Standard paths in another composition poset

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Submitted: Oct 8, 2003; Accepted: Oct 17, 2004; Published: Oct 26, 2004

Abstract

Bergeron, Bousquet-Mélou and Dulucq [1] enumerated paths in the Hasse diagram of the following poset: the underlying set is that of all compositions, and a composition μ covers another composition λ if μ can be obtained from λ by adding 1 to one of the parts of λ, or by inserting a part of size 1 into λ.

We employ the methods they developed in order to study the same problem for the following poset, which is of interest because of its relation to non-commutative term orders : the underlying set is the same, but μ covers λ if μ can be obtained from λ by adding 1 to one of the parts of λ, or by inserting a part of size 1 at the left or at the right of λ. We calculate generating functions for standard paths of fixed width and for standard paths of height ≤ 2.

1 Definition of standard paths

By a *composition* P we mean a sequence of positive integers $(p_1, p_2, \ldots, p_k)$, which are the *parts* of P. We define the *length* $\ell(P)$ of P as the number of parts, and the *weight* $|P| = \sum_{i=1}^{k} p_k$ as the sum of its parts. If P has weight n then P is a composition of n, and we write $P \vDash n$.

We say that a composition Q covers a composition P if Q is obtained from P either by adding 1 to a part of P, or by inserting a part of size 1 to the left, or by inserting a part of size 1 to the right. Thus, $P = (p_1, p_2, \ldots, p_k)$ is covered by

1. $(1, p_1, \ldots, p_k)$,
2. $(p_1, \ldots, p_k, 1)$,
3. and, for $1 \leq i \leq k$, $(p_1, \ldots, p_i + 1, \ldots, p_k)$.

Extending this relation by transitivity makes the set of all compositions into a partially ordered set, which we denote by $\mathfrak{N}$. This is in accordance with the notations in the author's article **A poset classifying non-commutative term orders** [3], where $\mathfrak{N}$ was used for the following isomorphic poset of words: the underlying set is X^*, the free associative monoid on $X = \{x_1, x_2, x_3, \ldots\}$, and $m_1 = x_{i_1} \cdots x_{i_r}$ is smaller than m_2 if m_2 can be obtained from m_1 by a sequence of operations of the form

(i) Multiply by a word to the left,

(ii) Multiply by a word to the right,

(iii) Replace an occurring x_i with an x_j, with $j > i$.

The bijection $(p_1, p_2, \ldots, p_k) \mapsto x_{p_1} \cdots x_{p_k}$ is an order isomorphism between these two partially ordered sets.

On the other hand, the partial order Γ on compositions studied by Bergeron, Bousquet-Mélou and Dulucq in **Standard paths in the composition poset** [1] is different, since in Γ the composition $P = (p_1, p_2, \ldots, p_k)$ is covered by

1. $(1, p_1, \ldots, p_k)$,
2. $(p_1, \ldots, p_k, 1)$,
3. for $1 \leq i \leq k$, $(p_1, \ldots, p_i + 1, \ldots, p_k)$,
4. for $1 \leq i < k$, $(p_1, \ldots, p_i, 1, p_{i+1}, \ldots, p_k)$.

Γ and $\mathfrak{N}$ coincide for compositions of weight ≤ 4. In Figure 1 this part of the Hasse diagram is depicted. We have that $(2, 2) \leq (2, 1, 2)$ in Γ but not in $\mathfrak{N}$, so the rest of the respective Hasse diagrams differ.

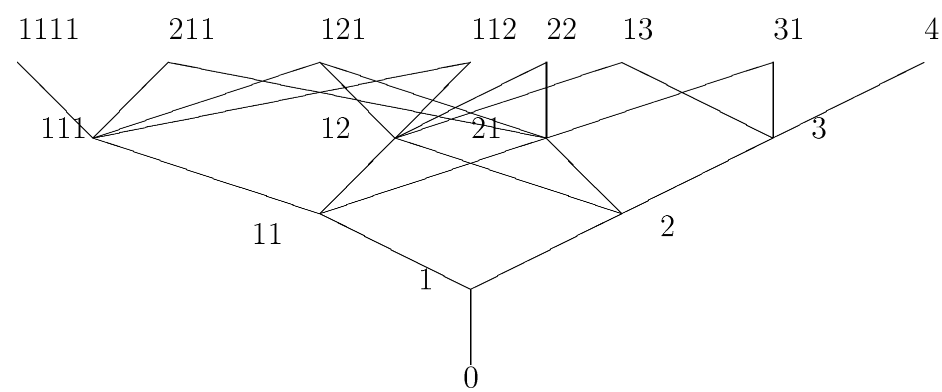

Figure 1: The Hasse diagram of $\mathfrak{N}$.

Following [1] we define a *standard path of length* n to be a sequence $\gamma = (P_0, P_1, P_2, \ldots, P_n)$ of compositions such that

$$P_0 \prec P_1 \prec P_2 \prec \cdots \prec P_n, \qquad P_i \vDash i. \tag{1}$$

The partial order is now that of $\mathfrak{N}$. For instance,

$$\rho = ((), (1), (1,1), (1,2), (1,1,2)) \tag{2}$$

is a standard path of length 4, corresponding to a saturated chain in Hasse diagram of $\mathfrak{N}$ between the minimal element $()$ and the element $(1,1,2)$.

We furthermore define the *diagram* of a composition $P = (p_1, \ldots, p_k)$ to be the set of points $(i,j) \in \mathbb{Z}^2$ with $1 \le j \le p_i$. Alternatively, we can replace the node (i,j) by the square with corners $(i-1, j-1)$,$(i-1, j)$,$(i, j-1)$ and (i,j). So the composition $(1,1,2)$ has diagram $\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & \ \\ \hline \ & \ & \ \\ \hline\end{array}$. For a standard path $\gamma = (P_1, \ldots, P_n)$ ending at P_n we label the boxes in the diagram of P_n in the order that they appear in the path. To avoid ambiguity, we use the convention that whenever P_i consists of i ones and P_{i+1} consists of $i+1$ ones, the extra one is considered to have been added to the left. So for the path ρ the corresponding tableau is $\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 3 \\ \hline 4 & 2 & 1 \\ \hline\end{array}$.

Clearly, two different standard paths give rise to different tableau. Furthermore, the tableau that occurs as tableau of standard paths must be increasing in every column, and have the additional property that whenever the numbers $1, 2, \ldots, k$ occur as a contiguous sequence on the bottom row, then that sequence is $k, k-1, \ldots, 2, 1$. This is a necessary but not sufficient condition.

The underlying diagram of a tableau is called its *shape*, and we define the shape of a standard path to be the shape of its tableau. We define the *height* and *width* of a diagram to be the height and width of the smallest rectangle containing it. Hence, the standard path ρ has width 2 and height 1.

2 Enumeration of standard paths of fixed width

Let $\mathfrak{N}_{(k)}$ denote the subposet of compositions of width k. For a path γ of shape $(p_1, p_2, \ldots, p_k)$ we set

$$v(\gamma) = x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k} \tag{3}$$

We want to compute the generating function

$$f_k(x_1, \ldots, x_k) = \sum_{\gamma \text{ path of width } k} v(\gamma) \tag{4}$$

Theorem 1. The generating function $f_k(x_1, \ldots, x_k)$ of standard paths of width k is a rational function given by the following recursive relation: $f_0 = 1$, $f_1(x_1) = x_1(1-x_1)^{-1}$, and for $k > 1$

$$f_k(x_1, \ldots, f_k) = \frac{x_1 f_{k-1}(x_2, \ldots, x_k) + x_k f_{k-1}(x_1, \ldots, x_{k-1}) - x_1 \cdots x_k}{1 - x_1 - \ldots - x_k} \tag{5}$$

Proof. A tableau of width k can be obtained by adding a new cell either

- at the top of a column of another tableau of width k,

- at the beginning of a tableau of width $k-1$,

- or at the end of a tableau of width $k-1$.

These three cases correspond respectively to $(x_1+x_2+\ldots+x_k)f_k$, to $x_1 f_{k-1}(x_2,\ldots,x_k)$, and to $x_k f_{k-1}(x_1,\ldots,x_{k-1})$. However, if the tableau has shape $(1,\ldots,1)$ then the last two operations give the same result. Hence

$$f_k = (x_1+x_2+\ldots+x_k)f_k + x_1 f_{k-1}(x_2,\ldots,x_k) + x_k f_{k-1}(x_1,\ldots,x_{k-1}) - x_1\cdots x_k,$$

from which (5) follows. $\square$

We obtain successively

$$\begin{aligned}
f_0 &= 1\\
f_1 &= \frac{x_1}{1-x_1}\\
f_2 &= \frac{x_1x_2(1-x_1x_2}{(1-x_1)(1-x_2)(1-x_1-x_2)}\\
f_3 &= \Big({x_1}^2{x_2}^2x_3+{x_1}^2x_2{x_3}^2+x_1{x_2}^3x_3+x_1{x_2}^2{x_3}^2-{x_1}^2{x_2}^2\\
&\quad -4\,{x_1}^2x_2x_3-{x_1}^2{x_3}^2-x_1{x_2}^3-7\,x_1{x_2}^2x_3-4\,x_1x_2{x_3}^2-\\
&\quad {x_2}^3x_3-{x_2}^2{x_3}^2+2\,{x_1}^2x_2+2\,{x_1}^2x_3+5\,x_1{x_2}^2+12\,x_1x_2x_3\\
&\quad +2\,x_1{x_3}^2+{x_2}^3+5\,{x_2}^2x_3+2\,x_2{x_3}^2-5\,x_1x_2-4\,x_1x_3\\
&\quad -3\,{x_2}^2-5\,x_2x_3+x_2+1\Big)\\
&\quad \times x_1x_2x_3\\
&\quad \times(1-x_1)^{-1}(1-x_2)^{-1}(1-x_3)^{-1}(1-x_1-x_2)^{-1}(1-x_2-x_3)^{-1}\\
&\quad \times(1-x_1-x_2-x_3)^{-1}
\end{aligned} \tag{6}$$

Theorem 2. For each k,

$$f_k(x_1,\ldots,x_k) = \frac{x_1\cdots x_k}{\prod_{i=1}^k\prod_{j=i}^k(1-x_i-x_{i+1}-\ldots-x_j)}\tilde{f}_k(x_1,\ldots,x_k) \tag{7}$$

where $\tilde{f}_k$ is a polynomial.

Proof. This is true for $k=0,1$. Assume that f_{k-1} has the above form. Then

$$\begin{aligned}
f_k(1-x_1-\cdots-x_k) &= x_1f_{k-1}(x_2,\ldots,x_k)+x_kf_{k-1}(x_1,\ldots,x_{k-1})-x_1\cdots x_k\\
&= x_1x_2\cdots x_k\tilde{f}_{k-1}(x_2,\ldots,x_k)\prod_{i=2}^{k}\prod_{j=i}^{k}(1-x_i-\cdots-x_j)^{-1}\\
&\quad + x_kx_1\cdots x_{k-1}\tilde{f}_{k-1}(x_1,\ldots,x_{k-1})\prod_{i=1}^{k-1}\prod_{j=i}^{k-1}(1-x_i-\cdots-x_j)^{-1}-x_1\cdots x_k
\end{aligned} \tag{8}$$

hence

$$\frac{f_k(1-x_1-\cdots-x_k)\prod_{i=1}^{k}\prod_{j=i}^{k}(1-x_i-\cdots-x_j)}{x_1\cdots x_k}$$
$$= \tilde{f}_{k-1}(x_2,\ldots,x_k)\prod_{j=1}^{k}(1-x_1-\cdots-x_j) + \tilde{f}_{k-1}(x_1,\ldots,x_{k-1})\prod_{i=1}^{k}(1-x_i-\cdots-x_k)$$
$$-\prod_{i=1}^{k}\prod_{j=i}^{k}(1-x_i-\cdots-x_j) \quad (9)$$

□

Let $a_{n,k}$ denote the number of standard paths of width k and length n, and let

$$L_k(t) = \sum_{n\geq 0} a_{n,k}t^n$$

be the generating function for the number of standard paths of width k and length n. Then $L_k(t) = f_k(t,\ldots,t)$. This substitution results in some cancellation in the numerator and denominator; we have that

$$\begin{aligned}
L_1(t) &= \frac{t}{1-t}\\
L_2(t) &= \frac{t^2(1+t)}{(1-t)(1-2t)}\\
L_3(t) &= \frac{t^3(1+5t-2t^2)}{(1-t)(1-2t)(1-3t)}\\
L_4(t) &= \frac{t^4(1+16t-15t^2+6t^3)}{(1-t)(1-2t)(1-3t)(1-4t)}\\
L_5(t) &= \frac{t^5(1+42t-65t^2+62t^3-24t^4)}{(1-t)(1-2t)(1-3t)(1-4t)(1-5t)}
\end{aligned} \quad (10)$$

Proposition 3.

$$L_k(t) = \frac{t^k\tilde{L}_k(t)}{\prod_{i=1}^{k}(1-it)} \quad (11)$$

where $\tilde{L}_k(t)$ is a polynomial of degree $k-1$ with $\tilde{L}_k(1) = 2^{k-1}$.

Proof. The recursive relation (5) specializes to

$$L_k = \frac{2tL_{k-1}-t^k}{1-kt} \quad (12)$$

Assume (11) for a fixed k; then (12) gives

$$L_{k+1} = \frac{2t^{k+1}\tilde{L}_k - t^{k+1}\prod_{i=1}^{k}(1-it)}{\prod_{i=1}^{k+1}(1-it)}$$

Since $\tilde{L}_k$ has degree $k-1$ and evaluates to 2^{k-1} at 1, we get that $\tilde{L}_{k+1} = 2\tilde{L}_k - \prod_{i=1}^{k}(1-it)$ has degree k and evaluates to 2^k at 1. The assertion now follows by induction. $\square$

Corollary 4. For a fixed k,

$$a_{n+k,k} \sim \frac{k^{k-1}}{(k-1)!}k^n \qquad \text{as } n \to \infty \tag{13}$$

Proof. This follows from the previous Proposition, and the partial fraction decomposition

$$\begin{aligned} \prod_{i=1}^{k}(1-it)^{-1} &= \sum_{j=1}^{k} v_{j,k}(1-it)^{-1} \\ v_{k,k} &= \frac{k^{k-1}}{(k-1)!} \end{aligned} \tag{14}$$

$\square$

3 Enumeration of standard paths of height at most two

Let $\mathfrak{N}^{(k)}_{n,i,j}$ denote the poset of compositions of n with height $\leq k$, having i parts of size 1 and j parts of size ≥ 2. Let $\gamma^{(k)}_{n,i,j}$ be the number of standard paths with endpoint in $\mathfrak{N}^{(k)}_{n,i,j}$.

We will derive a recurrence relation for $\gamma^{(2)}_{n,i,j}$. Note that a tableau of height ≤ 2, with i parts of size 1 and j parts of size 2, has a total of $n = i + 2j$ boxes, so $\gamma^{(2)}_{n,i,j} = 0$ unless $n = i + 2j$. Put $c^{(2)}_{i,j} = \gamma^{(2)}_{i+2j,i,j}$. A tableau with i parts of size 1 and j parts of size 2, can be obtained

- from a tableau with $i-1$ parts of size 1 and j parts of size 2, by adding a part of size 1 to the left,
- or from a tableau with $i-1$ parts of size 1 and j parts of size 2, by adding a part of size 1 to the right,
- or from a tableau with $i+1$ parts of size 1 and $j-1$ parts of size 2, by adding a box to a part of size 1.

For the composition consisting of n ones the first two ways are identical, which gives the recurrence

$$\begin{aligned} \gamma^{(2)}_{n,i,j} &= 2\gamma^{(2)}_{n-1,i-1,j} + (i+1)\gamma^{(2)}_{n-1,i+1,j-1} - \delta^0_j \\ c^{(2)}_{i,j} &= 2c^{(2)}_{i-1,j} + (i+1)c^{(2)}_{i+1,j-1} - \delta^0_j \end{aligned} \tag{15}$$

where δ^j_i is the Kronecker delta. We get that $c^{(2)}_{n,0} = \gamma^{(2)}_{n,n,0} = 1$, $\gamma^{(2)}_{n,i,0} = 0$ for $i \neq n$. For small values of i, j, $c^{(2)}_{i,j}$ is as in table 1

j / i	0	1	2	3	4	5	6	7
0	1	1	4	30	336	5040	95040	2162160
1	1	4	30	336	5040	95040	2162160	57657600
2	1	11	138	2184	42480	986040	26666640	824503680
3	1	26	504	10800	265320	7447440	236396160	8393898240
4	1	57	1608	45090	1368840	45765720	-	-
5	1	120	4698	167640	6174168	242686080	-	-
6	1	247	12910	572748	25192440	1151011680	-	-
7	1	502	33924	1834872	95091360	4999942080	-	-
8	1	1013	86172	5588310	337239840	-	-	-

Table 1: Values of $c^{(2)}_{i,j}$ for small i, j

Theorem 5. Put

$$P_k(x) = \sum_{n=0}^{\infty} c^{(2)}_{n,k} x^n \tag{16}$$

Then $P_0(x) = (1-x)^{-1}$ and

$$P_k(x) = \frac{\frac{d}{dx} P_{k-1}(x)}{1-2x} \tag{17}$$

Proof. Since $c^{(2)}_{n,0} = 1$ it follows that $P_0(x) = \sum_{n=0}^{\infty} c^{(2)}_{n,0} x^n = (1-x)^{-1}$.

Now, multiply (15) with x^i and sum over all $i \geq 0$ to get that

$$\sum_{i\geq 0} c^{(2)}_{i,j} x^i = 2 \sum_{i \geq 1} c^{(2)}_{i-1,j} x^i + \sum_{i\geq 0} (i+1) c^{(2)}_{i+1,j-1} x^i \tag{18}$$

which means that

$$P_j(x) = 2xP_j(x) + P'_{j-1}(x) \tag{19}$$

□

We get that

$$\begin{aligned} P_1(x) &= (1-x)^{-2}(1-2x)^{-1} \\ P_2(x) &= 2!(1-x)^{-3}(1-2x)^{-3}(2-3x) \\ P_3(x) &= 3!(1-x)^{-4}(1-2x)^{-5}(5-14x+10x^2x) \\ P_4(x) &= 4!(1-x)^{-5}(1-2x)^{-7}(14-56x+76x^2-35x^3) \end{aligned} \tag{20}$$

and in general

$$P_k(x) = k!(1-x)^{-1-k}(1-2x)^{1-2k} Q_k(x) \tag{21}$$

where $Q_k(x)$ is a polynomial of degree $k-1$, with $Q_k(1) = (-1)^{k+1}$.

Theorem 6. Put

$$P(x,y) = \sum_{i,j\geq 0} c_{i,j}^{(2)} x^i \frac{y^j}{j!} \tag{22}$$

Then

$$P(x,y) = \frac{2}{1+\sqrt{1-4(y+x-x^2)}} \tag{23}$$

Proof. We get from the recurrence relation (16) that

$$(1-2x)\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} \tag{24}$$

Furthermore, $P_0(x) = P(x,0) = (1-x)^{-1}$. The proposed $P(x,y)$ satisfies (24) and the initial condition, so it is the solution. $\square$

Theorem 7. With the notations above,

$$\begin{aligned} c_{0,n}^{(2)} &= \frac{(2n)!}{(n+1)!} \\ c_{1,n}^{(2)} &= c_{0,n+1}^{(2)} = \frac{(2(n+1))!}{(n+2)!} \\ c_{2,n}^{(2)} &= \frac{1}{2}c_{0,n+2}^{(2)} - c_{0,n+1}^{(2)} = \frac{1}{16}\frac{(2\,n^2+6\,n+3)\,2^{2\,n+6}\Gamma(n+3/2)}{(n+3)\sqrt{\pi}\,(n+2)} \end{aligned} \tag{25}$$

Thus, the sequences $(c_{0,n}^{(2)})_{n=0}^{\infty}$ and $(c_{1,n}^{(2)})_{n=0}^{\infty}$ are translations of the sequence A001761 in The On-Line Encyclopedia of Integer Sequences [2] (OEIS).

Proof. We have that

$$P(0,y) = \frac{2}{1+\sqrt{1-4y}} \tag{26}$$

which is the well-known ordinary generating function for the Catalan numbers. This proves the formula for $c_{0,n}^{(2)}$. The recurrence (15) gives $c_{1,n}^{(2)} = c_{0,n+1}^{(2)}$ and $c_{2,n}^{(2)} = \frac{1}{2}c_{0,n+2}^{(2)} - c_{0,n+1}^{(2)}$. Combining these two results, and simplifying, yields the theorem. $\square$

References

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