On the Graphs of Hoffman-Singleton and Higman-Sims

Paul R. Hafner

Department of Mathematics
University of Auckland
Auckland (New Zealand)
hafner@math.auckland.ac.nz

Submitted: Mar 2, 2004; Accepted: Aug 27, 2004; Published: Nov 3, 2004
MR Subject Classifications: 05C62, 05C25; 05B25, 51E10, 51E26
Keywords: Hoffman-Singleton graph, Higman-Sims graph, Higman-Sims group, biaffine plane, S(3,6,22)

Abstract

We propose a new elementary definition of the Higman-Sims graph in which the 100 vertices are parametrised with $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and adjacencies are described by linear and quadratic equations. This definition extends Robertson's pentagon-pentagram definition of the Hoffman-Singleton graph and is obtained by studying maximum cocliques of the Hoffman-Singleton graph in Robertson's parametrisation. The new description is used to count the 704 Hoffman-Singleton subgraphs in the Higman-Sims graph, and to describe the two orbits of the simple group HS on them, including a description of the doubly transitive action of HS within the Higman-Sims graph. Numerous geometric connections are pointed out. As a by-product we also have a new construction of the Steiner system $S(3,6,22)$.

1 Introduction

The Higman-Sims graph is the unique strongly regular graph whose parameters are $(100, 22, 0, 6)$, i.e. it is a graph of order 100, regular of degree 22; it is triangle-free (any two adjacent vertices have 0 common neighbours), and any two non-adjacent vertices have exactly 6 neighbours in common. This graph made its first official appearance [23] in the context of the construction of the sporadic simple group HS which is a subgroup of index 2 in the automorphism group of the graph (note Section 13 for a comment on the history).

In this paper we provide a new and elementary construction of the Higman-Sims graph, combining a geometric interpretation [16] of Robertson's pentagon-pentagram construction of the Hoffman-Singleton graph with the known construction of the Higman-Sims graph via maximum cocliques in the Hoffman-Singleton graph. We demonstrate the flavour of the construction by exploring some automorphisms, counting the Hoffman-Singleton subgraphs and describing the doubly transitive action of degree 176 of the sporadic simple group HS from within the Higman-Sims graph. These applications build also on the combinatorial studies of the Hoffman-Singleton graph by James [30]. The key result of [30] is that any 5-cycle in the Hoffman-Singleton graph determines a split of the graph into two sets of five 5-cycles which are related by Robertson's pentagon-pentagram equations. It will become apparent that this extends to the Higman-Sims graph: every 5-cycle determines a split into four sets of five 5-cycles.

Structure of the paper. We give some background information about the Hoffman-Singleton graph in Section 2. Section 3 contains our new construction of the Higman-Sims graph. A first verification that the given graph is the Higman-Sims graph is given as Theorem 1 whose proof is left as an exercise. Section 4 introduces some of the automorphisms of the graph which can be used to show that the Higman-Sims graph is in fact a Cayley graph. These automorphisms also give a hint of the remarkable symmetries of this graph. Sections 5 and 6 show how to derive the new definition from the description of the Higman-Sims graph as (modified) incidence graph of the vertices of the Hoffman-Singleton graph and one family of its maximum cocliques. This is achieved by extending the parametrisation of the Hoffman-Singleton graph in Definition 1 to a parametrisation of the maximum cocliques, allowing adjacencies (incidences and certain intersection properties) to be expressed in the form of simple equations (Fig. 2) without any reference to cocliques. Along the way we highlight some properties of maximum cocliques in the Hoffman-Singleton graph. The well-known existence of two families of maximum cocliques (containing 50 cocliques each) is captured very effectively by our parametrisation. Section 7 extends our definition of the Higman-Sims graph to a graph of order 150 which encapsulates everything about maximum cocliques of the Hoffman-Singleton graph. In Section 8 we show how to count the Hoffman-Singleton subgraphs in the Higman-Sims graph and characterise their two orbits under HS by means of certain intersection numbers. Section 9 is a brief sidetrack to demonstrate that some classical gems are explicitly present in the Higman-Sims graph: from the correspondence between lines of $\mathrm{PG}(3,2)$ and triples of a 7-element set to (almost) the exceptional isomorphism between the alternating group A_8 and $PSL(4,2)$, as well as the $\mathrm{Alt}(7)$ and $\mathrm{Alt}(8)$ geometries. In Section 10 we demonstrate the doubly transitive action of HS on 176 points as it manifests itself within the Higman-Sims graph. Section 11 picks up the geometric theme again, showing that the adjacencies of the Higman-Sims graph can be understood in terms of geometric relationships between points, lines, conics and dual conics in a biaffine plane, with strong connections to Wild's semibiplanes [49]. In Section 12 we highlight a decomposition of the Higman-Sims graph into 5 isomorphic subgraphs of order 20, concluding with a brief historical note in Section 13.

In the remainder of this introduction, we give a brief overview of some constructions of the Higman-Sims graph, and establish the notational conventions for the rest of the paper.

Constructions of the Higman-Sims graph. The original construction by Higman and Sims [23] is based on the Steiner system $S(3,6,22)$. This construction is visible in Fig. 4, if one considers only $H_3 \cup H_2$. In a variation on this theme, [2] begins with the projective plane of order 4 and effectively incorporates some of the construction steps of $S(3,6,22)$ into the construction of the Higman-Sims graph. Elsewhere [18], we will describe the extension of $S(2,5,21)$ to $S(5,8,24)$ from within the Higman-Sims graph.

It is known that the maximum cocliques of the Hoffman-Singleton graph form a graph with two connected components (each isomorphic to the Hoffman-Singleton graph) if adjacency is defined by disjointness. This allows to construct the Higman-Sims graph either by introducing additional edges between those cocliques which meet in 8 vertices, or else one can take the original Hoffman-Singleton graph together with one of the connected components of the max-coclique graph, defining further adjacencies by incidence. This latter approach is the basis of our new construction. A neat unification of these methods leads to a graph of order 150 ([5], p.108, [6], p.394, cf. Section 7 below).

In [35] Mathon and Street present 'the first elementary construction of the Higman-Sims graph, starting from scratch without having to refer to cocliques in the Hoffman-Singleton graph.' Their interesting construction should be seen as describing an occurrence of the Higman-Sims graph in an unexpected place, perhaps stretching the meaning of the word 'elementary'. Another elementary description of the Higman-Sims graph is its representation as a Cayley graph, found independently by Heinze [21], Jørgensen-Klin [32] and Praeger-Schneider [44] (cf. Theorem 3).

Apart from the Cayley graph construction, there are other group-theoretic approaches to the Higman-Sims graph, for example [10]. In Remark 28 we will indicate a construction based on an incidence graph combined with a group action.

Hughes [27] uses semisymmetric 3-designs, while Yoshiara [52] has a construction of the Higman-Sims graph with vertices in the Leech lattice. A comprehensive description of the Higman-Sims graph (and G. Higman's related geometry) in the Leech lattice appears in R.A. Wilson's paper [51].

Notation and Terminology. The following notation will be used throughout this paper:

$\mathbb{Z}_5$ denotes the field of order 5, $\mathbb{Z}_5^*$ its multiplicative group.

G will always denote the graph defined in Definition 2 (which is the Higman-Sims graph, cf. Theorem 1 and Remark 6).

V_i $(i = 0, \ldots, 5)$ are sets of 25 elements $(i, x, y), x, y \in \mathbb{Z}_5$; elements $(0, x, y) \in V_0$ will sometimes be referred to as *point vertices*, and in Section 5 just as points (x, y). Similarly for *line vertices* $(1, m, c) \in V_1$; these are also referred to as "the line $y = mx + c$" in Section 5.

H, H_1, H_2, H_3 denote Hoffman-Singleton graphs: H and H_1 will have $V_0 \cup V_1$ as vertex set; the vertex set of H_2 is $V_2 \cup V_3$ and for H_3 it is $V_4 \cup V_5$.

K denotes the supergraph of order 150, defined in Section 7; vertex set: $V_0 \cup \cdots \cup V_5$.

$\mathrm{Aut}(X)$ denotes the automorphism group of a graph X.

HS denotes the index 2 subgroup of $\mathrm{Aut}(G)$, consisting of all *even* permutations of the vertices of G (cf. Remark 14 and Section 10); this is the Higman-Sims group.

g, h are special automorphisms of the Higman-Sims graph, defined in Lemma 12.

In [16] we introduced the term *affine automorphism* to denote automorphisms of H which preserve the partition $\{V_0, V_1\}$ (they are induced by collineations or correlations of the biaffine plane).
A set of five disjoint 5-cycles with no further edges between any vertices will be denoted by $5C_5$.

Numbering of items: there are three distinct numbering schemes: Remarks and Lemmas are in one sequence; Definitions and Theorems each have a sequence of their own.

Web resources for this paper: some MAGMA [3] files and links related to this paper are available at [15].

2 The Hoffman-Singleton Graph

The Hoffman-Singleton graph is the unique Moore graph of degree 7 [26, 6]. There are essentially three constructions of this graph which may be described succinctly as "1 + 7 + 42", "15+35", and "25+25". For our purposes, Robertson's [45] pentagon-pentagram construction ("$25 + 25$") with the geometric interpretation in the affine plane AG(2,5) from [16] is pivotal and given as Definition 1 below. The "$15 + 35$" construction is related to the projective space PG$(3, 2)$ and will come into focus in Section 8 and Remark 45, whilst the Moore graph definition ("$1 + 7 + 42$") is visible in Fig. 4 (H_3).

Definition 1. The Hoffman-Singleton graph H has vertex set $\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and the following edges:

$$
\begin{array}{lllll}
(0,x,y) & \text{is adjacent to} & (0,x,y') & \text{if and only if} & y-y'=\pm 1; \qquad (1)\\
(1,m,c) & \text{is adjacent to} & (1,m,c') & \text{if and only if} & c-c'=\pm 2; \qquad (2)\\
(0,x,y) & \text{is adjacent to} & (1,m,c) & \text{if and only if} & y=mx+c. \qquad (3)
\end{array}
$$

In [16] we showed that the geometry of the pentagon-pentagram construction does not lie in the pentagons and pentagrams but in the adjacency rules $y = mx + c$. Under this geometric point of view the Hoffman-Singleton graph is the incidence graph of a biaffine plane with pentagons and pentagrams as additional edges. (A biaffine plane is an affine plane with one parallel class of lines — the 'vertical' lines, in our coordinatisation — omitted. These structures inherit the best features of both projective and affine geometry: duality and parallelism.) In this spirit, we refer to vertices $(0, x, y)$ as *points* and to vertices $(1, m, c)$ as *lines*. Fig. 1 summarises Definition 1, introducing also the notation $V_0 = \{(0, x, y) \colon x, y \in \mathbb{Z}_5\}, V_1 = \{(1, m, c) \colon m, c \in \mathbb{Z}_5\}$.

We recall from [16] that two parallel lines $(1, m, c)$ and $(1, m, c')$ of H are adjacent if and only if their points of intersection $(0, x, y)$ and $(0, x, y')$ with any vertical line are non-adjacent. A particular consequence of this is the existence of 125 5-cycles in H which consist of two adjacent points on a vertical line and three consecutive lines, e.g. $(0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 0, 3), (1, 0, 0)$ (and the same with 3 points and 2 lines). Each of these 5-cycles determines a distinct split of H into a pair of $5C_5$ (cf. [30] or [16]) which

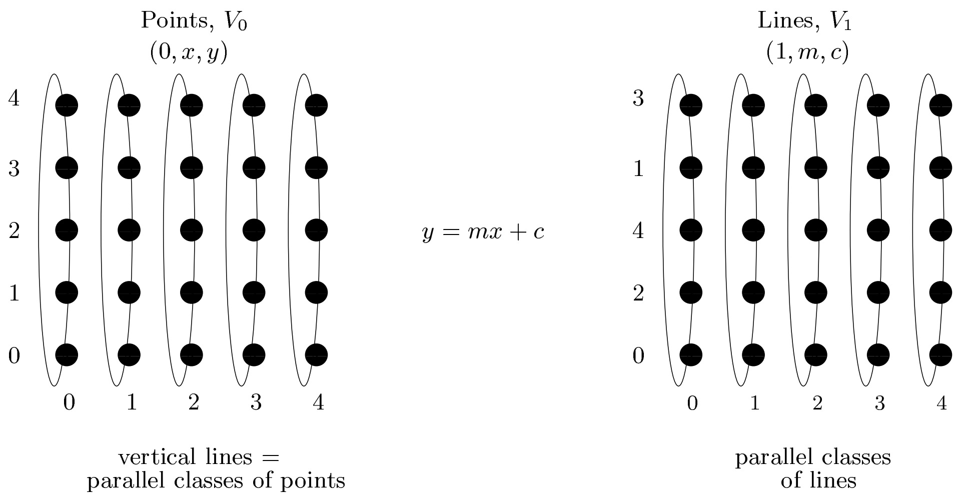

Figure 1: Geometric interpretation of Robertson's description of the Hoffman-Singleton graph

can be labelled as in Fig. 1 with the same adjacency rules. The 2-fold transitivity of the automorphism group of the Hoffman-Singleton graph on the 126 splits now follows from the transitivity on these special 5-cycles of the group of affine automorphisms (which stabilises the obvious split into the given pair of $5C_5$).

Remark 1. We note at this point that the enumeration of 5-cycles in [16] shows that any 5-cycle of H has 0, 2, 3, or 5 vertices in common with V_0. Now assume that $\{V_0', V_1'\} \neq \{V_0, V_1\}$ is a split of H into a pair of complementary $5C_5$ and that V_0' contains one of the above-mentioned special 5-cycles with exactly two consecutive vertices u, v on a vertical line, with the remaining three consecutive vertices r, s, t all in a fixed parallel class of lines (in V_1). Then r, s, t have distinct Neighbours on each of the 5-cycles in V_0, and therefore no vertical line can contribute more than 2 vertices to V_0'. Therefore $|V_0' \cap V_0| = 10$ in this case. Dually, if V_0' contains a 5-cycle with 3 consecutive points on a vertical line then $|V_0' \cap V_0| = 15$. This will be useful in the proof of Lemma 34.

The biaffine plane underlying our description of the Hoffman-Singleton graph inherits a duality from the projective geometry into which it can be embedded. An example of such a mapping ψ which interchanges points and lines and preserves all adjacencies is

$$(0, x, y) \overset{\psi}{\mapsto} (1, x, 2y), \qquad (1, m, c) \overset{\psi}{\mapsto} (0, 3m, 2c). \tag{4}$$

Whenever we need to interchange points and lines, we might use a phrase like 'by duality'.

Remark 2. This paper as well as its precursor [16] can be seen under the following general viewpoint. When the Petersen graph is viewed as a pair of 5-cycles, one immediately sees 20 of its automorphisms (dihedral group for the cycle, and swapping the cycles). The full automorphism group, however, has order 120, due to the fact that there are 6 distinct ways of choosing a pair of 'opposite' 5-cycles. The same holds for the Hoffman-Singleton graph: looking at the split into points and lines of a biaffine plane, one immediately sees 2000 affine automorphisms; the full automorphism group, however, has order 252 000 because there are 126 distinct splits into points and lines of a biaffine plane. We will note the same for the Higman-Sims graph later in this paper: when we consider the Higman-Sims graph as a pair of Hoffman-Singleton graphs, we can immediately see 252 000 automorphisms. But the total number of automorphisms is $352 \cdot 252\,000$, since there are 352 ways of splitting the Higman-Sims graph into a pair of Hoffman-Singleton graphs. The same phenomenon was observed [19] on a graph of order 32, the smallest of the McKay-Miller-Širáň graphs for $q = 2$.

The remainder of this section deals with non-affine automorphisms of the Hoffman-Singleton graph, showing how they arise from automorphisms of the Petersen graph. It is obvious that any of the 5-cycles of V_0 together with any of the 5-cycles of V_1 induce a Petersen graph in H. When considering automorphisms of H, we might therefore look at extending automorphisms of a Petersen graph.

Lemma 3. *Let P be a Petersen subgraph of H. Then every automorphism of P can be extended to an automorphism of H in exactly four ways.*

Proof. Implicit in the uniqueness proof [30] of the Hoffman-Singleton graph H is a proof that $\mathrm{Aut}(H)$ is transitive on the 525 Petersen subgraphs of H and that we may assume the vertices of P to be $(0,0,0), \ldots, (0,0,4)$, $(1,0,0), \ldots, (1,0,4)$. Then it follows from the orbit-stabiliser theorem that the stabiliser of P in $\mathrm{Aut}(H)$ has order $252\,000/525 = 480$. The identity of P has 4 extensions to an automorphism of H, since we are free to choose an eigenvalue in the horizontal direction (4 possibilities). Therefore the stabiliser of P induces 120 distinct automorphism of P, i.e. every automorphism of P can be extended to an automorphism of H. □

Remark 4. We give an example of a (non-affine) automorphism of P, and an extension to H, since this will be useful later on. It is easy enough to construct an automorphism of P: just choose any 5-cycle, and find its complementary cycle. We indicate this by listing the images of the vertices of P in a scheme according to Fig. 1. We also list the image of the additional vertex $(0,1,3)$. This image was determined as follows: the unique neighbour of $(0,1,3)$ in P is $(1,0,3)$ which is mapped to $(1,0,4)$ under the automorphism of P. Therefore the image of $(0,1,3)$ must be one of the 4 neighbours of $(1,0,4)$ outside P; our choice was $(0,4,4)$. For better orientation we have labelled the rows as they are labelled in Fig. 1, cycles in the left hand block V_0 being labelled differently from those in the right hand block V_1.

$$\begin{array}{cccccccccccccc}
 & {\scriptstyle (4)} & 103 & . & . & . & . & 104 & . & . & . & . & {\scriptstyle (3)} & \\
 & {\scriptstyle (3)} & 101 & 044 & . & . & . & 004 & . & . & . & . & {\scriptstyle (1)} & \\
(V_0) & {\scriptstyle (2)} & 001 & . & . & . & . & 003 & . & . & . & . & {\scriptstyle (4)} & (V_1) \\
 & {\scriptstyle (1)} & 000 & . & . & . & . & 002 & . & . & . & . & {\scriptstyle (2)} & \\
 & {\scriptstyle (0)} & 100 & . & . & . & . & 102 & . & . & . & . & {\scriptstyle (0)} &
\end{array} \tag{5}$$

The construction of the automorphism of H is now mechanical (and best left to a computer, although it is easy enough to do it by hand). The key ingredient is that H is an $\mathrm{srg}(50,7,0,\mathbf{1})$, and that if one starts with a subgraph X of H which contains P and at least one more vertex, one obtains all of H by successively adding common neighbours of pairs of non-adjacent vertices. (The Petersen graph, being an $\mathrm{srg}(10,3,0,1)$, is closed under the operation of taking 'midpoints' of non-adjacent vertices.) For example, to determine the image of $v=(1,3,0)$, note that v is the unique common neighbour of $(0,0,0)$ and $(0,1,3)$, both of whose images are already known: $(1,0,0)$ and $(0,4,4)$. The image of v must therefore be the unique common neighbour of these two vertices, i.e. $(0,4,0)$. After a bit of work one obtains the following automorphism of the graph H. The significance of the boldface entries will be explained in Section 10.

	(4)	**103**	**143**	123	133	**113**	**104**	021	**011**	**041**	031	(3)		
	(3)	101	044	034	024	014	**004**	**130**	110	140	**120**	(1)		
(V_0)	(2)	001	**132**	**142**	**112**	**122**	003	013	033	023	043	(4)	(V_1)	
	(1)	000	134	144	114	124	002	**121**	**141**	**111**	**131**	(2)		
	(0)	**100**	012	**022**	**032**	042	102	020	010	040	030	(0)		
														(6)

3 A New Definition of the Higman-Sims Graph

Definition 2. Throughout this paper, G is the graph with vertex set $\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_5$ and adjacencies defined as follows (cf. Figure 2):

$$
\begin{array}{rlrcl r}
(0,x,y) & \text{is adjacent to} & (0,x,y') & \Leftrightarrow & y-y'=\pm1; & (7)\\
(1,m,c) & \text{is adjacent to} & (1,m,c') & \Leftrightarrow & c-c'=\pm2; & (8)\\
(2,A,B) & \text{is adjacent to} & (2,A,B') & \Leftrightarrow & B-B'=\pm1; & (9)\\
(3,a,b) & \text{is adjacent to} & (3,a,b') & \Leftrightarrow & b-b'=\pm2; & (10)\\
(0,x,y) & \text{is adjacent to} & (1,m,c) & \Leftrightarrow & y=mx+c; & (11)\\
(1,m,c) & \text{is adjacent to} & (2,A,B) & \Leftrightarrow & c=2(m-A)^2+B; & (12)\\
(2,A,B) & \text{is adjacent to} & (3,a,b) & \Leftrightarrow & B=2A^2+3aA-a^2+b; & (13)\\
(3,a,b) & \text{is adjacent to} & (0,x,y) & \Leftrightarrow & y=(x-a)^2+b; & (14)\\
(0,x,y) & \text{is adjacent to} & (2,A,B) & \Leftrightarrow & y=3x^2+Ax+B\pm1; & (15)\\
(1,m,c) & \text{is adjacent to} & (3,a,b) & \Leftrightarrow & c=m^2-am+b\pm2. & (16)
\end{array}
$$

We further define

$$V_i=\{i\}\times\mathbb{Z}_5\times\mathbb{Z}_5 \quad (i=0,\dots,3). \tag{17}$$

Remark 5. The definition is summarised in Fig. 2; each of the four sets $V_0,\dots,V_3$ consists of five 5-cycles. They are indicated in the corners of the square, with labels '(±1)'

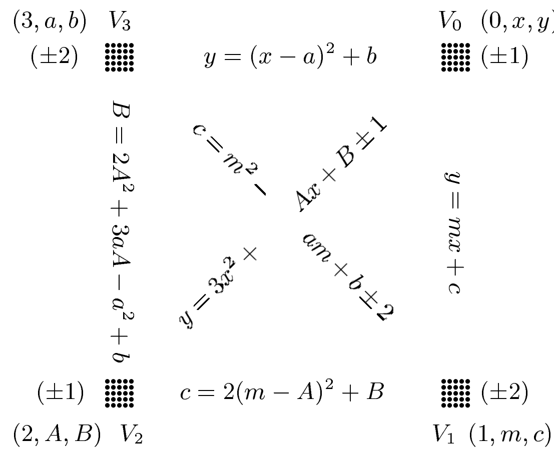

Figure 2: Higman-Sims 'à la Robertson'

to indicate pentagon 5-cycles, and labels '(± 2)' to indicate pentagram 5-cycles (cf. equations (1), (2), (7)–(10)). Equations between the four sets contain the rules of adjacency. The sets V_0 and V_1 together induce a Hoffman-Singleton graph as described in Definition 1. This subgraph is denoted by H_1 throughout the paper.

Theorem 1. *The graph G is strongly regular with parameters $(100, 22, 0, 6)$. This implies that G is the Higman-Sims graph, by the uniqueness theorem of Gewirtz [12].*

Remark 6. The proof of Theorem 1 is an exercise in solving quadratic equations over $\mathbb{Z}_5$ and can be tackled head-on. We leave the details to the reader. In Section 5 we will take a more gentle approach which indicates how the description given above is obtained, relating it to maximum cocliques in the Hoffman-Singleton graph. This shows that G is the Higman-Sims graph, without having to rely on the characterisation by Gewirtz. Alternatively, one can avoid the use of the theorem of Gewirtz by establishing that given a vertex x of G, the edges between vertices at distance 1 and 2 from x form the incidence graph of a $S(3,6,22)$; as shown in [1], p. 273, this can be achieved by an ingenious application of a result by Majindar [34] on block intersections. We note that our construction of the Higman-Sims graph provides also a new construction of $S(3,6,22)$.

As a further alternative, Corollary 22 proves that G is the Higman-Sims graph based on its construction from maximum cocliques in the Hoffman-Singleton graph. The construction from $S(3,6,22)$ is visible in Fig. 4, $H_1 \cup H_3$.

It should be pointed out that the proof of Theorem 1 becomes simpler if one makes use of Remark 8 below, as well as taking advantage of the automorphisms which we describe in Section 4. To show that G is triangle-free, one invokes the fact that the Hoffman-Singleton graph is triangle-free and proves by a simple calculation that there do not exist 3 vertices v_0, v_1, v_2 with $v_i \in V_i$ which form a triangle, nor do there exist any $v_0, v_1 \in V_0, v_2 \in V_2$ forming a triangle. Similarly, when proving that non-adjacent vertices u, v have 6 common neighbours, only the following cases need to be considered: (1) $v, w \in V_0$, belonging to the same 5-cycle of V_0; (2) $v, w \in V_0$, belonging to distinct 5-cycles of V_0; (3) $v \in V_0$, $w \in V_1$; (4) $v \in V_0$, $w \in V_2$.

For a different angle on this, we refer to Remark 25 and Lemma 26.

Remark 7. Alerted by the geometric interpretation of Definition 1, the attentive reader will have noted that for $(i, j) \in \{(0, 2), (0, 3), (1, 2), (1, 3)\}$ adjacencies between V_i and V_j correspond to incidences of certain points or lines with certain parabolas or dual parabolas. Less obvious is that adjacencies between V_2 and V_3, as well as those within V_2 and V_3, indicate disjointness of certain sets (cf. Corollary 22). Geometric interpretations of *all* adjacencies between V_i and V_j $(i \neq j)$ are found in Theorem 6.

Remark 8. Any two consecutive sets V_i and V_{i+1} (where subscripts are taken modulo 4) induce a subgraph of G which is isomorphic to the Hoffman-Singleton graph. We demonstrate this for $i = 2$: the two sets V_2 and V_3 each induce five 5-cycles, the first one arranged as pentagons, the second one arranged as pentagrams, as in the case of V_0 and V_1. The vertices $(2, A, B)$ and $(3, a, b)$ are adjacent if and only if $Y = MA + C$ where $Y = B - 2A^2$, $C = b - a^2$, $M = 3a$. Thus, after choosing the 0-point on each 5-cycle appropriately (additive adjustments), and after permuting the 5-cycles in V_3 (multiplication by 3), we get the equations which define the Hoffman-Singleton graph in Definition 1.

Remark 9. The 'diagonal' subgraphs of order 50 induced in G by $V_0 \cup V_2$ and by $V_1 \cup V_3$ have automorphism groups of order 2000, isomorphic to the group of the affine transformations of the Hoffman-Singleton graph (cf. [16]). See Section 11 for more.

4 Some Automorphisms of G

Automorphisms ϕ of H which map V_0 to itself are mappings $(0, x, y) \mapsto (0, x', y')$ where $(x, y) \mapsto (x', y')$ is an affine transformation whose linear part has $(0, 1)$ as eigenvector with eigenvalue ± 1:

$$(x, y) \stackrel{\phi}{\mapsto} (x, y) \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} + (e, f) = (rx + e, sx + ty + f), \tag{18}$$

where $r, t \in \mathbb{Z}_5^*$, $s, e, f \in \mathbb{Z}_5, t = \pm 1$. Such transformations can be extended readily to automorphisms of the Hoffman-Singleton graph H_1 (cf. [16]). If we stipulate further that $r^2 = t$, the mapping ϕ can be extended to an automorphism of G, preserving each of the sets V_i $(i = 0, \ldots, 3)$. Note that $(0, x, y) \mapsto (0, x, -y)$ can be extended to an automorphism of H_1, but not to an automorphism of G.

Theorem 2. *Let $r, s, t, e, f \in \mathbb{Z}_5$, $t = \pm 1$ and $r^2 = t$. The mapping $\phi\colon V_0 \to V_0$ defined by*

$$(0, x, y)^\phi = (0, rx + e, sx + ty + f) \tag{19}$$

can be extended to an automorphism ϕ of G by defining:

$$(1,m,c)^\phi = (1, rm+rst, -rme+tc+f-rest), \tag{20}$$
$$(2,A,B)^\phi = (2, rA-e+rst, -rAe+tB+f-rest-2e^2), \tag{21}$$
$$(3,a,b)^\phi = (3, ra+e+2rst, sa+tb+f+s^2t). \tag{22}$$

Proof. Verifications are by direct calculation and are left to the reader. □

Remark 10. The formulas are found by determining how the lines $y = mx + c$ and parabolas $y = (x-a)^2 + b$ and $c = 2(m-A)^2 + B$ transform when the points are transformed as in (19). Then one only needs to check that the adjacencies between V_0 and V_2 and between V_1 and V_3 are preserved.

We note that the condition $t = \pm 1$ is needed in order to preserve the (vertical) 5-cycles in V_0, and the condition $r^2 = t$ is needed to preserve the family of parabolas $y = (x-a)^2 + b, (a, b \in \mathbb{Z}_5)$, and thus the adjacencies between V_0 and V_3. After sections 5 and 6 we will see this in a different light: preservation of a family of maximum cocliques of H_1.

Remark 11. The square of the duality ψ of H introduced in (4) is $(0,x,y) \overset{\psi^2}{\mapsto} (0,3x,-y)$, $(1,m,c) \overset{\psi^2}{\mapsto} (1,3m,-c)$ and satisfies the hypotheses of Theorem 2. Therefore ψ^2 can be extended to an automorphism of G: $(2,A,B) \overset{\psi^2}{\mapsto} (2,3A,-B)$, $(3,a,b) \overset{\psi^2}{\mapsto} (3,3a,-b)$. Clearly, ψ^2 and its extension to G have order 4. It is not hard to find that ψ itself can be extended to an automorphism of G (of order 8) which interchanges V_0 with V_1 and V_2 with V_3 by defining $(2,A,B) \overset{\psi}{\mapsto} (3,3A,2B)$, $(3,a,b) \overset{\psi}{\mapsto} (2,a,2b)$. The automorphism ψ^4 is an involution whose fixed-point set of order 20 is the set W_0 defined in Section 12.

The following Lemma introduces two further automorphisms which will allow us to show that G is a Cayley graph (Theorem 2).

Lemma 12. *Define mappings $g, h\colon G \to G$ by*

$(0,x,y)$	$\overset{g}{\mapsto}$	$(0,x+1,y-x)$	$(0,x,y)$	$\overset{h}{\mapsto}$	$(1,2x,2y-2x^2)$
$(1,m,c)$	$\overset{g}{\mapsto}$	$(1,m-1,c-m+1)$	$(1,m,c)$	$\overset{h}{\mapsto}$	$(2,m,2c-2m^2)$
$(2,A,B)$	$\overset{g}{\mapsto}$	$(2,A-2,-A+B-1)$	$(2,A,B)$	$\overset{h}{\mapsto}$	$(3,-A,2B)$
$(3,a,b)$	$\overset{g}{\mapsto}$	$(3,a-1,-a+b+1)$	$(3,a,b)$	$\overset{h}{\mapsto}$	$(0,2a,2b+2a^2)$

Then g is an automorphism of order 5 which fixes each of the sets $V_0, \ldots, V_3$, and h is an automorphism of order 4 of G which cyclically permutes $V_0, \ldots, V_3$.

The proof is left as a computational exercise. The automorphism h confirms our earlier observation that the four sides of the square in Fig. 2 are Hoffman-Singleton graphs. In conjunction with Theorem 2 it shows that G is vertex transitive. The automorphism g is an extension of an affine self-mapping of V_0 as described in Theorem 2.

Remark 13. Considering the automorphism h and its powers, we note that for $v \in V_i \cup V_{i+1}$, the neighbours of v in $V_{i+2} \cup V_{i+3}$ (subscripts modulo 4) form a coclique of order 15 in the Hoffman-Singleton subgraph induced in G by $V_{i+2} \cup V_{i+3}$.

Theorem 3. *(Heinze [21], Jørgensen-Klin [32], Praeger and Schneider [44]) The Higman-Sims graph is a Cayley graph.*

Proof. We can obtain a direct proof of this result from our explicit knowledge of the automorphisms g and h. Note first that g and $k = h^{-1}gh$ together generate an elementary abelian group of order 25 which acts transitively on V_0. (It may help to observe that $h^{-1}kh = g^{-1}$.) It is easy to see that $\langle g, h \rangle$ has order 100 and is a regular group of automorphisms of G. An abstract definition of this group by generators and relations as well as a suitable generator set are given on [15]. $\square$

Remark 14. We note that, as a permutation, h is product of 25 cycles of length 4 and hence an *odd* permutation. This implies that the automorphism group of the Higman-Sims graph contains a subgroup of index 2, consisting of the automorphisms which are even permutations. This subgroup is the sporadic simple group HS.

Remark 15. Anticipating notation and results that will be introduced later, we note that an automorphism of H_1 can be extended to an automorphism of all of G if and only if it preserves the families $\mathcal{F}_1$ and $\mathcal{F}_2$ of maximum cocliques of H_1.

5 Maximum Cocliques in the Hoffman-Singleton Graph

We will now derive a description of the maximum cocliques in the Hoffman-Singleton graph as sets of parabolas in the biaffine plane. It is well-known that maximum cocliques in the Hoffman-Singleton graph are of order 15; this can be proved via eigenvalues ([14], Theorem 2.12) or via an application of the Cauchy-Schwarz inequality ([31, 13]). It will also be a by-product of Lemma 16.

In this section we will use geometric notation and terminology as much as possible. In particular, we will refer to the 'point vertices' $(0, x, y)$ of the Hoffman-Singleton graph as *points* (x, y), and a 'line vertex' $(1, m, c)$ will be referred to as *the line* $y = mx + c$. We remind the reader that 'a vertical line' consists of vertices $(0, x, y)$ on a 5-cycle of V_0, with x constant, $y \in \mathbb{Z}_5$ (and that vertical lines are not lines of our biaffine plane).

Lemma 16. *A coclique of order $n \geq 15$ in the Hoffman-Singleton graph consists either of 5 points, one from each vertical line, and 5 pairs of non-adjacent parallel lines, or of 5 pairs of non-adjacent points on vertical lines and 5 lines, one from each parallel class. In particular, the order of a maximum coclique in the Hoffman-Singleton graph is 15.*

Proof. Let C be a coclique of order $n \geq 15$ in the Hoffman-Singleton graph. Since each parallel class of lines is a 5-cycle, there can be at most two elements of each class in a coclique. In particular, C must contain at least 5 points and at least 5 lines.

Case 1: C contains one pair of non-adjacent parallel lines. We may assume that they are the horizontal lines $y = 0$ and $y = 1$, since otherwise an adjacency-preserving affine transformation can bring us into this situation. Then none of the points $(x, 0)$ and $(x, 1)$, $x \in \mathbb{Z}_5$, can belong to C. Now C contains at least 3 more lines, and we may assume that one of them is $y = mx, (m \neq 0)$ —represented by the vertex $(1, m, 0)$— otherwise we perform a translation $x \mapsto x + r$. The line $y = mx$ is incident with the points $(2/m, 2)$ and $(4/m, 4)$. Consequently, there can be at most one point in C whose first coordinate is $2/m$ because $(2/m, 0)$, $(2/m, 1)$, $(2/m, 2)$ do not belong to C, and the points $(2/m, 3)$ and $(2/m, 4)$ are adjacent. Similarly, C can contain only one point with first coordinate $4/m$.

It follows that if C contains a pair of non-adjacent parallel lines then C contains at most 8 points, and therefore at least two pairs of non-adjacent parallel lines. By duality, the analogous statement with points and lines interchanged is also valid.

Case 2: C contains two pairs of non-adjacent parallel lines, say $y = 0$, $y = 1$, $y = mx$ and $y = mx + 1$. The line $y = mx$ meets the horizontal lines $y = 0, 1$ in $(0, 0)$ and $(1/m, 1)$ respectively. Then the line $y = mx + 1$ passes through $(1/m, 2)$, allowing only one of the adjacent points $(1/m, 3)$ and $(1/m, 4)$ to belong to C. Similarly, only one of the points on the vertical line $x = 4/m$ can belong to C. Our four lines have 4 distinct points of intersection with each of the vertical lines $x = 2/m$ and $x = 3/m$, so that C cannot contain a pair of points from these vertical lines either.

It follows that if C contains two non-adjacent parallel lines then C can contain at most 1 pair of points from a vertical line. Since we established above that it is impossible for a maximum coclique to contain precisely one such pair of points, we conclude that if C contains one pair of parallel lines then C contains 5 pairs of parallel lines, but at most one point from each vertical line. By duality, if C contains 2 points on a vertical line, then C contains 5 such pairs and no pairs of parallel lines. In particular, we have established that the order of a maximum coclique in the Hoffman-Singleton graph is 15. □

Lemma 17. *Let C be a maximum coclique in the Hoffman-Singleton graph, and assume that C consists of 5 points p_i and 10 lines ℓ_j ($i = 1, \ldots, 5$, $j = 1, \ldots, 10$). Then no three of the points are collinear in the biaffine plane.*

Proof. We note that the 10 lines must be partitioned into pairs of non-adjacent parallel lines. Assume that p_1, p_2, p_3 are collinear points of C, incident with the line ℓ with equation $y = 0$. This is no loss of generality since we can always use an admissible affine transformation to transform a given line into ℓ. Then the lines $y = mx + c$ $(m \neq 0)$ in C must pass through the remaining two points of ℓ, say $(x_4, 0)$ and $(x_5, 0)$. If $y = mx + c_1$ and $y = mx + c_2$ are two such parallel lines, their adjacency is governed by the difference $c_1 - c_2 = m(x_4 - x_5)$. There are exactly two values for m which will make this a square and two which make it a non-square in $\mathbb{Z}_5^*$. This means that C contains at most 3 pairs of non-adjacent lines, a contradiction. □

Corollary 18. *Assume that C is a coclique in the Hoffman-Singleton graph consisting of 5 points and 10 lines. Then the 5 points form one of the sets with equation $y = \pm(x-a)^2+b$ in the biaffine plane.*

Proof. The 5 points together with the point of intersection of the vertical lines form an oval in the projective plane over $\mathbb{Z}_5^*$. By Segre's theorem [46], this is a conic. Since the line at infinity is a tangent, this conic is a parabola $y = r(x-a)^2+b, r \neq 0$ (in the affine plane). The 10 lines in the coclique represented by vertices $(1, m, c)$ are non-adjacent in the graph H if and only if $r = \pm 1$, because the 3 values $(b,\, b+r,\, b-r)$ of $r(x-a)^2+b$ $(x \in \mathbb{Z}_5)$ are consecutive mod 5 if and only if $r = \pm 1$. In that case, any pair of parallel lines which do not meet the parabola intersect a vertical line in adjacent points, and consequently the lines are non-adjacent in the graph (cf. the comment in the introduction). □

Corollary 19. *There are exactly 100 distinct cocliques of order 15 in the Hoffman-Singleton graph.*

Proof. We know from Corollary 18 that there are at most $50 = 2 \cdot 5 \cdot 5$ cocliques of order 15 containing exactly 5 points and 10 lines, and by duality there are at most 50 containing exactly 5 lines. It is an easy exercise to verify that the passants of the parabola $y = (x-a)^2 + b$ are the 10 lines $y = mx + c$, where $c = m^2 - ma + b \pm 2$; in other words: for any $a, b \in \mathbb{Z}_5$ the 15 vertices

$$\begin{array}{ll} (0, x, (x-a)^2 + b), & x \in \mathbb{Z}_5 \\ (1, m, m^2 - ma + b + 2), & m \in \mathbb{Z}_5, \\ (1, m, m^2 - ma + b - 2), & m \in \mathbb{Z}_5, \end{array}$$

form a 15-coclique in the Hoffman-Singleton graph. Similarly, the passants of the parabola $y = -(x-e)^2 + f$ are given by the 10 lines $y = mx + c$ where $c = -m^2 - me + f \pm 2$.

Dually, the 5 lines $y = mx + c$ where $c = 2(m-E)^2 + F$ avoid all the points (x, y) where $y = 3x^2 + Ex + F \pm 1$; and the 5 lines $y = mx + c$ where $c = 3(m+E)^2 - F$ avoid all the points (x, y) where $y = 2x^2 - Ex - F \pm 1$. □

This leads us to define the following 15-element sets for $a, b, A, B, e, f, E, F \in \mathbb{Z}_5$:

$$P(a,b) = \{(0, x, (x-a)^2 + b)\colon x \in \mathbb{Z}_5\} \cup \{(1, m, m^2 - ma + b \pm 2)\colon m \in \mathbb{Z}_5\}, \tag{23}$$

$$Q(A,B) = \{(1, m, 2(m-A)^2 + B)\colon m \in \mathbb{Z}_5\} \cup \{(0, x, 3x^2 + Ax + B \pm 1)\colon x \in \mathbb{Z}_5\}; \tag{24}$$

$$P'(e,f) = \{(0, x, -(x-e)^2 + f)\colon x \in \mathbb{Z}_5\} \cup \{(1, m, -m^2 - me + f \pm 2)\colon m \in \mathbb{Z}_5\}, \tag{25}$$

$$Q'(E,F) = \{(1, m, 3(m+E)^2 - F)\colon m \in \mathbb{Z}_5\} \cup \{(0, x, 2x^2 - Ex - F \pm 1)\colon x \in \mathbb{Z}_5\}. \tag{26}$$

These sets $P(a,b)$ etc. are the 100 maximum cocliques of the Hoffman-Singleton graph, grouped into 4 sets of 25.

6 Two Families of Maximum Cocliques and the Max-coclique Graph

We define 2 families of 50 maximum cocliques of the Hoffman-Singleton graph as follows:

$$\mathcal{F}_1 \quad = \{P(a,b) \colon a,b \in \mathbb{Z}_5\} \cup \{Q(A,B) \colon A,B \in \mathbb{Z}_5\}, \tag{27}$$

$$\mathcal{F}_2 \quad = \{P'(e,f) \colon e,f \in \mathbb{Z}_5\} \cup \{Q'(E,F) \colon E,F \in \mathbb{Z}_5\}. \tag{28}$$

These two families can be transformed into each other by means of the affine automorphism of H induced by $(0,x,y) \mapsto (0,x,-y)$. Each of them is invariant under dualities of the Hoffman-Singleton graph. An intrinsic characterisation of these two families is obtained by looking at the cardinality of intersections of their members:

Lemma 20. *Let $X \in \mathcal{F}_1$, and let Y be any maximum coclique of the Hoffman-Singleton graph. Then*

$$Y \in \mathcal{F}_1 \qquad \textit{if and only if} \quad |X \cap Y| \in \{0,5\}, \tag{29}$$

$$Y \in \mathcal{F}_2 \qquad \textit{if and only if} \quad |X \cap Y| \in \{3,8\}. \tag{30}$$

The corresponding result with $\mathcal{F}_1$ and $\mathcal{F}_2$ interchanged also holds.

Proof. We give a sample calculation. Let $a,b,e,f \in \mathbb{Z}_5$ and consider the two cocliques $X = P(a,b)$ and $Y = P'(e,f)$ (cf. (23) and (25)). One constructs two quadratic equations to determine their common points; after minimal manipulations they are:

$$2x^2 - 2(e+a)x + a^2 + e^2 + b - f = 0,$$
$$2m^2 + m(e-a) + b - f + \nu + \mu = 0, \quad \text{where} \quad \nu,\mu \in \{2,-2\}.$$

The discriminants are $\Delta_1 = (a-e)^2 + 2(b-f)$ and $\Delta_2 = (e-a)^2 + 2(b-f) + 2(\nu+\mu) = \Delta_1 + 2\nu + 2\mu$, respectively.

It is impossible that Δ_1 and Δ_2 are both ± 2 for all choices of ν,μ, hence X and Y cannot be disjoint.

When $\Delta_1 = \pm 1$ then X and Y have 2 vertices of V_0 in common, and consideration of the possibilities for Δ_2 reveals 3 combinations of μ,ν such that $\Delta_2 = \pm 1$ (and one combination yielding $\Delta_2 = \pm 2$). Hence X and Y have $3 \cdot 2 = 6$ vertices of V_1 in common, for an intersection of cardinality 8 in total.

When $\Delta_1 = 0$ then X and Y have 1 common neighbour in V_0, and consideration of the possibilities for Δ_2 reveals 2 combinations of μ,ν such that $\Delta_2 = 0$ (and 2 combinations yielding $\Delta_2 = \pm 2$). Hence X and Y have 2 vertices of V_1 in common for an intersection of cardinality 3 in total. $\square$

Remark 21. If one goes through all the detail of the preceding proof, the following stronger result is obtained: for given $X \in \mathcal{F}_1$ the number of maximum cocliques Y with $|X \cap Y| = 0,3,5,8,15$ is $7,35,42,15,1$, respectively.

Theorem 4 (The max-coclique graph). *Let Z be the graph whose vertex set is $\mathcal{F}_1 \cup \mathcal{F}_2$ and adjacency is defined by disjointness. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ each induce a connected component of Z, each of the components being isomorphic to the Hoffman-Singleton graph.*

Proof. Lemma 20 and Remark 21 imply that there are two connected components. It remains to show that $\mathcal{F}_1$ induces a Hoffman-Singleton graph. The case for $\mathcal{F}_2$ then follows by applying the automorphism induced by the affine transformation $(0, x, y) \mapsto (0, x, -y)$. Thinking of the sets $P(a, b)$ and $Q(A, B)$ as unions of parabolas in the (x, y)- and (m, c)-coordinate systems, it is obvious that $P(a, b) \cap P(r, s) = \emptyset$ if and only if $r = a$ and $s = b \pm 2$. Similarly, $Q(A, B) \cap Q(R, S) = \emptyset$ if and only if $R = A$ and $S = B \pm 1$. Next we observe that $P(a, b) \cap Q(A, B) = \emptyset$ if and only if none of the following 4 equations (in x or m) has a solution in $\mathbb{Z}_5$:

$$(x - a)^2 + b = 3x^2 + Ax + B \pm 1,$$
$$2(m - A)^2 + B = m^2 - am + b \pm 2.$$

The discriminants of these equations equal ± 2 if and only if $B = 2A^2 + 3aA - a^2 + b$.

Now we see that $\mathcal{F}_1$ induces a component which is isomorphic to the subgraph of G induced by $V_2 \cup V_3$: just identify the cocliques $P(a, b)$, $Q(A, B)$ with the vertices $(3, a, b)$ and $(2, A, B)$ of G, respectively. We have seen in Remark 8 that this graph is isomorphic to the Hoffman-Singleton graph. □

Corollary 22. *(1) The graph G is isomorphic to the graph whose vertex set is $H \cup \mathcal{F}_1$, where adjacency in H is defined as in Definition 1, adjacency in $\mathcal{F}_1$ is disjointness of cocliques, and $v \in H$ is adjacent to $X \in \mathcal{F}_1$ if and only if $v \in X$.*

(2) The graph G is isomorphic to the graph whose vertex set is $H \cup \mathcal{F}_2$, where adjacency in H is defined as in Definition 1, adjacency in $\mathcal{F}_2$ is disjointness of cocliques, and $v \in H$ is adjacent to $X \in \mathcal{F}_2$ if and only if $v \in X$.

(3) Each coclique in $\mathcal{F}_i, i \in \{1, 2\}$, contains 15 vertices of H, and each vertex of H is contained in 15 cocliques of $\mathcal{F}_i$.

Proof. For (1), recall the definition of $\mathcal{F}_1$ (cf. (27)) in equations (23) and (24), and note that membership of $(0, x, y)$ or $(1, m, c)$ in a coclique of $\mathcal{F}_1$ is governed by the same equations as those defining adjacency between vertices of $V_0 \cup V_1$ and $V_2 \cup V_3$ in G (cf. Fig. 2 and equations (7)–(16)). Adjacency between vertices of $\{P(a, b)\colon a, b \in \mathbb{Z}_5\}$ and $\{Q(A, B)\colon A, B \in \mathbb{Z}_5\}$ was shown in the proof of Theorem 4 to be governed by the same equations that apply in $V_2 \cup V_3$. The reader will have no difficulty confirming the 5-cycle adjacencies (labelled ± 1 and ± 2 in Fig. 2) within $\{Q(A, B)\colon A, B \in \mathbb{Z}_5\}$ and $\{P(a, b)\colon a, b \in \mathbb{Z}_5\}$.

(2) Application of the automorphism of H induced by $(0, x, y) \mapsto (0, x, -y)$ maps $\mathcal{F}_1$ to $\mathcal{F}_2$ and preserves disjointness of cocliques and incidence of vertices with cocliques.

(3) One part of this statement is evident from the definition of $\mathcal{F}_i$, and the other follows by applying the automorphism h^2 (cf. Lemma 12), bearing in mind Theorem 4. □

Remark 23. It follows from James [30] that every 5-cycle of the Hoffman-Singleton graph H_1 determines a split of H_1 into a pair of $5C_5$, and that whenever $\{V_0', V_1'\}$ is a split of H_1 into a pair of $5C_5$ then there exists an automorphism of H_1 mapping V_0 to V_0' and V_1 to V_1'. This implies that given a 15-coclique C of H_1 and *any* split of H_1 into a pair $\{V_0', V_1'\}$ of $5C_5$, the number of elements of C in V_0' will be 5 or 10; and the intersection of C with *any* 5-cycle of H_1 will consist of 1 or 2 vertices.

Corollary 22 shows that there is not only an automorphism of H_1 but also of G which maps V_0 to V_0' and V_1 to V_1' (because V_2 and V_3 are identified with cocliques in $\mathcal{F}_1$ and adjacencies determined by membership and disjointness).

Remark 24. Corollary 22 characterises the cocliques of $\mathcal{F}_1$ as the sets of neighbours in H_1 of vertices in H_2. Application of h and its powers shows more generally that if $v \in V_i \cup V_{i+1}$ then the neighbours of v in $V_{i+2} \cup V_{i+3}$ (indices mod 4) form a coclique in the Hoffman-Singleton subgraph induced by $V_{i+2} \cup V_{i+3}$; i.e. edges between $V_i \cup V_{i+1}$ and $V_{i+2} \cup V_{i+3}$ correspond to incidence of vertices of a Hoffman-Singleton subgraph and its maximum cocliques.

Remark 25. We now look back at Theorem 1 in the light of Corollary 22. To see that G is triangle-free, consider three vertices v_1, v_2, v_3 of G. If all three are contained in H_1 or in its complement H_2, they cannot form a triangle because the Hoffman-Singleton graph has girth 5. If not all three belong to H_1 or to H_2, we may assume that $v_1 \in H_1$, $v_2, v_3 \in H_2$ (apply the automorphism h^2 if need be). If v_2 and v_3 are adjacent, then their neighbourhoods in H_1 are disjoint cocliques, which means that v_1 cannot belong to both of them.

To see that any two non-adjacent vertices v_1, v_2 of G have exactly 6 common neighbours, assume first that both vertices belong to H_1. There is a unique common neighbour v in H_1, and we may assume that v_1, v, v_2 belong to a vertical 5-cycle. Now our knowledge of maximum cocliques makes it evident that there are exactly 5 of them which contain v_1 and v_2 (i.e. there are exactly 5 common neighbours of v_1 and v_2 in H_2). If, secondly, $v_1 \in H_1$, $v_2 \in H_2$, we prove the following result (the first part is contained in Neumaier's Proposition 3 or Jeurissen's Lemma 6.2) by a simple calculation.

Lemma 26. *(Neumaier [38], Jeurissen [31]) Let v be a vertex of the Hoffman-Singleton graph and C a maximum coclique in $\mathcal{F}_1$ not containing v.*

(1) There exist exactly 3 cocliques in $\mathcal{F}_1$ which contain v and which are disjoint from C.

(2) There are exactly 3 vertices in C which are adjacent to v.

The same is true if $\mathcal{F}_1$ is replaced by $\mathcal{F}_2$ throughout.

Proof. We first consider the case where $v = (0, x, y) \in V_0$, assuming that C consists of all neighbours in H_1 of $w = (2, A, B) \in V_2$, with $y = 3x^2 + Ax + B + \delta$ and $\delta \in \{0, \pm 2\}$ (non-adjacency). If $\delta = 0$, then v and C have two common neighbours $(2, A, B \pm 1)$ in V_2, and otherwise only one. Any further common neighbours in $V_2 \cup V_3$ must belong to V_3 and satisfy the equation

$$-x^2 + 2ax - a^2 + y = B - 2A^2 + 2aA + a^2$$

(refer to Fig. 2). Substitute $y = 3x^2 + Ax + B + \delta$ into this quadratic equation for a, and note that its discriminant is -2δ. This shows that there are 1 resp. 2 further neighbours common to v and w in H_2, according to whether $\delta = 0$ or $\delta = \pm 2$. In the language of maximum cocliques of H_1 this means that there are exactly three cocliques of $\mathcal{F}_1$ which contain v and which are disjoint from C.

The proof of (1) in the case where $v = (0, x, y) \in V_0$, $w = (3, a, b) \in V_3$, with $y = (x - a)^2 + b + \delta$ and $\delta \neq 0$ is similar: if $\delta = \pm 1$ then v and w have no common neighbours in V_3, and if $\delta = \pm 2$, v and w have 1 common neighbour in V_3. Vertices $(2, A, B)$ which are adjacent to v as well as w must satisfy the equation

$$(x - a)^2 + b + \delta = 3x^2 + Ax + B \pm 1$$

where $B = 2A^2 + 3aA - a^2 + b$. The discriminant of the resulting quadratic equation for A is $3(\delta \pm 1)$. This shows that there are 3 solutions in the case when $\delta = \pm 1$ and 2 solutions in the case when $\delta = \pm 2$, so that in each case we have 3 common neighbours of v and w in H_2.

To see that part (2) is merely a 'dual' of part (1), consider the Hoffman-Singleton graph H_1 in G. The vertex v has 15 neighbours in H_2, forming a coclique D (and each vertex of D representing a coclique in H_1). The coclique C in H_1 consists of all neighbours in H_1 of a vertex $w \in H_2$. Now apply part (1) in H_2: there exist precisely 3 vertices in D which are adjacent to w. These three vertices represent cocliques containing v which are disjoint from C.

To get the same results for $C \in \mathcal{F}_2$, we extend the affine mapping $(0, x, y) \mapsto (0, x, -y)$ to an automorphism of H which swaps $\mathcal{F}_1$ and $\mathcal{F}_2$. $\square$

Remark 27. The previous lemma looks very much like a parallel axiom in a geometry made up of the vertices of H as points and maximum cocliques from $\mathcal{F}_1$ as blocks. This incidence structure, a partial 5-geometry, was first observed by Neumaier [38]. In the terminology of [28] it is a semisymmetric design D with parameters $(50, 15, [5])$ while [35] speaks of an SPBIBD$(50, 15; 0, 5)$. We note that [38] has a realisation of the incidence graph of this design in the Leech lattice, but also describes it in terms of the 100 maximum cocliques in the Hoffman-Singleton graph.

Remark 28. Here is another approach to constructing the Higman-Sims graph: let T be the incidence graph of the semisymmetric design D defined in Remark 27 (whose automorphism group A is isomorphic to the automorphism group of the Hoffman-Singleton graph); now add one of the edges $\{x, y\}$ of H as well as all the images of $\{x, y\}$ under A. The result is the Higman-Sims graph — the new edges producing a vast increase in the order of the automorphism group.

For future reference, we include the following simple result about vertex stabilisers without proof.

Lemma 29. *Let H be the Hoffman-Singleton graph and denote by $Aut^+(H)$ the subgroup of $Aut(H)$ consisting of those automorphisms which preserve the two families $\mathcal{F}_1$ and $\mathcal{F}_2$.*

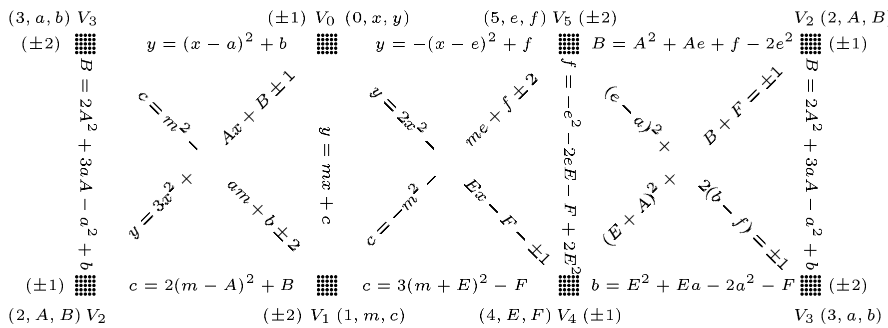

Figure 3: The supergraph K (note the wraparound, and twist)

1. *The stabiliser of a vertex of H in Aut(H) is the symmetric group S_7 in its natural action on the neighbours of the vertex.*

2. *The stabiliser of a vertex of H in Aut$^+(H)$ is the alternating group A_7.*

7 The Supergraph

We will now give an explicit construction of a 'supergraph' K of order 150, constructed from 3 Hoffman-Singleton graphs which are linked cyclically, so that removal of any one of them produces a graph isomorphic to G. This graph K is mentioned in [5], p.108, [6], p. 394. It provides an ideal environment for the study of maximum cocliques in the Hoffman-Singleton graph (and therefore for the study of the Higman-Sims graph), as we shall see.

The vertex set of the graph K is $V_0 \cup \cdots \cup V_5$, with adjacencies on $V_0 \cup \cdots \cup V_3$ defined as before (vertices in $V_2 \cup V_3$ representing 15-cocliques of $\mathcal{F}_1$). A second Higman-Sims graph is constructed on $V_0 \cup V_1 \cup V_4 \cup V_5$ (vertices in $V_4 \cup V_5$ representing 15-cocliques of $\mathcal{F}_2$), using equations (25) and (26). Finally, we define vertices of $u \in V_2 \cup V_3$ and $v \in V_4 \cup V_5$ to be adjacent when they have 8 common neighbours in H_1 (i.e. when their corresponding 15-cocliques intersect in 8 vertices). The resulting graph is described in Fig. 3; note that the figure wraps around, but a twist is needed when identifying the left and right. To see that the graph induced by $V_2 \cup \cdots \cup V_5$ is isomorphic to the Higman-Sims graph G, one shows that the neighbours of vertices in H_2 form 15-cocliques in H_3, and that cocliques corresponding to adjacent vertices of H_2 are disjoint. We omit the calculations. (H_1, H_2, H_3 are the subgraphs of order 50 induced by $V_0 \cup V_1$, $V_2 \cup V_3$, and $V_4 \cup V_5$ respectively. All three are isomorphic to the Hoffman-Singleton graph.)

Remark 30. In the graph G, only those automorphisms of H_1 which preserve the families $\mathcal{F}_1$, $\mathcal{F}_2$ can be extended to an automorphism of G. By contrast, every automorphisms of H_1 can be extended to an automorphism of K. Those automorphisms of H_1 which interchange $\mathcal{F}_1$ and $\mathcal{F}_2$ will swap H_2 and H_3. The full group of automorphisms of K has order $3 \cdot 252\,000 = 756\,000$.

8 Hoffman-Singleton Subgraphs of G

In this section we study the Hoffman-Singleton subgraphs of the Higman-Sims graph G. The structures revealed in the process will be discussed further in Section 9. Lemmas 31 and 32 show that $\mathrm{Aut}(G)$ is transitive on Hoffman-Singleton subgraphs; but there are two orbits under the action of the subgroup HS (which consists of the even permutations amongst the automorphisms).

Lemma 31. *Let Z be an srg$(100, 22, 0, 6)$ and assume that X is an induced Hoffman-Singleton subgraph of Z. Then $Y = Z \setminus X$ is also a Hoffman-Singleton graph. The neighbours in X of a vertex $v \in Y$ form maximum cocliques in X which intersect in 0 or 5 vertices. Therefore they all belong to the same family of maximum cocliques of X and hence there exists an automorphism of Z mapping X to Y.*

Proof. The statement of the lemma gives sufficient indication of the proof. We note that the lemma is a slight modification of Exercise 2 in [8], p. 113, but with a different approach to the proof. □

Lemma 32. *Let X, Y be Hoffman-Singleton subgraphs of G. If $\tau \in Aut(G)$ is an even permutation of degree 100 such that $X^\tau = Y$ then* all *automorphisms of G which map X to Y are even.*

Proof. Let $\sigma \in \mathrm{Aut}(G)$ be another automorphism of G with $X^\sigma = Y$. Then $\sigma\tau^{-1}$ belongs to the stabiliser of X in $\mathrm{Aut}(G)$, which is a simple group (cf. Remark 49) and therefore contains only even permutations. □

Theorem 5. *The graph G contains exactly 704 Hoffman-Singleton subgraphs.*

We split the proof into a sequence of lemmas which at the same time will help us become more familiar with the graphs G and K.

The following lemma allows the reconstruction of the Higman-Sims graph as in Fig. 2, starting from a 5-cycle. It is stated without proof.

Lemma 33. *Let $F = \{v_0, \ldots, v_4\}$ be a 5-cycle in V_0, where v_i is adjacent to $v_{i\pm1}$ (subscripts modulo 5). Then V_2 is the union of the sets of common neighbours outside F of v_i and v_{i+2}, $i = 0, \ldots, 4$.*

Lemma 34. *Let τ be an automorphism of G which preserves H_1 and H_2. Then $|V_3^\tau \cap V_3| = |V_0^\tau \cap V_0|$.*

Proof. If $V_0^\tau = V_0$ or V_1, the statement is obvious. Assume therefore that $|V_0^\tau \cap V_0| = 15$ (cf. Remark 1; the case where this intersection contains 10 vertices follows by duality). In view of Lemma 33 we find V_0^τ by choosing a 5-cycle F in V_0^τ and collecting the common neighbours outside F of vertices at distance 2 in F. Now we show that each 5-cycle of V_2^τ has 2 vertices in V_2. To this end, consider the case when τ maps some 5-cycle of V_0 to

$$(0, 0, 0), (0, 0, 1), (0, 0, 2), (1, 0, 2), (1, 0, 0).$$

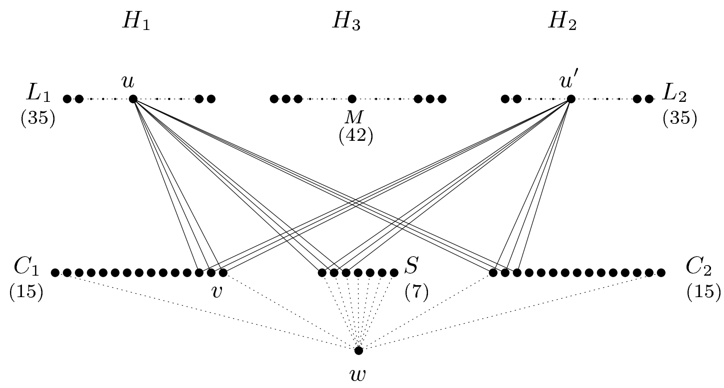

Figure 4: The graph K, as seen from the vertex w.

All other cases follow from this one by extending affine automorphisms of H_1 which fix V_0. Applying Lemma 33 one finds readily that one of the 5-cycles in V_2^τ is

$$(2,0,0),(2,0,1),(2,0,2),(3,0,2),(3,0,0).$$

Therefore $|V_2^\tau \cap V_2| = 15$ and hence $|V_3^\tau \cap V_3| = 15 = |V_0^\tau \cap V_0|$. $\square$

Lemma 35. *There are (at least) 125 pairs $\{X_1, X_2\}$ of Hoffman-Singleton subgraphs of G such that*

$$X_1 \cup X_2 = G, \quad |X_1 \cap V_0| = |X_1 \cap V_3| = 15, \quad \textit{and} \quad |X_1 \cap V_1| = |X_1 \cap V_2| = 10.$$

Proof. By Remark 23 there exists an automorphism τ of G such that $V_0^\tau = V_0'$, and $V_1^\tau = V_1'$. By Lemma 34, $X_1 = V_3^\tau \cup V_0^\tau$ and $X_2 = V_2^\tau \cup V_3^\tau$ have the required properties. Since there are 125 splits of H_1 into pairs of $5C_5$ other than $\{V_0, V_1\}$, we have the numerical result. We add that our final census (after Lemma 39) of Hoffman-Singleton subgraphs will show that the estimate of 125 pairs with the desired properties is sharp (hence the parentheses). $\square$

Together with the 2 Hoffman-Singleton subgraphs H_1 and H_2, Lemma 35 produces 252 Hoffman-Singleton subgraphs. A further 252 of them are obtained by applying the automorphism h. To find another 200 Hoffman-Singleton subgraphs, we consider Fig. 4. It shows the supergraph K as seen from a vertex $w \in H_3$; the graph H_3 appears as the Moore graph of degree 7, with S as neighbours of w. The remaining $30 = 15 + 15$ neighbours of w form maximum cocliques $C_1 \subset H_1$ and $C_2 \subset H_2$ (C_1 is an $\mathcal{F}_2$-coclique). The (set-theoretic) complements of these cocliques in H_1 and H_2 respectively are L_1 and L_2. Note that for $i = 1, 2$ the subgraph $S \cup C_i$ is a coclique, since $H_i \cup H_3$ induces a Higman-Sims graph and thus is triangle-free.

Lemma 36. *For $u' \in L_2$ denote by $N(u')$ the set of neighbours of u' in S. Let T be the set of triples of elements in S. Then*

$$u' \overset{\phi}{\mapsto} N(u')$$

determines a bijection ϕ between L_2 and T. A similar bijection can be defined between L_1 and T.

Proof. Consider a vertex $u' \in L_2$. Since the subgraph of K induced by $H_2 \cup H_3$ is isomorphic to G and therefore an srg$(100, 22, 0, 6)$ (Theorem 1), the vertices u' and w have 6 neighbours in common, with exactly 3 of them in C_2 by Lemma 26. Consequently, u' has precisely 3 neighbours in S.

By Remark 30, if τ is an automorphism of H_3 which fixes w and which preserves the two families of maximum cocliques of H_3 then τ can be extended to an automorphism of K which maps H_1 to H_1 and H_2 to H_2. The restriction of τ to S is A_7 (Lemma 29), and therefore transitive on triples of elements of S. Since there are 35 elements of L_2 and $\binom{7}{3} = 35$ triples of elements of S, the Higman-Sims graph induced by $H_2 \cup H_3$ in K provides an explicit representation of a one-to-one correspondence between L_2 and triples from S. The same reasoning applies to L_1. $\square$

Remark 37. We can explore the action of A_7 on S a little further, considering orbits of pairs of triples of elements of S (which correspond to edges or non-edges in L_2). One counts 70 pairs of disjoint triples, 315 pairs of triples which intersect in one point, and 210 pairs of triples which intersect in two points. Since the subgraph induced by L_2 is regular of degree 4, and therefore has 70 edges, we conclude that two vertices of L_2 are adjacent precisely then when their corresponding triples in S are disjoint (edges must correspond to a union of orbits of A_7 on pairs of triples from S, since A_7 acts as a group of automorphisms of L_2).

We anticipate Lemma 43 which identifies L_2 with the set of lines of PG$(3, 2)$ and note that PG$(3, 2)$ contains 315 pairs of intersecting lines. Our list of orbit cardinalities for pairs of triples of elements from S now implies that the bijection ϕ associates intersecting lines with triples having exactly one point in common. In other words: this famous correspondence [20] is hardwired into the Higman-Sims graph.

Composing the two bijections from Lemma 36 we obtain a bijection $u \leftrightarrow u'$ between L_1 and L_2 which we introduce formally in the following definition.

Definition 3. For $u \in L_1$ we define $u' \in L_2$ to be the vertex which has the same 3 neighbours in S as u.

Lemma 38. *Let $u \in L_1$. Then u and u' have the same 3 neighbours in each of C_1, S, and C_2. Moreover, if $z \in L_1$, then u and z' are adjacent if and only if u' and z are adjacent.*

Proof. Let $w = (4, 0, 0)$ and consider 3 of its neighbours in H_3, say $v_1 = (4, 0, -1), v_2 = (4, 0, 1), v_3 = (5, 0, 0)$ (any three neighbours will do, but they can always be transformed into these by an automorphism of K fixing w, since A_7 is transitive on triples from S). One finds easily that $u = (0, 0, 0)$ and $u' = (2, 0, 0)$ are the two vertices of L_1 resp. L_2 having v_1, v_2, v_3 as common neighbours. As one calculates the three neighbours of u, u' in C_1, C_2 respectively, it turns out that $(0, 0, -1), (0, 0, 1), (1, 0, 0)$ and $(2, 0, -1), (2, 0, 1), (3, 0, 0)$ are common neighbours of u and u', and that there are no other edges between any of the vertices considered.

Now we let A_7 operate. C_1 and C_2 are invariant, the orbit of u is L_1, and u and u' are not adjacent. Therefore there is never an edge between a vertex $x \in L_1$ and its counterpart $x' \in L_2$, and x and x' always share the same neighbours in C_1 and in C_2. In addition, observe that for any $u, z \in L_1$ there is an edge between u and z' if and only if there is an edge between u' and z (an even permutation of S which swaps the triples corresponding to u and z can be extended to an automorphism of K which must interchange u and z, as well as u' and z', preserving the presence or absence of any edges). □

When considering the graph $G = H_1 \cup H_2$, we can think of it embedded in the supergraph K. The neighbours of $w \in H_3$ form an $\mathcal{F}_2$-coclique C_1 in H_1, together with a maximum coclique C_2 of H_2. These two cocliques are paired in a natural way: each vertex of C_2 has precisely 8 neighbours in C_1, and each vertex in C_1 has precisely 8 neighbours in C_2 (cf. Lemma 20—note that the neighbours in H_1 of a vertex in H_2 form an $\mathcal{F}_1$-coclique).

Lemma 39. *Let C_1 be an $\mathcal{F}_2$-coclique in H_1, $w \in H_3$ the corresponding vertex of H_3, and C_2 the associated maximum coclique of H_2. Then the mapping $\tau\colon G \to G$ which interchanges u and u' for all $u \in L_1$ and fixes each vertex of C_1 and of C_2 is an automorphism of G. Moreover, τ is an odd permutation, hence $\tau \notin HS$.*

Proof. This follows immediately from Lemma 38: adjacencies between L_1 resp. L_1^τ and C_1, C_2 are unaffected, and τ is compatible with adjacencies between L_1 and L_2. □

Proof of Theorem 5. We have seen that $L_2 \cup C_1$ is a Hoffman-Singleton subgraph of G. Since there are 50 possibilities to choose w, and hence 50 possibilities to choose C_1 and C_2, we have found 50 more Hoffman-Singleton graphs. They are all distinct from the 504 which we have already, since the intersections with V_0, V_1 have cardinalities $5, 10$ or $10, 5$ respectively (C_1, like any 15-coclique, meets V_0 in 5 or 10 vertices and V_1 in 10 or 5). Now apply h, h^2, h^3 to finally obtain the total of 704 Hoffman-Singleton subgraphs. The next two Lemmas show that there are no further Hoffman-Singleton subgraphs: the total number of 5-cycles in G is 443 520 (Lemma 41), each contained in two Hoffman-Singleton subgraphs. Therefore the number of Hoffman-Singleton subgraphs is at most $2 \cdot 443\,520/1\,260 = 704$. □

Lemma 40. *Let Z be an srg$(100, 22, 0, 6)$ and let F be a 5-cycle of Z. Then F is contained in 0 or 2 Hoffman-Singleton subgraphs of Z.*

Proof. Assume that Z contains a Hoffman-Singleton subgraph H which contains F. In view of Lemma 31 we may assume that $Z = G$, that H is the subgraph induced by $V_0 \cup V_1$, and that F is one of the five 5-cycles in V_0 (otherwise we apply an automorphism of H which can be extended to G). Then it is clear that F belongs to two Hoffman-Singleton subgraphs, namely to H and also to the subgraph induced by $V_0 \cup V_3$. To see that there are no other Hoffman-Singleton subgraphs containing F, we note first that every vertex of G outside V_0 has a neighbour in F, and each vertex of V_2 has 2 neighbours in F. The latter means that no vertex of V_2 can be part of a Hoffman-Singleton graph X containing F; the former shows that X must contain V_0, since X must contain four 5-cycles without neighbours in F. Additional 5-cycles of X which are disjoint from V_0

must therefore consist of vertices in $V_1 \cup V_3$ and must contain at least one edge $\{u, v\}$ from V_1 (without loss of generality). Since each 5-cycle in V_3 contains 4 neighbours of this edge, it is impossible to construct five disjoint 5-cycles forming a $5C_5$ which are not entirely contained in V_1 or V_3. $\square$

Lemma 41. *Assume that Z is a strongly regular graph with parameters $(100, 22, 0, 6)$.*

1. *Z contains $443\,520$ pentagons, $22\,176$ through each vertex.*

2. *Z contains $28\,875$ 4-cycles, 1155 through each vertex.*

Proof. Let v be a vertex of Z. To count the 5-cycles through v, choose two neighbours of v, a and b. Since Z is triangle-free, there are exactly 6 neighbours of a which are also adjacent to b: these must be avoided, otherwise the pentagon v, a, x, y, b contains a triangle x, y, a. That leaves $22 - 6 = 16$ neighbours of a which are not adjacent to b, each of which allows 6 ways to finish off a pentagon. Hence there are $\binom{22}{2} \cdot 16 \cdot 6 = 22\,176$ pentagons through a given vertex v. The total number of pentagons in Z is therefore $22\,176 \cdot 100/5 = 443\,520$.

Now we count the 4-cycles. Any two non-adjacent vertices in Z have 6 common neighbours. Choosing two of these neighbours determines a 4-cycle with the given non-edge as diagonal. Z is regular of degree 22, and therefore has 1100 edges and $\binom{100}{2}/2 - 1100 = 3850$ non-edges. Each of these non-edges is a diagonal in $\binom{6}{2} = 15$ 4-cycles. As each 4-cycle has two diagonals, we find that the total number of 4-cycles is $3\,850 \cdot 15/2 = 28\,875$ (or 1155 such cycles through each vertex). $\square$

To conclude this section, we list the intersections of the 704 Hoffman-Singleton subgraphs of G with the sets $V_0, \ldots, V_3$.

Lemma 42. *The following table lists the cardinalities of the intersections of the 704 Hoffman-Singleton subgraphs of G with the sets $V_0, \ldots, V_3$. Rows 1 and 2 belong to one of the HS-orbits, rows 3 and 4 to the other. The last row lists the number of Hoffman-Singleton graphs with each intersection pattern above it. It is evident that members X of the HS-orbit of H_1 are characterised by the fact that $(|X \cap H_1|, |X \cap H_2|) \in \{(20, 30),\ (30, 20),\ (0, 50),\ (50, 0)\}$, whilst the corresponding cardinalities are*

$(15,35),(35,15),(25,25)$ *for the other HS-orbit.*

$$
\begin{array}{|c||c||c|c||c|}
\hline
 & (a) & (b) & (c) & (d) \\
\hline
\begin{array}{|c|c|}\hline V_3 & V_0 \\ \hline V_2 & V_1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 15 & 10 \\ \hline 15 & 10 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 10 & 5 \\ \hline 20 & 15 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 20 & 15 \\ \hline 10 & 5 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 0 & 25 \\ \hline 0 & 25 \\ \hline\end{array} \\
\hline
\begin{array}{|c|c|}\hline V_3 & V_0 \\ \hline V_2 & V_1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 10 & 15 \\ \hline 10 & 15 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 15 & 20 \\ \hline 5 & 10 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 5 & 10 \\ \hline 15 & 20 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 25 & 0 \\ \hline 25 & 0 \\ \hline\end{array} \\
\hline\hline
\begin{array}{|c|c|}\hline V_3 & V_0 \\ \hline V_2 & V_1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 10 & 10 \\ \hline 15 & 15 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 5 & 15 \\ \hline 10 & 20 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 15 & 5 \\ \hline 20 & 10 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 25 & 25 \\ \hline\end{array} \\
\hline
\begin{array}{|c|c|}\hline V_3 & V_0 \\ \hline V_2 & V_1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 15 & 15 \\ \hline 10 & 10 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 15 & 5 \\ \hline 20 & 10 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 20 & 10 \\ \hline 15 & 5 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 25 & 25 \\ \hline 0 & 0 \\ \hline\end{array} \\
\hline\hline
\# & 125 & 25 & 25 & 1 \\
\hline
\end{array}
\tag{31}
$$

(The first column of the table indicates that the intersection numbers are listed in accordance with Fig. 2.)

Proof. The description of the Hoffman-Singleton subgraphs in the proof of Theorem 5 leads immediately to all the entries of the table. Together with Lemma 32 it also justifies the claim about the two orbits. Addition of the entries for V_0 and V_1 does the rest. □

9 Actions of A_7, $PSL(4,2)$, and A_8

Fig. 4 and its description in Section 8 show the alternating group A_7 acting simultaneously on sets of cardinalities 7, 15, 35, and 42 (S; C_1 and C_2; L_1 and L_2; M). In this section we point out some well-known facts which manifest themselves in this view of the Higman-Sims graph. For additional guidance, we include the distance distribution diagram around w as Fig. 5.

We will indicate proofs of the various statements, to make clear how they all flow from this source. Alternative approaches to these themes can be found in [9, 11, 20].

Lemma 43. *The edges between C_2 and L_2 give the point-line incidence graph of the projective geometry $PG(3,2)$ with 15 points and 35 lines.*

Proof. Since H_2 is an srg$(50,7,0,1)$ and C_2 is a coclique in it, any two distinct vertices in C_2 have a unique common neighbour in L_2. To verify Veblen's axiom, one can proceed as follows. Note first that $v=(1,0,0)\in C_1$ has 8 neighbours in C_2 (v and w are adjacent, hence their respective cocliques in H_2 meet in 8 vertices). It follows that v has 7 neighbours in L_2; they are

$$(2,0,0),(3,0,2),(3,0,-2),(3,1,2),(3,2,-2),(3,3,-2),(3,4,2).$$

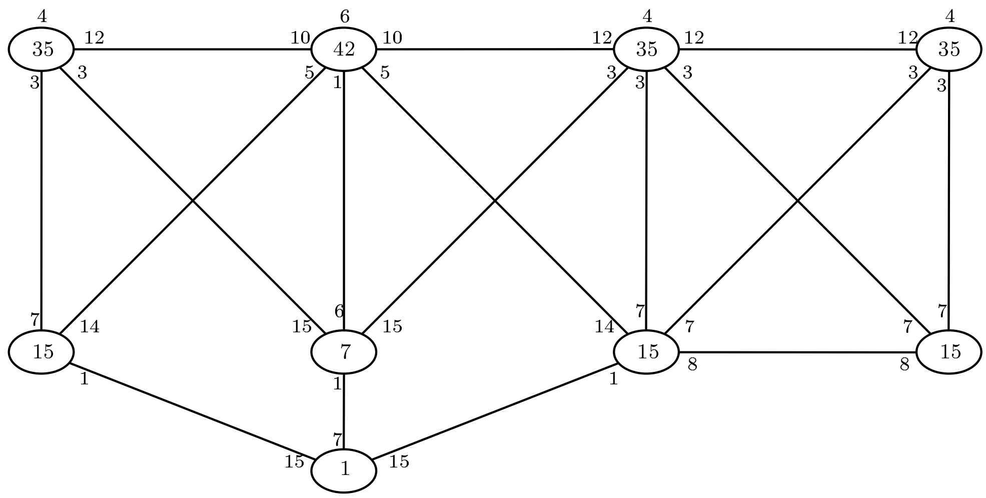

Figure 5: Distance distribution diagram in K

It turns out that these 7 vertices have just 7 neighbours in C_2, namely:

$$(2,0,4),(2,3,0),(2,1,0),(2,4,0),(2,2,0),(2,0,1),(3,0,0).$$

Now it is easy to see that these 7 points and 7 lines form a Fano plane. As one lets A_7 act, one obtains the validity of Veblen's axiom in general. $\square$

Remark 44. In the preceding proof, each vertex of C_1 represents a Fano plane: its 7 neighbours in L_2 as lines, and vertices of S as points. A further 15 such structures on S are obtained starting with vertices of C_2. This demonstrates the well-known fact that there are 30 possible ways of defining a Fano plane on a 7-element set (S), all equivalent under the action of the symmetric group S_7, but splitting into two orbits of length 15 under the action of A_7.

Remark 45. The construction of the Hoffman-Singleton graph from the 15 points and 35 lines of $\mathrm{PG}(3,2)$, where edges between points and lines indicate incidence, and edges between lines indicate disjointness of their corresponding triples, is also evident in the proof of Lemma 43. In addition, one sees again the bijection between lines of $\mathrm{PG}(3,2)$ and triples of a 7-element set, in which intersecting lines correspond to triples with intersections of cardinality 1.

It is but a short step from here to establish the exceptional isomorphism of $\mathrm{PSL}(4,2)$ and A_8: the automorphisms induced by A_7 are a subgroup of index 8 in the simple group $\mathrm{PSL}(4,2)$. The reader is encouraged to complete the story.

Remark 46. We note that the graph induced by $S \cup L_2 \cup C_2$ is a trivial modification of Neumaier's Alt(7)-geometry ([38], see also [40], p.153, or [7], p.523; the usual convention is to have all edges $\{s,c\}$ for $s \in S$, $c \in C_2$; here, no such edges are present).

Remark 47. In view of the presence of the group A_8 it is natural to ask if there is a way to construct the Alt(8)-geometry [38] (cf. also [40], p. 217) from the Higman-Sims graph G. The answer is indeed positive: one finds that the stabiliser in HS of an $\mathcal{F}_2$-coclique C of H_1 is A_8, and that the HS-orbit of the set complement of C in H_1 has length 8. The bijection between lines of PG(3, 2) and partitions of type 4^2 of an 8-element set becomes conspicuous. We will consider this in detail elsewhere.

10 The Doubly Transitive Action of HS on 176 Points

Since the Higman-Sims graph stood at the cradle of the sporadic simple group HS, a few words about the automorphism group are in order. Firstly, we note that as usual we can obtain the order of $\mathrm{Aut}(G)$ via the orbit-stabiliser theorem; we consider the action of $\mathrm{Aut}(G)$ on the Hoffman-Singleton subgraphs: $|\mathrm{Aut}(G)| = 704 \cdot 126\,000 = 88\,704\,000$. The index 2 subgroup HS therefore has order $44\,352\,000$. The simplicity of HS is a consequence of the simplicity of the stabiliser (cf. Remark 49) of a Hoffman-Singleton subgraph when HS acts on one of its two orbits of Hoffman-Singleton subgraphs. We also note that whilst HS contains a subgroup which is isomorphic to the full automorphism group of the Hoffman-Singleton graph, there is no such subgroup of HS which acts on a Hoffman-Singleton subgraph of G.

G. Higman [24] discovered a doubly transitive permutation representation of the group HS (at the time it was still undecided whether the group he considered was in fact isomorphic to HS, though). We will show such an action in the framework of the Higman-Sims graph G.

Lemma 48. *Let S be the set of the 176 pairs of complementary Hoffman-Singleton subgraphs of G in one of the two HS-orbits. Then the group HS acts doubly transitively on S.*

Proof. We recall (cf. Lemma 42) that there are two orbits of Hoffman-Singleton graphs under the action of HS, and that the graphs occur in complementary pairs in each orbit. The two orbits are distinguished by the cardinalities of their intersection with a fixed Hoffman-Singleton subgraph of G. We also recall that the stabiliser of a Hoffman-Singleton graph H_1 in HS is doubly transitive on the splits of H_1 into two $5C_5$.

Looking at the table in (31), rows 1 and 2, we must show that any Hoffman-Singleton subgraph of G with one of the intersection patterns in columns (a)–(c) can be transformed into any other by an automorphism of G which stabilises H_1. This will be established if we can show that for any Hoffman-Singleton subgraphs X, Y of G such that $|X \cap H_1| = |Y \cap H_1| = 20$ there exists an automorphism τ in the stabiliser of H_1 such that $X^\tau = Y$. In other words: we need only consider row 1, columns (a)–(c). Remembering our automorphism ψ from (4), we note that it suffices to prove transitivity of the stabiliser of H_1 on the patterns of columns (a) and (b).

Looking at column (a), note that the 15+10 vertices of X in $V_3 \cup V_0$ are one half of a split of $V_3 \cup V_0$ into a pair of $5C_5$. This means that amongst the 10 vertices in V_0 there is a unique pair of adjacent ones, and affine transformations of V_0 induce a transitive action on the 125 sets of 10 vertices.

Considering column (b), note that the 5 vertices of $X \cap V_0$ are part of an $\mathcal{F}_2$-coclique, and affine transformations with vertical eigenvalue 1 operate transitively on these.

Finally we must show that a pattern from (a) can be transformed into one from column (b). To this end we return to the example in Remark 4. The automorphism (of H_1; but note that it can be extended to G) defined by (6) will turn the pattern of 20 boldface positions into a pattern of 5 vertices in V_0 and 15 vertices of V_1. It remains to show that the 20 boldface positions are the intersection of a Hoffman-Singleton subgraph with H_1. To this end we define an automorphism τ of G which preserves $V_3 \cup V_0$ and such that V_3^τ and V_2^τ each have 10 vertices in V_0, resp. V_1. This can be achieved following the method of Remark 4: choose the Petersen graph consisting of the vertices $(3,0,0)$–$(3,0,4)$ and $(0,0,0)$–$(0,0,4)$ and define its image so that two V_3-vertices of P are mapped onto the bold positions. If we also require that $(0,1,0)$ maps to $(0,2,1)$, we obtain the following automorphism τ:

$$(V_3)\quad \begin{array}{cccccccccccc}
{\scriptstyle (3)} & 004 & 341 & 014 & 044 & 311 & 002 & 330 & 344 & 314 & 320 & {\scriptstyle (4)} \\
{\scriptstyle (1)} & 304 & 032 & 340 & 310 & 022 & 003 & 041 & 013 & 043 & 011 & {\scriptstyle (3)} \\
{\scriptstyle (4)} & 302 & 332 & 342 & 312 & 322 & 303 & 040 & 313 & 343 & 010 & {\scriptstyle (2)} \\
{\scriptstyle (2)} & 300 & 334 & 042 & 012 & 324 & 301 & 321 & 024 & 034 & 331 & {\scriptstyle (1)} \\
{\scriptstyle (0)} & 000 & 020 & 323 & 333 & 030 & 001 & 021 & 023 & 033 & 031 & {\scriptstyle (0)}
\end{array}\quad (V_0) \tag{32}$$

It is easy to verify that V_3^τ has 10 vertices in V_0, and that τ can be extended to an automorphism of G. $\square$

Remark 49. In [16] we showed how to use Definition 1 to obtain that the order of the automorphism group of the Hoffman-Singleton graph is 252 000. We now show that this automorphism group has a subgroup of index 2 which is simple. (This is of course a well-known fact, usually by reference to [22], but we want to show that all the arguments can proceed at a very elementary level.)

The automorphism of the Hoffman-Singleton graph H induced by $(0,x,y) \mapsto (0,x,-y)$ interchanges the two families of maximum cocliques. Therefore those automorphisms which preserve the two families of maximum cocliques form a subgroup U of index 2 in $\mathrm{Aut}(H)$, which is therefore of order 126 000. Since the transposition $(0,x,y) \mapsto (0,x,-y)$ does not belong to the vertex stabiliser of $(0,0,0)$ in this subgroup, this stabiliser must have index 2 in the vertex stabiliser of the full automorphism group, which is the symmetric group S_7. Since U is primitive on the vertices of H, and A_7 is simple, we conclude that U is a simple group of order 126 000. In order to identify the isomorphism type of this simple group ($PSU(3,5)$), we refer to O'Nan [39] or D.G. Higman [22].

11 Coordinate-free Description of G

We have noted that the edges between V_0 and V_1 describe the incidence graph of a biaffine plane of order 5, V_0 being the set of points, V_1 the set of lines. The edges between V_0

and V_3 form the incidence graph of points and the set of parabolas $y = (x-a)^2 + b$ in this biaffine plane. Further, the edges between V_1 and V_2 describe the incidence graph of 'dual points' and a certain set of 'dual parabolas' of the biaffine plane. (Dual points are, as usual, the lines of the geometry, and dual parabolas are sets of all tangents of a parabola—it is easy to show that the set of tangents to the parabola $y = 3(x-a)^2 + b$ consists of all lines $y = mx + c$ where $c = 2(m+a)^2 + b - 2a^2$.)

The following theorem provides geometric descriptions for all adjacencies between V_i and V_j $(i \neq j)$.

Theorem 6. *Interpret the sets $V_0, \ldots, V_3$ as above, with p, ℓ, P, Q be elements of $V_0, \ldots, V_3$ respectively. Then*

1. *the parabola P is adjacent (in G) to the dual parabola Q if and only if exactly one of the lines of Q is a tangent of P;*
2. *the point p is adjacent in G to the dual parabola Q if and only if p does not lie on any of the lines of Q (i.e. p is an internal point of the dual parabola Q);*
3. *the line ℓ is adjacent in G to the parabola P if and only if ℓ is a passant of P (cf. end of Section 5);*
4. *all other adjacencies in G, apart from the 5-cycles within each of $V_0, \ldots, V_3$, describe incidences.*

Proof. It suffices to indicate a proof of the first statement. Statements 2 and 3 are duals of each other, and statement 3 was established at the end of Section 5. To establish statement 1, assume that P has the equation $y = (x-a)^2 + b$ and Q is given by $c = 2(m-A)^2 + B$. Then Q consists of the lines $y = mx + 2(m-A)^2 + B$. Such a line is a tangent of P if it has a unique point of intersection with P; i.e. the equation $(x-a)^2 + b = mx + 2(m-A)^2 + B$ has discriminant 0 (as equation in x). This leads to the condition

$$m^2 + (a+A)m - b + 2A^2 + B = 0.$$

This quadratic equation in m has a unique solution if and only if its discriminant equals 0:

$$(a+A)^2 - b + 2A^2 + B = 0.$$

This is the condition of adjacency between vertices $(3, a, b)$ and $(2, A, B)$. □

Remark 50. This description opens the way to define families of generalised Higman-Sims graphs, starting from any McKay-Miller-Širáň graph instead of H (or more generally, starting from any graph based on the incidence graph of a biaffine plane).

Remark 51. As a point of interest we note that the sets $V_0, \ldots, V_3$ (as geometric entities in a biaffine plane) have been considered by Wild [49, 50]. The incidence graphs of the systems $S(C_1, C_2)$ of Wild are obtained by removing all edges within each of $V_0, \ldots, V_3$, and removing the (diagonal) edges between V_0 and V_2, and between V_1 and V_3.

Wild [50] also establishes the isomorphism of the graphs induced by $V_0 \cup V_1$ and $V_0 \cup V_3$ (points and lines vs points and conics, omitting the 5-cycles; the generalisation from $q = 5$ to general q is obvious). This is the biaffine analogue of results for projective planes [33, 41, 42]. See also [43] for recent work.

Remark 52. Further to Remark 51, we note that the isomorphism between the graphs induced by $V_0 \cup V_1$ and $V_0 \cup V_3$ manifested itself in [48] (cf. also [17]), disguised by the language of voltage assignments. In this paper, Šiagiová expressed certain adjacencies of the McKay-Miller-Širáň graphs [36, 19] by means of quadratic equations, whereas the original definition uses linear equations which are the direct analogue of the equations $y = mx + c$ in Robertson's definition of the Hoffman-Singleton graph. One might say that the original description operates in $V_0 \cup V_1$, whilst Šiagiová's description uses $V_0 \cup V_3$. (Of course, the present paper considers only the case $q = 5$; [48] deals with arbitrary $q \equiv 1$ mod 4.)

12 Decomposition of the Higman-Sims Graph into Five Isomorphic Subgraphs

In the course of our studies of the Higman-Sims graph, the following decomposition into 5 isomorphic subgraphs of order 20 appeared.

Lemma 53. *Let*

$$W_i = \bigcup_{r \in \mathbb{Z}_5} \{(0, i, r), (1, 3i, r), (2, 2i, r), (3, 2i, r)\}, \quad (i \in \mathbb{Z}_5).$$

For $i \in \mathbb{Z}_5$ the subgraphs of G induced by W_i are isomorphic to the Kronecker product of the Petersen graph with a coclique of order 2.

Proof. Each W_i consists of four 5-cycles, one from each $V_k, k = 1, \ldots, 4$. For each pair of (cyclically) consecutive values of k the two 5-cycles form a Petersen graph. One sees without difficulty that $(0, i, r)$ is adjacent to $(2, 2i, s)$ if and only if $r = s \pm 1$, i.e. if and only if $(0, i, r)$ is adjacent to $(0, i, s)$; and $(0, i, r)$ is adjacent to $(3, 2i, s)$ if and only if $r = s + i^2$, i.e. if and only if $(0, i, r)$ is adjacent to $(1, 3i, r + 2i^2)$. The remaining details can be left to the reader. □

Remark 54. The subgraph of order 80 induced by four of these subgraphs W_i is identified as the second orbit of the stabiliser of the remaining one of these graphs in [4]. What is new here is the observation that we are actually dealing with five isomorphic subgraphs of order 20.

As an aside, we add that the orbit of W_0 under $\mathrm{Aut}(G)$ as well as under HS has length 5775, the number of elements in one of the two conjugacy classes of involutions in HS. Indeed, each such involution has one of these 20-vertex subgraphs as fixed-point set. For example, W_0 is the fixed-point set of the automorphism ψ^4 as mentioned in Remark 11. (The second class of involutions of HS can be represented by h^2 (cf. Lemma 12) which is fixed-point-free.)

13 Historical Comment

T.B. Jajcayová and R. Jajcay, in their recent biographical note [29] on D.M. Mesner, report that the Higman-Sims graph made an early appearance in Mesner's 1956 dissertation, getting a brief mention as $NL_2(10)$ in [37] (designs of negative Latin square type). However, the automorphism group of the graph was not considered in either work. We add that in [47], p.107, the Higman-Sims graph is indeed identified as $NL_2(10)$.

An interesting first-hand account (in English) by C.C. Sims on how the Higman-Sims group was found is included in a forthcoming paper by G. Hiss [25].

Acknowledgements

R.H. Jeurissen's report [31] was a source of inspiration for this work. After the bulk of this paper was completed, I also became aware of A.E. Brouwer's suite of web pages [4] which are a mine of valuable information, presenting some of these results in a different light. The computer algebra system MAGMA [3] was used for the exploration of the graphs.

I wish to thank Cheryl Praeger for her interest in this work, and in particular for communicating her result in Theorem 3. Thanks also to M.A. Fiol who supplied me with a copy of [31]. Many thanks to Alice Karetai who worked with me in the framework of an undergraduate summer studentship, proving Theorem 1, counting cycles, and more. In particular she derived the equations for adjacency in H_3. I am grateful to my Department for this support.

Last but by no means least, I am grateful to an anonymous referee whose thorough reading of this paper lead to a number of improvements.

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