Matroid inequalities from electrical network theory

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Abstract

In 1981, Stanley applied the Aleksandrov–Fenchel Inequalities to prove a logarithmic concavity theorem for regular matroids. Using ideas from electrical network theory we prove a generalization of this for the wider class of matroids with the "half–plane property". Then we explore a nest of inequalities for weighted basis–generating polynomials that are related to these ideas. As a first result from this investigation we find that every matroid of rank three or corank three satisfies a condition only slightly weaker than the conclusion of Stanley's theorem.

Dedicated with great admiration
to Richard Stanley
on the occasion of his 60th birthday.

1 Introduction.

In 1981, Stanley [15] applied the Aleksandrov–Fenchel Inequalities to prove the following logarithmic concavity result:

Theorem 1.1 (Stanley, 1981) *Let $\mathcal{M}$ be a matroid, and let $\pi = (S, T, C_1, \ldots, C_k)$ be an ordered partition of the ground–set E of $\mathcal{M}$ into pairwise disjoint nonempty subsets, and fix nonnegative integers $c_1, \ldots, c_k$. For each $0 \leq j \leq |S|$, let $M_j(\pi)$ be the number of bases B of $\mathcal{M}$ such that $|B \cap S| = j$ and $|B \cap C_i| = c_i$ for all $1 \leq i \leq k$. If $\mathcal{M}$ is regular then for each $1 \leq j \leq |S| - 1$,*

$$\frac{M_j(\pi)^2}{\binom{|S|}{j}^2} \geq \frac{M_{j-1}(\pi)}{\binom{|S|}{j-1}} \cdot \frac{M_{j+1}(\pi)}{\binom{|S|}{j+1}}.$$

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Stanley's proof proceeds by constructing a set of zonotopes with the desired mixed volumes.

A few years later, answering a question raised by Stanley, Godsil [7] strengthened this conclusion as follows:

Theorem 1.2 (Godsil, 1984) *With the hypotheses and notation of Theorem 1.1, the polynomial* $\sum_{j=0}^{|S|} M_j(\pi)x^j$ *has only real (nonpositive) zeros.*

The well–known Newton's Inequalities (item (51) of [8]) show that Theorem 1.2 implies Theorem 1.1. Godsil's proof employs a determinantal theorem used by Schneider [13] to prove the Aleksandrov–Fenchel Inequalities.

The aim of this paper is to publicize a recent extension of these results (Theorem 4.5 of [4]), to collect the scattered details of its proof into a self–contained whole, and to present some preliminary findings on related inequalities. It can also be regarded as an idiosyncratic introduction to the literature on the half–plane property, Rayleigh monotonicity, and related concepts [1, 2, 3, 4, 5, 6, 10, 12, 14, 16]. We extend Theorem 1.2 in two directions – by relaxing the hypothesis and by strengthening the conclusion. For the hypothesis, we replace the condition that the matroid is regular by the weaker condition that the matroid has the half–plane property explained in Section 2. We strengthen the conclusion by introducing positive real weights $\mathbf{y} := \{y_e : \ e \in E\}$ on the elements of the ground–set $E(\mathcal{M})$. The weight of a basis B of $\mathcal{M}$ is then $\mathbf{y}^B := \prod_{e\in B} y_e$, and with the notation of Theorem 1.1 we let $M_j(\pi, \mathbf{y}) := \sum_B \mathbf{y}^B$ with the sum over all bases B of $\mathcal{M}$ such that $|B \cap S| = j$ and $|B \cap C_i| = c_i$ for all $1 \leq i \leq k$.

Theorem 1.3 (Theorem 4.5 of [4]) *With the above hypotheses and notation, if* $\mathcal{M}$ *has the half–plane property then the polynomial*

$$\sum_{j=0}^{|S|} M_j(\pi, \mathbf{y})x^j$$

has only real (nonpositive) zeros.

2 Electrical Networks and Matroids.

Our proof of Theorem 1.3 builds on ideas originating with Kirchhoff's formula for the effective conductance of a (linear, resistive, direct current) electrical network. Such a network is represented by a graph $G = (V, E)$ with real positive weights $\mathbf{y} := \{y_e : \ e \in E\}$ specifying the conductance of each edge of the graph. We use the notation

$$G(\mathbf{y}) := \sum_T \mathbf{y}^T$$

for the sum of $\mathbf{y}^T := \prod_{e\in T} y_e$ over all spanning trees T of G, and $\mathbf{y} > \mathbf{0}$ to indicate that every edge–weight is positive. Fixing two vertices $v, w \in V$, the effective conductance of the network measured between v and w is – by Kirchhoff's Formula [9] –

$$\mathcal{Y}_{vw}(G;\mathbf{y}) = \frac{H^f(\mathbf{y})}{H_f(\mathbf{y})}.$$

Here H denotes the graph obtained from G by adjoining a new edge f with ends v and w, $H^f = G$ is H with f deleted, and H_f is H with f contracted.

One intuitive property of electrical networks is that if the conductance y_e of some edge e of G is increased, then $\mathcal{Y}_{vw}(G;\mathbf{y})$ does not decrease. That is,

$$\frac{\partial}{\partial y_e}\mathcal{Y}_{vw}(G;\mathbf{y}) \geq 0.$$

This property is known as *Rayleigh monotonicity*. After some calculation using $H_f(\mathbf{y}) = H_f^e(\mathbf{y}) + y_e H_{ef}(\mathbf{y})$ *et cetera*, this is found to be equivalent to the condition that if $\mathbf{y} > \mathbf{0}$ then

$$H_e^f(\mathbf{y})H_f^e(\mathbf{y}) \geq H_{ef}(\mathbf{y})H^{ef}(\mathbf{y}).$$

(The deletion/contraction notation is extended in the obvious way.)

A less obvious property of the effective conductance is that if every y_e is a complex number with positive real part then $\mathcal{Y}_{vw}(G;\mathbf{y})$ is a complex number with nonnegative real part. Physically, this corresponds to the fact that if every edge of an alternating current circuit dissipates energy then the whole network cannot produce energy. Some minor hijinx with the equation $H(\mathbf{y}) = H^f(\mathbf{y}) + y_f H_f(\mathbf{y})$ shows that this is equivalent to the condition that if $\mathrm{Re}(y_e) > 0$ for all $e \in E(H)$ then $H(\mathbf{y}) \neq 0$. Such a polynomial $H(\mathbf{y})$ is said to have the *half–plane property*.

The combinatorics of the preceeding three paragraphs carries over *mutatis mutandis* to matroids in general. In place of a graph G we have a matroid $\mathcal{M}$. The edge–weights $\mathbf{y}$ become weights on the ground–set $E(\mathcal{M})$ of $\mathcal{M}$. In place of the spanning tree generating function $G(\mathbf{y})$ we have the basis generating function

$$M(\mathbf{y}) := \sum_{B\in\mathcal{M}} \mathbf{y}^B.$$

Since this is insensitive to loops we might as well think of $\mathcal{M}$ as given by its set of bases. For disjoint subsets $I, J \subseteq E$ the contraction of I and deletion of J from $\mathcal{M}$ is given by

$$\mathcal{M}_I^J := \{B \smallsetminus I : \ B \in \mathcal{M} \text{ and } I \subseteq B \subseteq E \smallsetminus J\}.$$

Rayleigh monotonicity corresponds to the inequalities

$$M_e^f(\mathbf{y})M_f^e(\mathbf{y}) \geq M_{ef}(\mathbf{y})M^{ef}(\mathbf{y})$$

for all $e, f \in E$ and $\mathbf{y} > \mathbf{0}$. A matroid satisfying these inequalities is called a *Rayleigh matroid*. If the basis generating polynomial $M(\mathbf{y})$ has the half–plane property then we say that $\mathcal{M}$ has the *half–plane property*, or is a *HPP matroid*.

3 The Half–Plane Property.

Our first item of business is to show that every regular matroid is a HPP matroid. For graphs, Proposition 3.1 is part of the "folklore" of electrical engineering. We take it from Corollary 8.2(a) and Theorem 8.9 of [3], but include the short and interesting proof for completeness.

A matrix A of complex numbers is a *sixth–root of unity* matrix provided that every nonzero minor of A is a sixth–root of unity. A matroid $\mathcal{M}$ is a *sixth–root of unity matroid* provided that it can be represented over the complex numbers by a sixth–root of unity matrix. For example, every regular matroid is a sixth–root of unity matroid. Whittle [17] has shown that a matroid is a sixth–root of unity matroid if and only if it is representable over both $GF(3)$ and $GF(4)$. (It is worth noting that Godsil's proof can be adapted to prove Theorem 1.3 in the special case of sixth–root of unity matroids.)

Proposition 3.1 *Every sixth–root of unity matroid is a HPP matroid.*

Proof. Let A be a sixth–root of unity matrix of full row–rank r, representing the matroid $\mathcal{M}$, and let A^* denote the conjugate transpose of A. Index the columns of A by the set E, and let $Y := \mathrm{diag}(y_e : \ e \in E)$ be a diagonal matrix of indeterminates. For an r–element subset $S \subseteq E$, let $A[S]$ denote the square submatrix of A supported on the set S of columns. Since A is a sixth–root of unity matrix, either $\det A[S] = 0$ or $|\det A[S]| = 1$. Thus, by the Binet–Cauchy formula,

$$\det(AYA^*) = \sum_{S \subseteq E:\ |S|=r} |\det A[S]|^2 \mathbf{y}^S = M(\mathbf{y})$$

is the basis–generating polynomial of $\mathcal{M}$.

Now we claim that if $\mathrm{Re}(y_e) > 0$ for all $e \in E$, then AYA^* is nonsingular. This suffices to prove the result. Consider any nonzero vector $\mathbf{v} \in \mathbb{C}^r$. Then $A^*\mathbf{v} \neq \mathbf{0}$ since the columns of A^* are linearly independent. Therefore

$$\mathbf{v}^* AYA^* \mathbf{v} = \sum_{e \in E} y_e |(A^*\mathbf{v})_e|^2$$

has strictly positive real part, since for all $e \in E$ the numbers $|(A^*\mathbf{v})_e|^2$ are nonnegative reals and at least one of these is positive. In particular, for any nonzero $\mathbf{v} \in \mathbb{C}^r$, the vector $AYA^*\mathbf{v}$ is nonzero. It follows that AYA^* is nonsingular, completing the proof. □

The same proof shows that for any complex matrix A of full row–rank r, the polynomial

$$\det(AYA^*) = \sum_{S \subseteq E:\ |S|=r} |\det A[S]|^2 \mathbf{y}^S$$

has the half–plane property. The weighted analogue of Rayleigh monotonicity in this case is discussed from a probabilistic point of view by Lyons [10]. It is a surprising fact that a complex matrix A of full row–rank r has $|\det A[S]|^2 = 1$ for all nonzero rank r minors if and only if A represents a sixth–root of unity matroid (Theorem 8.9 of [3]).

The proof of Theorem 1.3 is based on the following lemmas regarding the half–plane property. The paper [3] gives a much more thorough development of the theory.

Lemma 3.2 *Let $P(\mathbf{y})$ be a polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$, let $e \in E$, and let the degree of y_e in P be n. If $P(\mathbf{y})$ has the half–plane property then $y_e^n P(\{y_f : f \neq e\}, 1/y_e)$ has the half–plane property.*

Proof. This follows immediately from the fact that $\mathrm{Re}(1/y_e) > 0$ if and only if $\mathrm{Re}(y_e) > 0$. $\square$

Lemma 3.3 ([3], Proposition 2.8) *Let $P(\mathbf{y})$ be a polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$. For any $e \in E$, if P has the half–plane property then $\partial P/\partial y_e$ has the half–plane property.*

Proof. Fix complex values with positive real parts for every y_f with $f \in E \smallsetminus \{e\}$. The result of these substitutions is a univariate polynomial $F(y_e)$ all the roots of which have nonpositive real part. Thus

$$F(y_e) = C \prod_{j=1}^{n} (y_e + \theta_j)$$

with $\mathrm{Re}(\theta_j) \geq 0$ for all $1 \leq i \leq n$. It follows that if $\mathrm{Re}(y_e) > 0$ then the real part of

$$\frac{F'(y_e)}{F(y_e)} = \sum_{j=1}^{n} \frac{1}{y_e + \theta_j}$$

is also strictly positive. In particular $F'(y_e) \neq 0$. It follows that $\partial P/\partial y_e$ has the half–plane property. $\square$

Corollary 3.4 ([5], Theorem 18**, or [3], Proposition** 3.4.**)** *Let $P(\mathbf{y})$ be a polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$, fix $e \in E$, and let $P(\mathbf{y}) = \sum_{j=0}^{n} P_j(\{y_f : f \neq e\}) y_e^j$. If P has the half–plane property then each P_j has the half–plane property.*

Proof. Let n be the degree of $P(\mathbf{y})$ in the variable y_e. Let $A := \partial^j P/\partial y_e^j$, $B := y_e^{n-j} A(\{y_f : f \neq e\}, 1/y_e)$, and $C := \partial^{n-j-1} B/\partial y_e^{n-j-1}$. Then $C(\mathbf{y})$ is a nonzero multiple of P_j, and has the half–plane property by Lemmas 3.2 and 3.3. $\square$

Lemma 3.5 ([3], Proposition 5.2**)** *Let $P(\mathbf{y})$ be a homogeneous polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$. For sets of nonnegative real numbers $\mathbf{a} = \{a_e : e \in E\}$ and $\mathbf{b} = \{b_e : e \in E\}$, let $P(\mathbf{a}x + \mathbf{b})$ be the polyomial obtained by substituting $y_e = a_e x + b_e$ for each $e \in E$. The following are equivalent:*
(a) *$P(\mathbf{y})$ has the half–plane property;*
(b) *for all sets of nonnegative real numbers $\mathbf{a}$ and $\mathbf{b}$, $P(\mathbf{a}x+\mathbf{b})$ has only real (nonpositive) zeros.*

Proof. To see that (a) implies (b), suppose that ξ is a zero of $P(\mathbf{a}x + \mathbf{b})$ that is not a nonpositive real. Then there are complex numbers z and w with positive real part such that $z/w = \xi$. If $P(\mathbf{y})$ is homogeneous of degree r then $P(\mathbf{a}z + \mathbf{b}w) = w^r P(\mathbf{a}\xi + \mathbf{b}) = 0$, showing that $P(\mathbf{y})$ fails to have the half–plane property.

To see that (b) implies (a), consider any set of values $\{y_e : e \in E\}$ with $\mathrm{Re}(y_e) > 0$ for all $e \in E$. There are complex numbers z and w with positive real parts such that all the y_e are in the convex cone generated by z and w. That is, for each $e \in E$ there are nonnegative reals a_e and b_e such that $y_e = a_e z + b_e w$. Now $P(\mathbf{y}) = w^r P(\mathbf{a}\xi + \mathbf{b})$ in which $\xi = z/w$ is not in the interval $(-\infty, 0]$, and so $P(\mathbf{y}) \neq 0$. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{M}$ be a HPP matroid and fix $\mathbf{y} > \mathbf{0}$. Let s, t, and $z_1, \ldots, z_k$ be indeterminates, and for $e \in E$ put

$$u_e := \begin{cases} y_e s & \text{if } e \in S, \\ y_e t & \text{if } e \in T, \\ y_e z_i & \text{if } e \in C_i. \end{cases}$$

Then $M(\mathbf{u})$ is a homogeneous polynomial with the half–plane property in the variables $s, t, z_1, \ldots, z_k$. By repeated application of Corollary 3.4, the coefficient $M_{\mathbf{c}}(s,t)$ of $z_1^{c_1} \cdots z_k^{c_k}$ in $M(\mathbf{u})$ also has the half–plane property, and is homogeneous. In fact,

$$M_{\mathbf{c}}(s,t) = \sum_{j=0}^{|S|} M_j(\pi, \mathbf{y}) s^j t^{d-j},$$

in which $d = \mathrm{rank}(\mathcal{M}) - (c_1 + \cdots + c_k)$. Upon substituting $s = x$ and $t = 1$ in $M_{\mathbf{c}}(s,t)$, Lemma 3.5 implies that $\sum_{j=0}^{|S|} M_j(\pi, \mathbf{y}) x^j$ has only real (nonpositive) zeros, as claimed. □

4 Between HPP and Rayleigh.

This updated version of Stanley's theorem provides a link between the half–plane property and Rayleigh monotonicity in the context of matroids. Only the $k = 0$ case of the theorem is needed. (In fact the $k = 0$ case is equivalent to the general case by various properties of HPP matroids.) For a subset $S \subseteq E(\mathcal{M})$ and natural number j, let $M_j(S, \mathbf{y}) := \sum_B \mathbf{y}^B$, with the sum over all bases B of $\mathcal{M}$ such that $|B \cap S| = j$. For each positive integer m, consider the following conditions on a matroid $\mathcal{M}$:

RZ$[m]$**:** If $\mathbf{y} > \mathbf{0}$ then for all $S \subseteq E$ with $|S| \leq m$ the polynomial $\sum_{j=0}^{|S|} M_j(S, \mathbf{y}) x^j$ has only real zeros.

BLC$[m]$**:** If $\mathbf{y} > \mathbf{0}$ then for all $S \subseteq E$ with $|S| = n \leq m$ and all $1 \leq j \leq n-1$,

$$M_j(S, \mathbf{y})^2 \geq \left[1 + \frac{n+1}{j(n-j)}\right] M_{j-1}(S, \mathbf{y}) M_{j+1}(S, \mathbf{y}).$$

The mnemonics are for "real zeros" and "binomial logarithmic concavity", respectively. We also say that a matroid satisfies RZ if it satisfies RZ$[m]$ for all m, and that it satisfies

BLC if it satisfies BLC[m] for all m. Our BLC is a weighted strengthening of Stanley's "Property P".

The $k = 0$ case of Theorem 1.3 implies that a HPP matroid satisfies RZ, and Newton's Inequalities show that RZ[m] implies BLC[m] for every m. The implications RZ[m] $\Longrightarrow$ RZ[$m-1$] and BLC[m] $\Longrightarrow$ BLC[$m-1$] are trivial, as are the conditions RZ[1] and BLC[1]. Thus, the weakest nontrivial condition among these is BLC[2]. This is in fact **equivalent** to Rayleigh monotonicity, as the remarks after Theorem 4.3 show.

Proposition 4.1 ([4], Corollary 4.9) *Every HPP matroid is a Rayleigh matroid.*

Proof. Theorem 1.3 shows that every HPP matroid satisfies BLC[2]. We show here that if $\mathcal{M}$ satisfies BLC[2] then it is Rayleigh. So, let $\mathbf{y} > \mathbf{0}$ be positive weights on $E(\mathcal{M})$ and let $S = \{e, f\} \subseteq E$. To prove the Rayleigh inequality $M_e^f(\mathbf{y})M_f^e(\mathbf{y}) \geq M_{ef}(\mathbf{y})M^{ef}(\mathbf{y})$ it suffices to consider the case in which both e and f are neither loops nor coloops. In this case, define another set of weights by $w_e := M_f^e(\mathbf{y})$ and $w_f := M_e^f(\mathbf{y})$ and $w_g := y_g$ for all $g \in E \smallsetminus \{e, f\}$. Then, since $\mathbf{w} > 0$ and $\mathcal{M}$ satisfies BLC[2], the inequality

$$M_1(S, \mathbf{w})^2 \geq 4M_0(S, \mathbf{w})M_2(S, \mathbf{w})$$

holds. This can be expanded to

$$(w_e M_e^f(\mathbf{w}) + w_f M_f^e(\mathbf{w}))^2 \geq 4M^{ef}(\mathbf{w})w_e w_f M_{ef}(\mathbf{w}),$$

and finally to

$$4(M_e^f(\mathbf{y})M_f^e(\mathbf{y}))^2 \geq 4M_{ef}(\mathbf{y})M^{ef}(\mathbf{y})M_e^f(\mathbf{y})M_f^e(\mathbf{y}).$$

Cancellation of common (positive) factors from both sides yields the desired inequality. □

In view of the implication BLC[2] $\Longrightarrow$ Rayleigh, it is interesting to look for conditions (other than HPP) which imply BLC[m] for various m. This is further motivated by Stanley's application of "Property P" to Mason's Conjecture – see Theorem 2.9 of [15]. The following hierarchies of *strict root–binomial logarithmic concavity* and *strict logarithmic concavity* conditions are also interesting:

$\sqrt{\mathbf{BLC}}[m]$ If $\mathbf{y} > \mathbf{0}$ then for all $S \subseteq E$ with $|S| = n \leq m$ and all $1 \leq j \leq n-1$, if $M_j(S, \mathbf{y}) \neq 0$ then

$$M_j(S, \mathbf{y})^2 > \left[1 + \frac{1}{\min(j, n-j)}\right] M_{j-1}(S, \mathbf{y})M_{j+1}(S, \mathbf{y}).$$

SLC[m]**:** If $\mathbf{y} > \mathbf{0}$ then for all $S \subseteq E$ with $|S| \leq m$ and all $1 \leq j \leq |S|-1$, if $M_j(S, \mathbf{y}) \neq 0$ then

$$M_j(S, \mathbf{y})^2 > M_{j-1}(S, \mathbf{y})M_{j+1}(S, \mathbf{y}).$$

We also say that a matroid satisfies $\sqrt{\mathrm{BLC}}$ if it satisfies $\sqrt{\mathrm{BLC}}[m]$ for all m, and satisfies SLC if it satisfies SLC$[m]$ for all m. The inequalities

$$1+\frac{1}{\min(j,n-j)}<1+\frac{n+1}{j(n-j)}\leq\left[1+\frac{1}{\min(j,n-j)}\right]^2$$

for $1\leq j\leq n-1$ show that BLC$[m]$ implies $\sqrt{\mathrm{BLC}}[m]$ for every m, and motivate this somewhat odd terminology. Clearly $\sqrt{\mathrm{BLC}}[m]$ implies SLC$[m]$ for every m.

In the following calculations we will usually omit explicit reference to the variables $\mathbf{y}$ unless a particular substitution must be emphasized. For $\mathcal{M}$ a matroid, $S\subseteq E(\mathcal{M})$, and k a natural number, let

$$\Psi_k MS := \sum_{A\subseteq S:\ |A|=k} M_A^{S\smallsetminus A} M_{S\smallsetminus A}^{A}.$$

For example,

$$\Psi_2 M\{a,b,c\} := M_{ab}^{c}M_{c}^{ab}+M_{ac}^{b}M_{b}^{ac}+M_{bc}^{a}M_{a}^{bc}.$$

Notice that in general if $|S|=n$ then $\Psi_k MS=\Psi_{n-k}MS$. The Rayleigh inequality is that

$$\Psi_1 M\{e,f\}\geq 2\Psi_2 M\{e,f\}.$$

This suggests several possible generalizations, among which we will concentrate here on the following. For each integer $k\geq 1$ and real $\lambda>0$, say that $\mathcal{M}$ is *k–th level Rayleigh of strength* λ provided that:

λ-Ray$[k]$: If $\mathbf{y}>\mathbf{0}$ then for every $S\subseteq E(\mathcal{M})$ with $|S|=2k$,

$$\Psi_k MS\geq\lambda\Psi_{k+1}MS.$$

The condition 2-Ray[1] is exactly the Rayleigh condition. In general, each term on the LHS of λ-Ray$[k]$ occurs twice. Proposition 4.6 below shows that $(1+1/k)$-Ray$[k]$ is an especially natural strength for these conditions. Interestingly, this lies right between two of the most **useful** strengths for these conditions. As an example, the inequality for (3/2)-Ray[2] is

$$\begin{aligned} & 4\left[M_{ab}^{cd}M_{cd}^{ab}+M_{ac}^{bd}M_{bd}^{ac}+M_{ad}^{bc}M_{bc}^{ad}\right] \\ \geq\ & 3\left[M_{bcd}^{a}M_{a}^{bcd}+M_{acd}^{b}M_{b}^{acd}+M_{abd}^{c}M_{c}^{abd}+M_{abc}^{d}M_{d}^{abc}\right]. \end{aligned}$$

Lemma 4.2 (a) *For each $k\geq 1$ and $\lambda>0$, the class of matroids satisfying* λ-Ray$[k]$ *is closed by taking duals and minors.*
(b) *For each $m\geq 1$, the class of matroids satisfying* BLC$[m]$ *is closed by taking duals and minors.*
(c) *For each $m\geq 1$, the class of matroids satisfying* $\sqrt{\mathrm{BLC}}[m]$ *is closed by taking duals and minors.*
(d) *For each $m\geq 1$, the class of matroids satisfying* SLC$[m]$ *is closed by taking duals and minors.*

Sketch of proof. For the matroid $\mathcal{M}^*$ dual to $\mathcal{M}$ we have $M^*(\mathbf{y}) = \mathbf{y}^E M(\mathbf{1}/\mathbf{y})$. From this it follows that for $0 \leq k \leq n = |S|$,

$$\Psi_k M^* S(\mathbf{y}) = \left(\mathbf{y}^{E \smallsetminus S}\right)^2 \Psi_{n-k} MS(\mathbf{1}/\mathbf{y}).$$

Since $\mathbf{y} > \mathbf{0}$ is arbitrary, one sees that λ-Ray[k] for $\mathcal{M}$ implies λ-Ray[k] for $\mathcal{M}^*$. Similarly

$$M_j(S, \mathbf{y})^2 > M_{j-1}(S, \mathbf{y}) M_{j+1}(S, \mathbf{y})$$

implies

$$(\mathbf{y}^E)^2 M^*_{n-j}(S, \mathbf{1}/\mathbf{y})^2 > (\mathbf{y}^E)^2 M^*_{n-j+1}(S, \mathbf{1}/\mathbf{y}) M^*_{n-j-1}(S, \mathbf{1}/\mathbf{y}).$$

From this it follows that SLC[m] for $\mathcal{M}$ implies SLC[m] for $\mathcal{M}^*$. Analogous arguments work for BLC[m] and $\sqrt{\text{BLC}}[m]$.

For a set $S \subseteq E(\mathcal{M})$ and $g \in E \smallsetminus S$, we have

$$\Psi_k MS(\mathbf{y}) = y_g^2 \Psi_k M_g S(\mathbf{y}) + y_g Q(\mathbf{y}) + \Psi_k M^g S(\mathbf{y})$$

for some polynomial $Q(\mathbf{y})$. If $\mathcal{M}$ satisfies λ-Ray[k] then taking the limit as $y_g \to 0$ shows that the deletion $\mathcal{M}^g$ satisfies λ-Ray[k]. Similarly, multiplying by $1/y_g^2$ and taking the limit as $y_g \to \infty$ shows that the contraction $\mathcal{M}_g$ satisfies λ-Ray[k]. The case of a general minor is obtained by iteration of these two cases. Analogous arguments work for BLC[m], $\sqrt{\text{BLC}}[m]$, and SLC[m]. $\square$

Theorem 4.3 *If* $\mathcal{M}$ *satisfies* 2-Ray[1] *and* $(1 + 1/k)^2$-Ray[k] *for all* $2 \leq k \leq m$ *then* $\mathcal{M}$ *satisfies* BLC[$2m+1$].

Proof. The proof uses the following elementary inequality: for $N \geq 2$ real numbers $R_1, ..., R_N$,

$$(R_1 + \cdots + R_N)^2 \geq \frac{2N}{N-1} \sum_{\{i,j\} \subseteq \{1,\ldots,N\}} R_i R_j. \tag{1}$$

Assume that $\mathcal{M}$ satisfies the hypothesis, and fix positive real weights $\mathbf{y} > \mathbf{0}$. To show that $\mathcal{M}$ satisfies BLC[$2m+1$], consider a subset $S \subseteq E$ with $|S| = n \leq 2m+1$ and an index $1 \leq j \leq n-1$. We must show that

$$M_j(S)^2 \geq \left[1 + \frac{n+1}{j(n-j)}\right] M_{j-1}(S) M_{j+1}(S).$$

From the definition we have

$$M_j(S) = \sum_{A \subset S:\ |A|=j} \mathbf{y}^A M_A^{S \smallsetminus A}.$$

The inequality (1) implies that

$$M_j(S)^2 \geq \frac{2N}{N-1} \sum_{\{A,B\}} \mathbf{y}^A \mathbf{y}^B M_A^{S \smallsetminus A} M_B^{S \smallsetminus B}, \tag{2}$$

with $N := \binom{n}{j}$ and the sum over all pairs of distinct j–element subsets of S. We collect terms on the right side according to the intersection $I := A \cap B$ and union $U := A \cup B$ of the indexing pair of sets $\{A, B\}$. With $i := |I|$ and $k := j - i$ we have $|U \smallsetminus I| = 2k$ and the sum of the terms on the RHS of (2) with this fixed I and U is

$$\begin{aligned} & \frac{N}{N-1} \mathbf{y}^I \mathbf{y}^U \sum_{C \subset U \smallsetminus I:\ |C|=k} \left(M_I^{S \smallsetminus U}\right)_C^{U \smallsetminus I \smallsetminus C} \left(M_I^{S \smallsetminus U}\right)_{U \smallsetminus I \smallsetminus C}^C \\ = \ & \frac{N}{N-1} \mathbf{y}^I \mathbf{y}^U \Psi_k M_I^{S \smallsetminus U}(U \smallsetminus I). \end{aligned}$$

Therefore

$$M_j(S)^2 \geq \frac{N}{N-1} \sum_{k=1}^{h} \sum_{(I,U)} \mathbf{y}^I \mathbf{y}^U \Psi_k M_I^{S \smallsetminus U}(U \smallsetminus I)$$

in which $h := \min(j, n-j)$ and the inner sum is over all pairs of sets $I \subset U \subseteq S$ with $|U| + |I| = 2j$ and $|U \smallsetminus I| = 2k$. Let $\lambda_1 := 2$ and $\lambda_k := (1 + 1/k)^2$ for all $k \geq 2$. Since $k \leq h \leq m$ we are assuming that $\mathcal{M}$ satisfies λ_k-Ray$[k]$, and hence each minor $\mathcal{M}_I^{S \smallsetminus U}$ satisfies λ_k-Ray$[k]$ by Lemma 4.2(a). It follows that

$$M_j(S)^2 \ \geq \ \frac{N}{N-1} \sum_{k=1}^{h} \lambda_k \sum_{(I,U)} \mathbf{y}^I \mathbf{y}^U \Psi_{k+1} M_I^{S \smallsetminus U}(U \smallsetminus I). \tag{3}$$

If $h \geq 3$ then for all $1 \leq k \leq h$,

$$\lambda_k \geq \lambda_h = \left(1 + \frac{1}{h}\right)^2 \geq 1 + \frac{n+1}{j(n-j)}.$$

If $h = 2$ and $n \geq 5$ then

$$\lambda_2 = \frac{9}{4} > 2 = \lambda_1 \geq 1 + \frac{n+1}{j(n-j)}.$$

In these cases we conclude from (3) that

$$\begin{aligned} M_j(S)^2 \ & \geq \ \frac{N}{N-1} \left[1 + \frac{n+1}{j(n-j)}\right] \sum_{k=1}^{h} \sum_{(I,U)} \mathbf{y}^I \mathbf{y}^U \Psi_{k+1} M_I^{S \smallsetminus U}(U \smallsetminus I) \\ & = \ \frac{N}{N-1} \left[1 + \frac{n+1}{j(n-j)}\right] M_{j-1}(S) M_{j+1}(S). \end{aligned}$$

This implies the desired inequality in these cases.

If $h = 1$ then either $j = 1$ or $j = n-1$, so $k = 1$ and $N = n$, and we conclude from (3) that

$$\begin{aligned} M_j(S)^2 \ & \geq \ \frac{2n}{n-1} \sum_{(I,U)} \mathbf{y}^I \mathbf{y}^U \Psi_2 M_I^{S \smallsetminus U}(U \smallsetminus I) \\ & = \ \frac{2n}{n-1} M_{j-1}(S) M_{j+1}(S). \end{aligned}$$

This is the desired inequality when $j = 1$ or $j = n - 1$.

If $n \leq 3$ then $h = 1$, so the only remaining case is $n = 4$ and $h = 2$. In this case $j = 2$ and $N = 6$, and since $\lambda_2 = 9/4 > 2 = \lambda_1$, from (3) we conclude that

$$\begin{aligned} M_2(S)^2 &\geq \frac{6}{5}\sum_{k=1}^{2} 2 \sum_{(I,U)} \mathbf{y}^I \mathbf{y}^U \Psi_{k+1} M_I^{S \smallsetminus U}(U \smallsetminus I) \\ &\geq \frac{9}{4} M_1(S) M_3(S). \end{aligned}$$

This is the desired inequality in this case.

This completes the verification that $\mathcal{M}$ satisfies BLC$[2m+1]$. □

Notice that Proposition 4.1 and the $m = 1$ case of Theorem 4.3 show that the conditions BLC[3], BLC[2], and 2-Ray[1] are in fact equivalent – this is part of Theorem 4.8 of [4].

By examining the proof of Theorem 4.3 one sees the following.

Porism 4.4 (a) *If $\mathcal{M}$ satisfies $(1 + 1/k)$-Ray$[k]$ for all $1 \leq k \leq m$ then $\mathcal{M}$ satisfies $\sqrt{\text{BLC}}[2m + 1]$.*
(b) *If $\mathcal{M}$ satisfies 1-Ray$[k]$ for all $1 \leq k \leq m$ then $\mathcal{M}$ satisfies SLC$[2m + 1]$.*

Corollary 4.5 *Let $\mathcal{M}$ be a matroid of rank r with $|E| = \ell$ elements.*
(a) *If $\mathcal{M}$ satisfies 2-Ray[1] and $(1 + 1/k)^2$-Ray$[k]$ for all $2 \leq k \leq r - 1$ then $\mathcal{M}$ satisfies BLC.*
(b) *If $\mathcal{M}$ satisfies $(1 + 1/k)$-Ray$[k]$ for all $1 \leq k \leq r - 1$ then $\mathcal{M}$ satisfies $\sqrt{\text{BLC}}$.*
(c) *If $\mathcal{M}$ satisfies 1-Ray$[k]$ for all $1 \leq k \leq \min(r - 1, (\ell - 2)/4)$ then $\mathcal{M}$ satisfies SLC.*

Proof. If $\mathcal{M}$ has rank r then $\mathcal{M}$ satisfies any condition λ-Ray$[k]$ trivially for all $k \geq r$ because for any $S \subseteq E(\mathcal{M})$ with $|S| = 2k$, $\Psi_{k+1} M S = 0$ identically. In part (a) Theorem 4.3 thus implies that $\mathcal{M}$ satisfies BLC$[2m+1]$ for all m – that is BLC. In part (b) Porism 4.4(a) supports the analogous implication. Similarly, in part (c) if $r - 1 \leq (\ell - 2)/4$ then Porism 4.4(b) implies that $\mathcal{M}$ satisfies SLC. If $(\ell - 2)/4 < r - 1$ then Porism 4.4(b) implies that $\mathcal{M}$ satisfies SLC$[\lfloor \ell/2 \rfloor]$. For any $S \subset E$ with $|S| = n \leq \ell/2$ and any $1 \leq j \leq n$ we have $M_j(S) = M_{r-j}(E \smallsetminus S)$. The SLC inequalities for the set S thus imply the SLC inequalities for the set $E \smallsetminus S$. It follows that $\mathcal{M}$ satisfies SLC. □

One of course wants examples of matroids satisfying these higher level Rayleigh conditions. So far there are no very substantial results in this direction. Direct computations (with the aid of MAPLE) and *ad hoc* arguments have shown that the Kuratowski graphs K_5 and $\mathsf{K}_{3,3}$ satisfy (3/2)-Ray[2] but not (9/4)-Ray[2]. The hypothesis of Theorem 4.3 thus seems to be extremely restrictive. The corresponding calculation for K_6 is so far intractible. Similar computations for λ-Ray$[k]$ with $k \geq 3$ have not yet been tried.

One easy sufficient condition is the following naively optimistic generalization of the Rayleigh condition.

Proposition 4.6 *Assume that $\mathcal{M}$ is such that for all disjoint subsets A and B of $E(\mathcal{M})$ with $|A| = |B| = k$ and $b \in B$, if $\mathbf{y} > \mathbf{0}$ then*

$$\mathcal{M}_A^B M_B^A \geq M_{Ab}^{B\smallsetminus b} M_{B\smallsetminus b}^{Ab}.$$

Then $\mathcal{M}$ satisfies $(1 + 1/k)$-Ray$[k]$.

Proof. Let $\mathcal{M}$ be as in the hypothesis, let $\mathbf{y} > \mathbf{0}$, and fix a set $S \subseteq E(\mathcal{M})$ with $|S| = 2k$. To verify the inequality $k\Psi_k MS \geq (k+1)\Psi_{k+1}MS$, form a graph G with bipartition (X, Y) as follows. The vertices in X are the k–element subsets of S and the vertices of Y are the $(k+1)$ element subsets of S. There is an edge from $A \in X$ to $A' \in Y$ whenever $A \subset A'$. Thus, every vertex of X has degree k and every vertex of Y has degree $k+1$. To each edge $\{A, Ab\}$ of G associate the weight $M_A^{S\smallsetminus A} M_{S\smallsetminus A}^A$. The total weight assigned to the edges is thus

$$k \sum_{A \in X} M_A^{S\smallsetminus A} M_{S\smallsetminus A}^A = k\Psi_k MS.$$

On the other hand, for each $A' \in Y$ the sum of the weights of edges incident with A' is

$$\begin{aligned} & \sum_{b \in A'} M_{A'\smallsetminus b}^{S\smallsetminus(A'\smallsetminus b)} M_{S\smallsetminus(A'\smallsetminus b)}^{A'\smallsetminus b} \\ \geq \ & \sum_{b \in A'} M_{A'}^{S\smallsetminus A'} M_{S\smallsetminus A'}^{A'} = (k+1) M_{A'}^{S\smallsetminus A'} M_{S\smallsetminus A'}^{A'}. \end{aligned}$$

It follows that the sum of the weights of the edges of G is at least $(k+1)\Psi_{k+1}MS$, completing the proof. $\square$

Unfortunately, it is not hard to find **planar graphs** which fail to satisfy the hypothesis of Proposition 4.6 in the case $k = 2$. An easy example is the 4–wheel W_4 formed by a cycle of four "rim" edges with the vertices joined to a new "hub" vertex by four "spoke" edges. The spoke edges are labelled a, b, c, d in cyclic order, and the rim edges have weights y, 1, y, 1 in cyclic order starting with the edge forming a triangle with a and b. With $\mathcal{W}$ denoting the graphic matroid of W_4, we have $W_{ab}^{cd} = W_{cd}^{ab} = 2y + 1$ and $W_{abc}^d = y + 1$ and $W_d^{abc} = 2y(y+1)$, so that the inequality

$$W_{ab}^{cd} W_{cd}^{ab} \geq W_{abc}^d W_d^{abc}$$

is equivalent to

$$(2y+1)^2 \geq 2y(y+1)^2.$$

This inequality $2y^3 \leq 2y + 1$ is not satisfied for all $y > 0$, so that $\mathcal{W}$ does not meet the hypothesis of Proposition 4.4. Note that $\mathcal{W}$ does satisfy $(3/2)$-Ray[2], however, since W_4 is a minor of K_5.

In general, the condition $(1+1/k)$-Ray$[k]$ asserts that the inequalities in the hypothesis of Proposition 4.6 hold "locally on average" in some sense.

To close this section we show that the condition $\sqrt{\mathrm{BLC}}$ has consequences for Mason's Conjecture [11] – this is inspired by Theorem 2.9 of Stanley [15]. For a matroid $\mathcal{M}$ of rank r and $0 \le j \le r$, let $I_j(\mathcal{M})$ denote the number of j–element independent sets of $\mathcal{M}$. Mason's Conjecture is that for any matroid, $I_j^2 \ge I_{j-1}I_{j+1}$ for all $1 \le j \le r-1$. Let $\mathcal{M} \oplus \mathcal{U}_{\ell,\ell}$ denote the free extension of $\mathcal{M}$ by ℓ points, and let $T_r(\mathcal{M} \oplus \mathcal{U}_{\ell,\ell})$ denote the truncation of this matroid down to rank r.

Theorem 4.7 *Let $\mathcal{M}$ be a matroid of rank r. Assume that, for infinitely many integers $\ell \ge 1$, $T_r(\mathcal{M} \oplus \mathcal{U}_{\ell,\ell})$ satisfies $\sqrt{\mathrm{BLC}}[\ell]$. Then $\mathcal{M}$ satisfies Mason's Conjecture.*

Proof. Let $\mathcal{L} := T_r(\mathcal{M} \oplus \mathcal{U}_{\ell,\ell})$ for some $\ell \ge 1$, and let $S := E(\mathcal{L}) \smallsetminus E(\mathcal{M})$. For any $0 \le j \le r$ we have $L_j(S, \mathbf{1}) = \binom{\ell}{j} I_{r-j}(\mathcal{M})$. If $1 \le j \le r-1$ and $\ell \ge 2j$ is such that $\mathcal{L}$ satisfies $\sqrt{\mathrm{BLC}}[\ell]$ then

$$L_j(S,\mathbf{1})^2 > \left[1 + \frac{1}{j}\right] L_{j-1}(S,\mathbf{1})L_{j+1}(S,\mathbf{1}).$$

From this we obtain

$$I_{r-j}(\mathcal{M})^2 > \frac{\ell - j}{\ell - j + 1} I_{r-j+1}(\mathcal{M}) I_{r-j-1}(\mathcal{M}).$$

Since this holds for a sequence of $\ell \to \infty$ we conclude that

$$I_{r-j}(\mathcal{M})^2 \ge I_{r-j+1}(\mathcal{M}) I_{r-j-1}(\mathcal{M}).$$

As this holds for all $1 \le j \le r-1$, $\mathcal{M}$ satisfies Mason's Conjecture. □

5 Matroids of rank three.

In [16] it is shown that every matroid of rank (at most) three satisfies the Rayleigh condition 2-Ray[1].

Theorem 5.1 *Every matroid of rank three satisfies* (3/2)-Ray[2].

Proof. Let $\mathcal{M}$ be a matroid of rank three and let $\mathbf{y} > \mathbf{0}$ be positive weights on $E(\mathcal{M})$. Consider any four–element subset $S = \{a,b,c,d\}$ of E. There are three cases (up to permuting the elements of S):
(i) $\{a,b,c,d\}$ are collinear;
(ii) $\{a,b,c\}$ are collinear and d is not on this line;
(iii) $\{a,b,c,d\}$ are in general position.

In case (i) it is easy to see that $\Psi_3 MS = 0$, so that the inequality $2\Psi_2 MS \ge 3\Psi_3 MS$ is trivially satisfied.

For cases (ii) and (iii) we consider the coefficient of each monomial in $\Psi_2 MS$ and in $\Psi_3 MS$ in turn.

First consider monomials of the form y_e^2 with $e \in E \smallsetminus S$. The coefficient of y_e^2 in $\Psi_2 MS$ is nonnegative. On the other hand, the coefficient of y_e^2 in $M^a_{bcd} M^{bcd}_a$ is zero, and

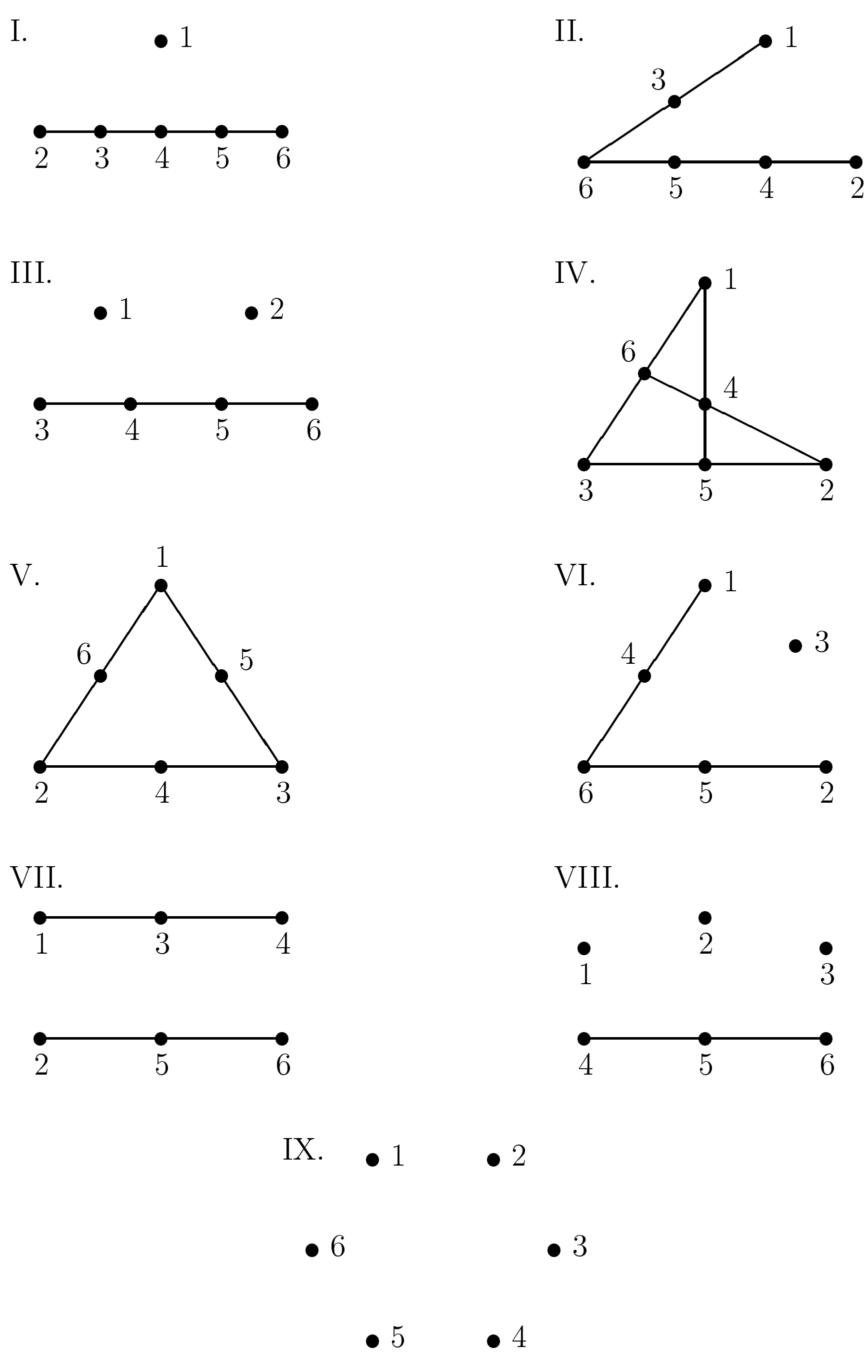

Figure 1: The six–element rank three matroids.

$\mathcal{N}\{a,b,c,d\}$	Ψ_2	Ψ_3	$2\Psi_2 - 3\Psi_3$
I$\{2,3,4,1\}$	0	0	0
II$\{1,3,6,2\}$	8	2	10
II$\{2,4,5,1\}$	6	3	3
II$\{2,4,6,1\}$	6	3	3
III$\{3,4,5,1\}$	6	3	3
IV$\{1,3,6,2\}$	8	2	10
V$\{1,2,6,3\}$	8	3	7
V$\{1,2,6,4\}$	10	3	11
VI$\{1,4,6,2\}$	10	3	11
VI$\{1,4,6,3\}$	12	3	18
VII$\{1,3,4,2\}$	12	3	15
VIII$\{4,5,6,1\}$	12	3	15

Table 1: Monomials of shape $y_e y_f$ in case (ii).

similarly the coefficient of y_e^2 in $\Psi_3 MS$ is zero. It follows that in both cases (i) and (ii), the coefficient of y_e^2 in $2\Psi_2 MS - 3\Psi_3 MS$ is nonnegative.

Next consider monomials of the form $y_e y_f$ with $e \neq f$ in $E \setminus S$. The coefficiens of $y_e y_f$ in $\Psi_2 MS$ and in $\Psi_3 MS$ depend only on the restriction $\mathcal{N} := \mathcal{M}|\{a,b,c,d,e,f\}$ and the position of $\{a,b,c,d\}$ within $\mathcal{N}$. Figure 1 lists all the rank three matroids on six elements, up to isomorphism. Table 1 lists the possible cases for $\mathcal{N}$ and $\{a,b,c,d\}$, up to isomorphism, in case (ii). Table 2 lists the possible cases for $\mathcal{N}$ and $\{a,b,c,d\}$, up to isomorphism, in case (iii). In each table the first column specifies $\mathcal{N}$ and $\{a,b,c,d\}$, the second column gives the coefficient of $y_e y_f$ in $\Psi_2 MS$, and the third column gives the coefficient of $y_e y_f$ in $\Psi_3 MS$. The fourth column computes the coefficient of the corresponding monomial of $2\Psi_2 MS - 3\Psi_3 MS$. Only in case (iii) subcase IV$\{1,2,3,4\}$ do we obtain a negative coefficient.

Given a set $S = \{a,b,c,d\}$ in general position in $\mathcal{M}$, there are at most three points in $E \setminus S$ that can appear in a restriction of $\mathcal{M}$ containing S in subcase IV$\{1,2,3,4\}$; these are the points $e := \overline{ab} \cap \overline{cd}$, $f := \overline{ac} \cap \overline{bd}$, and $g := \overline{ad} \cap \overline{bc}$. Notice that (when they exist) each of y_e^2, y_f^2 and y_g^2 occurs in $2\Psi_2 MS - 3\Psi_3 MS$ with coefficient 8. If at most one of e, f, and g exists in $E(\mathcal{M})$ then subcase IV$\{1,2,3,4\}$ does not arise. If two exist – say e and f – then

$$8y_e^2 + 8y_f^2 - 4y_e y_f = 6y_e^2 + 6y_f^2 + 2(y_e - y_f)^2$$

can be used to absorb the term with negative coefficient into a square of a binomial. If all of e, f, and g exist in $E(\mathcal{M})$ then

$$\begin{aligned} & 8y_e^2 + 8y_f^2 + 8y_g^2 - 4y_e y_f - 4y_e y_g - 4y_f y_g \\ = \; & 4y_e^2 + 4y_f^2 + 4y_g^2 + 2(y_e - y_f)^2 + 2(y_e - y_g)^2 + 2(y_f - y_g)^2 \end{aligned}$$

can be used to absorb the terms with negative coefficients into a sum of squares of binomials.

$\mathcal{N}\{a,b,c,d\}$	Ψ_2	Ψ_3	$2\Psi_2 - 3\Psi_3$
$\mathrm{II}\{1,2,3,4\}$	8	2	10
$\mathrm{III}\{1,2,3,4\}$	8	2	10
$\mathrm{IV}\{1,2,3,4\}$	4	4	-4
$\mathrm{V}\{1,2,4,5\}$	6	4	0
$\mathrm{V}\{1,4,5,6\}$	8	3	7
$\mathrm{VI}\{1,2,4,5\}$	8	4	4
$\mathrm{VI}\{1,2,3,4\}$	10	3	11
$\mathrm{VI}\{1,2,3,6\}$	10	4	8
$\mathrm{VII}\{1,2,3,5\}$	8	4	4
$\mathrm{VIII}\{1,2,4,5\}$	10	4	8
$\mathrm{IX}\{1,2,3,4\}$	12	4	12

Table 2: Monomials of shape $y_e y_f$ in case (iii).

In each of cases (i), (ii), and (iii) we can thus write $2\Psi_2 MS - 3\Psi_3 MS$ as a positive sum of monomials and squares of binomials in the variables $\mathbf{y}$. It follows that $\mathcal{M}$ satisfies the condition (3/2)-Ray[2]. $\square$

Theorem 5.1, the result of [16], Lemma 4.2(c), and Corollary 4.5(b) immediately imply the following.

Theorem 5.2 *Every matroid of rank three or corank three satisfies* $\sqrt{\mathrm{BLC}}$.

As seen in [3], there are many matroids of rank three that are not HPP matroids – the Fano and Pappus matroids are two familiar examples. Theorem 5.2 begins to explore situations in which the hypothesis of Theorem 1.3 does not apply, but something close to its conclusion does hold. Unfortunately Theorem 5.2 implies nothing new about Mason's Conjecture, which is trivial in rank three.

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