Bonferroni-Galambos Inequalities for Partition Lattices

Klaus Dohmen and Peter Tittmann

Department of Mathematics
Mittweida University of Applied Sciences
09648 Mittweida, Germany
e-mail: dohmen@htwm.de, peter@htwm.de

Submitted: Jul 16, 2004; Accepted: Nov 18, 2004; Published: Nov 30, 2004
Mathematics Subject Classification: 05A18, 05A20, 05C65, 60C05, 60E15

Abstract

In this paper, we establish a new analogue of the classical Bonferroni inequalities and their improvements by Galambos for sums of type $\sum_{\pi \in \mathbb{P}(U)} (-1)^{|\pi|-1}(|\pi|-1)! f(\pi)$ where U is a finite set, $\mathbb{P}(U)$ is the partition lattice of U and $f : \mathbb{P}(U) \to \mathbb{R}$ is some suitable non-negative function. Applications of this new analogue are given to counting connected k-uniform hypergraphs, network reliability, and cumulants.

1 Introduction

The classical Bonferroni inequalities of probability theory state that for any probability space $(\Omega, \mathcal{E}, P)$ and any finite family of events $\{E_u\}_{u \in U} \subseteq \mathcal{E}$,

$$(-1)^r P\left(\bigcap_{u \in U} \overline{E_u}\right) \leq (-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} P\left(\bigcap_{i \in I} E_i\right) \quad (r = 0, 1, 2, \ldots). \tag{1}$$

Thus, for even r the sum on the right-hand side of (1) provides an upper bound on the probability $P\left(\bigcap_{u \in U} \overline{E_u}\right)$ that none of the events E_u, $u \in U$, happen, while for odd r it provides a lower bound on this probability. Note that for $r \geq |U|$ the preceding inequality becomes an identity, which is known as the *inclusion-exclusion principle*. This principle and its associated truncation inequalities (1) have many applications in statistics and reliability theory (see [7] for a detailed survey and [4] for some recent developments).

Galambos [6] sharpened the classical Bonferroni bounds by including additional terms based on the $(r+1)$-subsets of U in case that $U \neq \emptyset$:

$$(-1)^r P\left(\bigcap_{u \in U} \overline{E_u}\right) \leq (-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} P\left(\bigcap_{i \in I} E_i\right) - \frac{r+1}{|U|} \sum_{\substack{I \subseteq U \\ |I| = r+1}} P\left(\bigcap_{i \in I} E_i\right).$$

Evidently, the preceding inequality can equivalently be stated in the form

$$(-1)^r \sum_{I\subseteq U} (-1)^{|I|} f(I) \le (-1)^r \sum_{\substack{I\subseteq U \\ |I|\le r}} (-1)^{|I|} f(I) - \frac{r+1}{|U|} \sum_{\substack{I\subseteq U \\ |I|=r+1}} f(I) \tag{2}$$

where $f(I) = P\left(\bigcap_{i\in I} E_i\right)$ for any $I \subseteq U$. Recently, in [5], this latter inequality has been generalized to a broader class of functions $f : 2^U \to \mathbb{R}$.

In this paper, we establish an analogue of (2) for non-negative real-valued functions f defined on the partition lattice $\mathbb{P}(U)$ of some finite set U and thus obtain approximate estimations for Möbius inversions on the lattice of partitions of a set. The need for such estimations has been pointed out by Gian-Carlo Rota in his famous Fubini Lectures [8, Problem 11].

2 Main result

Let U be a finite set. A *partition* of U is a set of pairwise disjoint non-empty subsets of U whose union is U. We use $\mathbb{P}(U)$ to denote the set of partitions of U. The elements of a partition are called *blocks*. The number of blocks in a partition π is denoted by $|\pi|$. The set of partitions $\mathbb{P}(U)$ is given the structure of a lattice by imposing $\sigma \le \pi$ if and only if σ is a *refinement* of π, which means that each block of σ is a subset of a block of π.

By $\mathbb{R}^+$ we denote the set of non-negative reals, and by $\mathbb{Z}^+$ the set of non-negative integers. For $n, k \in \mathbb{Z}^+$ we use $\left\{ n \atop k \right\}$ to denote the number of k-block partitions of an n-set, the so-called *Stirling number of the second kind*.

Theorem 2.1 *Let U be a non-empty finite set, and let $f, g : \mathbb{P}(U) \to \mathbb{R}^+$ such that $f(\pi) = \sum_{\sigma\le\pi} g(\sigma)$ for any $\pi \in \mathbb{P}(U)$. Then, for any $r \in \mathbb{Z}^+$,*

$$\begin{aligned}(-1)^r &\sum_{\pi\in\mathbb{P}(U)} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi) \\ &\ge (-1)^r \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|\le r}} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi) + \frac{r!}{\left\{ |U| \atop r+1 \right\}} \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|=r+1}} f(\pi)\end{aligned}$$

or equivalently, by means of Möbius inversion,

$$(-1)^r g(\hat{1}) \ge (-1)^r \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|\le r}} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi) + \frac{r!}{\left\{ |U| \atop r+1 \right\}} \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|=r+1}} f(\pi)$$

where $\hat{1}$ denotes the largest element $\{U\}$ of $\mathbb{P}(U)$.

Proof. It suffices to prove that

$$(-1)^r \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|>r}} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi) \ge \frac{r!}{\left\{ |U| \atop r+1 \right\}} \sum_{\substack{\pi\in\mathbb{P}(U) \\ |\pi|=r+1}} f(\pi)\,. \tag{3}$$

Let S denote the left-hand side of (3). We obtain

$$
\begin{aligned}
S &= (-1)^r \sum_{\substack{\pi\in\mathbb{P}(U)\\ |\pi|>r}} (-1)^{|\pi|-1}(|\pi|-1)! \sum_{\sigma\le\pi} g(\sigma) \\
&= (-1)^r \sum_{\sigma\in\mathbb{P}(U)} g(\sigma) \sum_{\substack{\pi\ge\sigma\\ |\pi|>r}} (-1)^{|\pi|-1}(|\pi|-1)! \\
&= (-1)^r \sum_{\sigma\in\mathbb{P}(U)} g(\sigma) \sum_{k=r+1}^{|\sigma|} \left\{ {|\sigma| \atop k} \right\} (-1)^{k-1}(k-1)!\,. \qquad (4)
\end{aligned}
$$

It is well-known that (see e.g., [10])

$$
\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \quad (n,k=1,2,3,\dots).
$$

Thus, for the inner sum in (4) we obtain

$$
\begin{aligned}
\sum_{k=r+1}^{|\sigma|} \left\{ {|\sigma|-1 \atop k-1} \right\} (-1)^{k-1}(k-1)! &+ \sum_{k=r+1}^{|\sigma|} \left\{ {|\sigma|-1 \atop k} \right\} (-1)^{k-1}k! \\
&= \sum_{k=r}^{|\sigma|-1} \left\{ {|\sigma|-1 \atop k} \right\} (-1)^{k}k! - \sum_{k=r+1}^{|\sigma|-1} \left\{ {|\sigma|-1 \atop k} \right\} (-1)^{k}k!\,.
\end{aligned}
$$

After cancelling out, we are left with $\left\{ {|\sigma|-1 \atop r} \right\}(-1)^r r!$. Thus, we find that

$$
S = (-1)^r \sum_{\sigma\in\mathbb{P}(U)} g(\sigma) \left\{ {|\sigma|-1 \atop r} \right\} (-1)^r r! = \sum_{\sigma\in\mathbb{P}(U)} g(\sigma) \left\{ {|\sigma|-1 \atop r} \right\} r!\,.
$$

Therefore, for any $\omega\in\mathbb{P}(U)$,

$$
S \ge \sum_{\substack{\sigma\in\mathbb{P}(U)\\ \sigma\le\omega}} g(\sigma) \left\{ {|\sigma|-1 \atop r} \right\} r! \ge \sum_{\substack{\sigma\in\mathbb{P}(U)\\ \sigma\le\omega}} g(\sigma) \left\{ {|\omega|-1 \atop r} \right\} r! = f(\omega) \left\{ {|\omega|-1 \atop r} \right\} r!\,.
$$

By choosing ω uniformly at random among all $(r+1)$-block partitions of U and taking the expectation we obtain

$$
S \ge \mathrm{E}(f(\omega))r! = \sum_{\substack{\pi\in\mathbb{P}(U)\\ |\pi|=r+1}} \mathrm{Prob}[\omega=\pi] f(\pi) r! = \frac{r!}{\left\{ {|U| \atop r+1} \right\}} \sum_{\substack{\pi\in\mathbb{P}(U)\\ |\pi|=r+1}} f(\pi)\,,
$$

which finally proves (3). Thus, the proof of the theorem is complete. □

The following weaker bounds obtained from Theorem 2.1 may be considered as a partition lattice analogue of the classical Bonferroni inequalities.

Corollary 2.2 *Under the requirements of Theorem 2.1,*

$$(-1)^r \sum_{\pi\in\mathbb{P}(U)} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi) \;\geq\; (-1)^r \sum_{\substack{\pi\in\mathbb{P}(U)\\ |\pi|\leq r}} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi)$$

respectively,

$$(-1)^r g(\hat{1}) \;\geq\; (-1)^r \sum_{\substack{\pi\in\mathbb{P}(U)\\ |\pi|\leq r}} (-1)^{|\pi|-1}(|\pi|-1)!f(\pi)\,.$$

3 Connected k-uniform hypergraphs

It is well-known (cf. [1, 10]) that for any $n, k \in \mathbb{N}$ the number $c_{n,k}$ of connected k-uniform hypergraphs on vertex-set $\{1,\ldots,n\}$ is given by the formula

$$c_{n,k} \;=\; \sum_{\pi\in\mathbb{P}(V)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{X\in\pi} 2^{\binom{|X|}{k}},$$

which can equivalently be stated as

$$c_{n,k} \;=\; \sum_{\lambda\vdash n} (-1)^{|\lambda|-1}\binom{n}{\lambda}\binom{|\lambda|}{\kappa(\lambda)}\frac{1}{|\lambda|}\prod_{i=1}^{|\lambda|} 2^{\binom{\lambda_i}{k}},$$

where $\lambda \vdash n$ means that λ is a *number partition* of n, that is, $\lambda = (\lambda_1,\ldots,\lambda_m)$ for some $m \in \mathbb{N}$ such that $\lambda_1 + \cdots + \lambda_m = n$, and $\kappa(\lambda)$ is an n-tuple whose i-th component counts the number of occurrences of i in λ for $i = 1,\ldots,n$. We use $|\lambda|$ to denote the number of parts in λ (that is, the length of λ when considered as a tuple), and for any $m, i \in \mathbb{N}$ and any m-tuple $j = (j_1,\ldots,j_m)$ of non-negative integers we use $\binom{i}{j}$ to denote the multinomial coefficient

$$\binom{i}{j_1,\ldots,j_m} \;=\; \frac{i!}{j_1!\cdots j_m!}\,.$$

From Theorem 2.1 we now deduce the following bounds on $c_{n,k}$, some of which are listed in Table 1.

Theorem 3.1 *For any $n, k \in \mathbb{N}$ and $r \in \mathbb{Z}^+$,*

$$(-1)^r c_{n,k} \;\geq\; (-1)^r \sum_{\substack{\pi\in\mathbb{P}(V)\\ |\pi|\leq r}} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{X\in\pi} 2^{\binom{|X|}{k}} \;+\; \frac{r!}{\left\{{n \atop r+1}\right\}} \sum_{\substack{\pi\in\mathbb{P}(V)\\ |\pi|=r+1}} \prod_{X\in\pi} 2^{\binom{|X|}{k}},$$

or equivalently, in terms of number partitions,

$$(-1)^r c_{n,k} \;\geq\; (-1)^r \sum_{\substack{\lambda\vdash n\\ |\lambda|\leq r}} (-1)^{|\lambda|-1}\binom{n}{\lambda}\binom{|\lambda|}{\kappa(\lambda)}\frac{1}{|\lambda|}\prod_{i=1}^{|\lambda|} 2^{\binom{\lambda_i}{k}}$$
$$+\; \frac{1}{(r+1)\left\{{n \atop r+1}\right\}} \sum_{\substack{\lambda\vdash n\\ |\lambda|=r+1}} \binom{n}{\lambda}\binom{r+1}{\kappa(\lambda)}\prod_{i=1}^{r+1} 2^{\binom{\lambda_i}{k}}.$$

n,k	bounds on $c_{n,k}$ for $r=1,\ldots,n-1$ (last bound in each line gives the exact value)
5,2	992, 555, 812, 728
5,3	1017, 927, 988, 958
5,4	31, 14, 56, 26
6,2	32487, 24109, 28113, 26152, 26704
6,3	1048369, 1042160, 1042894, 1042416, 1042632
6,4	32761, 32538, 32740, 32380, 32596
7,2	2092544, 1807132, 1892306, 1853896, 1870336, 1866256
7,3	34359621500, 34352375869, 34352423041, 34352416580, 34352420630, 34352418950
7,4	34359734715, 34359508257, 34359510357, 34359508078, 34359511294, 34359509614
7,5	2097144, 2096629, 2097265, 2095195, 2098411, 2096731

Table 1: Bounds on $c_{n,k}$ for different values of n,k and r.

Proof. Every hypergraph H on vertex-set $V=\{1,\ldots,n\}$ induces a partition of V, where each block in the partition corresponds to a connected component of H. For any partition $\pi\in\mathbb{P}(V)$ let $g(\pi)$ resp. $f(\pi)$ be the number of k-uniform hypergraphs on V whose induced partition is π resp. a refinement of π. Then, $f(\pi)=\sum_{\sigma\leq\pi}g(\sigma)$ for any $\pi\in\mathbb{P}(V)$, and $g(\hat{1})=c_{n,k}$ where $\hat{1}=\{V\}$ denotes the largest element of $\mathbb{P}(V)$. By Theorem 2.1,

$$(-1)^r c_{n,k}\geq(-1)^r\sum_{\substack{\pi\in\mathbb{P}(V)\\|\pi|\leq r}}(-1)^{|\pi|-1}(|\pi|-1)!f(\pi)+\frac{r!}{\left\{{n\atop r+1}\right\}}\sum_{\substack{\pi\in\mathbb{P}(V)\\|\pi|=r+1}}f(\pi).\tag{5}$$

Since for any partition $\pi\in\mathbb{P}(V)$ and any block $X\in\pi$ there are exactly $2^{\binom{|X|}{k}}$ k-uniform hypergraphs having vertex-set X, we find that $f(\pi)=\prod_{X\in\pi}2^{\binom{|X|}{k}}$, which in combination with (5) proves the first inequality of this theorem. Since for any number partition $\lambda=(\lambda_1,\ldots,\lambda_m)$ of n there are exactly $\binom{n}{\lambda}\binom{|\lambda|}{\kappa(\lambda)}/|\lambda|!$ different set partitions in $\mathbb{P}(V)$ whose block sizes agree with $\lambda_1,\ldots,\lambda_m$, we can re-write the right-hand side of the first inequality as a sum over integer partitions. Thus, the second inequality is proved. $\square$

4 Network reliability

Let $G=(V,E)$ be a finite undirected graph having vertex-set V and edge-set E. We assume that the edges of G are subject to random and independent failures, while the nodes are perfectly reliable. The failure probabilities of the edges are assumed to be known and denoted by q_e for each edge $e\in E$. Under this random graph model, the *all-terminal reliability* $R(G)$ is the probability that each pair of vertices of G is joined by a path of operating (that is, non-failing) edges. This reliability measure has been studied extensively, see e.g., Colbourn [3] for a survey. A well-known result due to Buzacott and Chang [2], which is often referred to as the *node partition formula*, states that

$$R(G)=\sum_{\pi\in\mathbb{P}(V)}(-1)^{|\pi|-1}\,(|\pi|-1)!\prod_{e\in E(G,\pi)}q_e\tag{6}$$

where $E(G,\pi)$ denotes the set of edges of G whose endpoints belong to different blocks of π (see also [11]). The following theorem states that by restricting the sum in (6) to partitions having at most r blocks lower bounds and upper bounds to $R(G)$ are obtained depending on whether r is even or odd. As in Theorem 2.1 an additional term is included in these bounds which sharpens the estimates and which can be omitted if convenient.

Theorem 4.1 *Let $G = (V, E)$ be a finite undirected graph whose nodes are perfectly reliable and whose edges fail randomly and independently with probability q_e for each edge $e \in E$. Then, for any $r \in \mathbb{Z}^+$,*

$$(-1)^r R(G) \ \geq\ (-1)^r \sum_{\substack{\pi\in\mathbb{P}(V)\\ |\pi|\leq r}} (-1)^{|\pi|-1}\,(|\pi|-1)! \prod_{e\in E(G,\pi)} q_e \ +\ \frac{r!}{\left\{ {|V| \atop r+1} \right\}} \sum_{\substack{\pi\in\mathbb{P}(V)\\ |\pi|=r+1}} \prod_{e\in E(G,\pi)} q_e\,.$$

Proof. The state of the network induces a partition $\pi \in \mathbb{P}(V)$, where two nodes are in the same block of π if and only if they are joined by a path of operating edges in G. Let $g(\pi)$ denote the probability that π is the partition induced by the state of the network and $f(\pi)$ denote the probability that the induced partition is a refinement of π. Then, $f(\pi) = \sum_{\sigma\leq\pi} g(\sigma)$ for any $\pi \in \mathbb{P}(V)$, and $g(\hat{1}) = R(G)$ where $\hat{1} = \{V\}$ denotes the largest element of $\mathbb{P}(V)$. It is easily seen that, on the other hand, $f(\pi) = \prod_{e\in E(G,\pi)} q_e$ for any $\pi \in \mathbb{P}(V)$. Thus, the result follows by applying Theorem 2.1. $\square$

Example 4.2 For $G = K_n$ (the complete graph on n nodes), $r = 2$ and $q_e = q$ for each edge $e \in E$ the inequality in Theorem 4.1 specializes to

$$R(K_n) \ \geq\ 1 - \frac{1}{2}\sum_{k=1}^{n-1} \binom{n}{k} q^{k(n-k)} \ =\ 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)}$$

where the last term in the estimate of Theorem 4.1 has been omitted. Thus, for $n = 3, \ldots, 6$ the following lower bounds on $R(K_n)$ are obtained:

$$\begin{aligned} &R(K_3) \geq 1 - 3q^2, && R(K_5) \geq 1 - 5q^4 - 10q^6,\\ &R(K_4) \geq 1 - 4q^3 - 3q^4, && R(K_6) \geq 1 - 6q^5 - 15q^8 - 10q^9. \end{aligned}$$

Figure 1 compares the bound for $R(K_6)$ with the exact reliability given by

$$R(K_6) = 1 - 6q^5 - 15q^8 + 20q^9 + 120q^{11} - 90q^{12} - 270q^{13} + 360q^{14} - 120q^{15}.$$

It turns out that the bound for $R(K_6)$ is close to the exact reliability if the common edge failure probability q is small. Fortunately, this is the typical case in real-world computer and communications networks.

5 Cumulants

Let $X_1, \ldots, X_n$ be random variables. Due to Speed [9] (see also Rota [8]) the *multilinear cumulant* of these random variables can be expressed as

$$\kappa(X_1, \ldots, X_n) \ =\ \sum_{\pi\in\mathbb{P}(\{1,\ldots,n\})} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B\in\pi} E\left(\prod_{b\in B} X_b\right) \tag{7}$$

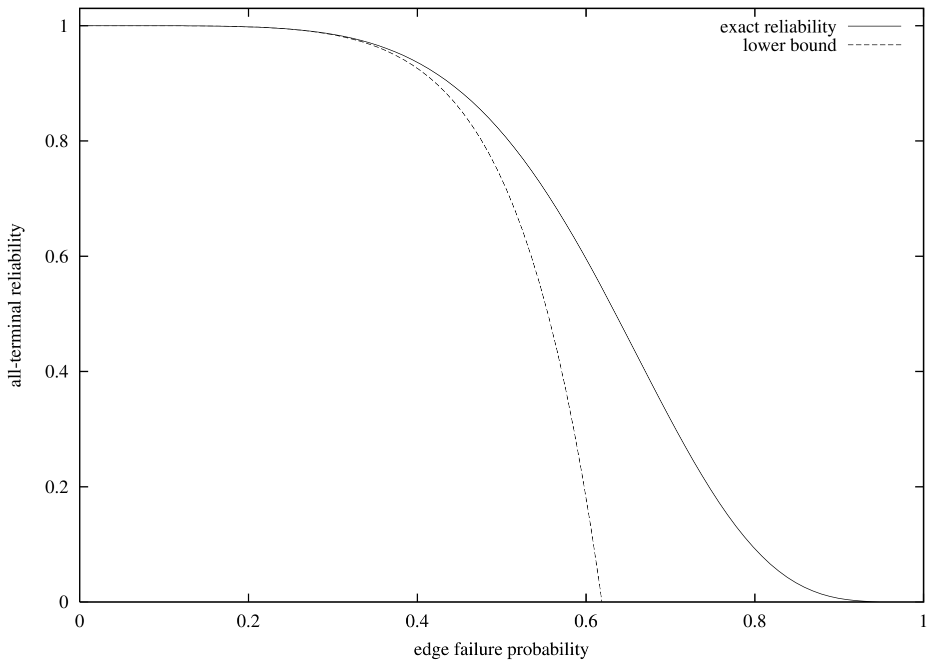

Figure 1: Exact and approximate reliability of K_6.

which, for simplicity, is considered here as a definition. We now generalize the notion of a multilinear cumulant to what we call a *partition cumulant*: For any partition $\sigma \in \mathbb{P}(\{1, \ldots, n\})$ we define

$$\kappa(\sigma) \;=\; \sum_{\substack{\pi \in \mathbb{P}(\{1,\ldots,n\}) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \prod_{B \in \pi} E\left(\prod_{b \in B} X_b\right) \tag{8}$$

where μ denotes the Möbius function of $\mathbb{P}(\{1, \ldots, n\})$ (see [10] for details). Since $\mu(\pi, \hat{1}) = (-1)^{|\pi|-1}(|\pi| - 1)!$, where $\hat{1}$ denotes the largest element of $\mathbb{P}(\{1, \ldots, n\})$, we find that $\kappa(\hat{1}) = \kappa(X_1, \ldots, X_n)$, so (8) generalizes (7).

Theorem 5.1 *Let $X_1, \ldots, X_n$ be random variables such that all partition cumulants of $X_1, \ldots, X_n$ are non-negative. Then, for any $r \in \mathbb{Z}^+$,*

$$\begin{aligned}(-1)^r \kappa(X_1, \ldots, X_n) \;\geq\; (-1)^r & \sum_{\substack{\pi \in \mathbb{P}(\{1,\ldots,n\}) \\ |\pi| \leq r}} (-1)^{|\pi|-1}(|\pi| - 1)! \prod_{B \in \pi} E\left(\prod_{b \in B} X_b\right) \\ & + \frac{r!}{\left\{ {n \atop r+1} \right\}} \sum_{\substack{\pi \in \mathbb{P}(\{1,\ldots,n\}) \\ |\pi| = r+1}} \prod_{B \in \pi} E\left(\prod_{b \in B} X_b\right).\end{aligned}$$

Proof. For any $\pi \in \mathbb{P}(\{1,\ldots,n\})$ define $f(\pi) = \prod_{B\in\pi} E\left(\prod_{b\in B} X_b\right)$ and $g(\pi) = \kappa(\pi)$. By (8), $g(\sigma) = \sum_{\pi\leq\sigma} \mu(\pi,\sigma) f(\pi)$ for any $\sigma \in \mathbb{P}(\{1,\ldots,n\})$, whence by Möbius inversion, $f(\pi) = \sum_{\sigma\leq\pi} g(\sigma)$ for any $\pi \in \mathbb{P}(\{1,\ldots,n\})$. By this and the assumption that all partition cumulants are non-negative, the requirements of Theorem 2.1 are satisfied, and the result follows. $\square$

Remark. In statistics, one often considers the *nth cumulant* $\kappa_n(X)$ of a random variable X which is related to the multilinear cumulant via $\kappa_n(X) = \kappa(X_1,\ldots,X_n)$ with $X_i = X$ for $i = 1,\ldots,n$. Theorem 5.1 provides bounds for the nth cumulant of a random variable X in terms of the binomial moments $E(X), E(X^2),\ldots,E(X^n)$, provided the associated partition cumulants are non-negative. As an example, consider the fifth cumulant of a random variable X. By applying Theorem 5.1 for $r = 1,2$ we obtain the inequality

$$\kappa_5(X) \;\leq\; E(X^5) - \frac{1}{3}\,E(X)E(X^4) - \frac{2}{3}\,E(X^2)E(X^3),$$

respectively

$$\begin{aligned}\kappa_5(X) \;\geq\; & E(X^5) - 5E(X)E(X^4) - 10E(X^2)E(X^3) \\ & + \frac{4}{5}\,E(X)^2E(X^3) + \frac{6}{5}\,E(X)E(X^2)^2.\end{aligned}$$

References

[1] E.A. Bender and J.R. Goldman, *On the applications of Möbius inversion in combinatorial analysis*, Amer. Math. Monthly, **82** (1975), 789–803.

[2] J.A. Buzacott and S.K. Chang, *Cut set intersections and node partitions*, IEEE Trans. Reliab., **R-33** (1984), 385–389.

[3] C.J. Colbourn, *The Combinatorics of Network Reliability*, Oxford University Press, 1987.

[4] K. Dohmen, *Improved Bonferroni Inequalities via Abstract Tubes, Inequalities and Identities of Inclusion-Exclusion Type*, Lecture Notes in Mathematics, No. 1826, Springer-Verlag, Berlin Heidelberg, 2003.

[5] K. Dohmen and P. Tittmann, *Inequalities of Bonferroni-Galambos type with applications to the Tutte polynomial and the chromatic polynomial*, JIPAM, J. Inequal. Pure Appl. Math., **5**(3) (2004), Article 64.

[6] J. Galambos, *Methods for proving Bonferroni type inequalities*, J. London Math. Soc., **2** (1975), 561–564.

[7] J. Galambos and I. Simonelli, *Bonferroni-type Inequalities with Applications*, Springer Series in Statistics, Probability and Its Applications, Springer-Verlag, New York, 1996.

[8] G.-C. Rota, *Twelve problems in probability no one likes to bring up, Problem eleven: cumulants*, The Fubini Lectures, in: Algebraic Combinatorics Computer Science (H. Crapo and D. Senato, Eds.), Springer-Verlag Italia, Milano, 2001, pp. 57–93.

[9] T.P. Speed, *Cumulants and partition lattices*, Austral. J. Statist., **25** (1983), 378–388.

[10] P. Tittmann, *Einführung in die Kombinatorik*, Spektrum Akademischer Verlag, Heidelberg Berlin, 2000.

[11] P. Tittmann, *Partitions and network reliability*, Discrete Appl. Math., **95** (1999), 445–453.