Rook Theory, Generalized Stirling Numbers and (p, q)-analogues

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Abstract

In this paper, we define two natural (p,q)-analogues of the generalized Stirling numbers of the first and second kind $S^1(\alpha,\beta,r)$ and $S^2(\alpha,\beta,r)$ as introduced by Hsu and Shiue [17]. We show that in the case where $\beta = 0$ and α and r are nonnegative integers both of our (p,q)-analogues have natural interpretations in terms of rook theory and derive a number of generating functions for them.

We also show how our (p,q)-analogues of the generalized Stirling numbers of the second kind can be interpreted in terms of colored set partitions and colored restricted growth functions. Finally we show that our (p,q)-analogues of the generalized Stirling numbers of the first kind can be interpreted in terms of colored permutations and how they can be related to generating functions of permutations and signed permutations according to certain natural statistics.

1 Introduction

In this paper we present a new rook theory interpretation of a certain class of generalized Stirling numbers and their (p,q)-analogues. Our starting point is to develop two natural (p,q)-analogues of the generalized Stirling numbers as defined by Hsu and Shiue in [17]. That is, Hsu and Shiue gave a unified approach to many extensions of the Stirling numbers that had appeared in the literature by defining analogues of the Stirling numbers of the first and second kind which depend on three parameters α, β and r as follows. First define $(z|\alpha)_0 = 1$ and $(z|\alpha)_n = z(z-\alpha)\cdots(z-(n-1)\alpha)$ for each integer $n > 0$. We write $(z)\downarrow_n$ for $(z|\alpha)_n$ when $\alpha = 1$

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and $(z)_n$ for $(z|\alpha)_n$ when $\alpha = -1$. Then Hsu and Shiue defined $\overline{S}^1_{n,k}(\alpha,\beta,r)$ and $\overline{S}^2_{n,k}(\alpha,\beta,r)$ for $0 \le k \le n$ via the following equations:

$$(x|\alpha)_n = \sum_{k=0}^{n} \overline{S}^1_{n,k}(\alpha,\beta,r)(x-r|\beta)_k \text{ and} \tag{1}$$

$$(x|\beta)_n = \sum_{k=0}^{n} \overline{S}^2_{n,k}(\alpha,\beta,r)(x+r|\alpha)_k. \tag{2}$$

It is easy to see that when $\alpha = 1$, $\beta = 0$ and $r = 0$, equations (1) and (2) become

$$(x)\downarrow_n = \sum_{k=0}^{n} \overline{S}^1_{n,k}(1,0,0)x^k \text{ and} \tag{3}$$

$$x^n = \sum_{k=0}^{n} \overline{S}^2_{n,k}(1,0,0)(x)\downarrow_k \tag{4}$$

which are the usual defining equations for the Stirling numbers of the first and second kind. Thus $\overline{S}^1_{n,k}(1,0,0)$ is the usual Stirling number of the first kind $s_{n,k}$ and $\overline{S}^2_{n,k}(1,0,0)$ is the usual Stirling number of the second kind $S_{n,k}$. In addition, it is easy to see from equations (1) and (2) that for all $0 \le k \le n$,

$$\overline{S}^1_{n,k}(\alpha,\beta,r) = \overline{S}^2_{n,k}(\beta,\alpha,-r). \tag{5}$$

q-Analogues of the Stirling numbers of the first and second kind were first considered by Gould [14] and further studied by Milne [21][20], Garsia and Remmel [11], and others, who gave interpretations in terms of rook placements and restricted growth functions. A more general two parameter, (p,q)-analogue of the Stirling number of the second kind was introduced and studied by Wachs and White [26], who also gave interpretations in terms of rook placements and restricted growth functions.

We shall define two natural (p,q)-analogues of the $\overline{S}^i_{n,k}(\alpha,\beta,r)$'s, one of which reduces to the (p,q)-analogue of Wachs and White when $i=2$ and $(\alpha,\beta,r) = (1,0,0)$. To do this we shall find it more convenient to modify equations (1) and (2) slightly. That is, we let

$$S^1_{n,k}(\alpha,\beta,r) = \overline{S}^1_{n,k}(\alpha,\beta,-r) \text{ and} \tag{6}$$

$$S^2_{n,k}(\alpha,\beta,r) = \overline{S}^2_{n,k}(\alpha,\beta,-r) \tag{7}$$

Then if we replace x by $t-r$ in equation (1) and x by t in equation (2), we obtain the following pair of equations.

$$(t-r|\alpha)_n = \sum_{k=0}^{n} S^1_{n,k}(\alpha,\beta,r)(t|\beta)_k \tag{8}$$

and

$$(t|\beta)_n = \sum_{k=0}^{n} S^2_{n,k}(\alpha,\beta,r)(t-r|\alpha)_k. \tag{9}$$

It is easy to see from equations (8) and (9) that for all $0 \le m \le n$,

$$\sum_{k=m}^{n} S^2_{n,k}(\alpha,\beta,r)S^1_{k,m}(\alpha,\beta,r) = \chi(m=n) \tag{10}$$

where we use that convention that for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Hsu and Shiue [17] proved a number of fundamental formulas for the $S^i_{n,k}(\alpha,\beta,r)$'s. We shall state just a few examples of these formulas. First they showed that the $S^i_{n,k}(\alpha,\beta,r)$'s satisfy the following recursions. Let $S^1_{0,0}(\alpha,\beta,r) = 1$ and $S^1_{n,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$. Then for all $0 \leq k \leq n+1$,

$$S^1_{n+1,k}(\alpha,\beta,r) = S^1_{n,k-1}(\alpha,\beta,r) + (k\beta - n\alpha - r)S^1_{n,k}(\alpha,\beta,r). \tag{11}$$

Similarly if we let $S^2_{0,0}(\alpha,\beta,r) = 1$ and $S^2_{n,,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$, then for all $0 \leq k \leq n+1$,

$$S^2_{n+1,k}(\alpha,\beta,r) = S^2_{n,k-1}(\alpha,\beta,r) + (k\alpha - n\beta + r)S^2_{n,k}(\alpha,\beta,r). \tag{12}$$

Next they proved the following generating functions.

$$k!\sum_{n\geq 1} S^1_{n,k}(\alpha,\beta,r)\frac{t^n}{n!} = (1+\alpha t)^{-r/\alpha}\Big(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\Big)^k \text{ if } \alpha\beta \neq 0, \tag{13}$$

and

$$\sum_{n\geq 0} S^1_n(x) = \left(\frac{1}{e}\right)^{x/\beta} \sum_{k\geq 0} \frac{(x/\beta)^k}{k!}(k\beta - r|\alpha)_n \tag{14}$$

where

$$S^1_n(x) = \sum_{k=0}^{n} S^1_{n,k}(\alpha,\beta,r)x^k. \tag{15}$$

We now present two natural ways to give (p,q)-analogues of (8) and (9) which we shall call type I and type II (p,q)-analogues. We shall see that both of the (p,q)-analogues arise naturally in our rook theory interpretations for certain values of α, β and r.

First for any γ, let

$$[\gamma]_{p,q} = \frac{p^\gamma - q^\gamma}{p-q}. \tag{16}$$

Thus in the case where $\gamma = n$ is a non-negative integer,

$$[n]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$$

is the usual (p,q)-analogue of n. We also let

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$$

and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

We shall write $[n]_q$, $[n]_q!$ and $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ for $[n]_{1,q}$, $[n]_{1,q}!$ and $\left[\begin{array}{c} n \\ k \end{array}\right]_{1,q}$ respectively.

For the type I (p,q)-analogues of (8) and (9), we replace $(t-r|\gamma)_n$ by $\langle t-r|\gamma\rangle_n$ where

$$\langle t-r|\gamma\rangle_0 = 1 \tag{17}$$

and for $n > 0$,

$$\langle t-r|\gamma\rangle_n = ([t]_{p,q}-[r]_{p,q})([t]_{p,q}-[r+\gamma]_{p,q})\cdots([t]_{p,q}-[r+(n-1)\gamma]_{p,q}). \tag{18}$$

That is, we define $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ for $0 \leq k \leq n$ via the following equations:

$$\langle t-r|\alpha\rangle_n = \sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)\langle t|\beta\rangle_k \tag{19}$$

and

$$\langle t|\beta\rangle_n = \sum_{k=0}^{n} S^{2,p,q}_{n,k}(\alpha,\beta,r)\langle t-r|\alpha\rangle_k. \tag{20}$$

We then have the following basic recursions for the $S^{i,p,q}_{n,k}(\alpha,\beta,r)$'s.

Theorem 1. *If $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ are defined according to equations (19) and (20) respectively for $0 \leq k \leq n$, then $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $S^{2,p,q}_{n,k}(\alpha,\beta,r)$ satisfy the following recursions.*

$$S^{1,p,q}_{0,0}(\alpha,\beta,r) = 1 \ \textit{and} \ \ S^{1,p,q}_{n,k}(\alpha,\beta,r) = 0 \ \textit{if}\ k<0 \ \textit{or}\ k>n \tag{21}$$

and

$$S^{1,p,q}_{n+1,k}(\alpha,\beta,r) = S^{1,p,q}_{n,k-1}(\alpha,\beta,r) + ([k\beta]_{p,q}-[n\alpha+r]_{p,q})S^{1,p,q}_{n,k}(\alpha,\beta,r). \tag{22}$$

$$S^{2,p,q}_{0,0}(\alpha,\beta,r) = 1 \ \textit{and} \ \ S^{2,p,q}_{n,k}(\alpha,\beta,r) = 0 \ \textit{if}\ k<0 \ \textit{or}\ k>n \tag{23}$$

and

$$S^{2,p,q}_{n+1,k}(\alpha,\beta,r) = S^{2,p,q}_{n,k-1}(\alpha,\beta,r) + ([k\alpha+r]_{p,q}-[n\beta]_{p,q})S^{2,p,q}_{n,k}(\alpha,\beta,r). \tag{24}$$

Proof. To prove (22), we start with (19). That is,

$$\begin{aligned}
&\sum_{k=0}^{n+1} S^{1,p,q}_{n+1,k}(\alpha,\beta,r)\langle t|\beta\rangle_k = \langle t-r|\alpha\rangle_{n+1} \qquad (25)\\
&= ([t]_{p,q}-[r+n\alpha]_{p,q})\langle t-r|\alpha\rangle_n\\
&= ([t]_{p,q}-[r+n\alpha]_{p,q})(\sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)\langle t|\beta\rangle_k)\\
&= \sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)\langle t|\beta\rangle_k([t]_{p,q}-[k\beta]_{p,q}+[k\beta]_{p,q}-[r+n\alpha]_{p,q})\\
&= \sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)\langle t|\beta\rangle_{k+1}\\
&+ \sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)([k\beta]_{p,q}-[r+n\alpha]_{p,q})\langle t|\beta\rangle_k.
\end{aligned}$$

Taking the coefficient of $\langle t|\beta\rangle_k$ on both sides of (25) yields (22).

Similarly to prove (24), we start with (20). That is,

$$\sum_{k=0}^{n+1} S_{n+1,k}^{2,p,q}(\alpha,\beta,r)\langle t-r|\alpha\rangle_k = \langle t|\beta\rangle_{n+1} \tag{26}$$
$$= ([t]_{p,q}-[n\beta]_{p,q})\langle t|\beta\rangle_n$$
$$= ([t]_{p,q}-[n\beta]_{p,q})(\sum_{k=0}^{n} S_{n,k}^{2,p,q}(\alpha,\beta,r)\langle t-r|\alpha\rangle_k)$$
$$= \sum_{k=0}^{n} S_{n,k}^{2,p,q}(\alpha,\beta,r)\langle t-r|\alpha\rangle_k([t]_{p,q}-[r+k\alpha]_{p,q}+[r+k\alpha]_{p,q}-[n\beta]_{p,q})$$
$$= \sum_{k=0}^{n} S_{n,k}^{2,p,q}(\alpha,\beta,r)\langle t-r|\beta\rangle_{k+1}$$
$$+ \sum_{k=0}^{n} S_{n,k}^{2,p,q}(\alpha,\beta,r)([r+k\alpha]_{p,q}-[n\beta]_{p,q})\langle t-r|\alpha\rangle_k$$

Taking the coefficient of $\langle t-r|\alpha\rangle_k$ on both sides of (26) yields (24). ■

We shall then show that when $\beta=0$ and $\alpha=j$ and $r=i$ are non-negative integers such that $i\geq 0$ and $j>0$, the polynomials

$$c_{n,k}^{i,j}(p,q) = (-1)^{n-k}S_{n,k}^{1,p,q}(j,0,i) \tag{27}$$

and

$$S_{n,k}^{i,j}(p,q) = S_{n-k}^{2,p,q}(j,0,i) \tag{28}$$

have natural interpretations in terms of p,q-counting rooks placements on certain boards. It follows from (21), (22), (23) (24) that these polynomials satisfy the following recursions.

$$c_{0,0}^{i,j}(p,q)=1 \text{ and } c_{n,k}^{i,j}(p,q)=0 \text{ if } k<0 \text{ or } k>n \tag{29}$$

and

$$c_{n+1,k}^{i,j}(p,q) = c_{n,k-1}^{i,j}(p,q)+[i+nj]_{p,q}c_{n,k}^{i,j}(p,q). \tag{30}$$

$$S_{0,0}^{i,j}(p,q)=1 \text{ and } S_{n,k}^{i,j}(p,q)=0 \text{ if } k<0 \text{ or } k>n \tag{31}$$

and

$$S_{n+1,k}^{i,j}(p,q) = S_{n,k-1}^{i,j}(p,q)+[i+jk]_{p,q}S_{n,k}^{i,j}(p,q). \tag{32}$$

Moreover, it easily follows from (19) and (20) that

$$([t]_{p,q}+[i]_{p,q})([t]_{p,q}+[i+j]_{p,q})\cdots([t]_{p,q}+[i+(n-1)j]_{p,q}) = \sum_{k=0}^{n} c_{n,k}^{i,j}(p,q)[t]_{p,q}^k \tag{33}$$

and

$$[t]_{p,q}^n = \sum_{k=0}^{n} S_{n,k}^{i,j}(p,q)([t]_{p,q}-[i]_{p,q})([t]_{p,q}-[i+j]_{p,q})\cdots([t]_{p,q}-[i+(k-1)j]_{p,q}). \tag{34}$$

Thus if we let $s_{n,k}^{i,j}(p,q) = (-1)^{n-k} c_{n,k}^{i,j}(p,q) = S_{n,k}^{1,p,q}(j,0,i)$, it follows from (19) that

$$([t]_{p,q} - [i]_{p,q})([t]_{p,q} - [i+j]_{p,q}) \cdots ([t]_{p,q} - [i+(n-1)j]_{p,q}) = \sum_{k=0}^{n} s_{n,k}^{i,j}(p,q)[t]_{p,q}^{k} \tag{35}$$

from which it easily follows that the matrices $||s_{n,k}^{i,j}(p,q)||_{n,k\geq 0}$ and $||S_{n,k}^{i,j}(p,q)||_{n,k\geq 0}$ are inverses of each other.

For the type II (p,q)-analogues of (8) and (9), we replace $(t-r|\gamma)_n$ by $[t-r|\gamma]_n$ where

$$[t-r|\gamma]_0 = 1 \tag{36}$$

and for $n > 0$,

$$[t-r|\gamma]_n = ([t-r]_{p,q})([t-r-\gamma]_{p,q}) \cdots ([t-r-(n-1)\gamma]_{p,q}). \tag{37}$$

By analogy with our type I (p,q)-analogues of the generalized Stirling numbers, the type II (p,q)-analogues of the generalized Stirling numbers, $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)$, should be solutions to the following equations:

$$[t-r|\alpha]_n = \sum_{k=0}^{n} \tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)[t|\beta]_k \tag{38}$$

and

$$[t|\beta]_n = \sum_{k=0}^{n} \tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)[t-r|\alpha]_k. \tag{39}$$

However, as we shall see shortly, (38) and (39) do not completely determine $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)$. Instead we will define the type II (p,q)-analogues of the generalized Stirling numbers, $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)$, by the following recursions:

$$\tilde{S}_{0,0}^{1,p,q}(\alpha,\beta,r) = 1 \text{ and } \tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r) = 0 \text{ if } k<0 \text{ or } k>n \tag{40}$$

and

$$\tilde{S}_{n+1,k}^{1,p,q}(\alpha,\beta,r) = q^{(k-1)\beta-n\alpha-r}\ \tilde{S}_{n,k-1}^{1,p,q}(\alpha,\beta,r) + p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r). \tag{41}$$

$$\tilde{S}_{0,0}^{2,p,q}(\alpha,\beta,r) = 1 \text{ and } \tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r) = 0 \text{ if } k<0 \text{ or } k>n \tag{42}$$

and

$$\tilde{S}_{n+1,k}^{2,p,q}(\alpha,\beta,r) = q^{r+(k-1)\alpha-n\beta}\ \tilde{S}_{n,k-1}^{2,p,q}(\alpha,\beta,r) + p^{t-r-k\alpha}[k\alpha+r-n\beta]_{p,q}\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r). \tag{43}$$

Here t is an extra parameter and technically we should use the notation $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r,t)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r,t)$ to specify the dependence on the paramater t. However since we will not vary the parameter t, we will instead use the less cumbersome notation $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)$.

Our next result will show that $\tilde{S}_{n,k}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}_{n,k}^{2,p,q}(\alpha,\beta,r)$ do satisfy (38) and (39).

Theorem 2. *If we define $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)$ for $0 \le k \le n$ by (40), (41), (42), and (43), then (38) and (39) hold.*

Proof. To prove (41), we first observe the following identity:

$$\begin{aligned}
[t-r-n\alpha]_{p,q} &= \frac{p^{t-rn\alpha} - q^{t-r-n\alpha}}{p-q} \\
&= \frac{q^{k\beta-n\alpha-r}(p^{t-k\beta} - q^{t-k\beta}) + p^{t-k\beta}(p^{k\beta-n\alpha-r} - q^{k\beta-n\alpha-r})}{p-q} \\
&= q^{k\beta-n\alpha-r}[t-k\beta]_{p,q} + p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}. \qquad (44)
\end{aligned}$$

We then prove (38) by induction. Clearly (38) holds for $n=0$. Next assume that (38) holds for n. Then

$$\begin{aligned}
&[t-r|\alpha]_{n+1} = ([t-r-n\alpha]_{p,q})[t-r|\alpha]_n \\
&= ([t-r-n\alpha]_{p,q})(\sum_{k=0}^{n} \tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t|\beta]_k) \\
&= \sum_{k=0}^{n} \tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t|\beta]_k(q^{k\beta-n\alpha-r}[t-k\beta]_{p,q} + p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}) \\
&= \sum_{k=0}^{n} q^{k\beta-n\alpha-r}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t|\beta]_{k+1} \\
&+ \sum_{k=0}^{n} p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t|\beta]_k \\
&= \sum_{k=0}^{n+1} \left(q^{(k-1)\beta-n\alpha-r}\ \tilde{S}^{1,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-k\beta}[k\beta-n\alpha-r]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)\right)[t|\beta]_k \\
&= \sum_{k=0}^{n+1} \tilde{S}^{1,p,q}_{n+1,k}(\alpha,\beta,r)[t|\beta]_k.
\end{aligned}$$

Similarly to prove (39), we observe the following identity:

$$\begin{aligned}
[t-n\beta]_{p,q} &= \frac{p^{t-n\beta} - q^{t-n\beta}}{p-q} \\
&= \frac{q^{r+k\alpha-n\beta}(p^{t-r-k\alpha} - q^{t-r-k\alpha}) + p^{t-r-k\alpha}(p^{r+k\alpha-n\beta} - q^{r+k\alpha-n\beta})}{p-q} \\
&= q^{r+k\alpha-n\beta}[t-r-k\alpha]_{p,q} + p^{t-r-k\alpha}[r+k\alpha-n\beta]_{p,q}. \qquad (45)
\end{aligned}$$

We then prove (39) by induction. Clearly (39) holds for $n=0$. Next assume that (39) holds for

n. Then

$$
\begin{aligned}
&[t|\beta]_{n+1} = ([t-n\beta]_{p,q})[t|\beta]_n \\
&= ([t-n\beta]_{p,q})(\sum_{k=0}^{n} \tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)[t-r|\alpha]_k) \\
&= \sum_{k=0}^{n} \tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)[t-r|\alpha]_k(q^{r+k\alpha-n\beta}[t-r-k\alpha]_{p,q} + p^{t-r-k\alpha}[r+k\alpha-n\beta]_{p,q}) \\
&= \sum_{k=0}^{n} q^{r+k\alpha-n\beta}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)[t-r|\alpha]_{k+1} \\
&+ \sum_{k=0}^{n} p^{t-r-k\alpha}[r+k\alpha-n\beta]_{p,q})\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)[t-r|\alpha]_k \\
&= \sum_{k=0}^{n+1} \left(q^{r+(k-1)\alpha-n\beta}\ \tilde{S}^{2,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-r-k\alpha}[k\alpha+r-n\beta]_{p,q}\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)\right)[t-r|\alpha]_k \\
&= \sum_{k=0}^{n+1} \tilde{S}^{2,p,q}_{n+1,k}(\alpha,\beta,r)[t-r|\alpha]_k.
\end{aligned}
$$

■

We can now see why there there are multiple solutions to (38) and (39). That is, by symmetry, it must be the case that $\tilde{S}^{1,q,p}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{2,q,p}_{n,k}(\alpha,\beta,r)$ are also solutions to (38) and (39). However it is not the case that $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r) = \tilde{S}^{1,q,p}_{n,k}(\alpha,\beta,r)$ and $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r) = \tilde{S}^{2,q,p}_{n,k}(\alpha,\beta,r)$ due to the extra parameter t.

Again we shall be able to give a rook theory interpretation to $\tilde{S}^{1,p,q}(\alpha,\beta,r)$ and $\tilde{S}^{2,p,q}(\alpha,\beta,r)$ in the special case when $\beta = 0$ and $r = i$ and $\alpha = j$ are integers such that $i \geq 0$ and $j > 0$. For later developments, it will be convenient to replace t by $x+i$ so that the basic recursions (41) and (43) become the following:

$$\tilde{S}^{1,p,q}_{0,0}(j,0,i) = 1 \text{ and } \tilde{S}^{1,p,q}_{n,k}(j,0,i) = 0 \text{ if } k < 0 \text{ or } k > n \tag{46}$$

and

$$\tilde{S}^{1,p,q}_{n+1,k}(j,0,i) = q^{-nj-i}\tilde{S}^{1,p,q}_{n,k-1}(j,0,i) - p^{x+i}(pq)^{-nj-i}([nj+i]_{p,q})\tilde{S}^{1,p,q}_{n,k}(j,0,i). \tag{47}$$

(Here we have used the fact that
$[-nj-i]_{p,q} = \frac{p^{-nj-i}-q^{-nj-i}}{p-q} = (pq)^{-nj-i}\frac{q^{nj+i}-p^{nj+i}}{p-q} = -(pq)^{-nj-i}[nj+i]_{p,q}$.)

$$\tilde{S}^{2,p,q}_{0,0}(j,0,i) = 1 \text{ and } \tilde{S}^{2,p,q}_{n,k}(j,0,i) = 0 \text{ if } k < 0 \text{ or } k > n \tag{48}$$

and

$$\tilde{S}^{2,p,q}_{n+1,k}(j,0,i) = q^{i+(k-1)j}\tilde{S}^{2,p,q}_{n,k-1}(j,0,i) + p^{x-kj}([kj+i]_{p,q})\tilde{S}^{2,p,q}_{n,k}(j,0,i). \tag{49}$$

Moreover, it follows from (38) and (39) that

$$[x]_{p,q}\downarrow_{n,j} = \sum_{k=0}^{n} \tilde{S}^{1,p,q}_{n,k}(j,0,i)[x+i]^k_{p,q} \tag{50}$$

and

$$[x+i]_{p,q}^n = \sum_{k=0}^{n} \tilde{S}_{n,k}^{2,p,q}(j,0,i)[x]_{p,q}\downarrow_{k,j} \tag{51}$$

where $[x]_{p,q}\downarrow_{k,j}= 1$ if $k=0$ and $[x]_{p,q}\downarrow_{k,j}= [x]_{p,q}[x-j]_{p,q}\cdots[x-(k-1)j]_{p,q}$ if k is a positive integer.

As we shall see later, the most natural thing to do in terms of rook theory is to define

$$\tilde{s}_{n,k}^{i,j}(p,q) = p^{-(x+i)(n-k)}(qp)^{\binom{n}{2}j+ni}\ \tilde{S}_{n,k}^{1,p,q}(j,0,i) \tag{52}$$

and

$$\tilde{S}_{n,k}^{i,j}(p,q) = p^{-x(n-k)-\binom{n-k+1}{2}j}\ \tilde{S}_{n,k}^{2,p,q}(j,0,i). \tag{53}$$

It then follows that

$$\tilde{s}_{0,0}^{i,j}(p,q) = 1 \text{ and } \tilde{s}_{n,k}^{i,j}(p,q) = 0 \text{ if } k<0 \text{ or } k>n \tag{54}$$

and

$$\tilde{s}_{n+1,k}^{i,j}(p,q) = p^{nj+i}\ \tilde{s}_{n,k-1}^{i,j}(p,q) - [nj+i]_{p,q}\tilde{s}_{n,k}^{i,j}(p,q). \tag{55}$$

Similarly

$$\tilde{S}_{0,0}^{i,j}(p,q) = 1 \text{ and } \tilde{S}_{n,k}^{i,j}(p,q) = 0 \text{ if } k<0 \text{ or } k>n \tag{56}$$

and

$$\tilde{S}_{n+1,k}^{ij}(p,q) = q^{i+(k-1)j}\ \tilde{S}_{n,k-1}^{i,j}(p,q) + p^{-(n+1)j}[kj+i]_{p,q}\tilde{S}_{n,k}^{i,j}(p,q). \tag{57}$$

Moreover, it follows from (50) and (51) that

$$[x]_{p,q}\downarrow_{n,j}= \sum_{k=0}^{n} p^{(x+i)(n-k)}(pq)^{-\binom{n}{2}j-ni}\tilde{s}_{n,k}^{i,j}(p,q)[x+i]_{p,q}^k \tag{58}$$

and

$$[x+i]_{p,q}^n = \sum_{k=0}^{n} \tilde{S}_{n,k}^{i,j}(p,q)p^{x(n-k)+\binom{n-k+1}{2}j}[x]_{p,q}\downarrow_{k,j}\ . \tag{59}$$

It happens that the type II generalized (p,q)-Stirling numbers $\tilde{s}_{n,k}^{i,j}(p,q)$ and $\tilde{S}_{n,k}^{i,j}(p,q)$ can be expressed in terms of the type I generalized q-Stirling numbers. The relationship is as follows:

$$\tilde{s}_{n,k}^{i,j}(p,q) \quad = \quad p^{n(i-1)+\binom{n}{2}j+k}s_{n,k}^{i,j}(1,q/p) \tag{60}$$

$$\tilde{S}_{n,k}^{i,j}(p,q) \quad = \quad p^{-\binom{n-k+1}{2}j+(n-k)(i-1)}q^{ki+\binom{k}{2}j}S_{n,k}^{i,j}(1,q/p). \tag{61}$$

This can be proved by using the recurrences (30) and (32) to show that the expressions on the right side of the equations satisfy the recurrences (55) and (57), respectively.

In this case, the orthogonality relations between the $\tilde{s}_{n,k}^{i,j}(p,q)$'s and $\tilde{S}_{n,k}^{i,j}(p,q)$'s are more complicated than the orthogonality relations between the $s_{n,k}^{i,j}(p,q)$'s and $S_{n,k}^{i,j}(p,q)$'s. Thus we will state them explicitly.

Theorem 3. *The matrices* $||(pq)^{-\binom{n}{2}j}p^{-ik}q^{-ni}\tilde{s}_{n,k}^{i,j}(p,q)||_{n,k\geq 0}$ *and* $||p^{\binom{n-k+1}{2}j}\tilde{S}_{n,k}^{i,j}(p,q)||_{n,k\geq 0}$ *are inverses of each other.*

Proof. Since the matrices $||S^{i,j}_{n,k}(1,q/p)||$ and $||s^{i,j}_{n,k}(1,q/p)||$ are inverses of each other, we have for any $0 \le k \le n$,

$$\begin{aligned}
\chi(n=k) &= \sum_{l=k}^{n} S^{i,j}_{n,l}(1,q/p) s^{i,j}_{l,k}(1,q/p) \\
&= \sum_{l=k}^{n} p^{\binom{n-l+1}{2}j-(n-l)(i-1)} q^{-li-\binom{l}{2}j} \tilde{S}^{i,j}_{n,l}(p,q) p^{-l(i-1)-\binom{l}{2}j-k} \tilde{s}^{i,j}_{l,k}(p,q) \\
&= p^{-n(i-1)-k} \sum_{l=k}^{n} p^{\binom{n-l+1}{2}j-\binom{l}{2}j} q^{-li-\binom{l}{2}j} \tilde{S}^{i,j}_{n,l}(p,q) \tilde{s}^{i,j}_{l,k}(p,q).
\end{aligned}$$

Multiplying both sides of the equation by $p^{(n-k)(i-1)}$ we get,

$$\begin{aligned}
\chi(n=k) &= p^{(n-k)(i-1))} \chi(n=k) \\
&= p^{-ki} \sum_{l=k}^{n} p^{\binom{n-l+1}{2}j-\binom{l}{2}j} q^{-li-\binom{l}{2}j} \tilde{S}^{i,j}_{n,l}(p,q) \tilde{s}^{i,j}_{l,k}(p,q) \\
&= \sum_{l=k}^{n} \left(p^{\binom{n-l+1}{2}j} \tilde{S}^{i,j}_{n,l}(p,q)\right) \left(p^{-ki-\binom{l}{2}j} q^{-li-\binom{l}{2}j} \tilde{s}^{i,j}_{l,k}(p,q)\right),
\end{aligned}$$

which proves the result. ■

Having defined our two families of (p,q)-analogues of generalized Stirling numbers of the first and second kind, $(s^{i,j}_{n,k}(p,q), S^{i,j}_{n,k}(p,q))$ and $(\tilde{s}^{i,j}_{n,k}(p,q), \tilde{S}^{i,j}_{n,k}(p,q))$, the main result of this paper is to define a rook theory interpretation of these two families by modifying the set up of Garsia and Remmel [11]. That is, in section 2 we shall develop a rook theory interpretation of the families $(s^{i,j}_{n,k}(p,q), S^{i,j}_{n,k}(p,q))$ and give a combinatorial proof that the matrices $||s^{i,j}_{n,k}(p,q)||$ and $||S^{i,j}_{n,k}(p,q))||$ are inverses of each other. Then in section 3, we shall develop a rook theory interpretation of the families $(\tilde{s}^{i,j}_{n,k}(p,q), \tilde{S}^{i,j}_{n,k}(p,q))$. In section 4, we shall prove a number of generating function results for our two families. In section 5, we shall develop other combinatorial interpretations of our two families in terms of permutations statistics, colored partitions and restricted growth functions.

The (p,q)-Stirling numbers of the second kind, introduced by Wachs and White [26], are defined by the recursion

$$S_{0,0}(p,q) = 1 \text{ and } S_{n,k}(p,q) = 0 \text{ if } k < 0 \text{ or } k > n \tag{62}$$

and

$$S_{n+1,k}(p,q) = p^{k-1} S_{n,k-1}(p,q) + [k]_{p,q} S_{n,k}(p,q). \tag{63}$$

In the special case when $i = 0$ and $j = 1$, the recursion given in (31) and (32) for the type I (p,q)-Stirling number of the second kind $S^{0,1}_{n,k}(p,q)$ becomes

$$S^{0,1}_{0,0}(p,q) = 1 \text{ and } S^{0,1}_{n,k}(p,q) = 0 \text{ if } k < 0 \text{ or } k > n \tag{64}$$

and

$$S^{0,1}_{n+1,k}(p,q) = S^{0,1}_{n,k-1}(p,q) + [k]_{p,q} S^{0,1}_{n,k}(p,q). \tag{65}$$

It is easy to see that the polynomials $p^{\binom{k}{2}} S^{0,1}_{n,k}(p,q)$ also satisfy (62) and (63) so that $S_{n,k}(p,q) = p^{\binom{k}{2}} S^{0,1}_{n,k}(p,q)$.

We should also note that in the case when $i = 0$ and $j = 1$, our type I (p,q)-Stirling numbers of the first and second kind, $s^{0,1}_{n,k}(p,q)$ and $S^{0,1}_{n,k}(p,q)$, have been studied by a number of other authors, see [18], [19], [27], [28] and [23]. The case $i = p = q = 1$ has also appeared in the literature as Whitney numbers for Dowling lattices, see [2], [3], [13]. Moreover an alternative approach to combinatorially interpreting a different family of generalized (p,q)-Stirling numbers which includes our (p,q)-Stirling numbers $s^{i,j}_{n,k}(p,q)$ and $S^{i,j}_{n,k}(p,q)$ can be found in [19] where the authors interpret generalized (p,q)-Stirling numbers via 0-1 tableaux. However, our (p,q)-Stirling numbers of type II, $\tilde{s}^{i,j}_{n,k}(p,q)$ and $\tilde{S}^{i,j}_{n,k}(p,q)$, appear to be new.

2 Rook theory interpretation of $s^{i,j}_{n,k}(p,q)$ and $S^{i,j}_{n,k}(p,q)$

In this section, we shall give a rook theory interpretation of $s^{i,j}_{n,k}(p,q)$ and $S^{i,j}_{n,k}(p,q)$ and use our interpretation to give a combinatorial proof that the matrices $||s^{i,j}_{n,k}(p,q)||$ and $||S^{i,j}_{n,k}(p,q))||$ are inverses of each other.

Given a sequence $(a_1,\ldots,a_n)$ of non-negative integers, let $B(a_1,\ldots,a_n)$ denote a board with n columns whose column heights from left to right are $a_1,\ldots,a_n$ respectively. If $a_1 \leq \ldots \leq a_n$, then we say that $B(a_1,\ldots,a_n)$ is a Ferrers board. For example, $B(0,1,1,3)$ is pictured in Figure 1.

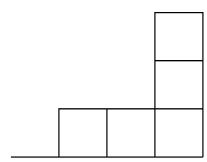

Figure 1: The board $B(0,1,1,3)$.

We say that $B(a_1,\ldots,a_n)$ is a j-*attacking* board if for all $1 \leq i < n$, $a_i \neq 0$ implies $a_{i+1} \geq a_i + j - 1$. Suppose that $B(a_1,\ldots,a_n)$ is a j-attacking board and $\mathcal{P}$ is a placement of rooks in $B(a_1,\ldots,a_n)$ which has at most one rook in each column of $B(a_1,\ldots,a_n)$. Then for any individual rook $r \in \mathcal{P}$, we say that r j-*attacks* cell $c \in B(a_1,\ldots,a_n)$ if c lies in a column which is strictly to the right of the column of r and c lies in the first j rows which are weakly above the row of r and which are not j-attacked by any rook which lies in a column that is strictly to the left of r.

For example, suppose $j = 2$ and $\mathcal{P}$ is the placement in $B(1,2,3,5,7,8,10)$ pictured in Figure 2. Here the rooks are indicated by placing an x in each cell that contains a rook. We place a 2 in each cell attacked by the rook r_2 in column 2. In this case, since there are no rooks to the left of r_2, the cells c which are 2-attacked by r_2 lie in the first two rows which are weakly above the row of r_2, i.e., all the cells in rows 2 and 3 that are in columns 3,4,5,6 and 7. Next consider the rook r_4 which lies in column 4. Again we place a 4 in each of the cells that are 2-attacked by r_4. In this case, the first two rows which lie weakly above r_4 that are not 2-attacked by any rook to the left of r_4 are rows 1 and 4. Thus r_4 2-attacks all the cells in rows 1 and 4 that lie in columns 5, 6 and 7. Finally the rook r_6, which lies in column 6, 2-attacks the cells (6,7) and

(7,7) and we place a 6 in these cells. We say that a placement $\mathcal{P}$ is *j-non-attacking* if no rook in $\mathcal{P}$ is j-attacked by a rook to its left and there is at most one rook in each row and column.

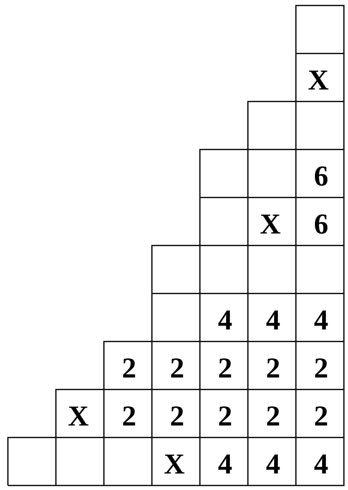

Figure 2: Cells that are 2 attacked

Note that the condition that $B(a_1, \ldots, a_n)$ is j-attacking ensures that for any placement $\mathcal{P}$ of j-non-attacking rooks in $B(a_1, \ldots, a_n)$, with at most one rook in each column, has the property that, for any rook $r \in \mathcal{P}$ which lies in a column $k < n$, there are j rows which lie weakly above r and which have no cells which are j-attacked by a rook to the left of r, namely, the row of r plus the top $j-1$ rows in column $k+1$ since $a_{k+1} \geq a_k + j - 1$.

Given a j-attacking board $B = B(a_1, \ldots, a_n)$, we let $\mathcal{N}_k^j(B)$ be the set of all placements $\mathcal{P}$ of k j-nonattacking rooks in B. For example, if $j = 2$ and $B = B(0, 2, 3, 4)$, then $|\mathcal{N}_1^2(B)| = 9$ since there are 9 cells in B, $|\mathcal{N}_2^2(B)| = 6$ and these 12 placements are pictured in Figure 3, and $|\mathcal{N}_3^2(B)| = 0$ since any placement $\mathcal{P}$ which has one rook in each nonempty column of B and at most one rook in each row has the property that the rooks in columns 2 and 3 would 2-attack 4 cells in column 4 and hence there would be no place to put a rook in column 4 that is not 2-attacked by a rook to its left. We then define the *k-th j-rook number* of B, $r_k^j(B)$, by setting $r_k^j(B) = |\mathcal{N}_k^j(B)|$.

For any board $B(a_1, \ldots, a_n)$, we let $\mathcal{F}_k(B)$ denote the set of all placements of k rooks in B such that there is at most one rook in each column. We then define the *k-th file number* of B, $f_k(B)$, to be $f_k(B) = |\mathcal{F}_k(B)|$.

Next we define what we call the type I (p,q)-analogues of $r_k^j(B)$ and $f_k(B)$ when $B = B(a_1, \ldots, a_n)$ is a j-attacking board. First suppose that we are given a placement $\mathcal{P}$ in $\mathcal{F}_k(B)$. Then let

(a) $a_B(\mathcal{P})$ = the number of cells in B that lie directly above some rook r in $\mathcal{P}$,

(b) $b_B(\mathcal{P})$ = the number of cells in B that lie directly below some rook r in $\mathcal{P}$, and

(c) $w_{p,q,B}(\mathcal{P}) = q^{a_B(\mathcal{P})} p^{b_B(\mathcal{P})}$.

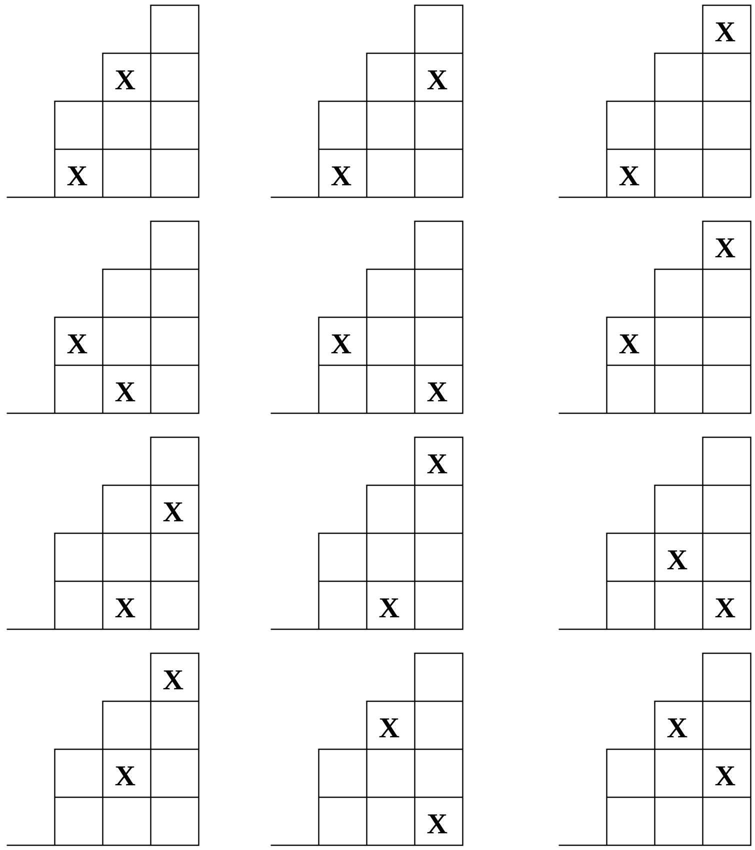

Figure 3: The placements in $\mathcal{N}_2^2(B(0,2,3,4)$

Then we define $f_k(B,p,q)$ by

$$f_k(B,p,q) = \sum_{\mathcal{P}\in\mathcal{F}_k(B)} w_{p,q,B}(\mathcal{P}). \tag{66}$$

Next suppose that we are given a placement $\mathcal{P}$ in $\mathcal{N}_k^j(B)$. Then let

(A) $\alpha_B(\mathcal{P})$ = the number of cells in B that lie directly above some rook r in $\mathcal{P}$ which are not j attacked by any rook in $\mathcal{P}$ to the left of r,

(B) $\beta_B(\mathcal{P})$ = the number of cells in B that lie directly below some rook r in $\mathcal{P}$ which are not j attacked by any rook in $\mathcal{P}$ to the left of r, and

(C) $W_{p,q,B}(\mathcal{P}) = q^{\alpha_B(\mathcal{P})}p^{\beta_B(\mathcal{P})}$.

Then we define $r_k^j(B,p,q)$ by

$$r_k^j(B,p,q) = \sum_{\mathcal{P}\in\mathcal{N}_k^j(B)} W_{p,q,B}(\mathcal{P}). \tag{67}$$

For example, in Figure 4, we have pictured an element $\mathcal{P} \in \mathcal{F}_3(B)$ where $B = B(1,2,3,5,7,8,10)$ such that $w_{p,q,B}(\mathcal{P}) = q^5p^7$. Here we have placed a q in each cell that contributes to $a_B(\mathcal{P})$ and a p in each cell that contributes to $b_B(\mathcal{P})$. In Figure 5, we have pictured an element $\mathcal{Q} \in \mathcal{N}_3^2(B)$ where $B = B(1,2,3,5,7,8,10)$ such that $W_{p,q,B}(\mathcal{Q}) = q^4p^2$. Again we have placed a q in each cell that contributes to $\alpha_B(\mathcal{P})$, a p in each cell that contributes to $\beta_B(\mathcal{P})$, and a $\cdot$ in each cell that is 2-attacked by some rook in $\mathcal{Q}$.

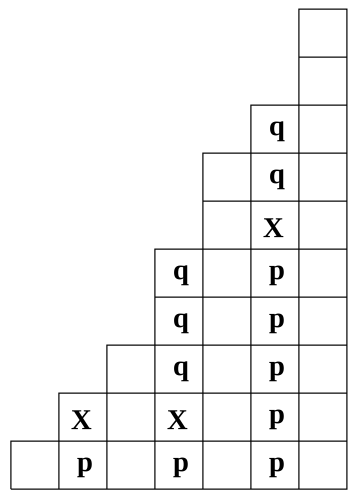

Figure 4: $w_{p,q,B}(\mathcal{P})$ for a placement in $\mathcal{F}_3(B(1,2,3,5,7,8,10))$

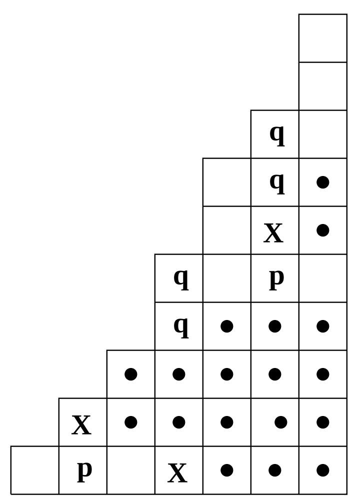

Figure 5: $W_{p,q,B}(\mathcal{Q})$ for a placement in $\mathcal{N}_3^2(B(1,2,3,5,7,8,10))$

Given any board $B = B(a_1, \ldots, a_n)$, we let B_x denote the board that results by placing x rows of size n below B. Here we call the line that separates B from the extra x rows, the *bar*; see Figure 6. This given, we have the following.

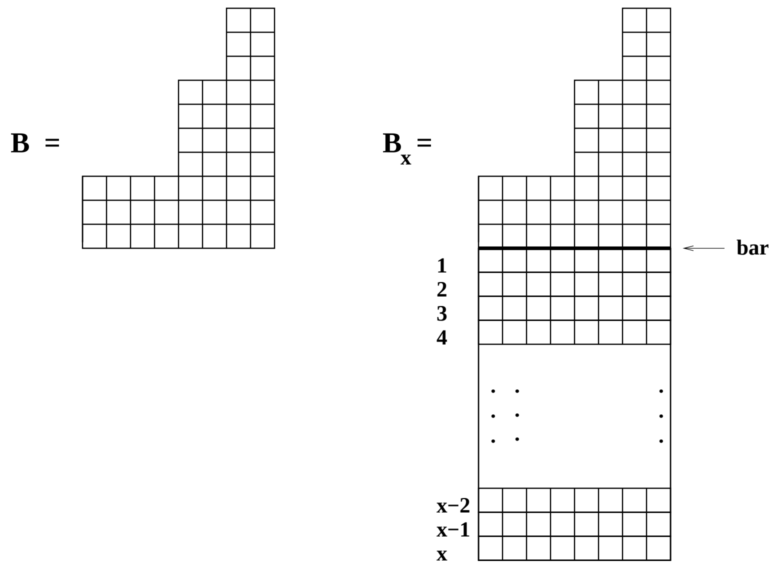

Figure 6: The board B_x

Theorem 4. *Let $B = B(a_1, \ldots, a_n)$. Then*

$$(x + [a_1]_{p,q}) \cdots (x + [a_n]_{p,q}) = \sum_{k=0}^{n} f_k(p, q, B) x^{n-k}. \tag{68}$$

Proof. We claim that for each positive integer x, the identity (68) arises from two ways of counting

$$S = \sum_{\mathcal{P} \in \mathcal{F}_n(B_x)} w_{p,q,B}(\mathcal{P} \cap B). \tag{69}$$

That is, each $\mathcal{P} \in \mathcal{F}_n(B_x)$ has exactly one rook in each column of B_x. If we consider the placement of rook r_k in the k-th column, then the possible contribution of r_k to (69) is $p^{a_k - 1}$ if we place it at the top of the column, qp^{a_k-2} if we place it in second row from the top, $\ldots$, q^{a_k-1} if we place it in the a_k-th row from the top since all these cells are in B. We also have a contribution of x to (69) which corresponds to placing r_k in rows $1, \ldots x$ below the bar in column k. It then easily follows that

$$S = (x + [a_1]_{p,q}) \cdots (x + [a_n]_{p,q}).$$

We can calculate S in second way by classifying $\mathcal{P}$ according to the number of rooks that fall in B. For any $\mathcal{Q} \in \mathcal{F}_k(B)$, we can complete $\mathcal{Q}$ to a placement $\mathcal{P} \in \mathcal{F}_n(B_x)$ such that $\mathcal{P} \cap B = \mathcal{Q}$ in exactly x^{n-k} ways corresponding to the ways of placing the $n - k$ rooks below the bar in columns which contain no rook in $\mathcal{Q}$. Thus

$$\sum_{\mathcal{P} \in \mathcal{F}_n(B_x) : \mathcal{P} \cap B = \mathcal{Q}} w_{p,q,B}(\mathcal{P} \cap B) = w_{p,q,B}(\mathcal{Q}) x^{n-k}.$$

Then

$$\begin{aligned} S &= \sum_{k=0}^{n} \sum_{\mathcal{Q}\in\mathcal{F}_k(B)} \sum_{\mathcal{P}\in\mathcal{F}_n(B_x):\mathcal{P}\cap B=\mathcal{Q}} w_{p,q,B}(\mathcal{P}\cap B) \\ &= \sum_{k=0}^{n} \sum_{\mathcal{Q}\in\mathcal{F}_k(B)} w_{p,q,B}(\mathcal{Q})x^{n-k} \\ &= \sum_{k=0}^{n} x^{n-k} \sum_{\mathcal{Q}\in\mathcal{F}_k(B)} w_{p,q,B}(\mathcal{Q}) \\ &= \sum_{k=0}^{n} f_k(p,q,B)x^{n-k}. \end{aligned}$$

Hence (68) holds for all positive integers x and since it is a polynomial identity, it must hold for all x. ■

If we replace x by $[t]_{p,q}$ in (68), we get the following.

Corollary 5.

$$([t]_{p,q}+[a_1]_{p,q})\cdots([t]_{p,q}+[a_n]_{p,q}) = \sum_{k=0}^{n} f_k(p,q,B)[t]_{p,q}^{n-k}. \tag{70}$$

Proof. We note that we can have direct combinatorial proof (70) by using the same type of reasoning as in the proof of Theorem 4 and computing the sum

$$S = \sum_{\mathcal{P}\in\mathcal{F}_n(B_x)} \overline{w}_{p,q,B_x}(\mathcal{P}). \tag{71}$$

where

$$\overline{w}_{p,q,B_x}(\mathcal{P}) = \prod_{r\in\mathcal{P}} \overline{w}_{p,q,B_x}(r)$$

where for any rook r,

$$\overline{w}_{p,q,B_x}(r) = \begin{cases} q^{a(r,B)}p^{b(r,B)} & \text{if } r\in B \\ q^{k-1}p^{x-k} & \text{if } r \text{ is in row } k \text{ below the bar} \end{cases}$$

and $a(r,B)$ is the number of cells directly above r in B and $b(r,B)$ is the number of cells directly below r in B. ■

We are now in a position to give our combinatorial interpretations of $c_{n,k}^{i,j}(p,q)$ and $S_{n,k}^{i,j}(p,q)$ defined in the introduction. Let $i\geq 0$ and $j>0$ be integers and let $B_{i,j,n}$ be the board $B(i,i+j,i+2j,\ldots,i+(n-1)j)$. Then we have the following.

Theorem 6. *If n is a positive integer and k is an integer such that $0\leq k\leq n$, then*

$$c_{n,k}^{i,j}(p,q) = f_{n-k}(p,q,B_{i,j,n}) \tag{72}$$

and

$$S_{n,k}^{i,j}(p,q) = r_{n-k}^{j}(p,q,B_{i,j,n}). \tag{73}$$

Proof. It is easy to check that $f_{n-k}(p,q,B_{i,j,n})$ and $r_{n-k}(p,q,B_{i,j,n})$ satisfy the appropriate recursions. That is, $B_{i,j,1} = B((i))$ so that it immediately follows form our definitions that for all $i \geq 0$ and $j > 0$,

$$\begin{aligned} f_1(p,q,B_{i,j,1}) &= r_1^j(p,q,B_{i,j,1}) = [i]_{p,q} \text{ and} \\ f_0(p,q,B_{i,j,1}) &= r_0^j(p,q,B_{i,j,1}) = 1. \end{aligned}$$

It follows from (30) and (32) that

$$\begin{aligned} c_{1,0}^{i,j}(p,q) &= S_{1,0}^{i,j}(p,q) = [i]_{p,q} \text{ and} \\ c_{1,1}^{i,j}(p,q) &= S_{1,1}^{i,j}(p,q) = 1. \end{aligned}$$

Thus for $k \in \{0,1\}$,

$$\begin{aligned} c_{1,k}^{i,j}(p,q) &= f_{1-k}(p,q,B_{i,j,1}) \text{ and} \\ S_{1,k}^{i,j}(p,q) &= r_{1-k}^j(p,q,B_{i,j,1}). \end{aligned}$$

Clearly $f_k(p,q,B_{i,j,n}) = 0$ and $r_k^j(p,q,B_{i,j,n}) = 0$ if $k > n$ or $k < 0$ since there are no placements in $\mathcal{F}_k(B_{i,j,n})$ or $\mathcal{N}_k^j(B_{i,j,n})$ if $k > n$ or $k < 0$. Thus to verify that (72) and (73) hold we need only verify that for all $n \geq 1$ and $0 \leq k \leq n$,

$$f_{n+1-k}(p,q,B_{i,j,n+1}) = f_{n-(k-1)}(p,q,B_{i,j,n}) + [i+nj]_{p,q} f_{n-k}(p,q,B_{i,j,n}) \tag{74}$$

and

$$r_{n+1-k}^j(p,q,B_{i,j,n+1}) = r_{n-(k-1)}^j(p,q,B_{i,j,n}) + [i+kj]_{p,q} r_{n-k}^j(p,q,B_{i,j,n}). \tag{75}$$

Both recursions can be proved in the same way. That is, to prove (74), we simply partition the elements of $\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ into two sets No and $Last$ where No consists of the placements of $\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ which have no rook in the last column and $Last$ consists of the placements of $\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ which have a rook in the last column. It is easy to see that a placement in No has $n-(k-1)$ rooks to the left of the last column and the weight of any placement $\mathcal{P} \in No$ is the same as the placement $\mathcal{Q}$ in $\mathcal{F}_{n-(k-1)}(B_{i,j,n})$ that results by eliminating the last column. Thus

$$\sum_{\mathcal{P} \in No} w_{p,q,B_{i,j,n+1}}(\mathcal{P}) = \sum_{\mathcal{Q} \in \mathcal{F}_{n-(k-1)}(B_{i,j,n})} w_{p,q,B_{i,j,n}}(\mathcal{Q}) = f_{n-(k-1)}(p,q,B_{i,j,n}).$$

To compute $\sum_{\mathcal{P} \in Last} w_{p,q,B_{i,j,n+1}}(\mathcal{P})$, observe that if we fix a placement $\mathcal{Q} \in \mathcal{F}_{n-k}(B_{i,j,n})$, then we can extend $\mathcal{Q}$ to a placement $\mathcal{P} \in Last$ by placing an additional rook in the last column. Since the height of the last column of $B_{i,j,n+1}$ is $i+nj$, there will be $i+nj$ such placements. Moreover, it is easy to see that if we place the rook in the s-th row of the last column, where we label the row with $1,\ldots,i+nj$ reading from bottom to top, then the weight of the corresponding placement $\mathcal{P}^s$ is $q^{i+nj-s}p^{s-1}w_{p,q,B_{i,j,n}}(\mathcal{Q})$. It follows that

$$\begin{aligned} &\sum_{\mathcal{P} \in Last} w_{p,q,B_{i,j,n+1}}(\mathcal{P}) \\ &= \sum_{\mathcal{Q} \in \mathcal{F}_{n-k}(B_{i,j,n})} (q^{i+nj-1} + q^{i+nj-2}p + \cdots + qp^{i+nj-2} + p^{i+nj-1}) w_{p,q,B_{i,j,n}}(\mathcal{Q}) \\ &= [i+nj]_{p,q} f_{n-k}(p,q,B_{i,j,n}). \end{aligned}$$

The argument to prove (75) is essentially the same. That is, again partition the elements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ into two sets No and $Last$ where No consists of the placements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ which have no rook in the last column and $Last$ consists of the placements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ which have a rook in the last column. Again it is easy to see that

$$\sum_{\mathcal{P}\in No} W_{p,q,B_{i,j,n+1}}(\mathcal{P}) = \sum_{\mathcal{Q}\in\mathcal{N}^j_{n-(k-1)}(B_{i,j,n})} W_{p,q,B_{i,j,n+1}}(\mathcal{Q}) = r^j_{n-(k-1)}(p,q,B_{i,j,n}).$$

To compute $\sum_{\mathcal{P}\in Last} W_{p,q,B_{i,j,n+1}}(\mathcal{P})$ observe that if we fix a placement $\mathcal{Q}\in\mathcal{N}^j_{n-k}(B_{i,j,n})$, then we can extend $\mathcal{Q}$ to a placement $\mathcal{P}\in Last$ by placing an additional rook in the last column. In this case, the $n-k$ rooks in $\mathcal{Q}$ will j-attack exactly $(n-k)j$ cells in the last column. Since the height of the last column of $B_{i,j,n+1}$ is $i+nj$, there will be $i+nj-(n-k)j = i+kj$ cells in the last column of $B_{i,j,n+1}$ which are not j-attacked and hence there will be $i+kj$ such placements. Moreover, it is easy to see that if we place the rook in the s-th row of the non-j-attacked cells of the last column, where we label such rows with $1,\ldots,i+kj$ reading from bottom to top, then the weight of the corresponding placement $\mathcal{P}^s$ is $q^{i+kj-s}p^{s-1}W_{p,q,B_{i,j,n}}(\mathcal{Q})$. It follows that

$$\begin{aligned}
&\sum_{\mathcal{P}\in Last} W_{p,q,B_{i,j,n+1}}(\mathcal{P}) \\
&= \sum_{\mathcal{Q}\in\mathcal{N}^j_{n-k}(B_{i,j,n})} (q^{i+kj-1}+q^{i+kj-2}p+\cdots+qp^{i+kj-2}+p^{i+kj-1})W_{p,q,B_{i,j,n}}(\mathcal{Q}) \\
&= [i+kj]_{p,q} r^j_{n-k}(p,q,B_{i,j,n}).
\end{aligned}$$

■

We point out here that if we modify our combinatorial interpretation of $S^{0,1}_{n,k}(p,q)$ to include a factor of p for each uncancelled cell in an empty column, we will get one of the rook theoretic combinatorial interpretations of $S_{n,k}(p,q)$ given by Wachs and White [26].

We note that when $B = B_{i,j,n}$, then Corollary 5 becomes

$$([t]_{p,q}+[i]_{p,q})\cdots([t]_{p,q}+[i+(n-1)j]_{p,q}) = \sum_{k+0}^{n} c^{i,j}_{n,k}[t]^k_{p,q}$$

which is just (33). Thus (33) has a combinatorial proof. Then replacing $[t]_{p,q}$ by $-[t]_{p,q}$, multiplying by $(-1)^n$ and using the fact that $s^{i,j}_{n,k}(p,q) = (-1)^{n-k}c^{i,j}_{n,k}(p,q)$ clearly yields (35). Then we can derive (34) from (35) by using the fact that the the matrices $||s^{i,j}_{n,k}(p,q)||$ and $||S^{i,j}_{n,k}(p,q)||$ are inverses of each other. A direct combinatorial proof of (34) was found by Briggs and Remmel in [8]. We give a direct combinatorial proof the matrices $||s^{i,j}_{n,k}(p,q)||$ and $||S^{i,j}_{n,k}(p,q)||$ are inverses of each other. That is, if we start with our combinatorial interpretations of $c^{i,j}_{n,k}(p,q)$ and $S^{i,j}_{n,k}(p,q)$, then we can give a combinatorial proof of the following for all $0\le r\le n$.

$$\sum_{k=r}^{n} S^{i,j}_{n,k}(p,q)s^{i,j}_{k,r}(p,q) = \chi(r=n). \tag{76}$$

Note that if $r=n$, then (76) reduces down to

$$1 = S^{i,j}_{n,n}(p,q)s^{i,j}_{n,n}(p,q). \tag{77}$$

But (77) holds since both $\mathcal{N}^j_{n-n}(B_{i,j,n})$ and $\mathcal{F}_{n-n}(B_{i,j,n})$ consist solely of the empty placement $\mathcal{E}$. Since $W_{p,q,B_{i,j,n}}(\mathcal{E}) = w_{p,q,B_{i,j,n}}(\mathcal{E}) = 1$, it follows that $S^{i,j}_{n,n}(p,q) = s^{i,j}_{n,n}(p,q) = 1$ and hence (77) holds.

Now suppose that $n > r$. Then

$$\begin{aligned}
&\sum_{k=r}^{n} S^{i,j}_{n,k}(p,q) s^{i,j}_{k,r}(p,q) \\
&= \sum_{k=r}^{n} (-1)^{k-r} \sum_{(\mathcal{P},\mathcal{Q}) \in \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})} W_{p,q,B_{i,j,n}}(\mathcal{P}) w_{p,q,B_{i,j,k}}(\mathcal{Q}) \\
&= \sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q}) \in \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})} W_{p,q,B_{i,j,n}}(\mathcal{P}) sgn(\mathcal{Q}) w_{p,q,B_{i,j,k}}(\mathcal{Q})
\end{aligned}$$

where $sgn(\mathcal{Q}) = (-1)^{\text{no. of rooks in } \mathcal{Q}}$. Then consider the elements

$$(\mathcal{P},\mathcal{Q}) \in \bigcup_{k=r}^{n} \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k}).$$

We can partition these elements into three classes.

Class I. There is a rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$.

Class II. There is no rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$, but there is a rook of $\mathcal{Q}$ in the last column of $B_{i,j,k}$.

Class III. There is no rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$ and there is no rook of $\mathcal{Q}$ in the last column of $B_{i,j,k}$.

Next we define a weight preserving sign-reversing bijection f from Class I to Class II. Given an element $(\mathcal{P},\mathcal{Q}) \in \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})$ in Class I, note that there are a total of $n-k-1$ rooks in $\mathcal{P}$ to the left of the last column of $B_{i,j,n}$ and these rooks j-attack a total of $(n-k-1)j$ cells in the last column. Thus in the last column of $B_{i,j,n}$, there are a total of $i+(n-1)j-(n-k-1)j = i+kj$ cells in the last column of $B_{i,j,n}$ which are not j-attacked by a rook in $\mathcal{P}$. Then define $f((\mathcal{P},\mathcal{Q})) = (\mathcal{P}',\mathcal{Q}')$ where

(i) $\mathcal{P}'$ is the result of taking the placement $\mathcal{P}$ and removing the rook in the last column of $B_{i,j,n}$ and

(ii) $\mathcal{Q}'$ is the result of adding an extra column of height $i+kj$ to the right of the placement $\mathcal{Q}$ and placing a rook f_k in that column which is in row t if the rook r_n in $\mathcal{P}$ in the last column of $B_{i,j,n}$ was in the t-th cell, reading from bottom to top, which was not j-attacked by a rook in $\mathcal{P}$ to the left of r_n.

See Figure 7 for an example of this map when $n=6$, $k=3$ and $r=1$. Our definitions ensure that r_n contributes a factor of $q^{i+jk-t}p^{t-1}$ to $W_{p,q,B_{i,j,n}}(\mathcal{P})$ and that f_k contributes a factor of

$q^{i+jk-t}p^{t-1}$ to $w_{p,q,B_{i,j,k+1}}(\mathcal{Q}')$. Thus

$$\begin{aligned} W_{p,q,B_{i,j,n}}(\mathcal{P})w_{p,q,B_{i,j,k}}(\mathcal{Q}) &= W_{p,q,B_{i,j,n}}(\mathcal{P}')q^{i+jk-t}p^{t-1}W_{p,q,B_{i,j,n}}(\mathcal{Q}) \\ &= W_{p,q,B_{i,j,n}}(\mathcal{P}')w_{p,q,B_{i,j,k+1}}(\mathcal{Q}'). \end{aligned}$$

Clearly $sgn(\mathcal{Q}) = (-i)^{k-r} = -sgn(\mathcal{Q}') = (-1)^{k-r+1}$ so that f is a sign reversing weight preserving function which for each $r \leq k \leq n$ maps the elements of $\mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})$ in Class I to the elements $\mathcal{N}^j_{n-k-1}(B_{i,j,n}) \times \mathcal{F}_{k+1-r}(B_{i,j,k+1})$ in Class II. Moreover f^{-1} is easily defined. That is, if $(\mathcal{P}', \mathcal{Q}') \in \mathcal{N}^j_{n-k-1}(B_{i,j,n}) \times \mathcal{F}_{k+1-r}(B_{i,j,k+1})$ is in Class II and the rook in the last column of $\mathcal{Q}'$ is in row t, then $f^{-1}(\mathcal{P}', \mathcal{Q}')) = (\mathcal{P}, \mathcal{Q})$ where $\mathcal{Q}$ is results from $\mathcal{Q}'$ by removing the last column of $B_{i,j,k+1}$ and $\mathcal{P}$ results from $\mathcal{P}'$ by adding a rook in the last column of $B_{i,j,n}$ in the t-th cell from the bottom which is not j-attacked by any rook in $\mathcal{P}'$. Thus f is a bijection which shows that

$$\sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q})\in\mathcal{N}^j_{n-k}(B_{i,j,n})\times\mathcal{F}_{k-r}(B_{i,j,k})} W_{p,q,B_{i,j,n}}(\mathcal{P})sgn(\mathcal{Q})w_{p,q,B_{i,j,k}}(\mathcal{Q}) =$$

$$\sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q})\in Class\ III} W_{p,q,B_{i,j,n}}(\mathcal{P})sgn(\mathcal{Q})w_{p,q,B_{i,j,k}}(\mathcal{Q})$$

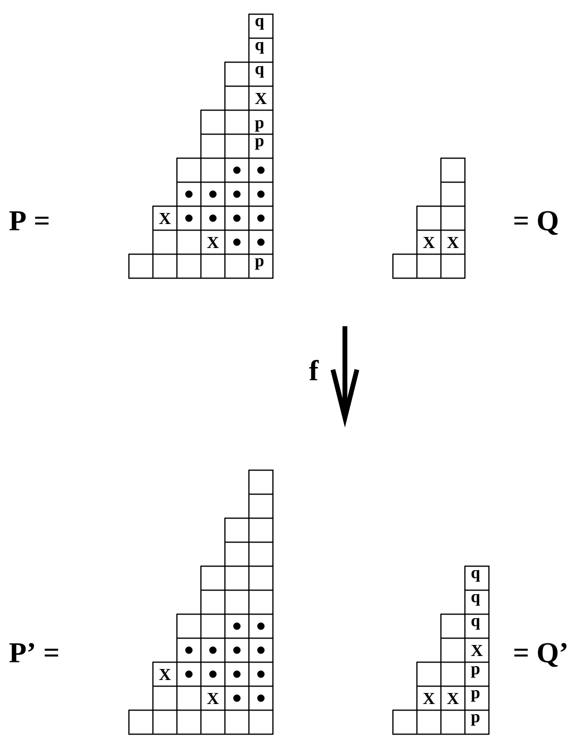

Figure 7: An example of the map f from Class I to Class II

Note if $r = 0$, then there are no elements in Class III since every element of $(\mathcal{P}, \mathcal{Q}) \in \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-0}(B_{i,j,k})$ has a rook of $\mathcal{Q}$ in the last column of $B_{i,j,k}$. Thus if $r = 0$, then

f shows that $\sum_{k=0}^{n} S_{n,k}^{i,j}(p,q)s_{k,0}^{i,j}(p,q) = 0$. Finally if $r \geq 1$, then there is a weight preserving bijection g which maps Class III onto $\bigcup_{k=r-1}^{n-1} \mathcal{N}_{n-1-k}^{j}(B_{i,j,n-1}) \times \mathcal{F}_{k-(r-1)}(B_{i,j,k})$. That is, if $(\mathcal{P},\mathcal{Q})$ is in Class III, then $g((\mathcal{P},\mathcal{Q})) = (\mathcal{P}'',\mathcal{Q}'')$ where $\mathcal{P}''$ is obtained from $\mathcal{P}$ by removing its last column and $\mathcal{Q}''$ is obtained from $\mathcal{Q}$ by removing its last column. See Figure 8 for an example.

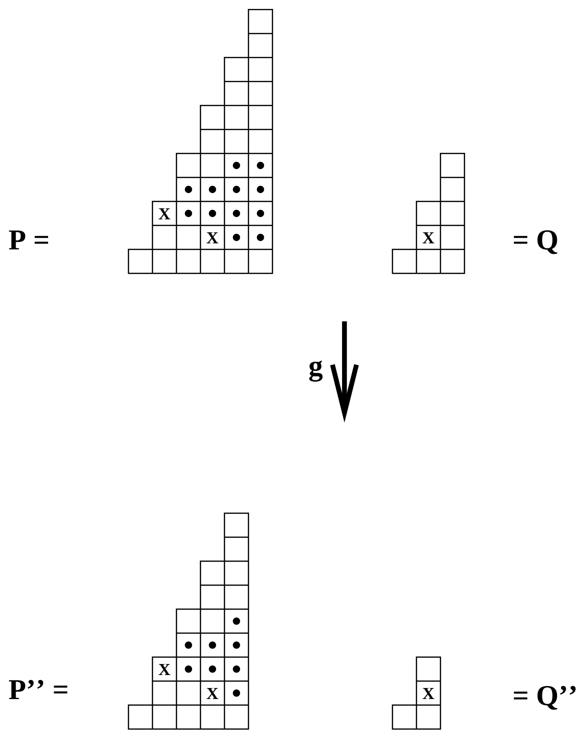

Figure 8: An example of the map g

Thus if $r \geq 1$, then our bijections f and g show that

$$\sum_{k=r}^{n} S_{n,k}^{i,j}(p,q)s_{k,r}^{i,j}(p,q) = \sum_{k=r-1}^{n-1} S_{n-1,k}^{i,j}(p,q)s_{k,r-1}^{i,j}(p,q) = \chi(r-1=n-1) \tag{78}$$

where the last equality follows by induction. Thus we have proved that

$$\sum_{k=r}^{n} S_{n,k}^{i,j}(p,q)s_{k,r}^{i,j}(p,q) = \chi(r=n)$$

as desired.

3 A combinatorial interpretation of $\tilde{s}_{n,k}^{i,j}(p,q)$ and $\tilde{S}_{n,k}^{i,j}(p,q)$

The main purpose of this section is to develop alternative versions of (p,q)-rook numbers and (p,q)-file numbers which are suitable to be specialized to give combinatorial interpretations of $\tilde{s}_{n,k}^{i,j}(p,q)$ and $\tilde{S}_{n,k}^{i,j}(p,q)$.

Let $B = B(a_1, \ldots, a_n)$ be a j-attacking board. Then for any placement $\mathcal{P} \in \mathcal{N}_k^j(B)$, we define

$$\tilde{W}^j_{p,q,B}(\mathcal{P}) = q^{a_B(\mathcal{P})} p^{b_B(\mathcal{P})} q^{e_B(\mathcal{P})} p^{-(c_1 + \cdots + c_n)j} \tag{79}$$

where

1. $a_B(\mathcal{P})$ equals the number of cells of B which lie above a rook in $\mathcal{P}$ and which are not j-attacked by any rook in $\mathcal{P}$,

2. $b_B(\mathcal{P})$ equals the number of cells of B which lie below a rook in $\mathcal{P}$ and which are not j-attacked by any rook in $\mathcal{P}$,

3. $e_B(\mathcal{P})$ equals the number of cells of B which lie in a column with no rook in $\mathcal{P}$ and which are not j-attacked by any rook in $\mathcal{P}$, and

4. $c_1 < \cdots < c_k$ are the columns which contain rooks in $\mathcal{P}$ where we label the columns of B with $1, \ldots, n$ reading from left to right.

For example, in Figure 9, we have pictured a placement $\mathcal{P} \in \mathcal{N}_3^3(B)$ where B is the 3-attacking board $B(2, 5, 8, 10, 12)$ such that $\mathcal{P}$ has rooks in columns 1, 3 and 4 and $a_B(\mathcal{P}) = 3$, $b_B(\mathcal{P}) = 5$, $e_B(\mathcal{P}) = 5$. Thus $\tilde{W}^3_{p,q,B}(\mathcal{P}) = q^3 p^5 q^5 p^{-(1+3+4)3} = q^8 p^{-19}$. Moreover, we have placed a p in each cell of B which contributes to the $b_B(\mathcal{P})$, a q in each cell that contributes to either $a_B(\mathcal{P})$ or $e_B(\mathcal{P})$, and a dot in each cell that is j-attacked by some rook in $\mathcal{P}$.

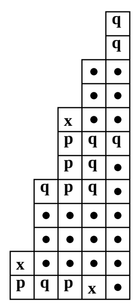

Figure 9: An example of $\tilde{W}_{p,q,B}(\mathcal{P})$

We then define the (p, q)-rook number of B (of type II) by

$$\tilde{r}^j_{k,B}(p, q) = \sum_{\mathcal{P} \in \mathcal{N}_k^j(B)} \tilde{W}^j_{p,q,B}(\mathcal{P}). \tag{80}$$

This given, we then have the following result.

Theorem 7. *Let $B = B(a_1, \ldots, a_n)$ be a j-attacking board. Then*

$$[x + a_1]_{p,q}[x + a_2 - j]_{p,q} \cdots [x + a_n - (n-1)j]_{p,q} = \sum_{k=0}^{n} \tilde{r}^j_{k,B}(p, q) p^{kx + \binom{k+1}{2}j} [x]_{p,q} \downarrow_{n-k,j} \tag{81}$$

where $[x]_{p,q} \downarrow_{0,j} = 1$ and for $k > 0$, $[x]_{p,q} \downarrow_{k,j} = [x]_{p,q}[x - j]_{p,q} \cdots [x - (k-1)j]_{p,q}$.

Proof. We note that when $j = 1$ and $p = 1$, (81) becomes

$$[x + a_1]_q[x + a_2 - 1]_q \cdots [x + a_n - (n-1)]_q = \sum_{k=0}^{n} \tilde{r}^1_{n-k,B}(1,q)[x]_q \downarrow_k$$

which was first proved by Garsia and Remmel [11]. Our proof is a generalization of their proof.

It is enough to prove (81) for all positive integers $x \geq jn$. So fix a positive integer $x \geq jn$ and let B_x be the board which results by adding x rows of length n below B as described in section 1. We shall consider placements of n rooks in B_x where there is at most one rook in each row and column. A rook r which lies above the bar will j-attack cells as described in section 1. Thus a rook r which lies above the bar will only j-attack cells which are above the bar. Similarly, we shall define the cells which a rook r' below the bar j-attacks so that each rook r' will only j-attack cells below the bar in B_x. We say that a rook r' which lies in column k and row l, where here we label the rows below the bar with $1, \ldots, x$ reading from top to bottom, j-attacks a cell $c \in B_x$ which is below the bar only if c lies in a column that is strictly to the right of column k and either

(i) c lies in the first j rows of B_x below the bar which are weakly above row l and which contain no cell that is j-attacked by some rook r'' to the left of r' or

(ii) there are $t < j$ rows below the bar which are weakly above row l and which contain no cell that is j-attacked by some rook r'' which is strictly to the left of column k and c is in the largest $j - t$ rows which are not j-attacked by any rook r'' which is strictly to the left of r'.

In other words, a rook in column k and row l below the bar j-attacks all cells below the bar which are not j-attacked by any rook r'' to the left of r', which are in a column strictly to the right of k and which lie in the first j such rows where we order the rows in the order $l, l-1, \ldots, 1, x, x-1, \ldots, l+1$. Thus when we look for rows for r' to j-attack, we only consider rows below the bar which are not j-attacked by any rook r'' to the left of r'. Then we first look at such rows which are weakly above l, but if there are not j such rows weakly above row l, then we cycle around starting at the bottom row until we find a total of j rows to attack. We then let $\mathcal{N}^j_k(B_x)$ denote the set of all placements $\mathcal{P}$ of n rooks in B_x such that there is at most one rook in each row and column and such that no rook j-attacks another rook. This given, we can then define $W_{p,q,B_x}(\mathcal{P})$ just as we did in section 1, namely,

$$W^j_{p,q,B_x}(\mathcal{P}) = q^{a_B(\mathcal{P})} p^{b_B(\mathcal{P})} \tag{82}$$

where

$a_B(\mathcal{P})$ equals the number of cells of B which lie above a rook in $\mathcal{P}$ and which are not j-attacked by any rook in $\mathcal{P}$ and

$b_B(\mathcal{P})$ equals the number of cells of B which lie below a rook in $\mathcal{P}$ and which are not j-attacked by any rook in $\mathcal{P}$.

For example, consider the placement $\mathcal{P} \in \mathcal{N}^3_4(B(1,3,5,7)_{10})$ pictured in Figure 10. We shall denote the positions of the four rooks, reading from left to right, by placing circled elements containing the numbers 1, 2, 3 and 4. We shall then indicate the cells which are 3-attacked by

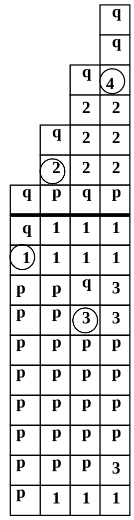

Figure 10: An example of $\tilde{W}_{p,q,B}(\mathcal{P})$

the circled rook with label i by placing i's in such cells. We shall place a q or a p in those cells which are not 3-attacked by any rook in $\mathcal{P}$ depending on whether the cell contributes a factor of q or p to $W_{p,q,B_x}(\mathcal{P})$ from which it will be clear that $W_{p,q,B_x}(\mathcal{P}) = q^8p^{25}$.

This given, we shall show that (81) results from two different ways of computing the sum

$$S = \sum_{\mathcal{P}\in\mathcal{N}_n^j(B_x)} W_{p,q,B_x}(\mathcal{P}). \tag{83}$$

That is, first consider the contribution to S of the possible placements of rooks in each column proceeding from left to right. For the first column, it is easy to see that the contribution to S by placing rooks in the cells starting at the top and going down to the bottom are, respectively, $p^{a_1+x-1}, qp^{a_1+x-2}, q^2p^{a_1+x-3}, \ldots, q^{a_1+x-2}p, q^{a_1+x-1}$. Thus the contribution to S from the first column is $[a_1+x]_{p,q}$. We can apply the same argument to the second column except that j-cells in that column will be j-attacked by the rook in column 1 so the the contribution to S from the second column is $[a_2+x-j]_{p,q}$. Similarly, the contribution to S from the third column is $[a_3+x-2j]_{p,q}$ since a total of $2j$ cells in column 3 will be j-attacked by the rooks in columns 1 and 2. Continuing on in this way, we see that

$$S = \prod_{r=1}^{n}[a_r+x-(r-1)j]_{p,q}. \tag{84}$$

Next fix a placement $\mathcal{Q}$ of k rooks in B. We want to compute

$$S(\mathcal{Q}) = \sum_{\mathcal{P}\in\mathcal{N}_n^j(B_x),\mathcal{P}\cap B=\mathcal{Q}} W_{p,q,B_x}(\mathcal{P}). \tag{85}$$

It is easy to see that $q^{a_B(\mathcal{Q})}b^{b_B(\mathcal{Q})}q^{e_B(\mathcal{Q})}$ is the contribution $W_{p,q,B_x}(\mathcal{P})$ of the cells above the bar. Now if $\mathcal{P}$ has rooks in columns $c_1,\ldots,c_k$ where $1 \le c_1 < \ldots < c_k \le n$, then the cells in those columns, which lie below the bar and which are not j-attacked by a rook in $\mathcal{P}$, each contribute a factor of p to $W_{p,q,B_x}(\mathcal{P})$. Note that there are $c_t - t$ rooks of $\mathcal{P}$ below the bar which lie to the left of column c_t, and each such rook will j-attack exactly j cells in column c_t. Thus the total number of cells below the bar in columns $c_1, \ldots, c_k$ which are not j-attacked by any rook in $\mathcal{P}$ is

$$(x-(c_1-1)j)+(x-(c_2-2)j)+\cdots+(x-(c_k-k)j) = kx + \binom{k+1}{2}j - (c_1+\cdots+c_k)j.$$

Thus such cells below the bar contribute a factor of $p^{kx+\binom{k+1}{2}j-(c_1+\cdots+c_k)j}$ to $W_{p,q,B_x}(\mathcal{P})$. Finally consider the contribution to $W_{p,q,B_x}(\mathcal{P})$ of the cells below the bar in the remaining $n-k$ columns. For the leftmost such column, it is easy to see that the contribution to S by placing rooks in the cells starting at the top and going down to the bottom are, respectively, $p^{x-1}, qp^{x-2}, q^2p^{x-3}, \ldots, q^{x-2}p, q^{x-1}$. Thus the contribution to $S(\mathcal{Q})$ from the leftmost column which contains a rook below the bar is is $[x]_{p,q}$. We can apply the same argument to the second leftmost column that contains a rook below the bar except that j-cells in that column will be j-attacked by the rook in the leftmost column which contains a rook below the bar. Thus the the contribution to $S(\mathcal{Q})$ from the second such column is $[x-j]_{p,q}$. Similarly, the contribution to S from the third such column is $[x-2j]_{p,q}$ since at total of $2j$ cells in that column will be j-attacked by the rooks in below the bar to its left. Continuing on in this way, we see that contribution to $S(\mathcal{Q})$ from the cells in the remains $n-k$ columns is

$$\prod_{r=1}^{n-k}[x-(r-1)j]_{p,q} = [x]_{p,q}\downarrow_{n-k,j}.$$

It follows that

$$\begin{aligned} S(\mathcal{Q}) &= q^{a_B(\mathcal{Q})}b^{b_B(\mathcal{Q})}q^{e_B(\mathcal{Q})}p^{-(c_1+\ldots+c_k)j}p^{kx+\binom{k+1}{2}}[x]_{p,q}\downarrow_{n-k,j} \\ &= \tilde{W}_{p,q,B}(\mathcal{Q})p^{kx+\binom{k+1}{2}j}[x]_{p,q}\downarrow_{n-k,j}. \end{aligned} \tag{86}$$

Thus

$$\begin{aligned} S &= \sum_{k=0}^{n} p^{kx+\binom{k+1}{2}j}[x]_{p,q}\downarrow_{n-k,j} \sum_{\mathcal{Q}\in\mathcal{N}_k^j(B)} \tilde{W}_{p,q,B}(\mathcal{Q}) \\ &= \sum_{k=0}^{n} \tilde{r}_{k,B}^{j}(p,q)p^{kx+\binom{k+1}{2}j}[x]_{p,q}\downarrow_{n-k,j}. \end{aligned} \tag{87}$$

Combining (84) and (87) yields (81) as desired. ■

Next we define the (p,q)-file number (of type II) for a j-attacking board $B = B(a_1, \ldots, a_n)$ by

$$\tilde{f}_{k,B}(p,q) = \sum_{\mathcal{P}\in\mathcal{F}_k(B)} \tilde{w}_{p,q,B}^{j}(\mathcal{P}) \tag{88}$$

where for any $\mathcal{P} \in \mathcal{F}_k(B)$, we define

$$\tilde{w}_{p,q,B}^{j}(\mathcal{P}) = q^{\alpha_B(\mathcal{P})}p^{\beta_B(\mathcal{P})}p^{\epsilon_B(\mathcal{P})} \tag{89}$$

where

1. $\alpha_B(\mathcal{P})$ equals the number of cells of B which lie above a rook in $\mathcal{P}$,

2. $\beta_B(\mathcal{P})$ equals the number of cells of B which lie below a rook in $\mathcal{P}$,

3. $\epsilon_B(\mathcal{P})$ equals the number of cells of B which lie in a column with no rook in $\mathcal{P}$.

We are now in a position to give our combinatorial interpretations of $\tilde{s}_{n,k}^{i,j}(p,q)$ and $\tilde{S}_{n,k}^{i,j}(p,q)$ defined in the introduction. Set $\tilde{c}_{n,k}^{i,j}(p,q) = (-1)^{n-k}\tilde{s}_{n,k}^{i,j}(p,q)$. Let $i \geq 0$ and $j > 0$ be integers and let $B_{i,j,n}$ be the board $B(i, i+j, i+2j, \ldots, i+(n-1)j)$. Then we have the following.

Theorem 8. *If n is a positive integer and k is an integer such that $0 \leq k \leq n$, then*

$$\tilde{c}_{n,k}^{i,j}(p,q) = \tilde{f}_{n-k}(p,q,B_{i,j,n}) \tag{90}$$

and

$$\tilde{S}_{n,k}^{i,j}(p,q) = \tilde{r}_{n-k}^{j}(p,q,B_{i,j,n}). \tag{91}$$

Proof. It is easy to check that $\tilde{f}_{n-k}(p,q,B_{i,j,n})$ and $\tilde{r}_{n-k}^{j}(p,q,B_{i,j,n})$ satisfy the appropriate recursions. That is, $B_{i,j,1} = B((i))$ so that it immediately follows from our definitions that for all i and j,

$$\begin{aligned}
\tilde{f}_1(p,q,B_{i,j,1}) &= \tilde{r}_1^{j}(p,q,B_{i,j,1}) = [i]_{p,q}, \\
\tilde{f}_0(p,q,B_{i,j,1}) &= p^i \text{ and} \\
\tilde{r}_0^{j}(p,q,B_{i,j,1}) &= q^i.
\end{aligned}$$

It follows from (30) and (32) that

$$\begin{aligned}
\tilde{c}_{1,0}^{i,j}(p,q) &= \tilde{S}_{1,0}^{i,j}(p,q) = [i]_{p,q}, \\
\tilde{c}_{1,1}^{i,j}(p,q) &= p^i \text{ and} \\
\tilde{S}_{1,1}^{i,j}(p,q) &= q^i.
\end{aligned}$$

Thus for $k \in \{0,1\}$,

$$\begin{aligned}
\tilde{c}_{1,k}^{i,j}(p,q) &= \tilde{f}_{1-k}(p,q,B_{i,j,1}) \text{ and} \\
\tilde{S}_{1,k}^{i,j}(p,q) &= \tilde{r}_{1-k}^{j}(p,q,B_{i,j,1}).
\end{aligned}$$

Clearly $\tilde{f}_k(p,q,B_{i,j,n}) = 0$ and $\tilde{r}_k^{j}(p,q,B_{i,j,n}) = 0$ if $k > n$ or $k < 0$ since there are no placements in $\mathcal{F}_k(B_{i,j,n})$ or $\mathcal{N}_k^{j}(B_{i,j,n})$ if $k > n$ or $k < 0$. Thus to verify that (90) and (91) hold we need only verify that for all $n \geq 1$ and $0 \leq k \leq n$,

$$\tilde{f}_{n+1-k}(p,q,B_{i,j,n+1}) = p^{nj+i}\tilde{f}_{n-k-1}(p,q,B_{i,j,n}) + [i+nj]_{p,q}\tilde{f}_{n-k}(p,q,B_{i,j,n}) \tag{92}$$

and

$$\tilde{r}_{n+1-k}^{j}(p,q,B_{i,j,n+1}) = q^{i+kj}\tilde{r}_{n-k-1}^{j}(p,q,B_{i,j,n}) + p^{-(n+1)j}[i+kj]_{p,q}\tilde{r}_{n-k}^{j}(p,q,B_{i,j,n}). \tag{93}$$

Both recursions can be proved in the same way. That is, to prove (92), we simply partition the elements of $\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ into two sets No and $Last$ where No consists of the placements of

$\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ which have no rook in the last column and *Last* consists of the placements of $\mathcal{F}_{n+1-k}(B_{i,j,n+1})$ which have a rook in the last column. It is easy to see that a placement in *No* has $n-k-1$ rooks to the left of the last column and the weight of any placement $\mathcal{P} \in No$ is $p^{nj+i}\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{P}')$ where $\mathcal{P}'$ is the placement in $\mathcal{F}_{n-(k-1)}(B_{i,j,n})$ that results by eliminating the last column of $\mathcal{P}$. Thus

$$\sum_{\mathcal{P}\in No} \tilde{w}_{p,q,B_{i,j,n+1}}(\mathcal{P}) = \sum_{\mathcal{P}'\in\mathcal{F}_{n-(k-1)}(B_{i,j,n})} p^{nj+i}\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{P}') = p^{nj+i}\tilde{f}_{n-(k-1)}(p,q,B_{i,j,n}).$$

To compute $\sum_{\mathcal{P}\in Last} \tilde{w}_{p,q,B_{i,j,n+1}}(\mathcal{P})$ observe that if we fix a placement $\mathcal{Q} \in \mathcal{F}_{n-k}(B_{i,j,n})$, then we can extend $\mathcal{Q}$ to a placement $\mathcal{P} \in Last$ by placing an additional rook in the last column. Since the height of the last column of $B_{i,j,n+1}$ is $i+nj$, there will be $i+nj$ such placements. Moreover, it is easy to see that if we place the rook in the s-th row of the last column, where we label the row with $1,\ldots,i+nj$ reading from bottom to top, then the weight of the corresponding placement $\mathcal{P}^s$ is $q^{i+nj-s}p^{s-1}w_{p,q,B_{i,j,n}}(\mathcal{Q})$. It follows that

$$\begin{aligned}
&\sum_{\mathcal{P}\in Last} \tilde{w}_{p,q,B_{i,j,n+1}}(\mathcal{P}) \\
&= \sum_{\mathcal{Q}\in\mathcal{F}_{n-k}(B_{i,j,n})} (q^{i+nj-1} + q^{i+nj-2}p + \cdots + qp^{i+nj-2} + p^{i+nj-1})\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{Q}) \\
&= [i+nj]_{p,q}\tilde{f}_{n-k}(p,q,B_{i,j,n}).
\end{aligned}$$

The argument to prove (93) is essentially the same. That is, again partition the elements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ into two sets *No* and *Last* where *No* consists of the placements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ which have no rook in the last column and *Last* consists of the placements of $\mathcal{N}^j_{n+1-k}(B_{i,j,n+1})$ which have a rook in the last column. If $\mathcal{P} \in No$, then there are $n+1-k$ rooks to the left of the last column in P. These rook j-attack a total of $(n+1-k)j$ cells in the last column. Thus there are a total of $nj+i-(n+1-k)j = (k-1)j+i$ cells in last column of $B_{i,j,n+1}$ which are not j-attacked by any rook in $\mathcal{P}$. Each such cell is counted in $\epsilon_{B_{i,j,n+1}}(\mathcal{P})$ so that these cells contribute a factor of $q^{(k-1)j+i}$ to $\tilde{W}_{p,q,B_{i,j,n+1}}(\mathcal{P})$. Thus

$$\begin{aligned}
\sum_{\mathcal{P}\in No} \tilde{W}_{p,q,B_{i,j,n+1}}(\mathcal{P}) &= q^{(k-1)j+i} \sum_{\mathcal{Q}\in\mathcal{N}^j_{n-(k-1)}(B_{i,j,n})} \tilde{W}_{p,q,B_{i,j,n}}(\mathcal{Q}) \\
&= q^{(k-1)j+i}\tilde{r}^j_{n-(k-1)}(p,q,B_{i,j,n}).
\end{aligned}$$

To compute $\sum_{\mathcal{P}\in Last} \tilde{W}_{p,q,B_{i,j,n+1}}(\mathcal{P})$ observe that if we fix a placement $\mathcal{Q} \in \mathcal{N}^j_{n-k}(B_{i,j,n})$, then we can extend $\mathcal{Q}$ to a placement $\mathcal{P} \in N$ by placing an additional rook in the last column. In this case, the $n-k$ rooks in $\mathcal{Q}$ will j-attack exactly $(n-k)j$ cells in the last column. Since the height of the last column of $B_{i,j,n+1}$ is $i+nj$, there will be $i+nj-(n-k)j = i+kj$ cells in the last column of $B_{i,j,n+1}$ which are not j-attacked and hence there will be $i+kj$ such placements. Moreover, it is easy to see that if we place the rook in the s-th row of the non-j-attacked cells of the last column, where we label such rows with $1,\ldots,i+kj$ reading from bottom to top, then the weight of the corresponding placement $\mathcal{P}^s$ is $q^{i+kj-s}p^{s-1}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{Q})$. Finally there is an extra factor of $p^{-(n+1)}$ in $\tilde{W}_{p,q,B_{i,j,n+1}}(\mathcal{P})$ that does not occur in $\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{Q})$ due to the rook in column $n+1$ in $\mathcal{P}$. It follows that

$$\begin{aligned}
&\sum_{\mathcal{P}\in Last} \tilde{W}_{p,q,B_{i,j,n+1}}(\mathcal{P}) \\
&= p^{-(n+1)} \sum_{\mathcal{Q}\in\mathcal{N}^j_{n-k}(B_{i,j,n})} (q^{i+kj-1}+q^{i+kj-2}p+\cdots+qp^{i+kj-2}+p^{i+kj-1})\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{Q}) \\
&= p^{-(n+1)}[i+kj]_{p,q}\tilde{r}^j_{n-k}(p,q,B_{i,j,n}).
\end{aligned}$$

■

We note that when $B=B_{i,j,n}$, then Theorem 7 becomes

$$[x+i]^n_{p,q} = \sum_{k=0}^{n} \tilde{S}^{i,j}_{n,k} p^{(n-k)x+\binom{n-k+1}{2}j}[x]_{p,q}\downarrow_k$$

which is just (59). Thus (59) has a combinatorial proof.

We can also give a combinatorial proof that the matrices $||p^{\binom{n-k+1}{2}j}\tilde{S}^{i,j}_{n,k}(p,q)||$ and $||(pq)^{-\binom{n}{2}j}p^{-ik}q^{-in}\tilde{s}^{i,j}_{n,k}(p,q)||$ are inverses of each other. In fact, we can use the same proof that we used to give a combinatorial proof that the matrices $||S^{i,j}_{n,k}(p,q)||$ and $||s^{i,j}_{n,k}(p,q)||$ are inverses of each other. That is, we must show that for all n and $0\leq r\leq n$,

$$\sum_{k=r}^{n} p^{\binom{n-k+1}{2}j}\tilde{S}^{i,j}_{n,k}(p,q)(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{s}^{i,j}_{k,r}(p,q) = \chi(n=r). \tag{94}$$

Note that if $r=n$, then (94) reduces down to

$$1 = \tilde{S}^{i,j}_{n,n}(p,q)(pq)^{-\binom{n}{2}j}p^{-in}q^{-in}\tilde{s}^{i,j}_{n,n}(p,q). \tag{95}$$

Now both $\mathcal{N}^j_{n-n}(B_{i,j,n})$ and $\mathcal{F}_{n-n}(B_{i,j,n})$ consist solely of the empty placement $\mathcal{E}$. Then it is easy to see that our definitions ensure that

$$\begin{aligned}
\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{E}) &= q^{\sum_{s=1}^n i+(s-1)j} = q^{\binom{n}{2}j+in} \\
\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{E}) &= p^{\sum_{s=1}^n i+(s-1)j} = p^{\binom{n}{2}j+in}.
\end{aligned}$$

It thus follows that $\tilde{S}^{i,j}_{n,n}(p,q)=q^{\binom{n}{2}j+in}$ and $\tilde{s}^{i,j}_{n,n}(p,q)=p^{\binom{n}{2}j+in}$ and hence (95) holds.

Now suppose that $n>r$. Then

$$\sum_{k=r}^{n} p^{\binom{n-k+1}{2}j}\tilde{S}^{i,j}_{n,k}(p,q)(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{s}^{i,j}_{k,r}(p,q) = \tag{96}$$

$$\sum_{k=r}^{n} \sum_{(\mathcal{P},\mathcal{Q})\in\mathcal{N}^j_{n-k}(B_{i,j,n})\times\mathcal{F}_{k-r}(B_{i,j,k})} p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})$$

where $sgn(\mathcal{Q})=(-1)^{k-r}=(-1)^{\text{no. of rooks in }\mathcal{Q}}$. We partition the elements

$$(\mathcal{P},\mathcal{Q})\in\bigcup_{k=r}^{n}\mathcal{N}^j_{n-k}(B_{i,j,n})\times\mathcal{F}_{k-r}(B_{i,j,k})$$

into three classes just as we did in section 2.

Class I. There is a rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$.

Class II. There is no rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$, but there is a rook of $\mathcal{Q}$ in the last column of $B_{i,j,k}$.

Class III. There is no rook of $\mathcal{P}$ in the last column of $B_{i,j,n}$ and there is no rook of $\mathcal{Q}$ in the last column of $B_{i,j,k}$.

Let f be the bijection from Class I to Class II defined at the end of section 1. Thus f is a sign reversing bijection which for each $r \leq k \leq n$ maps the elements of $\mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})$ in Class I to the elements $\mathcal{N}^j_{n-k-1}(B_{i,j,n}) \times \mathcal{F}_{k+1-r}(B_{i,j,k+1})$ in Class II. Now suppose that $(\mathcal{P},\mathcal{Q}) \in \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,k})$ and $f((\mathcal{P},\mathcal{Q})) = (\mathcal{P}',\mathcal{Q}')$. Thus there $i+kj$ cells of $\mathcal{P}$ in the last column which are not j-attacked by any rook to the right of the last column. Thus the effect of removing the last rook r_n in $\mathcal{P}$ and placing it in the corresponding position in a new column of height $i+kj$ to the left of $\mathcal{Q}$ means that we lose a factor of p^{-nj} and we gain a factor of q^{i+kj} since the last column of $\mathcal{P}'$ is now empty. Thus

$$q^{i+kj}p^{nj}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q}) = \tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P}')\tilde{w}_{p,q,B_{i,j,k+11}}(\mathcal{Q}'). \tag{97}$$

But then

$$\begin{aligned}
&p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})\\
&= p^{\binom{n-k}{2}j}p^{(n-k)j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k+1}{2}j}(pq)^{kj}p^{-ir}q^{-i(k+1)}q^{i}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})\\
&= p^{nj}q^{i+kj}(p^{\binom{n-k}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k+1}{2}j}p^{-ir}q^{-i(k+1)}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q}))\\
&= -(p^{\binom{n-k}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P}')(pq)^{-\binom{k+1}{2}j}p^{-ir}q^{-i(k+1)}\tilde{w}_{p,q,B_{i,j,k+1}}(\mathcal{Q}')sgn(\mathcal{Q}'))
\end{aligned}$$

which is precisely the sign-reversing weight preserving property required to show that f cancels all the elements in Classes I and II in the sum (96). If we let $\mathcal{T}_{n,k,r,i,j} = \mathcal{N}^j_{n-k}(B_{i,j,n}) \times \mathcal{F}_{k-r}(B_{i,j,n})$, then f shows that

$$\begin{aligned}
&\sum_{k=r}^{n}\sum_{(\mathcal{P},\mathcal{Q})\in\mathcal{T}_{n,k,r,i,j}} p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{Q})sgn(\mathcal{Q})\\
&= \sum_{k=r}^{n}\sum_{\substack{(\mathcal{P},\mathcal{Q})\in\mathcal{T}_{n,k,r,i,j}\\ (\mathcal{P},\mathcal{Q})\in Class\ III}} p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,n}}(\mathcal{Q})sgn(\mathcal{Q}).
\end{aligned}$$

Again if $r=0$, then there are no elements in Class III. Thus if $r=0$, then f shows that $\sum_{k=0}^{n} p^{\binom{n-k+1}{2}j}\tilde{S}^{i,j}_{n,k}(p,q)(pq)^{-\binom{k}{2}j}q^{-ik}s^{i,j}_{k,0}(p,q) = 0$. Finally if $r \geq 1$, we again use the bijection g defined in section 1 which maps Class III onto $\bigcup_{k=r-1}^{n-1}\mathcal{N}^j_{n-1-k}(B_{i,j,n-1}) \times \mathcal{F}_{k-(r-1)}(B_{i,j,k})$. That is, if $(\mathcal{P},\mathcal{Q})$ is in Class III, then $g((\mathcal{P},\mathcal{Q})) = (\mathcal{P}'',\mathcal{Q}'')$ where $\mathcal{P}''$ is obtained from $\mathcal{P}$ by removing its last column and $\mathcal{Q}''$ is obtained from $\mathcal{Q}$ by removing its last column. In this case, it is easy to see that since there $i+(k-1)j$ cells in the last column of $\mathcal{P}$ which are not j-attacked by any rook to the left of last column we lose a factor of $q^{i+(k-1)j}$ from the $\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})$ by removing the last column. Similarly we lose a factor of $p^{i+(k-1)j}$ from the $\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{Q})$ by removing the last column. Thus if $g((\mathcal{P},\mathcal{Q}))=(\mathcal{P}',\mathcal{Q}')$, then

$$(pq)^{-i-(k-1)j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})=\tilde{W}_{p,q,B_{i,j,n-1}}(\mathcal{P}')\tilde{w}_{p,q,B_{i,j,k-1}}(\mathcal{Q}'). \tag{98}$$

But then

$$\begin{aligned}
&p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})\\
&=p^{\binom{(n-k+1)}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k-1}{2}j}(pq)^{-(k-1)j}p^{-i(r-1)}p^{-i}q^{-i(k-1)}q^{-i}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})\\
&=(pq)^{-i-(k-1)j}(p^{\binom{(n-k+1)}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k-1}{2}j}p^{-i(r-1)}q^{-i(k-1)}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q}))\\
&=(p^{\binom{(n-1)-(k-1)+1}{2}j}\tilde{W}_{p,q,B_{i,j,n-1}}(\mathcal{P}')(pq)^{-\binom{k-1}{2}j}p^{-i(r-1)}q^{-i(k-1)}\tilde{w}_{p,q,B_{i,j,k-1}}(\mathcal{Q}')sgn(\mathcal{Q}')
\end{aligned}$$

Thus g shows that

$$\begin{aligned}
&\textstyle\sum_{k=r}^{n}\sum_{\substack{(\mathcal{P},\mathcal{Q})\in\mathcal{T}_{n,k,r,i,j}\\(\mathcal{P},\mathcal{Q})\in Class\ III}} p^{\binom{n-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n}}(\mathcal{P})(pq)^{-\binom{k}{2}j}p^{-ir}q^{-ik}\tilde{w}_{p,q,B_{i,j,k}}(\mathcal{Q})sgn(\mathcal{Q})\\
&=\textstyle\sum_{k=(r-1)}^{n-1}\sum_{(\mathcal{P}',\mathcal{Q}')\in\mathcal{T}_{n-1,k,r-1,i,j}} p^{\binom{(n-1)-k+1}{2}j}\tilde{W}_{p,q,B_{i,j,n-1}}(\mathcal{P}')(pq)^{-\binom{k}{2}j}p^{-i(r-1)}q^{-i(k-1)}\tilde{w}_{p,q,B_{i,j,k-1}}(\mathcal{Q}')sgn(\mathcal{Q}')\\
&=\chi(n-1=r-1)
\end{aligned}$$

where the last equality follows from our induction hypothesis. Thus (94) holds as claimed.

4 Generating Functions

In this section, we shall present some basic generating functions for sequences involving $c_{n,k}^{i,j}(p,q)$, $S_{n,k}^{i,j}(p,q)$, $\tilde{c}_{n,k}^{i,j}(p,q)$ and $\tilde{S}_{n,k}^{i,j}(p,q)$. In the case that $(p,q,i)=(1,1,1)$, our results reduce to results on Whitney numbers of Dowling lattices appearing in work of Benoumhani [2], [3]. The connection with Dowling lattices is discussed in Section 5.

First we consider exponential generating functions when $p=q=1$.

Theorem 9. *Let $F_k^{i,j}(x)=\sum_{n\geq k}c_{n,k}^{i,j}(1,1)\frac{x^n}{n!}$ and $R_k^{i,j}(x)=\sum_{n\geq k}S_{n,k}^{i,j}(1,1)\frac{x^n}{n!}$. Then for all $j>0$,*

$$F_k^{0,j}(x)=\frac{1}{k!}(\ln((1-jx)^{-1/j}))^k \tag{99}$$

$$F_k^{i,j}(x)=\frac{1}{i^k k!}(1-jx)^{-i/j}(\ln((1-jx)^{-i/j}))^k \text{ if } i>0 \text{ and } k\geq 0, \tag{100}$$

and

$$R_k^{i,j}(x)=\frac{1}{j^k k!}e^{ix}(e^{jx}-1)^k \text{ if } i\geq 0 \text{ and } k\geq 0, \tag{101}$$

Proof. All of these results can be obtained by taking appropriate limits in (13) which is Hsu and Shiue's exponential generating function for the sequence $\{S_{n,k}^1(\alpha,\beta,r)\}_{n\geq k}$ when $\alpha\beta\neq 0$. It is also easy to give more direct proofs.

For example, if $k=0$ and $i>0$, then it is easy to see that

$$\begin{aligned}
c_{n,0}^{i,j}(1,1) &= i(i+j)\cdots(i+(n-1)j) \text{ and}\\
S_{n,0}^{i,j}(1,1) &= i^n.
\end{aligned}$$

Thus in that case,

$$
\begin{aligned}
F_0^{i,j}(x) &= \sum_{n\geq 0} i(i+j)\cdots(i+(n-1)j)\frac{x^n}{n!} \\
&= \sum_{n\geq 0} \frac{-i}{j}(\frac{-i}{j}-1)\cdots(\frac{-i}{j}-(n-1))\frac{(-jx)^n}{n!} \\
&= (1-jx)^{-i/j}
\end{aligned} \tag{102}
$$

by Newton's binomial theorem. Similarly

$$R_0^{i,j}(x) = \sum_{n\geq 0} i^n \frac{x^n}{n!} = e^{ix}. \tag{103}$$

When $k = i = 0$, $c_{n,0}^{0,j}(1,1) = S_{n,0}^{0,j}(1,1) = 0$ when $n > 0$ since one cannot place n rooks on $B_{0,j,n} = B(0,j,2j,\ldots,(n-1)j)$ with out placing at least two rooks in the same column. Since $c_{0,0}^{i,j} = S_{0,0}^{i,j} = 1$, it follows that

$$F_0^{0,j}(x) = R_0^{0,j}(x) = 1. \tag{104}$$

Now if $k > 0$ and $i \geq 0$, then

$$
\begin{aligned}
\frac{d}{dx}F_k^{i,j}(x) &= \sum_{n\geq k} c_{n,k}^{i,j}\frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq k}(c_{n-1,k-1}^{i,j} + (i+(n-1)j)c_{n-1,k}^{i,j})\frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq k} c_{n-1,k-1}^{i,j}\frac{x^{n-1}}{(n-1)!} + i\sum_{n\geq k} c_{n-1,k}^{i,j}\frac{x^{n-1}}{(n-1)!} + j\sum_{n\geq k} c_{n-1,k}^{i,j}\frac{x^{n-1}}{(n-2)!} \\
&= F_{k-1}^{i,j}(x) + iF_k^{i,j}(x) + jx\frac{d}{dx}(F_k^{i,j}(x))
\end{aligned}
$$

so that

$$\frac{d}{dx}(F_k^{i,j}(x)) = \frac{1}{1-jx}F_{k-1}^{i,j}(x) + \frac{i}{1-jx}F_k^{i,j}(x) \tag{105}$$

Similarly if $k > 0$ and $i \geq 0$, then

$$
\begin{aligned}
\frac{d}{dx}R_k^{i,j}(x) &= \sum_{n\geq k} S_{n,k}^{i,j}\frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq k}(S_{n-1,k-1}^{i,j} + (i+kj)S_{n-1,k}^{i,j})\frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq k} S_{n-1,k-1}^{i,j}\frac{x^{n-1}}{(n-1)!} + (i+kj)\sum_{n\geq k} S_{n-1,k}^{i,j}\frac{x^{n-1}}{(n-1)!} \\
&= R_{k-1}^{i,j}(x) + (i+kj)R_k^{i,j}(x).
\end{aligned} \tag{106}
$$

If $k > 0$, then for all $i \geq 0$ and $j > 0$,

$$F_k^{i,j}(x)|_{x^k} = R_k^{i,j}(x)|_{x^k} = \frac{1}{k!}. \tag{107}$$

Now it is easy to see that when $i > 0$, (105), (107) and (102) completely determine the family $\{F_k^{i,j}(x)\}_{k\geq 0}$. Since the family $\{\frac{1}{i^k k!}(1-jx)^{-i/j}(\ln((1-jx)^{-i/j}))^k\}_{k\geq 0}$ satisfies (105), (107) and (102), it follows that for all $k \geq 0$,

$$F_k^{i,j}(x) = \frac{1}{i^k k!}(1-jx)^{-i/j}(\ln((1-jx)^{-i/j}))^k \text{ if } i > 0.$$

If $i = 0$, then (105), (107) and (104) completely determine the family $\{F_k^{0,j}(x)\}_{k\geq 0}$. Since the family $\{\frac{1}{k!}(\ln((1-jx)^{-1/j}))^k\}_{k\geq 0}$ satisfies (105), (107) and (104), it follows that for all $k \geq 0$,

$$F_k^{0,j}(x) = \frac{1}{k!}(\ln((1-jx)^{-1/j}))^k.$$

Similarly, it is easy to check that when $i > 0$, then (106), (107) and (103) completely determine the family $\{R_k^{i,j}(x)\}_{k>0}$. Since the $\{\frac{1}{j^k k!}e^{ix}(e^{jx}-1)^k\}_{k>0}$ satisfies (106), (107) and (103), it immediately follows that for all $k \geq 0$

$$R_k^{i,j}(x) = \frac{1}{j^k k!}e^{ix}(e^{jx}-1)^k \text{ if } i > 0.$$

If $i = 0$, then (106), (107) and (104) completely determine the family $\{F_k^{0,j}(x)\}_{k\geq 0}$. Since the family $\{\frac{1}{j^k k!}(e^{jx}-1)^k\}_{k\geq 0}$ satisfies (106), (107) and (104), it follows that for all $k \geq 0$,

$$R_k^{0,j}(x) = \frac{1}{j^k k!}(e^{jx}-1)^k.$$

■

Note that it follows from Theorem 9 that for $i > 0$,

$$\begin{aligned}
\sum_{n\geq 0}\frac{x^n}{n!}\sum_{k=0}^{n} c_{n,k}^{i,j}(1,1)u^k &= \sum_{k\geq 0}u^k \sum_{n\geq k} c_{n,k}^{i,j}(1,1)\frac{x^n}{n!} \qquad (108)\\
&= (1-jx)^{-i/j}\sum_{k\geq 0}\frac{(u/i)^k}{k!}(\ln((1-jx)^{-i/j})^k\\
&= (1-jx)^{-i/j}e^{\frac{u}{i}\ln((1-jx)^{-i/j})}\\
&= (1-jx)^{-i/j}(1-jx)^{-u/j} = (1-jx)^{-(i+u)/j}.
\end{aligned}$$

Thus replacing x by x/j in (108), we get

$$\sum_{n\geq 0}\frac{x^n}{j^n n!}\left(\sum_{k=0}^{n} c_{n,k}^{i,j}(1,1)u^k\right) = (1-x)^{-(i+u)/j}. \qquad (109)$$

There is a natural q-analogue of (109). That is, it follows from Theorem 4 that

$$\begin{aligned}
&\sum_{k=0}^{n} c_{n,k}^{i,j}(1,q)x^k = (x+[i]_q)(x+[i+j]_q)\cdots(x+[i+(n-1)j]_q)\\
&= (x+\frac{1-q^i}{1-q})(x+\frac{1-q^{i+j}}{1-q})\cdots(x+\frac{1-q^{i+(n-1)j}}{1-q})\\
&= \frac{1}{(1-q)^n}(x(1-q)+1-q^i)(x(1-q)+1-q^{i+j})\cdots(x(1-q)+1-q^{i+(n-1)j})\\
&= \frac{(x(1-q)+1)^n}{(1-q)^n}(1-\frac{q^i}{x(1-q)+1})(1-\frac{q^i}{x(1-q)+1}q^j)\cdots(1-\frac{q^i}{x(1-q)+1}q^{(n-1)j}).
\end{aligned}$$

Thus

$$1+\sum_{n\geq 1}\frac{u^n}{(1-q^j)\cdots(1-q^{nj})}\left(\sum_{k=0}^{n}c_{n,k}^{i,j}(1,q)x^k\right) \tag{110}$$
$$=1+\sum_{n\geq 1}\left(\frac{u(x(1-q)+1)}{1-q}\right)^n\frac{(1-\frac{q^i}{x(1-q)+1})(1-\frac{q^i}{x(1-q)+1}q^j)\cdots(1-\frac{q^i}{x(1-q)+1}q^{(n-1)j})}{(1-q^j)\cdots(1-q^{nj})}.$$

We can now apply Cauchy's formula, see [1],

$$1+\sum_{n\geq 1}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}t^n=\prod_{n=0}^{\infty}\frac{(1-atq^n)}{(1-tq^n)}. \tag{111}$$

Thus

$$1+\sum_{n\geq 1}\frac{u^n}{(1-q^j)\cdots(1-q^{nj})}\left(\sum_{k=0}^{n}c_{n,k}^{i,j}(1,q)x^k\right)=\prod_{n=0}^{\infty}\frac{(1-\frac{uq^i}{1-q}q^{nj})}{(1-\frac{u(x((1-q)+1)}{1-q}q^{nj})}. \tag{112}$$

Hence if we replace u by $u(1-q)$ in (112), we obtain the following theorem.

Theorem 10. *For $i>0$ and $k\geq 0$,*

$$1+\sum_{n\geq 1}\frac{u^n}{[j]_q[2j]_q\cdots[nj]_q}\left(\sum_{k=0}^{n}c_{n,k}^{i,j}(1,q)x^k\right)=\prod_{n=0}^{\infty}\frac{(1-uq^iq^{nj})}{(1-u(x((1-q)+1)q^{nj})}. \tag{113}$$

Next we consider an ordinary generating function for the $S_{n,k}^{i,j}(p,q)$'s.

Theorem 11. *For all $k\geq 0$ and $i\geq 0$,*

$$H_k^{i,j}(x)=\sum_{n\geq k}S_{n,k}^{i,j}(p,q)x^n=\frac{x^k}{(1-[i]_{p,q}x)(1-[i+j]_{p,q}x)\cdots(1-[i+kj]_{p,q}x)}. \tag{114}$$

Proof. We proceed by induction on k. For $k=0$, we have observed that $S_{n,0}^{i,j}(p,q)=[i]_{p,q}^n$ for all $n\geq 0$. Thus

$$H_0^{i,j}(x)=\sum_{n\geq 0}[i]_{p,q}^n x^n=\frac{1}{(1-[i]_{p,q}x)}. \tag{115}$$

For $k>0$,

$$\begin{aligned}
H_k^{i,j}(x) &= \sum_{n\geq k}S_{n,k}^{i,j}(p,q)x^n\\
&= \sum_{n\geq k}(S_{n-1,k-1}^{i,j}(p,q)+[i+kj]_{p,q}S_{n-1,k}^{i,j}(p,q))x^n\\
&= x\sum_{n\geq k}(S_{n-1,k-1}^{i,j}(p,q)x^{n-1}+[i+kj]_{p,q}x\sum_{n\geq k}S_{n-1,k}^{i,j}(p,q))x^{n-1}\\
&= xH_{k-1}^{i,j}(x)+[i+kj]_{p,q}xH_k^{i,j}(x).
\end{aligned}$$

Thus

$$H_k^{i,k}(x)=\frac{x}{(1-[i+kj]_{p,q}x)}H_{k-1}^{i,j}(x). \tag{116}$$

Thus (114) easily follows by induction. ■

We also obtain the following expression for $S^{i,j}_{n,k}(p,q)$ as a corollary of Theorem 11.

Corollary 12. *For all* $0 \le k \le n$,

$$S^{i,j}_{n,k}(p,q) = \sum_{\substack{a_0+a_1+\cdots a_k=n \\ a_0\ge 0, a_i\ge 1 \ \text{if}\ i>0}} [i]^{a_0}_{p,q}[i+j]^{a_1-1}_{p,q}\cdots[i+kj]^{a_k-1}_{p,q}. \tag{117}$$

Note that from (61) and (117), one can also get a closed expression for $\tilde{S}^{i,j}_{n,k}(p,q)$.

We can get another closed expression for $\tilde{S}^{i,j}_{n,k}(p,q)$ (and $S^{i,j}_{n,k}(1,q)$). To motivate our result, observe that $S^{i,j}_{n,k}(1,1) = \tilde{S}^{i,j}_{n,k}(1,1)$ so that it follows from (101) that

$$\begin{aligned}
\tilde{S}^{i,j}_{n,k}(1,1) &= \frac{1}{j^k k!} e^{ix}(e^{jx}-1)^k\Big|_{\frac{x^n}{n!}} \qquad (118)\\
&= \frac{1}{j^k k!} e^{ix}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}e^{jsx}\Bigg|_{\frac{x^n}{n!}}\\
&= \frac{1}{j^k k!}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}e^{(i+js)x}\Bigg|_{\frac{x^n}{n!}}\\
&= \frac{1}{j^k k!}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(i+js)^n.
\end{aligned}$$

We then have the following (p,q)-analogue of (118).

Theorem 13.

$$\tilde{S}^{i,j}_{n,k}(p,q) = \frac{p^{(2k-n)(n+1)j/2}}{[j]^k_{p,q}[k]_{p^j,q^j}!}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{p^j,q^j}(-1)^{k-s}p^{j(\binom{s}{2}-sn)}q^{\binom{k-s}{2}j}[i+sj]^n_{p,q} \tag{119}$$

Proof. We first prove the $p=1$ case,

$$\tilde{S}^{i,j}_{n,k}(1,q) = \frac{1}{[j]^k_q[k]_{q^j}!}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{q^j}(-1)^{k-s}q^{\binom{k-s}{2}j}[i+sj]^n_q. \tag{120}$$

We proceed by induction on n. Clearly the formula holds for $n=0$ since $\tilde{S}^{i,j}_{0,0}(1,q)=1$. Next assume that formula holds for n. Then

$$\begin{aligned}
\tilde{S}^{i,j}_{n+1,k}(1,q) &= q^{i+(k-1)j}\tilde{S}^{i,j}_{n,k-1}(1,q) + [i+kj]_q\tilde{S}^{i,j}_{n,k}(1,q)\\
&= \frac{q^{i+(k-1)j}}{[j]^{k-1}_q[k-1]_{q^j}!}\sum_{s=0}^{k-1}\begin{bmatrix} k-1 \\ s \end{bmatrix}_{q^j}(-1)^{k-1-s}q^{\binom{k-1-s}{2}j}[i+sj]^n_q\\
&+ [i+kj]_q\frac{1}{[j]^k_q[k]_{q^j}!}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_{q^j}(-1)^{k-s}q^{\binom{k-s}{2}j}[i+sj]^n_q\\
&= \frac{1}{[j]^k_q[k]_{q^j}!}\left([i+kj]^{n+1}_q + \sum_{s=0}^{k-1}\begin{bmatrix} k \\ s \end{bmatrix}_{q^j}(-1)^{k-s}q^{\binom{k-s}{2}j}[i+sj]^n_q Z_s\right)
\end{aligned}$$

where

$$\begin{aligned}
Z_s &= [i+kj]_q - q^{i+(k-1)j}[k]_{q^j}[j]_q q^{-j(k-1-s)} \frac{[k-s]_{q^j}}{[k]_{q^j}} \\
&= [i+kj]_q - q^{i+sj}[j]_q[k-s]_{q^j} \\
&= [i+kj]_q - q^{i+sj}[(k-s)j]_q = [i+sj]_q.
\end{aligned}$$

It thus follows that

$$\tilde{S}^{i,j}_{n+1,k}(1,q) = \frac{1}{[j]_q^k [k]_{q^j}!} \sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix}_{q^j} (-1)^{k-s} q^{\binom{k-s}{2}j} [i+sj]_q^{n+1}$$

as desired.

To prove the general result (119), we use (61) to express $\tilde{S}^{i,j}_{n,k}(p,q)$ as a power of p times $\tilde{S}^{i,j}_{n,k}(1,q/p)$. Then we apply (120) to $\tilde{S}^{i,j}_{n,k}(1,q/p)$. ■

Next we introduce a (p,q)-analogue of the Bell numbers in our setting by defining

$$\tilde{B}^{i,j}_n(p,q) = \sum_{k=0}^{n} \tilde{S}^{i,j}_{n,k}(p,q). \tag{121}$$

Since $S^{i,j}_{n,k}(1,1) = \tilde{S}^{i,j}_{n,k}(1,1)$, our next result immediately follows from (101).

Theorem 14. *For all $i,j \geq 0$,*

$$\sum_{n\geq 0} \tilde{B}^{i,j}_n(1,1)\frac{x^n}{n!} = e^{ix} e^{\frac{e^{jx}-1}{j}}. \tag{122}$$

Next let

$$\epsilon_{j,q}(x) = \sum_{s\geq 0} \frac{x^s}{[s]_{q^j}![j]_q^s}. \tag{123}$$

Then we have the following q-analogue of the Dobinski's equality for our generalized Bell numbers $\tilde{B}^{i,j}_n(1,q)$, which reduces to Milne's q-analogue [20] of Dobinski's equality when $(i,j) = (0,1)$.

Theorem 15. *For all $i \geq 0$, and $j \geq 1$,*

$$\tilde{B}^{i,j}_n(1,q) = \frac{1}{\epsilon_{j,q}(1)} \sum_{s\geq 0} \frac{[i+sj]_q^n}{[s]_{q^j}![j]_q^s}. \tag{124}$$

Proof. Let

$$A_{j,k}(x) = [jx]_q[j(x-1)]_q \cdots [j(x-k+1)]_q \tag{125}$$

and V_j be the vector space generated by the set of all $A_{j,k}(x)$ with $k \geq 0$ with coefficients in $R(q)$, the set of rational functions in q with real coefficients. We claim that the set $\{A_{j,k}(x)\}_{k\geq 0}$ is a basis for V_j. Clearly $\{A_{j,k}(x)\}_{k\geq 0}$ span V_j by our definition of V_j. If $\{A_{j,k}(x)\}_{k\geq 0}$ is not a linearly independent set, then we can find a linear combination

$$\sum_{k\in T} c_k(q) A_{j,k}(x) = 0 \tag{126}$$

where T is some finite set and $c_k(q)$ are non-zero polynomials in q for all $k \in T$ and $q-1$ does not divide $c_l(q)$ for some $l \in T$. But then when we set $q=1$ in (126), we get

$$\sum_{k \in T} c_k(1) j^k (x) \downarrow_k = 0$$

which would violate the linear independence of $\{(x)\downarrow_k\}$ since $c_l(1) \neq 0$. Thus $\{A_{j,k}(x)\}_{k\geq 0}$ are linearly independent and hence they form a basis for V_j.

Next it follows from Theorem 7 with $B = B_{i,j,n}$ and x replaced by jx that

$$\sum_{k=0}^{n} \tilde{S}^{i,j}_{n,k}(1,q)[jx]_q[j(x-1)]_q \cdots [(j(x-k+1)]_q = [i+jx]_q^n. \tag{127}$$

We then define a linear functional $L_j : V_j \to R(q)$ by setting $L_j(A_{j,k}(x)) = 1$ for all k. Note that

$$\begin{aligned}
\epsilon_{j,q}(1) &= \sum_{s\geq 0} \frac{1}{[s]_{q^j}![j]_q^s} \qquad (128)\\
&= \sum_{s\geq n} \frac{1}{[s-n]_{q^j}![j]_q^{s-n}}\\
&= \sum_{s\geq n} \frac{[sj]_q[(s-1)j]_q \cdots [[(s-n+1)j]_q}{[s]_{q^j}[j]_q[s-1]_{q^j}[j]_q \cdots [s-n+1]_{q^j}[j]_q([s-n]_{q^j}![j]_q^{s-n})}\\
&= \sum_{s\geq n} \frac{A_{j,n}(s)}{[s]_{q^j}![j]_q^s}\\
&= \sum_{s\geq 0} \frac{A_{j,n}(s)}{[s]_{q^j}![j]_q^s}.
\end{aligned} \tag{129}$$

Thus

$$L_j(A_{j,n}(x)) = 1 = \frac{1}{\epsilon_{j,q}(1)} \sum_{s\geq 0} \frac{A_{j,n}(s)}{[s]_{q^j}![j]_q^s} \tag{130}$$

for all n. Since L_j is linear and $\{A_{j,n}(x)\}_{m\geq 0}$ is a basis for V_j, it follows that for any $p(x) \in V_j$,

$$L_j(p(x)) = \frac{1}{\epsilon_{j,q}(1)} \sum_{s\geq 0} \frac{p(s)}{[s]_{q^j}![j]_q^s}. \tag{131}$$

Thus if we apply L_j to both sides of (127), we get

$$\begin{aligned}
\frac{1}{\epsilon_{j,q}(1)} \sum_{s\geq 0} \frac{[i+sj]_q^n}{[s]_{q^j}![j]_q^s} &= L_j([i+jx]_q^n)\\
&= L_j\left(\sum_{k=0}^{n} \tilde{S}^{i,j}_{n,k}(1,q) A_{j,k}(x)\right)\\
&= \sum_{k=0}^{n} \tilde{S}^{i,j}_{n,k}(1,q) = \tilde{B}^{i,j}_n(1,q).
\end{aligned}$$

■

Next we derive a formula for $\tilde{s}^{i,j}_{n,k}(p,q)$.

Theorem 16.

$$\tilde{s}^{i,j}_{n,k}(p,q) = \frac{1}{(q-p)^{n-k}} \sum_{s=k}^{n} (-1)^{n-s} \binom{s}{k} q^{\binom{n-s}{2}j+(n-s)i} p^{\binom{s}{2}j+is} \left[\begin{array}{c} n \\ s \end{array} \right]_{q^j,p^j}. \tag{132}$$

Proof. We can rewrite (58) as

$$\prod_{s=0}^{n-1} \frac{(q^{x-sj}-p^{x-sj})}{(q-p)} = \sum_{k=0}^{n} p^{(x+i)(n-k)} (pq)^{-\binom{n}{2}j-ni} \ \tilde{s}^{i,j}_{n,k}(p,q) \frac{(q^{x+i}-p^{x+i})^k}{(q-p)^k} \tag{133}$$

Multiplying both sides of (133) by $(q-p)^n$, we get

$$\begin{aligned} & p^{nx-\binom{n}{2}j} \prod_{s=0}^{n-1} ((\frac{q}{p})^{x-sj}-1) \qquad (134) \\ & = \sum_{k=0}^{n} p^{(x+i)(n-k)} (pq)^{-\binom{n}{2}j-ni} \ \tilde{s}^{i,j}_{n,k}(p,q)(q-p)^{n-k} p^{kx+ki} ((\frac{q}{p})^{x+i}-1)^k. \end{aligned}$$

Thus

$$\prod_{s=0}^{n-1} ((\frac{q}{p})^{(x-(n-1)j+sj)}-1) = \sum_{k=0}^{n} \tilde{s}^{i,j}_{n,k}(p,q) q^{-\binom{n}{2}j-ni} (q-p)^{n-k} ((\frac{q}{p})^{x+i}-1)^k. \tag{135}$$

By the q-binomial Theorem,

$$\begin{aligned} & \prod_{s=0}^{n-1} (\frac{q}{p})^{(x-(n-1)j+sj)}-1) \qquad (136) \\ & = \sum_{s=0}^{n} (-1)^{n-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{(\frac{q}{p})^j} (\frac{q}{p})^{\binom{s}{2}j} (\frac{q}{p})^{(x-(n-1)j)s} \\ & = \sum_{s=0}^{n} (-1)^{n-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{(\frac{q}{p})^j} (\frac{q}{p})^{\binom{s}{2}j} (\frac{q}{p})^{-(n-1)js-is} (\frac{q}{p})^{(x+i)s} \\ & = \sum_{s=0}^{n} (-1)^{n-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{(\frac{q}{p})^j} (\frac{q}{p})^{\binom{s}{2}j-(n-1)js-is} ((\frac{q}{p})^{(x+i)}-1)+1)^s \\ & = \sum_{s=0}^{n} (-1)^{n-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{(\frac{q}{p})^j} (\frac{q}{p})^{\binom{s}{2}j-(n-1)js-is} \sum_{t=0}^{s} \binom{s}{t} ((\frac{q}{p})^{(x+i)}-1)^t. \end{aligned}$$

Using (136), we can see that taking the coefficient of $((\frac{q}{p})^{(x+i)}-1)^k$ on both sides of (135), we get that

$$\tilde{s}^{i,j}_{n,k}(p,q) q^{-\binom{n}{2}j-ni} (q-p)^{n-k} = \sum_{s=k}^{n} (-1)^{n-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{(\frac{q}{p})^j} (\frac{q}{p})^{\binom{s}{2}j-(n-1)js-is} \binom{s}{k}. \tag{137}$$

Using the fact that $(n-1)s - \binom{s}{2} = \binom{n}{2} - \binom{n-s}{2}$ and that $\left[\begin{array}{c} n \\ s \end{array}\right]_{\frac{q}{p}} = p^{\binom{s}{2}+\binom{n-s}{2}-\binom{n}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{q,p}$, we can see that if we multiply the right hand side of (137) by $q^{\binom{n}{2}+ni}$, then the power of q in the k-th term in the sum is

$$\binom{n}{2}j + ni - \left(\binom{n}{2}j - \binom{n-s}{2}j\right) - is = \binom{n-s}{2}j + (n-s)i$$

and the power of p in the k-th term in the sum is

$$\binom{s}{2}j + \binom{n-s}{2}j - \binom{n}{2}j + (\binom{n}{2}j - \binom{n-s}{2}j) + si = \binom{s}{2}j + si.$$

Thus

$$\tilde{s}^{i,j}_{n,k}(p,q) = \frac{1}{(q-p)^{n-k}} \sum_{s=k}^{n} (-1)^{n-s} \binom{s}{k} q^{\binom{n-s}{2}j+(n-s)i} p^{\binom{s}{2}j+is} \left[\begin{array}{c} n \\ s \end{array}\right]_{q^j,p^j}.$$

which is what we wanted to prove. ■

5 Permutation Statistics, Colored Partitions, and Restricted Growth Functions

In this section we shall give alternative interpretations of our generalized (p,q)-Stirling numbers which are connected to a well-known generalization of the partition lattice called the Dowling lattice. We start by giving two closely related interpretations of $S^{i,j}_{n,k}(1,1)$. Through out this section, we shall assume that $0 \leq i \leq j$. Let $\mathcal{CP}$ be the collection of all set partitions of $\{0,1,\ldots,n\}$ whose nonzero elements are colored with colors from the set $\{0,\ldots,j-1\}$. We refer to the block of a colored partition that contains 0 as the *zero-block*. Define $\mathcal{CP}^{i,j}_{n,k}$ to be the subset of $\mathcal{CP}$ consisting of partitions with $k+1$ blocks where the elements are colored so that

(a) the nonzero elements of the zero-block have colors in $\{0,\ldots,i-1\}$

(b) the smallest element of each block other than the zero-block has color 0.

Remark 17. When $i=1$, the set $\mathcal{CP}^{i,j}_{n,k}$ consists of the elements of rank $n-k+1$ in the Dowling lattice $Q_n(\mathbb{Z}_j)$ (see, eg., [13] for the definition of a Dowling lattice). Hence the $|\mathcal{CP}^{1,j}_{n,k}|$ are the Whitney numbers of the second kind for the Dowling lattice $Q_n(\mathbb{Z}_j)$. When $j=1$, $Q_n(\mathbb{Z}_j)$ is the partition lattice Π_{n+1}. So in this case, the Whitney numbers of the second kind become the Stirling numbers of the second kind. Note that when $(i,j)=(0,1)$, the elements of $\mathcal{CP}^{i,j}_{n,k}$ correspond to the elements of rank $n-k+1$ in the partition lattice Π_n. So again we get the Stirling numbers of the second kind.

There is a natural way to encode the partitions of $[n]$ as restricted growth functions. A restricted growth function is a word $w_1\cdots w_n$ over alphabet $[n]$ such that $w_1 = 1$ and for $s = 2,\ldots,n$, we have $w_s \leq 1 + max\{w_1,\ldots,w_{s-1}\}$. To a partition $\pi = \langle \pi_1,\ldots,\pi_k\rangle$, where $min(\pi_1) < \cdots < min(\pi_k)$, we associate the restricted growth function $w_1w_2\ldots w_n$, where $w_s = t$ if $t \in \pi_s$. It is easy to generalize this encoding to colored partitions.

Let $\pi = \langle \pi_0,\ldots,\pi_k\rangle \in \mathcal{CP}^{i,j}_{n,k}$ where $min(\pi_0) < \cdots < min(\pi_k)$ and let $w(\pi) = w_0w_1\cdots w_n$ where for all $0 \leq s \leq n$, $w_s = t$ if $s \in \pi_t$. We then color w_s with same color that s was colored with in π. For example, if $\pi = \langle\{0, 1^1, 4^0\}, \{2^0, 5^1\}, \{3^0, 6^2\}\}\rangle \in \mathcal{CP}^{2,3}_{6,2}$, then $w(\pi) = 00^1 1^0 2^0 0^0 1^1 2^2$. We let $\mathcal{RG}^{i,j}_{n,k} = \{w(\pi) : \pi \in \mathcal{CP}^{i,j}_{n,k}\}$. Then it is easy to see that $\mathcal{RG}^{i,j}_{n,k}$ is the set of all colored words $w = w_0 \cdots w_n$ such that

(a) $w_0 = 0$ and w_0 is uncolored,

(b) for all $1 \le s \le n$, $w_s \le 1 + max\{w_0, \ldots, w_{s-1}\}$,

(c) for all $1 \le s \le n$, if $w_s > max\{w_0, \ldots, w_{s-1}\}$, then w_s is colored with 0,

(d) for all $1 \le s \le n$, if $w_s \le max\{w_0, \ldots, w_{s-1}\}$, then w_s is colored with a color from $\{0, \ldots, j-1\}$,

(e) for all $1 \le s \le n$, if $w_s = 0$ then w_s is colored with a color from $\{0, \ldots, i-1\}$,

(f) $max\{w_0, \ldots, w_n\} = k$.

We can express the colored word $w_0 w_1^{e_1} w_2^{e_2} \cdots w_n^{e_n}$ as a pair of words $(w_0 w_1 \cdots w_n;$ $e_1 \cdots e_n)$. The elements of $\mathcal{RG}^{i,j}_{n,k}$ will be referred to as *colored restricted growth functions.*

We have the following.

Theorem 18. *For all* $0 \le i \le j$, $S^{i,j}_{n,k}(1,1) = |\mathcal{RG}^{i,j}_{n,k}| = |\mathcal{CP}^{i,j}_{n,k}|$.

Proof. The second equation follows from the bijection $w : \mathcal{CP}^{i,j}_{n,k} \to \mathcal{RG}^{i,j}_{n,k}$ described above. To prove the first equation we shall construct a simple bijection $\phi : \mathcal{RG}^{i,j}_{n,k} \to \mathcal{N}^j_{n-k}(B_{i,j,n})$. Let $(w, e) \in \mathcal{RG}^{i,j}_{n,k}$. Starting from column 1 of $B_{i,j,n}$ we place rooks from left to right; so that in column s, we place a rook in the $i + w_s j - e_s$ available (i.e., not j-attacked) cell from the bottom. If no such cell is available then we leave column s empty. It is easy to see that this will happen if and only if $w_s > max\{w_0, \ldots, w_{s-1}\}$. Hence ϕ is well-defined. It is also straight forward to check that ϕ is bijective. ■

We can use the correspondence ϕ to define a weight function $U^{i,j}_{p,q}$ such that for each $\gamma \in \mathcal{RG}^{i,j}_{n,k}$, we have

$$U^{i,j}_{p,q}(\gamma) \quad = \quad W^{i,j}_{p,q,B_{i,j,n}}(\phi(\gamma)).$$

This weight function turns out to have a nice description in terms of natural statistics on colored restricted growth functions. For each $(w, e) \in \mathcal{RG}^{i,j}_{n,k}$ define

$$\begin{aligned}
\mathcal{MAX}(w,e) &= \{s \in [n] : w_s > max\{w_0, \ldots, w_{s-1}\}\} \\
max(w,e) &= |\mathcal{MAX}(w,e)| \\
\Sigma max(w,e) &= \sum_{s=1}^{n} s\chi(s \in \mathcal{MAX}(w,e)) \\
inv(w,e) &= j \sum_{1 \le s < t \le n} \chi(w_s > w_t \ \& \ s \in \mathcal{MAX}(w,e)) + \sum_{s=1}^{n} e_s
\end{aligned}$$

We remark that if we set $(i, j) = (0, 1)$, then inv becomes one of the statistics on restricted growth functions introduced by Milne [21].

Define

$$D^{i,j}_{n,k}(p,q) = \sum_{\gamma\in\mathcal{RG}^{i,j}_{n,k}} p^{\Sigma max(\gamma)} q^{inv(\gamma)}.$$

Theorem 19. *For each $\gamma \in \mathcal{RG}^{i,j}_{n,k}$, we have*

$$W^{i,j}_{p,q,B_{i,j,n}}(\phi(\gamma)) = p^{(n-k)(i-1)+j(nk-\binom{k}{2})}\ p^{-j\Sigma max(\gamma)}\ (\frac{q}{p})^{inv(\gamma)}.$$

Consequently,

$$S^{i,j}_{n,k}(p,q) = p^{(n-k)(i-1)+j(nk-\binom{k}{2})}\ D^{i,j}_{n,k}\left(\frac{1}{p^j}, \frac{q}{p}\right). \tag{138}$$

and

$$D^{i,j}_{n,k}(p,q) = p^{\frac{(n-k)(i-1)}{j}+nk-\binom{k}{2}}\ S^{i,j}_{n,k}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right). \tag{139}$$

Proof. For a rook placement P, recall that $\alpha(P)$ is defined to be the number of cells that lie directly above some rook and are not j attacked by any rook on the left and $\beta(P)$ is defined to be the number of cells that lie directly below some rook and are not j attacked by any rook on the left.

Let $\gamma = (w,e)$. First we compute $\alpha(\phi(\gamma))$. We observe that for each $s = 1,\ldots,n$, column s of $\phi(\gamma)$ has $i + jm_s(\gamma)$ cells that are not j-attacked by any rooks on the left, where $m_s(\gamma) = \max\{w_1,\ldots,w_{s-1}\}$. This implies that the number of cells α_s above a rook in column s that are not j-attacked by a rook to the left is

$$\begin{aligned} i + jm_s(\gamma) - (i + jw_s - e_s) &= j(m_s(\gamma) - w_s) + e_s \\ &= j|\{t < s : w_t > w_s \ \&\ t \in \mathcal{MAX}(\gamma)\}| + e_s. \end{aligned}$$

By summing over all s for which column s has a rook, we get

$$\alpha(\phi(\gamma)) = inv(\gamma).$$

Next we compute $\beta(\phi(\gamma))$. The number of non-j-attacked cells below a rook in column s is $i + jm_s(\gamma) - \alpha_s - 1$. By summing over all s for which column s has a rook, we get

$$\beta(\phi(\gamma)) = (n-k)(i-1) - inv(\gamma) + j\sum_{r=1}^{k}(t_{r+1} - t_r - 1)r, \tag{140}$$

where $\{t_1 < t_2 < \cdots < t_k\} = \mathcal{MAX}(\gamma)$ and $t_{k+1} = n+1$. We have

$$\begin{aligned} \sum_{r=1}^{k}(t_{r+1} - t_r - 1)r &= \sum_{r=2}^{k+1} t_r(r-1) - \sum_{r=1}^{k} t_r r - \binom{k+1}{2} \\ &= (n+1)k - \sum_{r=1}^{k} t_r - \binom{k+1}{2} \\ &= nk - \binom{k}{2} - \Sigma max(\gamma). \end{aligned}$$

By substituting this into (140), we get

$$\beta(\phi(\gamma)) = (n-k)(i-1) - inv(\gamma) + j(nk - \binom{k}{2} - \Sigma max(\gamma)).$$

It follows that

$$p^{\beta(\phi(\gamma))} q^{\alpha(\phi(\gamma))} = p^{(n-k)(i-1)+j(nk-\binom{k}{2}-\Sigma max(\gamma))} (\frac{q}{p})^{inv(\gamma)},$$

from which the result follows. ■

The following consequence of (139) and (32) also follows directly from the combinatorial definiton of $D^{i,j}_{n,k}(p,q)$.

Theorem 20. *For* $0 \leq k \leq n$,

$$D^{i,j}_{n,k}(p,q) = p^n D^{i,j}_{n-1,k-1}(p,q) + [i+jk]_q D^{i,j}_{n-1,k}(p,q). \tag{141}$$

Next we consider two closely related combinatorial interpretations for $c^{i,j}_{n,k}(1,1)$. Recall that the wreath product of the cyclic group $\mathbb{Z}_j$ and the symmetric group $\mathcal{S}_n$, $\mathbb{Z}_j \S \mathcal{S}_n$, consists of colored permutations (σ, e) where $\sigma \in \mathcal{S}_n$ and e is an n-tuple of elements from $\mathbb{Z}_j$. If $e = e_1 e_2 \cdots e_n$, then we say letter s in σ is colored with e_s for all s.

Let $\mathcal{CYC}^{i,j}_{n,k}$ denote the set of colored permutations $(\sigma, e) \in \mathbb{Z}_j \S \mathcal{S}_n$ such that

(I) the largest element in any cycle of σ is colored with a color from $\{0, \ldots, i\}$,

(II) there are exactly k-cycles of σ whose largest element is colored with 0.

There is a classical bijection on permutations called Foata's first fundamental transformation which takes permutations with k cycles to permutations with k left-to-right maxima, where $\sigma(t)$ is called a left-to-right maxima of $\sigma \in S_n$ if $\sigma(t) > \sigma(1), \ldots, \sigma(t-1)$. The permutation σ maps to the permutation $f(\sigma)$ obtained by first listing the cycles of σ in increasing order of largest element, then writing each cycle with largest element first, and then dropping the parenthesis. Foata's first fundamental transformation generalizes to $\mathbb{Z}_j \S \mathcal{S}_n$ in the obvious way by coloring each letter of $f(\sigma)$ with the same color that was used in σ, that is, (σ, e) maps to $(f(\sigma), e)$. Clearly, the image of $\mathcal{CYC}^{i,j}_{n,k}$ under this bijection is the set $\mathcal{LRM}^{i,j}_{n,k}$ of all colored permutations $(\sigma, e) \in \mathbb{Z}_j \S \mathcal{S}_n$ such that

(i) if $\sigma(t) > \sigma(1), \ldots, \sigma(t-1)$, then $\sigma(t)$ is colored with a color from $\{0, \ldots, i\}$,

(ii) $k = |\{t \in [n] : \sigma(t) > \sigma(1), \ldots, \sigma(t-1) \ \& \ \sigma(t) \text{ has color } 0\}|$.

Recall from Remark 17, that the Whitney numbers of the second kind for the Dowling lattice count colored restricted growth funtions. Colored permutations are also related to Dowling lattices. The signless Whitney numbers of the first kind for geometric lattices (or more generally Cohen-Macaulay posets) are the dimensions of Whitney homology of the poset. A basis for the Whitney homology of the Dowling lattice $Q_n(\mathbb{Z}_j)$ consisting of cycles that are naturally indexed by elements of $\mathcal{LRM}^{1,j}_{n,k}$ was constructed by Gottlieb and Wachs [13]. Hence, they gave a combinatorial interpretation of the signless Whitney numbers of the first kind for $Q_n(\mathbb{Z}_j)$ as $|\mathcal{LRM}^{1,j}_{n,k}|$. Below we give a p,q-analogue of this interpretation.

We start with the following.

Theorem 21. *For all* $0 \leq i \leq j$, $c^{i,j}_{n,k}(1,1) = |\mathcal{LRM}^{i,j}_{n,k}| = |\mathcal{CYC}^{i,j}_{n,k}|$.

Proof. The second equation follows from the bijection described above. To prove the first equation we shall construct a bijection $\theta : \mathcal{LRM}^{i,j}_{n-k} \to \mathcal{F}_{n-k}(B_{i,j,n})$. Let $(\sigma, e) \in \mathcal{LRM}^{i,j}_{n-k}$. Define

$$lb_t(\sigma) := |\{s < t : \sigma(s) > \sigma(t)\}|.$$

We construct $\theta(\sigma, e)$ as follows. For each $t = 1, \ldots, n$, place a rook in the $r_t th$ cell of column t reading from top to bottom, where

$$r_t = lb_t(\sigma) + e_t t - (e_t - i)\chi(e_t > i).$$

It is easy to check that θ is a well-defined bijection by listing the values of r_t. First note that lb_t takes on each value between 0 and $t-1$ exactly once for each $e_t \leq i$ and lb_t takes on each value between 1 and $t-1$ exactly once for each $e_t > i$. We list the values of r_t in increasing order first for $e_t = 0$, next for $e_t = 1$, and so on, ending with $e_t = j-1$. This produces the list $0, 1, \ldots, j(t-1)+i$. Note that column t is empty (i.e. $r_t = 0$) if and only if $t \in \mathcal{MAX}(\sigma, e)$ and $e_t = 0$. Hence θ is a well-defined bijection. ∎

Once again we can use the correspondence to define a weight function $u^{i,j}_{p,q}$ such that for each $(\sigma, e) \in \mathcal{LRM}^{i,j}_{n,k}$, we have

$$u^{i,j}_{p,q}(\pi) = w^{i,j}_{p,q,B_{i,j,n}}(\theta^{i,j}_{n,k}(\pi)).$$

As before the weight function has a nice description in terms of natural statistics on elements of the wreath product $\mathbb{Z}_j \S \mathcal{S}_n$.

For $(\sigma, e) \in \mathbb{Z}_j \S \mathcal{S}_n$ and $i \leq j$, define

$$\begin{aligned}
\mathcal{MAX}(\sigma, e) &= \{t \in [n] : \sigma(t) > \sigma(1), \ldots, \sigma(t-1) \ \ \& \ \ e_t = 0\} \\
max(\sigma, e) &= |\mathcal{MAX}(\sigma, e)| \\
\Sigma max(\sigma, e) &= \sum_{t=1}^{n} t \ \chi(t \in \mathcal{MAX}(\sigma, e)) \\
inv(\sigma, e) &= \sum_{1 \leq s < t \leq n} \chi(\sigma(s) > \sigma(t)) \quad + \quad \sum_{t=1}^{n} e_t t - (e_t - i)\chi(e_t > i).
\end{aligned}$$

Define

$$d^{i,j}_{n,k}(p,q) = \sum_{\gamma \in \mathcal{LRM}^{i,j}_{n,k}} p^{\Sigma max(\gamma)} q^{inv(\gamma)}.$$

Theorem 22. *For each* $\gamma \in \mathcal{LRM}^{i,j}_{n,k}$, *we have*

$$w^{i,j}_{p,q,B_{i,j,n}}(\theta(\gamma)) = p^{i(n-k)+j(\binom{n}{2}+k)} q^{k-n} p^{-j\Sigma max(\gamma)} (\frac{q}{p})^{inv(\gamma)}.$$

Consequently,

$$c^{i,j}_{n,k}(p,q) = p^{i(n-k)+j(\binom{n}{2}+k)} \ q^{k-n} \ d_{n,k}\left(\frac{1}{p^j}, \frac{q}{p}\right). \tag{142}$$

and

$$d^{i,j}_{n,k}(p,q) = p^{(i-1)(n-k)/j+\binom{n}{2}+k} \ q^{n-k} \ c_{n,k}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right). \tag{143}$$

Proof. For a rook placement P, recall that $a(P)$ is defined to be the number of cells that lie directly above some rook and $b(P)$ is defined to be the number of cells that lie directly below some rook.

First we compute $a(\theta(\gamma))$. Let $\gamma = (\sigma, e)$. First note that in the rook placement $\theta(\gamma)$, the number of cells above a rook in column t is $r_t - 1$. Note also that

$$\sum_{t=1}^{n} r_t = inv(\gamma).$$

Hence by summing over all columns that contain a rook, we get

$$a(\theta(\gamma)) = inv(\gamma) - (n-k).$$

Next we compute $b(\theta(\gamma))$. The number of cells below a rook in column t is $j(t-1)+i-r_t$. Summing over all columns that contain a rook, we get

$$\begin{aligned} b(\theta(\gamma)) &= \sum_{t \notin \mathcal{MAX}(\gamma)} j(t-1)+i \quad - \quad inv(\gamma) \\ &= i(n-k) + \sum_{t=1}^{n} j(t-1) - \sum_{t \in \mathcal{MAX}(\gamma)} j(t-1) \quad - \quad inv(\gamma) \\ &= i(n-k) + j\binom{n}{2} + jk - j\Sigma max(\gamma) - inv(\gamma). \end{aligned}$$

It follows that

$$p^{b(\theta(\gamma))} q^{a(\theta(\gamma))} = p^{i(n-k)+j\binom{n}{2}+jk-j\Sigma max(\gamma)} q^{k-n} (\frac{q}{p})^{inv(\gamma)},$$

from which the result follows. ∎

When $j = 1, 2$, the wreath product groups $\mathbb{Z}_j \S \mathcal{S}_n$ are the symmetric group and the hyperoctahedral group, respectively. These groups when viewed as Coxeter groups have a natural length function. For the symmetric group, the Coxeter length is just the usual inversion statistic $\sum_{1 \le s < t \le n} \chi(\sigma(s) > \sigma(t))$. This is precisely what inv reduces to when $(i, j) = (0, 1)$. For the hyperoctahedral group the Coxeter length is described as follows (cf. [24]):

$$\begin{aligned} l(\sigma, e) &= \sum_{1 \le s < t \le n} \chi(\sigma(s) > \sigma(t) \quad \& \quad e_t = 0) \\ &+ \sum_{1 \le s < t \le n} \chi(\sigma(s) < \sigma(t) \quad \& \quad e_t = 1) \\ &+ \sum_{t=1}^{n} e_t t \end{aligned}$$

Clearly, our inv statistic does not reduce to length when $(i, j) = (1, 2)$. However, we can modify our definition of inv to obtain a statistic on $\mathbb{Z}_j \S \mathcal{S}_n$ which does generalize the length function of both the hyperoctahedral group and the symmetric group. The important thing is that all the results (and proofs) of this paper pertaining to inv hold for the modified inv.

First define,

$$r'_t = \begin{cases} lb_t(\sigma) & \text{if } e_t = 0 \\ (t-1) - lb_t(\sigma) + e_t t & \text{if } e_t = 1, 2, \ldots, i \\ (t-1) - lb_t(\sigma) + e_t t - e_t + i + 1 & \text{if } e_t = i+1, i+2, \ldots, j-1 \end{cases}.$$

Next define the bijection $\theta' : \mathcal{LRM}^{i,j}_{n-k} \to \mathcal{F}_{n-k}(B_{i,j,n})$ exactly as θ, but with r' instead of r. Now define

$$\begin{aligned} inv'(\sigma, e) &= \sum_{1 \le s < t \le n} \chi(\sigma(s) > \sigma(t) \quad \& \quad e_t = 0) \\ &+ \sum_{1 \le s < t \le n} \chi(\sigma(s) < \sigma(t) \quad \& \quad e_t \neq 0) \\ &+ \sum_{t=1}^{n} e_t t - (e_t - i - 1)\chi(e_t > i) \end{aligned}$$

Clearly when $(i,j) = (0,1)$ and $(1,2)$, inv' reduces to the Coxeter length function for the symmetric group and the hyperoctahedral group. If we replace r, θ and inv with r', θ' and inv', respectively, all results and proofs pertaining to these notions go through unchanged. Consequently,

$$d^{i,j}_{n,k}(p,q) = \sum_{\gamma \in \mathcal{LRM}^{i,j}_{n,k}} p^{\Sigma max(\gamma)} q^{inv'(\gamma)}.$$

We chose our original definitions for the sake of simplicity.

The following consequence of (143) and (30) can also be proved directly from the combinatorial definition of $d^{i,j}_{n,k}(p,q)$.

Theorem 23. *For* $0 \le k \le n$,

$$d^{i,j}_{n,k}(p,q) = p^n d^{i,j}_{n-1,k-1}(p,q) + q[i + j(n-1)]_q d^{i,j}_{n-1,k}(p,q). \tag{144}$$

Theorem 24. *For fixed* $i \le j$, *the matrices* $||(-1)^{n-k} d^{i,j}_{n,k}(p,q)||$ *and* $||q^{n-k} p^{-n(k+1)} D^{i,j}_{n,k}(p,q)||$ *are inverses of each other.*

Proof. Note that since the matrices $||(-1)^{n-k} c^{i,j}_{n,k}(p,q)||$ and $||S^{i,j}_{n,k}(p,q)||$ are inverses of each other, we have for any $0 \le k \le n$,

$$\begin{aligned} &\sum_{l=k}^{n} p^{-n(l+1)} q^{n-l} D^{i,j}_{n,l}(p,q)(-1)^{l-k} d^{i,j}_{l,k}(p,q) \\ &= \sum_{l=k}^{n} p^{-n(l+1)} q^{n-l} p^{(n-l)(i-1)/j + nl - \binom{l}{2}} \ S^{i,j}_{n,l}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right) \times \\ &\qquad (-1)^{l-k} p^{(i-1)(l-k)/j + \binom{l}{2} + k} \ q^{l-k} \ c^{i,j}_{l,k}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right) \\ &= (\frac{q}{p})^{n-k} p^{(n-k)(i-1)/j} \sum_{l=k}^{n} S^{i,j}_{n,l}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right) (-1)^{l-k} c^{i,j}_{l,k}\left(\frac{1}{p^{\frac{1}{j}}}, \frac{q}{p^{\frac{1}{j}}}\right) \\ &= (\frac{q}{p^2})^{n-k} p^{(n-k)(i-1)/j} \chi(n = k) \end{aligned}$$

which verifies that the matrices $||p^{-n(k+1)}q^{n-k}D^{i,j}_{n,k}(p,q)||$ and $||(-1)^{n-k}d^{i,j}_{n,k}(p,q)||$ are inverses of each other. ∎

Remark 25. It is known that the matrix formed by the Whitney numbers of the first kind of the Dowling lattice is the inverse of the matrix formed by Whitney numbers of the second kind (see [25, Exercises 3.50 and 3.51]). Hence Theorem 24 provides a p,q-analogue of this result.

Define $\mathcal{S}^{i,j}_n$ to be the set of all colored permutations $(\sigma,e) \in \mathbb{Z}_j\S\mathcal{S}_n$ such that if $\sigma(t) > \sigma(1),\ldots,\sigma(t-1)$, then $\sigma(t)$ is colored with a color from $\{0,\ldots,i\}$. In other words $\mathcal{S}^{i,j}_n$ is the union of the $\mathcal{LRM}^{i,j}_{n,k}$ over all k.

Theorem 26. *For $i \le j$*

$$\sum_{\gamma\in\mathcal{S}^{i,j}_n} q^{inv(\gamma)} \prod_{t\in\mathcal{MAX}(\sigma)} x_t = (x_1+q[i]_q)(x_2+q[i+j]_q)(x_3+q[i+2j]_q)\cdots(x_n+q[i+(n-1)j]_q). \quad (145)$$

Proof. To prove (145), let $R_n = \{0,\ldots,i\}\times\{0,\ldots,i+j\}\times\cdots\times\{0,\ldots,i+j(n-1)\}$ and define a bijection $r : \mathcal{S}^{i,j}_n \to R_n$ by

$$r(\gamma) = (r_1,\ldots,r_n)$$

where r_t is as in the proof of Theorem 21 . Recall that

$$\begin{aligned} inv(\gamma) &= \sum_{t=1}^{n} r_t \\ \mathcal{MAX}(\gamma) &= \{t : r_t = 0\}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}^{i,j}_n} q^{inv(\gamma)} \prod_{t\in\mathcal{MAX}(\gamma)} x_t &= \sum_{(r_1,\ldots,r_n)\in R_n} q^{r_1+\ldots+r_n} \prod_{t:\ r_t=0} x_t \\ &= \prod_{t=1}^{n}(x_t + \sum_{r_t=1}^{i+j(t-1)} q^{r_t}) \\ &= \prod_{t=1}^{n}(x_t + q[i+j(t-1)]_q). \end{aligned}$$

∎

Setting $x_t = xp^t$ in (145), we obtain the following.

Corollary 27.

$$\sum_{k=0}^{n} d^{i,j}_{n,k}(p,q)x^k = (px+q[i]_q)(p^2x+q[i+j]_q)(p^3x+q[i+2j]_q)\cdots(p^nx+q[i+(n-1)j]_q).$$

By inverting we obtain,

Corollary 28.

$$\sum_{k=0}^{n} q^{n-k}p^{-n(k+1)}D^{i,j}_{n,k}(p,q)\prod_{t=1}^{k}(p^tx - q[i+(t-1)j]_q) = x^n.$$

6 Final Remarks

We note that it is a natural question to ask whether there are combinatorial interpretations of our (p,q)-analogues of the generalized Stirling numbers $s_{n,k}^{i,p,q}(\alpha,\beta,r)$, $S_{n,k}^{i,p,q}(\alpha,\beta,r)$, $\tilde{s}_{n,k}^{i,p,q}(\alpha,\beta,r)$, and $\tilde{S}_{n,k}^{i,p,q}(\alpha,\beta,r)$ for other vaules of α,β and r. In a forthcoming paper with K. Briggs [8], we show that when α,β and r are non-negative integers, then we can give such combinatorial interpretations in terms of pairs of rook placements on two boards B and B'. Our model allows rooks in a given board to cancel cells not only on its own board but also on its companion board.

We should also note that in the special case when $i=0$ and $j=1$, Briggs and Remmel [6] showed that there is a (p,q)-analogue of the hit polynomial corresponding to the rook number $\tilde{r}^1_{n-k,B}(p,q)$. That is, given a board B contained in the $n\times n$ board B_n, we define the *p,q-hit polynomial* of B, denoted $H_B(x,p,q)$, as follows:

$$\begin{aligned} H_B(x,p,q) &= \sum_{k=0}^{n} h_{k,n}(B,p,q)x^k \\ &= \sum_{k=0}^{n} \tilde{r}^1_{k,B}(P,q)[n-k]_{p,q}!p^{\binom{k+1}{2}+k(n-k)} \prod_{l=n-k+1}^{n}(x-q^l p^{n-l}). \end{aligned}$$

Briggs and Remmel [6] showed that when $B=B(a_1,\ldots,a_n)$ is a Ferrers board, i.e. $0\le a_1\le \ldots \le a_n \le n$, then $h_{k,n}(B,p,q)$ is polynomial in p and q with non-negative integer coefficients. Moreover, Briggs [4, 5] has shown that if $H_{k,n}(B)$ is the set of all placements $\mathcal{P}$ in $\mathcal{N}^1_N(B_n)$ such that $\mathcal{P}$ has exactly k rooks in B, then there are statistics $\alpha_B(\mathcal{P})$ and $\beta_B(\mathcal{P})$ such that

$$h_{k,n}(B,p,q) = \sum_{\mathcal{P}\in H_{k,n}(B)} p^{\alpha_B(\mathcal{P})} q^{\beta_B(\mathcal{P})}.$$

We should also note that Briggs and Remmel [6] showed that there is another connection between permutation statistics and our (p,q)-rook placements of type II. That is, they proved the following (p,q)-analogue of a formula of Frobenius:

$$\sum_{k=0}^{n} \frac{\tilde{S}^{0,1}_{n,k}(p,q)[k]_{p,q}!p^{\binom{n-k+1}{2}+k(n-k)}x^k}{\prod_{i=0}^{k}(1-xq^ip^{n-i})} = \frac{\sum_{\sigma\in\mathcal{S}_n} q^{maj(\sigma)}p^{comaj(\sigma)}x^{des(\sigma)+1}}{\prod_{i=0}^{n}(1-xq^ip^{n-i})} \tag{146}$$

where for any permutation $\sigma=\sigma_1\ldots\sigma_n\in\mathcal{S}_n$,

$$\begin{aligned} Des(\sigma) &= \{i:\sigma_i>\sigma_{i+1}\}, \\ Rise(\sigma) &= \{i:\sigma_i<\sigma_{i+1}\}, \\ des(\sigma) &= |Des(\sigma)|, \\ maj(\sigma) &= \sum_{i\in Des(\sigma)} i, \text{ and} \\ comaj(\sigma) &= \sum_{i\in Rise(\sigma)} i. \end{aligned}$$

Certain special cases of the rook numbers $\tilde{r}^2_{k,B}(1,q)$ also have shown up in another rook theory model due to Haglund and Remmel [16] where the rook placements naturally correspond to partial perfect matchings in the complete graph K_{2n}. Haglund and Remmel also develop a combinatorial theory of hit polynomials in that model. Finally, certain special cases of the more general (p,q) rook numbers $\tilde{r}^{j}_{k,B}(1,q)$ show up in yet another rook theory model due to Briggs and Remmel [4, 7] where the rook placements naturally correspond to elements of the wreath product of the cyclic group $\mathbb{Z}_k$ and the symmetric group $\mathcal{S}_n$, $\mathbb{Z}_k \S \mathcal{S}_n$. Again there is a natural combinatorial theory of hit polynomials in their model. We do not know, however, how to develop a natural theory of hit polynomials for either the type I or the type II (p,q)-rook numbers in the rook theory model presented in this paper.

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