\ \hline 2 & 3 & 8 & 9 \\ \hline 4 & 7 & & \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 & 8 & 9 \\ \hline 4 & \bullet & & \end{array}$$

Figure 5: Jeu de taquin slides

Two tableaux T and T' are called *jeu de taquin equivalent* if one can be obtained from another by a sequence of jeu de taquin slides. Given an SYT T of shape λ/μ, there is exactly one SYT P of straight shape, denoted $\mathrm{jdt}(T)$, that is jeu de taquin equivalent to T [2, Thm. A1.2.4].

The *reading word* of a (semi)standard Young tableau is the sequence of entries of T obtained by concatenating the rows of T bottom to top. For example, the tableau on the left in Figure 5 has the reading word 472389156. The *reverse reading word* of a tableau is simply the reading word read backwards.

A *lattice permutation* is a sequence $a_1a_2\cdots a_n$ such that in any initial factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $i+1$'s (for all i). For example 123112213 is a lattice permutation.

The *Littlewood-Richardson coefficients* $c^\lambda_{\mu\nu}$ are the coefficients in the expansion of a skew Schur function in the basis of Schur functions:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

The Littlewood-Richardson rule is a combinatorial description of the coefficients $c^\lambda_{\mu\nu}$. We will use two different versions of the rule.

Theorem 4.1 (Schützenberger, Thomas). *Fix an SYT P of shape ν. The Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$ is equal to the number of SYT of shape λ/μ that are jeu de taquin equivalent to P.*

For example, let $\lambda = (5,3,3,1)$, $\mu = (3,1)$, and $\nu = (3,3,2)$. Consider the tableau P of shape ν shown above. There are exactly two SYTs T of shape λ/μ such that $\mathrm{jdt}(T) = P$, namely,

$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \cline{1-2} \end{array}$$

$$\begin{array}{ccccc} & & & 2 & 3 \\ & 1 & 6 & & \\ 4 & 5 & 8 & & \\ 7 & & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & & 2 & 3 \\ & 5 & 6 & & \\ 1 & 7 & 8 & & \\ 4 & & & & \end{array}$$

Theorem 4.2 (Schützenberger, Thomas). *The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard Young tableaux of shape λ/μ and type ν whose reverse reading word is a lattice permutation.*

For example, with $\lambda = (5, 3, 3, 1), \mu = (3, 1)$, and $\nu = (3, 3, 2)$ as above, there are exactly two SSYTs T of shape λ/μ and type ν whose reverse reading word is a lattice permutation:

$$\begin{array}{ccccc} & & & 1 & 1 \\ & 1 & 2 & & \\ 2 & 2 & 3 & & \\ 3 & & & & \end{array} \qquad \begin{array}{ccccc} & & & 1 & 1 \\ & 2 & 2 & & \\ 1 & 3 & 3 & & \\ 2 & & & & \end{array}$$

Now we have enough machinery to state and prove this section's main theorem.

Theorem 4.3. *Fix n and k. The set $\{\hat{s}_\nu : \nu \vdash n,\ \mathrm{rank}(\nu) = k$ and $\ell(\nu) = k\}$ is a basis for the space $\mathrm{span}_{\mathbb{Q}}\{\hat{s}_\lambda : \lambda \vdash n$ and $\mathrm{rank}(\lambda) = k\}$.*

For example if $n = 12$ and $k = 3$, we have that $\{\hat{s}_{633}, \hat{s}_{543}, \hat{s}_{444}\}$ is a basis for $\mathrm{span}_{\mathbb{Q}}\{\hat{s}_{633}, \hat{s}_{543}, \hat{s}_{5331}, \hat{s}_{444}, \hat{s}_{4431}, \hat{s}_{4332}, \hat{s}_{43311}, \hat{s}_{3333}, \hat{s}_{33321}, \hat{s}_{333111}\}$.

Proof. First we prove that the $\hat{s}_\nu$ are linearly independent. We show that given any such ν, there is some term in the expansion of $\hat{s}_\nu$ which does not occur in the expansion of any $\hat{s}_{\nu'}$ for ν' lexicographically less than ν.

From [3, Theorem 5.2] we have that

$$\hat{s}_\nu = (-1)^{z(\nu)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i-u_i},$$

where $\mathcal{I}$ ranges over all interval sets of ν. Let $t = \pm p_{j_1 \geqslant \cdots \geqslant j_k}$ be the term corresponding to the noncrossing interval set $\mathcal{I}$ of the snake sequence of ν. We claim that t does not occur in the expansion of any $\hat{s}_{\nu'}$ for ν' lexicographically less than ν. Assume by way of contradiction that it does occur for some such ν' with corresponding interval set $\mathcal{I}'$.

Assume inductively that the first $i-1$ L's are matched with the last $i-1$ R's without crossings in $\mathcal{I}'$. Let r_j (and r'_j respectively) be the position of the jth R in the snake sequence of ν (ν' respectively). By Lemma 3.1 $r_j \geqslant r'_j$. But the length of the interval matching the ith R from the right in $\mathcal{I}$ is $r_{k-i+1} - i$. So for there to be an interval of this length in $\mathcal{I}'$ we must match the ith R from the right with the ith L; this interval has no crossing. Proceeding by induction we see that $\mathcal{I}'$ is also noncrossing, and so must equal $\mathcal{I}$. This shows that the snake sequences corresponding to ν and ν' are equal. Identical snake sequences and equal lengths guarantee that $\nu = \nu'$, a contradiction.

Now we prove that the $\hat{s}_\nu$ span the space of all $\hat{s}_\lambda$. We have shown that $\hat{s}_\lambda = \hat{s}_{\mu/\sigma}$. Expand $s_{\mu/\sigma}$ in terms of (straight) Schur functions using the Littlewood Richardson rule

$$s_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} s_\nu \ .$$

We need to show that $c^\mu_{\sigma\nu} = 0$ unless ν is of rank k and length k.

Fix an SYT P of shape ν. The Littlewood-Richardson coefficient $c^\mu_{\sigma\nu}$ is equal to the number of SYT of shape μ/σ that are jeu de taquin equivalent to P. Playing jeu de taquin on a straight-shape tableau of shape ν can only increase the length of the shape. Hence if $c^\mu_{\sigma\nu} \neq 0, \ell(\nu) \leqslant \ell(\mu) = k$.

The Littlewood-Richardson coefficient $c^\mu_{\sigma\nu}$ is also equal to the number of semistandard Young tableaux of shape μ/σ and type ν whose reverse reading word is a lattice permutation. But we know that μ/σ contains a square of size k (by Lemma 3.2). Therefore the bottom k boxes of this square must have labels at least k. Since $\ell(\nu) \leqslant k$, the labels are exactly k. So $\nu_k \geqslant k$, i.e. $\operatorname{rank}(\nu) \geqslant k$. Since $\ell(\nu) \leqslant k$, we must have $\operatorname{rank}(\nu) = \ell(\nu) = k$.

Taking terms of lowest degree on both sides of

$$s_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} s_\nu \ ,$$

we have that

$$\hat{s}_\lambda = \hat{s}_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} \hat{s}_\nu \ ,$$

where the sum is over ν of length k and rank k as required. □

5 Dimension of the space spanned by the bottom Schur functions

Let $p_{\leqslant k}(n)$ be the number of partitions of n with length at most k, and define $p_{\leqslant k}(0) = 1$. A partition $\nu \vdash n$ of length k and rank k decomposes into a $k \times k$ square of boxes and a partition of $n - k^2$ of length at most k.

Corollary 5.1. *The dimension of the space of bottom Schur functions* $\operatorname{span}_{\mathbb{Q}}\{\hat{s}_\lambda : \lambda \vdash n\}$ *is*

$$\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} p_{\leqslant k}(n - k^2).$$

For example, the first 27 terms in this sequence (beginning with $n = 1$) are

$$1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79.$$

There is a nice bijection between the above partitions and the set of partitions $\{\lambda \vdash n : \lambda_i - \lambda_{i+1} \geqslant 2\}$. For, given a k and a partition $\lambda^* \vdash n - k^2$ with fewer than k parts, we can set $\lambda_i = \lambda_i^* + 2k - 2i + 1$. This gives a partition of n with k rows with $\lambda_i - \lambda_{i+1} \geqslant 2$ as required. This is clearly a bijection.

This classical sequence also gives the number of partitions of n into parts congruent to 1 or 4 mod 5; equivalently these numbers are the coefficients in the expansion of the Rogers-Ramanujan identity

$$1 + \sum_{n \geqslant 1} \frac{t^{n^2}}{(1-t)(1-t^2)\cdots(1-t^n)} = \prod_{n \geqslant 1} \frac{1}{(1-t^{5n-1})(1-t^{5n-4})}$$

6 2-bottom Schur functions

We have shown that a basis for the space spanned by the bottom Schur functions consists of the $\hat{s}_\lambda$ where $\ell(\lambda) \leqslant \operatorname{rank}(\lambda)$. It is natural to define for fixed $j \geqslant 1$ the *j-bottom Schur function* $\hat{s}^j_\lambda$ to be the sum of those terms of degree at most $\operatorname{rank}(\lambda)+j-1$ in the expansion (2.1) (with $\deg p_i = 1$ as usual). When $j = 2$ we have verified (using Stembridge's SF package for Maple [4]) that for $n \leqslant 14$ the dimension of the space spanned by $\{\hat{s}^2_\lambda : \lambda \vdash n\}$ equals the number of $\lambda \vdash n$ satisfying $\ell(\lambda) \leqslant \operatorname{rank}(\lambda) + 1$. This suggests the following conjecture.

Conjecture 6.1. *A basis for the space spanned by the 2-bottom Schur functions consists of all 2-bottom Schur functions $\hat{s}^2_\lambda$, where λ is a partition of n satisfying $\ell(\lambda) \leqslant \mathit{rank}(\lambda)+1$.*

However in the $j = 3$ case, the dimensions of the spaces spanned by the 3-bottom Schur functions are $1, 2, 3, 4, 6, 9, 11, 15, 19, 24, 30, \ldots$. We have computed that the numbers of $\lambda \vdash n$ satisfying $\ell(\lambda) \leqslant \operatorname{rank}(\lambda) + 2$ are given by $1, 2, 3, 4, 5, 8, 10, 14, 17, 22, 27, \ldots$. Unfortunately these sequences do not agree.

7 A condition satisfied by bottom Schur functions

We prove a surprising identity satisfied by a variant of the bottom Schur functions related to their expansion in terms of power sum and monomial symmetric functions.

Fix a shape λ of rank k. Given an interval set $\mathcal{I} = \{(u_1, v_1), \ldots, (u_k, v_k)\}$, of λ and a labelling of the intervals $(\alpha_1, \ldots, \alpha_k)$ such that $\alpha_i \in \mathbb{P}$, define

$$x^{\mathcal{I}} = \prod_{i=1}^{k} x_{\alpha_i}^{v_i - u_i}.$$

Recall that $c(\mathcal{I})$ is the number of crossings of the interval set $\mathcal{I}$. Figure 6 shows a labelled interval set of the shape 533322 with the snake sequence *LLOOLORROOR*. For this interval set $c(\mathcal{I}) = 1$ and for this labelling $x^{\mathcal{I}} = x_4^{10} x_2^5 x_4^3 = x_2^5 x_4^{13}$.

Figure 6: A labelled interval set of the shape 533322

Example 7.1. For the shape $\lambda = (4, 4, 4)$ with snake sequence $LLLORRR$, Figure 7 depicts some of the labelled interval sets. In the top left we have $(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = (-1)^3 x_a^4 x_a^4 x_b^4 = -x_a^8 x_b^4$. In the top right we have $(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = (-1)^2 x_a^5 x_a^3 x_b^4 = x_a^8 x_b^4$. In fact in every row the term $(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$ in the left column is exactly the negative of the corresponding term $(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$ in the right column.

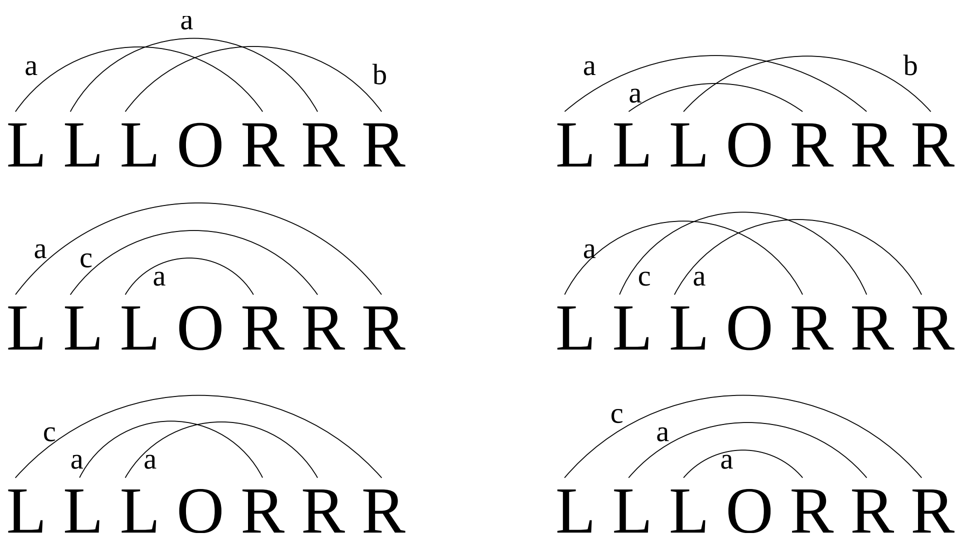


Figure 7: Some labelled interval sets of the shape 444

Lemma 7.1. *Fix a shape λ. Then $\sum(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = 0$, where the sum is over all labelled interval sets of λ with at least one label repeated.*

Proof. We give a sign reversing involution on these labelled interval sets. Examine a specific labelled interval set $\mathcal{I}$. Since we are dealing with straight (non-skew) shapes, we know by Corollary 2.2 that the snake sequence has all the L's before any of the R's, or $u_k < v_1$. So any two intervals i and j ($> i$ say) either intersect ($u_i < u_j < v_i < v_j$) or are nested ($u_i < u_j < v_j < v_i$).

Let a be the smallest label which is repeated. The intervals in $\mathcal{I}$ are ordered by where they start, so identifying the first two intervals i and j ($> i$) labelled by a is well-defined.

Simply change the interval (u_i, v_i) to (u_i, v_j) and the interval (u_j, v_j) to (u_j, v_i), while preserving the label a on both. Where the intervals start remains unchanged, so these intervals remain the first two intervals labelled by a. Hence this operation is an involution.

Note that if the two intervals initially nested, they now intersect, and if they initially intersected, they now nest. The parity of the number of crossings of these two with any other interval is preserved under this operation. So the parity of the total number of crossings has changed, and this involution is sign reversing. The rows of Figure 7 are some examples of this involution (if $a \neq c$).

Thus given any labelled interval set with repeated labels, there is a unique labelled interval set with one more (or fewer) crossings, and so the sum of all such terms $(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$ is zero. $\square$

Fix a shape λ. Recall that

$$\hat{s}_\lambda = (-1)^{z(\lambda)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i-u_i},$$

where $\mathcal{I}$ ranges over all interval sets of λ.

For another shape μ, define c_μ to be the coefficient of $\tilde{p}_\mu$ in the above sum, i.e.

$$c_\mu = (-1)^{z(\lambda)} \sum_{\mathcal{I}} (-1)^{c(\mathcal{I})},$$

where the sum is over all interval sets (of λ) of type μ.

Lemma 7.2. *$c_\mu p_\mu = (-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}}$, where the sum is over all labelled interval sets of type μ.*

Example 7.2. Consider as before the shape $\lambda = (4,4,4)$ with snake sequence $LLLORRR$. In particular, consider the interval set $\{(1,7),(2,5),(3,6)\}$ of type $(6,3,3)$ labelled by (a,b,c). This interval set is illustrated in Figure 8. If $a = b = c$ then $x^{\mathcal{I}} = x_a^{12}$ and the sum over all such labellings will give $x_1^{12} + x_2^{12} + \cdots = m_{(12)}$. If $a = b \neq c$ the sum over all such labellings will give $x_1^6 x_1^3 x_2^3 + x_1^6 x_1^3 x_3^3 + x_2^6 x_2^3 x_1^3 + \cdots = x_1^9 x_2^3 + x_1^9 x_3^3 + x_2^9 x_1^3 + \cdots = m_{93}$. Similarly if $a = c \neq b$ we will get m_{93}, and if $b = c \neq a$ we will get $x_1^6 x_2^3 x_2^3 + x_2^6 x_1^3 x_1^3 + \cdots = 2x_1^6 x_2^6 + \cdots = 2m_{66}$. Finally if the three labels are all different, the sum will give $x_1^6 x_2^3 x_3^3 + x_1^6 x_3^3 x_2^3 + \cdots = 2m_{633}$. So the sum over all such labellings is $m_{(12)} + 2m_{66} + 2m_{93} + 2m_{633} = p_{633}$.

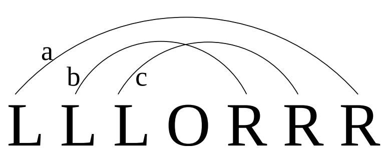


Figure 8: A labelled interval set of the shape 444

Proof. We need to show that for every interval set $\mathcal{I}$ of type μ, $\sum x^{\mathcal{I}} = p_\mu$, where the sum is over all labellings of $\mathcal{I}$. First note that the intervals can be ordered largest first and left to right among intervals of the same length. So the ith interval is well defined, and has length μ_i.

By definition $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}} = (x_1^{\mu_1} + x_2^{\mu_1} + \cdots)(x_1^{\mu_2} + x_2^{\mu_2} + \cdots) \cdots (x_1^{\mu_{\ell(\mu)}} + x_2^{\mu_{\ell(\mu)}} + \cdots)$. But if we expand this product into monomials $x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}$, each monomial corresponds uniquely to the labelling of $\mathcal{I}$ where the jth interval is labelled by i_j, and so occurs exactly once in $\sum x^{\mathcal{I}}$ as required. □

The *augmented monomial symmetric function* $\tilde{m}_\mu$ is defined by

$$\tilde{m}_\mu = m_1(\mu)! m_2(\mu)! \cdot m_\mu,$$

where m_μ denotes a monomial symmetric function.

Lemma 7.3. *$c_\mu \tilde{m}_\mu = (-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}}$, where the sum is over all labelled interval sets of λ of type μ with no label repeated.*

Note that we have already demonstrated this result in Example 7.2. Indeed for that interval set when the labels were all different we saw that $\sum x^{\mathcal{I}} = 2m_{633}$.

Proof. Fix a specific interval set $\mathcal{I}$ of type μ. We need to show $\tilde{m}_\mu = \sum x^{\mathcal{I}}$ where the sum is over all labellings with no label repeated. As before we can order the intervals and say that the ith interval is labelled by α_i. Note that if $\mu_j = \mu_{j+1}$, the two labellings $(\alpha_1, \alpha_2, \ldots, \alpha_j, \alpha_{j+1}, \ldots)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, \alpha_j, \ldots)$ both produce the same term $x^{\mathcal{I}} = x_{\alpha_1}^{\mu_1} x_{\alpha_2}^{\mu_2} \cdots$. So we have $\sum_{(\alpha_1, \alpha_2, \ldots)} x^{\mathcal{I}} = \sum_{(\beta_1, \beta_2, \ldots)} m_1(\mu)! m_2(\mu)! \cdots x^{\mathcal{I}}$ where we impose the condition that if $\mu_j = \mu_{j+1}$, then $\beta_j < \beta_{j+1}$. Recall that $m_\mu = \sum_{(\beta_1, \beta_2, \ldots)} x^{\mathcal{I}}$ by definition. So we have $\tilde{m}_\mu = \sum x^{\mathcal{I}}_{(\alpha_1, \alpha_2, \ldots)}$ as required. □

Theorem 7.4. *For each shape λ, write the bottom Schur function $\hat{s}_\lambda = \sum_\mu c_\mu \tilde{p}_\mu$. Then $\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu \tilde{m}_\mu$.*

Example 7.3. For $\lambda = (4, 4, 4)$ we have

$$\hat{s}_\lambda = -\tilde{p}_{642} + \tilde{p}_{633} + \tilde{p}_{552} - 2\tilde{p}_{543} + \tilde{p}_{444}.$$

So our result states that

$$-p_{642} + p_{633} + p_{552} - 2p_{543} + p_{444} = -m_{642} + 2m_{633} + 2m_{552} - 2m_{543} + 6m_{444}.$$

Proof. From Lemma 7.2 we have $c_\mu p_\mu = (-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}}$, where the sum is over all labelled interval sets of type μ. But by Lemma 7.1

$$(-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}} = 0$$

if we sum over all labelled interval sets with a repeated label, while by Lemma 7.3 we have

$$(-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}} = c_\mu m_1(\mu)! m_2(\mu)! \cdots m_\mu$$

if we sum over all labelled interval sets with no label repeated. So $\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu \tilde{m}_\mu$. □

Let Γ denote the space of all symmetric functions f with rational coefficients such that if $f = \sum_\mu c_\mu \tilde{p}_\mu$, then

$$\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu \tilde{m}_\mu.$$

Theorem 7.4 shows that $\hat{s}_\lambda \in \Gamma$. However, there are elements of Γ that are not linear combinations of $\hat{s}_\lambda$'s, such as $f = \tilde{p}_{511} - 3\tilde{p}_{421} + \tilde{p}_{331} + \tilde{p}_{322}$. Let R denote the transition matrix from monomial symmetric functions to the power sums, i.e.,

$$p_\lambda = \sum_\mu R_{\lambda\mu} m_\mu.$$

Let D denote the diagonal matrix whose diagonal coincides with that of R, so

$$D_{\lambda\lambda} = R_{\lambda\lambda} = \prod_{i \geqslant 1} m_i(\lambda)!.$$

It is easy to see that $\Gamma = \ker(R - D)$, the kernel (or null space) of $R - D$. Let Γ_n denote the elements of Γ that are homogeneous of degree n, and let $\gamma_n = \dim \Gamma_n$. We have computed that

$$(\gamma_1, \gamma_2, \dots) = (1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 11, 15, 19, 24, \dots).$$

Compare this with the dimension β_n of the space spanned by the bottom Schur functions of degree n, given in general by Corollary 5.1 and for $n \leqslant 27$ just below this corollary. In particular, the least n for which $\beta_n < \gamma_n$ is $n = 7$. We don't have a conjecture for the value of γ_n.

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