SIGNED WORDS AND PERMUTATIONS, II; THE EULER-MAHONIAN POLYNOMIALS

Dominique Foata

Institut Lothaire, 1, rue Murner
F-67000 Strasbourg, France
foata@math.u-strasbg.fr

Guo-Niu Han

I.R.M.A. UMR 7501, Université Louis Pasteur et CNRS
7, rue René-Descartes, F-67084 Strasbourg, France
guoniu@math.u-strasbg.fr

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Dans la théorie de Morse, quand on veut étudier un espace, on introduit une fonction numérique; puis on aplatit cet espace sur l'axe de la valeur de cette fonction. Dans cette opération d'aplatissement, on crée des singularités de la fonction et celles-ci sont en quelque sorte les vestiges de la topologie qu'on a tuée.
René Thom, *Logos et Théorie des catastrophes*, 1982.

Dedicated to Richard Stanley,
on the occasion of his sixtieth birthday.

Abstract

As for the symmetric group of ordinary permutations there is also a statistical study of the group of signed permutations, that consists of calculating multivariable generating functions for this group by statistics involving record values and the length function. Two approaches are here systematically explored, using the flag-major index on the one hand, and the flag-inversion number on the other hand. The MacMahon Verfahren appears as a powerful tool throughout.

1. Introduction

The elements of the hyperoctahedral group B_n $(n \geq 0)$, usually called *signed permutations*, may be viewed as words $w = x_1x_2 \ldots x_n$, where the letters x_i are positive or negative integers and where $|x_1|\,|x_2|\,\ldots\,|x_n|$ is a permutation of $1\,2\,\ldots\,n$ (see [Bo68] p. 252–253). For typographical reasons we shall use the notation $\bar{i} := -i$ in the sequel. Using the χ-notation that maps each statement A onto the value $\chi(A) = 1$ or 0 depending on whether A is true or not, we recall that the usual *inversion number*, $\operatorname{inv} w$, of the signed permutation $w = x_1x_2\dots x_n$ is defined by

$$\operatorname{inv} w := \sum_{1\le j\le n}\sum_{i<j}\chi(x_i > x_j).$$

It also makes sense to introduce

$$\overline{\operatorname{inv}}\, w := \sum_{1\le j\le n}\sum_{i<j}\chi(\overline{x}_i > x_j),$$

and verify that the *length function* (see [Bo68, p. 7], [Hu90, p. 12]), that will be denoted by "finv" (*flag-inversion number*) in the whole paper, can be defined, using the notation $\operatorname{neg} w := \sum_{1\le j\le n}\chi(x_j < 0)$, by

$$\operatorname{finv} w := \operatorname{inv} w + \overline{\operatorname{inv}}\, w + \operatorname{neg} w.$$

Another equivalent definition will be given in (7.1). The *flag-major index* "fmaj" and the *flag descent number* "fdes" were introduced by Adin and Roichman [AR01] and read:

$$\operatorname{fmaj} w := 2\operatorname{maj} w + \operatorname{neg} w;$$
$$\operatorname{fdes} w := 2\operatorname{des} w + \chi(x_1 < 0);$$

where $\operatorname{maj} w := \sum_j j\,\chi(x_j > x_{j+1})$ denotes the usual *major index* of w and $\operatorname{des} w$ the *number of descents* $\operatorname{des} w := \sum_j \chi(x_j > x_{j+1})$.

Another class of statistics needed here is the class of *lower records*. A letter x_i $(1 \le i \le n)$ is said to be a *lower record* of the signed permutation $w = x_1x_2\dots x_n$, if $|x_i| < |x_j|$ for all j such that $i+1 \le j \le n$. When reading the lower records of w from left to right we get a *signed subword*, called the *lower record subword*, denoted by $\operatorname{Lower} w$. Denote the number of *positive* (resp. *negative*) letters in $\operatorname{Lower} w$ by $\operatorname{lowerp} w$ (resp. $\operatorname{lowern} w$).

In our previous paper [FoHa05] we gave the construction of a transformation Ψ on (arbitrary) signed words, that is, words, whose letters are positive or negative with repetitions allowed. When applied to the group B_n, the transformation Ψ has the following properties:

(a) $\operatorname{fmaj} w = \operatorname{finv} \Psi(w)$ for every signed permutation w;

(b) Ψ is a bijection of B_n onto itself, so that "fmaj" and "finv" are equidistributed over the hyperoctahedral group B_n;

(c) $\operatorname{Lower} w = \operatorname{Lower} \Psi(w)$, so that $\operatorname{lowerp} w = \operatorname{lowerp} \Psi(w)$ and $\operatorname{lowern} w = \operatorname{lowern} \Psi(w)$.

Actually, the transformation Ψ has stronger properties than those stated above, but these restrictive properties will suffice for the following derivation. Having properties (a)–(c) in mind, we see that the two three-variable statistics (fmaj, lowerp, lowern) and (finv, lowerp, lowern) are equidistributed over B_n. Hence, the two generating polynomials

$${}^{\mathrm{fmaj}}B_n(q,X,Y) := \sum_{w\in B_n} q^{\operatorname{fmaj}} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w}$$
$${}^{\mathrm{finv}}B_n(q,X,Y) := \sum_{w\in B_n} q^{\operatorname{finv}} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w}$$

are *identical*. To derive the analytical expression for the common polynomial we have two approaches, using the "fmaj" interpretation, on the one hand, and the "finv" geometry, on the other. In each case we will go beyond the three-variable case, as we consider the generating polynomial for the group B_n by the five-variable statistic $(\mathrm{fdes}, \mathrm{fmaj}, \mathrm{lowerp}, \mathrm{lowern}, \mathrm{neg})$

$$(1.1)\qquad {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) := \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} X^{\mathrm{lowerp}\,w} Y^{\mathrm{lowern}\,w} Z^{\mathrm{neg}\,w}$$

and the generating polynomial for the group B_n by the four-variable statistic $(\mathrm{finv}, \mathrm{lowerp}, \mathrm{lowern}, \mathrm{neg})$

$$(1.2)\qquad {}^{\mathrm{finv}}B_n(q,X,Y,Z) := \sum_{w\in B_n} q^{\mathrm{finv}\,w} X^{\mathrm{lowerp}\,w} Y^{\mathrm{lowern}\,w} Z^{\mathrm{neg}\,w}.$$

Using the usual notations for the *q-ascending factorial*

$$(1.3)\qquad (a;q)_n := \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\,\ldots\,(1-aq^{n-1}), & \text{if } n\geq 1;\end{cases}$$

in its *finite* form and

$$(1.4)\qquad (a;q)_\infty := \lim_n (a;q)_n = \prod_{n\geq 0}(1-aq^n);$$

in its *infinite* form, we consider the products

$$(1.5)\qquad H_\infty(u) := \frac{\Bigl(uq\Bigl(\dfrac{Z+q}{1-q^2}-ZY\Bigr);q^2\Bigr)_\infty}{\Bigl(u\Bigl(\dfrac{q(Z+q)}{1-q^2}+X\Bigr);q^2\Bigr)_\infty},$$

in its infinite version, and

$$(1.6)\ H_{2s}(u) := \frac{1-q^2}{1-q^2+uq^{2s+1}(Z+q)}\,\frac{\Bigl(uq\dfrac{Z+q-ZY(1-q^2)}{1-q^2+uq^{2s+1}(Z+q)};q^2\Bigr)_s}{\Bigl(u\dfrac{q(Z+q)+X(1-q^2)}{1-q^2+uq^{2s+1}(Z+q)};q^2\Bigr)_{s+1}},$$

as well as

$$(1.7)\qquad H_{2s+1}(u) := \frac{\Bigl(uq\dfrac{Z+q-ZY(1-q^2)}{1-q^2+uq^{2s+2}(Zq+1)};q^2\Bigr)_{s+1}}{\Bigl(u\dfrac{q(Z+q)+X(1-q^2)}{1-q^2+uq^{2s+2}(Zq+1)};q^2\Bigr)_{s+1}},$$

in its graded version under the form $\sum_{s\geq 0} t^s H_s(u)$.

The purpose of this paper is to prove the following two theorems and derive several applications regarding statistical distributions over B_n.

Theorem 1.1 (the "fmaj" approach). *Let* ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ *be the generating polynomial for the group* B_n *by the five-variable statistic* (fdes, fmaj, lowerp, lowern, neg) *as defined in* (1.1). *Then*

$$\sum_{n\ge 0}(1+t)\,{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\ge 0}t^s H_s(u), \tag{1.8}$$

where $H_s(u)$ *is the finite product introduced in* (1.6) *and* (1.7).

Theorem 1.2 (the "finv" approach). *Let* ${}^{\mathrm{finv}}B_n(q,X,Y,Z)$ *be the generating polynomial for the group* B_n *by the four-variable statistic*
(finv, lowerp, lowern, neg), *as defined in* (1.2). *Then*

$$\begin{aligned}{}^{\mathrm{finv}}B_n(q,X,Y,Z)&=(X+q+\cdots+q^{n-1}+q^nZ+\cdots+q^{2n-2}Z+q^{2n-1}YZ)\\ &\cdots\times(X+q+q^2+q^3Z+q^4Z+q^5YZ)(X+q+q^2Z+q^3YZ)(X+qYZ).\end{aligned} \tag{1.9}$$

The proofs of those two theorems are very different in nature. For proving Theorem 1.1 we re-adapt the *MacMahon Verfahren* to make it work for *signed* permutations. René Thom's quotation that appears as an epigraph to this paper illustrates the essence of the *MacMahon Verfahren*. The topology of the signed permutations measured by the various statistics, "fdes", "fmaj", ... must be reconstructed when the group of the signed permutations is mapped onto a set of plain words for which the calculation of the associated statistic is easy. There is then a combinatorial bijection between signed permutations and plain words that describes the "flattening" ("aplatissement") process. This is the content of Theorem 4.1.

Another approach might have been to make use of the P-partition technique introduced by Stanley [St72] and successfully employed by Reiner [Re93a, Re93b, Re93c, Re95a, Re95b] in his statistical study of the hyperoctahedral group.

Theorem 1.2 is based upon another definition of the length function for B_n (see formula (7.1)). Notice that in the two theorems we have included a variable Z, which takes the number "neg" of *negative* letters into account. This allows us to re-obtain the classical results on the symmetric group by letting $Z=0$.

In the next section we show that the infinite product $H_s(u)$ first appears as the generating function for a class of *plain* words by a four-variable statistic (see Theorem 2.2). This theorem will be an essential tool in section 4 in the MacMahon Verfahren for signed permutations to handle the five-variable polynomial ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$. Section 3 contains an axiomatic definition of the *Record-Signed-Euler-Mahonian Polynomials* $B_n(t,q,X,Y,Z)$. They are defined, not only by (1.8) (with B_n replacing ${}^{\mathrm{fmaj}}B_n$), but also by a recurrence relation. The proof of Theorem 1.1 using the MacMahon Verfahren is found in Section 4. In Section 5 we show how to prove that the polynomials ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ satisfy the same recurrence as the polynomials $B_n(t,q,X,Y,Z)$, using an insertion technique. The specializations of Theorem 1.1 are numerous and described in section 6. We end the paper with the proof of Theorem 1.2 and its specializations.

2. Lower Records on Words

As mentioned in the introduction, Theorem 2.2 below, dealing with *ordinary words*, appears to be a *preparation lemma* for Theorem 1.1, that takes the geometry of *signed permutations* into account. Consider an ordinary word $c = c_1c_2\dots c_n$, whose letters belong to the alphabet $\{0, 1, \dots, s\}$, that is, a word from the free monoid $\{0, 1, \dots, s\}^*$. A letter c_i $(1 \le i \le n)$ is said to be an *even lower record* (resp. *odd lower record*) of c, if c_i is even (resp. odd) and if $c_j \ge c_i$ (resp. $c_j > c_i$) for all j such that $1 \le j \le i-1$. Notice the discrepancy between even and odd letters. Also, to define those even and odd lower records for words the reading is made *from left to right*, while for signed permutations, the lower records are read from *right to left* (see Sections 1 and 4). We could have considered a totally ordered alphabet with two kinds of letters, but playing with the parity of the nonnegative integers is more convenient for our applications. For instance, the even (resp. odd) lower records of the word $c = \mathit{5}\,\mathbf{4}\,\mathbf{4}\,\mathit{1}\,5\,2\,1\,\mathbf{0}\,4\,\mathbf{0}\,3$ are reproduced in boldface (resp. in italic).

For each word c let $\mathrm{evenlower}\,c$ (resp. $\mathrm{oddlower}\,c$) be the number of even (resp. odd) lower records of c. Also let $\mathrm{tot}\,c$ ("tot" stands for "total") be the *sum* $c_1 + c_2 + \cdots + c_n$ of the letters of c and $\mathrm{odd}\,c$ be the *number* of its odd letters. Also denote its *length* by $|c|$ and let $|c|_k$ be the number of letters in c equal to k. Our purpose is to calculate the generating function for $\{0, 1, \dots, s\}^*$ by the four-variable statistic $(\mathrm{tot}, \mathrm{evenlower}, \mathrm{oddlower}, \mathrm{odd})$.

Say that $c = c_1c_2\dots c_n$ is of *minimal index* k $(0 \le k \le s/2)$, if $\min c := \min\{c_1, \dots, c_n\}$ is equal to $2k$ or $2k+1$. Let c_j be the leftmost letter of c equal to $2k$ or $2k+1$. Then, c admits a unique factorization

$$c = c' c_j c'', \tag{2.1}$$

having the following properties:

$$c' \in \{2k+2, 2k+3, \dots, s\}^*,\ c_j = 2k \text{ or } 2k+1,\ c'' \in \{2k, 2k+1, \dots, s\}^*.$$

With the forementioned example we have the factorization $c' = 5\,4\,4$, $c_j = 1$, $c'' = 5\,2\,1\,0\,4\,0\,3$. In this example notice that $c_j = 1 \ne \min c = 0$.

Lemma 2.1. *The numbers of even and odd lower records of a word c can be calculated by induction as follows:* $\mathrm{evenlower}\,c = \mathrm{oddlower}\,c := 0$ *if c is empty; otherwise, let $c = c'c_jc''$ be its minimal index factorization (defined in* (2.1)*). Then*

$$\mathrm{evenlower}\,c = \mathrm{evenlower}\,c' + \chi(c_j = 2k) + |c''|_{2k}; \tag{2.2}$$

$$\mathrm{oddlower}\,c = \mathrm{oddlower}\,c' + \chi(c_j = 2k+1). \tag{2.3}$$

Proof. Keep the same notations as in (2.1). If $c_j = 2k$, then c_j is an even lower record, as well as all the letters equal to $2k$ to the right of c_j. On the other hand, there is no even lower record equal to $2k$ to the left of c_j, so that (2.2) holds. If $c_j = 2k+1$, then c_j is an odd lower record and there is no odd lower record equal to $2k+1$ to the right of c_j. Moreover, there is no odd lower record to the left of c_j equal to c_j. Again (2.3) holds. ∎

It is straightforward to verify that the fraction $H_s(u)$ displayed in (1.6) and (1.7) can also be expressed as

$$H_{2s}(u) = \prod_{0\le k\le s} \frac{1-u([q^{2k+1}(1-Y)Z+q^{2k+2}+\cdots+q^{2s-2}+q^{2s-1}Z+q^{2s}])}{1-u(q^{2k}X+[q^{2k+1}Z+q^{2k+2}+\cdots+q^{2s-2}+q^{2s-1}Z+q^{2s}])} \tag{2.4}$$

$$H_{2s+1}(u) = \prod_{0\le k\le s} \frac{1-u(q^{2k+1}(1-Y)Z+[q^{2k+2}+\cdots+q^{2s}+q^{2s+1}Z])}{1-u(q^{2k}X+q^{2k+1}Z+[q^{2k+2}+\cdots+q^{2s}+q^{2s+1}Z])}, \tag{2.5}$$

where the expression between brackets vanishes whenever $k=s$, and that the $H_s(u)$'s satisfy the recurrence formula

$$H_0(u) = \frac{1}{1-uX}; \quad H_1(u) = \frac{1-uqZ(1-Y)}{1-u(X+qZ)}; \quad \text{and for } s\ge 1 \tag{2.6}$$

$$H_{2s}(u) = \frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})} H_{2s-2}(uq^2);$$

$$H_{2s+1}(u) = \frac{1-u(q(1-Y)Z+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}{1-u(X+qZ+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)} H_{2s-1}(uq^2).$$

Theorem 2.2. *The generating function for the free monoid* $\{0,1,\ldots,s\}^*$ *by the four-variable statistic* $(\mathrm{tot}, \mathrm{evenlower}, \mathrm{oddlower}, \mathrm{odd})$ *is equal to* $H_s(u)$*, that is to say,*

$$\sum_{c\in\{0,1,\ldots,s\}^*} u^{|c|} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c} = H_s(u). \tag{2.7}$$

Proof. Let $H_s^*(u)$ denote the left-hand side of (2.7). Then,

$$H_0^*(u) = \sum_{c\in\{0\}^*} u^{|c|} q^0 X^{|c|} Y^0 Z^0 = \frac{1}{1-uX}.$$

When $s=1$ the minimal index factorization of each nonempty word c reads $c=c_j c''$, so that

$$\begin{aligned} H_1^*(u) &= 1+u(X+qYZ) \sum_{c''\in\{0,1\}^*} u^{|c''|} q^{|c''|_1} X^{|c''|_0} Y^0 Z^{|c''|_1} \\ &= 1+u(X+qYZ)\frac{1}{1-u(X+qZ)} = \frac{1-uqZ(1-Y)}{1-u(X+qZ)}. \end{aligned}$$

Consequently, $H_s^*(u)=H_s(u)$ for $s=0,1$. For $s\ge 2$ we write

$$H_s^*(u) = \sum_{0\le k\le s/2} H_{s,k}^*(u)$$

with

$$H_{s,k}^*(u) := \sum_{\substack{c\in\{0,1,\ldots,s\}^* \\ \min_i c_i = 2k \text{ or } 2k+1}} u^{|c|} q^{\mathrm{tot}\, c} X^{\mathrm{evenlower}\, c} Y^{\mathrm{oddlower}\, c} Z^{\mathrm{odd}\, c}.$$

From Lemma 2.1 it follows that

$$\begin{aligned}
H^*_{s,0}(u) &= \sum_{c'\in\{2,\dots,s\}^*} u^{|c'|}q^{\operatorname{tot} c'}X^{\operatorname{evenlower} c'}Y^{\operatorname{oddlower} c'}Z^{\operatorname{odd} c'} \times u(X+qYZ)\\
&\qquad\qquad \times \sum_{c''\in\{0,\dots,s\}^*} u^{|c''|}q^{\operatorname{tot} c''}X^{|c''|_0}Z^{\operatorname{odd} c''}\\
&= \sum_{c'\in\{0,\dots,s-2\}^*}(uq^2)^{|c'|}q^{\operatorname{tot} c'}X^{\operatorname{evenlower} c'}Y^{\operatorname{oddlower} c'}Z^{\operatorname{odd} c'}\times u(X+qYZ)\\
&\quad \times \sum_{c''\in\{0,\dots,s\}^*}(uX)^{|c''|_0}(uqZ)^{|c''|_1}(uq^2)^{|c''|_2}(uq^3Z)^{|c''|_3}(uq^4)^{|c''|_4}\cdots\\
&= H^*_{s-2}(uq^2)u(X+qYZ)\frac{1}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)},
\end{aligned}$$

the polynomial in the denominator ending with $\cdots+q^{s-1}Z+q^s$ or $\cdots+q^{s-1}+q^sZ$ depending on whether s is even or odd.

On the other hand,

$$\begin{aligned}
\sum_{1\le k\le s/2}H^*_{s,k}(u) &= \sum_{c\in\{2,3,\dots,s\}^*} u^{|c|}q^{\operatorname{tot} c}X^{\operatorname{evenlower} c}Y^{\operatorname{oddlower} c}Z^{\operatorname{odd} c}\\
&= \sum_{c\in\{0,1,\dots,s-2\}^*}(uq^2)^{|c|}q^{\operatorname{tot} c}X^{\operatorname{evenlower} c}Y^{\operatorname{oddlower} c}Z^{\operatorname{odd} c}\\
&= H^*_{s-2}(uq^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
H^*_s(u) &= \Bigl(1+\frac{u(X+qYZ)}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)}\Bigr)H^*_{s-2}(uq^2)\\
&= \frac{1-u(qZ(1-Y)+q^2+q^3Z+q^4+\cdots)}{1-u(X+qZ+q^2+q^3Z+q^4+\cdots)}H^*_{s-2}(uq^2).
\end{aligned}$$

As the fractions $H^*_s(u)$ satisfy the same induction relation as the $H_s(u)$'s, we conclude that $H^*_s(u)=H_s(u)$ for all s. ∎

When s tends to infinity, then $H_s(u)$ tends to $H_\infty(u)$, whose expression is shown in (1.5). In particular, we have the identity:

$$\sum_{c\in\{0,1,2,\dots\}^*} u^{|c|}q^{\operatorname{tot} c}X^{\operatorname{evenlower} c}Y^{\operatorname{oddlower} c}Z^{\operatorname{odd} c} = H_\infty(u). \tag{2.8}$$

3. The Record-Signed-Euler-Mahonian Polynomials

Our next step is to form the series $\sum_{s\ge0}t^sH_s(u)$ and show that the series can be expanded as a series in the variable u in the form

$$\sum_{n\ge0}C_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\ge0}t^sH_s(u), \tag{3.1}$$

where $B_n(t,q,X,Y,Z):=C_n(t,q,X,Y,Z)/(1+t)$ is a *polynomial* with nonnegative integral coefficients such that $B_n(1,1,1,1,1)=2^nn!$.

Definition. A sequence $\Big(B_n(t,q,X,Y,Z) = \sum_{k\ge 0} t^k B_{n,k}(q,X,Y,Z)\Big)$ $(n \ge 0)$ of polynomials in five variables t, q, X, Y and Z is said to be *record-signed-Euler-Mahonian*, if one of the following *equivalent* three conditions holds:

(1) The (t^2,q^2)-factorial generating function for the polynomials

$$C_n(t,q,X,Y,Z) := (1+t)B_n(t,q,X,Y,Z) \tag{3.2}$$

is given by identity (3.1).

(2) For $n \ge 2$ the recurrence relation holds:

$$\begin{aligned}
(3.3)\ (1-q^2)&B_n(t,q,X,Y,Z)\\
=&\Big(X(1-q^2)+(Zq+q^2)(1-t^2q^{2n-2})+t^2q^{2n-1}(1-q^2)ZY\Big)B_{n-1}(t,q,X,Y,Z)\\
&-\frac{1}{2}(1-t)q(1+q)(1+tq)(1+Z)B_{n-1}(tq,q,X,Y,Z)\\
&+\frac{1}{2}(1-t)q(1-q)(1-tq)(1-Z)B_{n-1}(-tq,q,X,Y,Z),
\end{aligned}$$

while $B_0(t,q,X,Y,Z)=1$, $\quad B_1(t,q,X,Y,Z)=X+tqYZ$.

(3) The recurrence relation holds for the coefficients $B_{n,k}(q,X,Y,Z)$:

$$\begin{gathered}
B_{0,0}(q,X,Y,Z)=1,\quad B_{0,k}(q,X,Y,Z)=0 \text{ for all } k\ne 0;\\
B_{1,0}(q,X,Y,Z)=X,\quad B_{1,1}(q,X,Y,Z)=qYZ,\\
B_{1,k}(q,X,Y,Z)=0 \text{ for all } k\ne 0,1;
\end{gathered} \tag{3.4}$$

$$\begin{aligned}
B_{n,2k}(q,X,Y,Z) &= (X+qZ+q^2+q^3Z+\cdots+q^{2k})B_{n-1,2k}(q,X,Y,Z)\\
&\quad + q^{2k}B_{n-1,2k-1}(q,X,Y,Z)\\
&\quad + (q^{2k}+q^{2k+1}Z+\cdots+q^{2n-1}YZ)B_{n-1,2k-2}(q,X,Y,Z),\\
B_{n,2k+1}(q,X,Y,Z) &= (X+qZ+q^2+\cdots+q^{2k}+q^{2k+1}Z)B_{n-1,2k+1}(q,X,Y,Z)\\
&\quad + q^{2k+1}ZB_{n-1,2k}(q,X,Y,Z)\\
&\quad + (q^{2k+1}Z+q^{2k+2}+\cdots+q^{2n-2}+q^{2n-1}YZ)B_{n-1,2k-1}(q,X,Y,Z),
\end{aligned}$$

for $n\ge 2$ and $0\le 2k+1\le 2n-1$.

Theorem 3.1. *The conditions* (1), (2) *and* (3) *in the previous definition are equivalent.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) requires a lengthy but elementary algebraic argument and will be omitted. The other equivalence (1) $\Leftrightarrow$ (2) involves a more elaborate q-series technique, which is now developed. Let $G_s(u) := H_s(u^2)$; then

$$G_0(u)=\frac{1}{1-u^2X};\quad G_1(u)=\frac{1-u^2qZ(1-Y)}{1-u^2(X+qZ)};$$

and by (2.6)

$$G_{2s}(u) = \frac{1-u^2(qZ(1-Y)+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}{1-u^2(X+qZ+q^2+q^3Z+\cdots+q^{2s-1}Z+q^{2s})}G_{2s-2}(uq),$$

$$G_{2s+1}(u) = \frac{1-u^2(qZ(1-Y)+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}{1-u^2(X+qZ+q^2+q^3Z+\cdots+q^{2s}+q^{2s+1}Z)}G_{2s-1}(uq),$$

for $s \geq 1$. Working with the series $\sum_{s\geq 0} t^s G_s(u)$ we obtain

$$\begin{aligned}
&\sum_{s\geq 0} t^{2s}G_{2s}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s}}{1-q^2}(Zq+q^2)\big)\Big)\\
&\qquad+\sum_{s\geq 0} t^{2s+1}G_{2s+1}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s+2}}{1-q^2}(Zq+1)\big)\Big)\\
&\quad=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{s\geq 1} t^{2s}G_{2s-2}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s}}{1-q^2}(Zq+q^2)\big)\Big)\\
&\qquad+\sum_{s\geq 1} t^{2s+1}G_{2s-1}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}-\frac{q^{2s+2}}{1-q^2}(Zq+1)\big)\Big),
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
&\sum_{s\geq 0} t^{s}G_{s}(u)\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}\big)\Big)=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{s\geq 0} t^{s+2}G_{s}(qu)\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}\big)\Big)\\
&\qquad-\sum_{s\geq 0}(tq)^{2s}\big(G_{2s}(u)-t^2q^2G_{2s}(qu)\big)u^2\frac{Zq+q^2}{1-q^2}\\
&\qquad-\sum_{s\geq 0}(tq)^{2s+1}\big(G_{2s+1}(u)-t^2q^2G_{2s+1}(qu)\big)u^2q\frac{Zq+1}{1-q^2}.
\end{aligned}$$

Now let $\sum_{n\geq 0} b_n(t)u^{2n} := \sum_{s\geq 0} t^s G_s(u)$. This gives:

$$\begin{aligned}
&\sum_{n\geq 0} b_n(t)u^{2n}\Big(1-u^2\big(X+\frac{Zq+q^2}{1-q^2}\big)\Big)=1+t(1-u^2qZ(1-Y))\\
&\qquad+\sum_{n\geq 0} b_n(t)t^2q^{2n}u^{2n}\Big(1-u^2\big(-qZY+\frac{Zq+q^2}{1-q^2}\big)\Big)\\
&\qquad-\sum_{n\geq 0}\frac{b_n(tq)+b_n(-tq)}{2}(1-t^2q^{2n+2})u^{2n+2}\frac{Zq+q^2}{1-q^2}\\
&\qquad-\sum_{n\geq 0}\frac{b_n(tq)-b_n(-tq)}{2}(1-t^2q^{2n+2})u^{2n+2}q\frac{Zq+1}{1-q^2}.
\end{aligned}$$

We then have $b_0(t)=\dfrac{1}{1-t}$, $\quad b_1(t)=\dfrac{X+tqYZ}{(1-t)(1-t^2q^2)}$ and for $n\ge 2$

$$\begin{aligned}b_n(t)(1-t^2q^{2n})&=\Bigl(X+\frac{Zq+q^2}{1-q^2}+t^2q^{2n-1}ZY-t^2q^{2n-2}\frac{Zq+q^2}{1-q^2}\Bigr)b_{n-1}(t)\\&\quad-\frac{b_{n-1}(tq)}{2(1-q^2)}(1-t^2q^{2n})q(1+q)(1+Z)+\frac{b_{n-1}(-tq)}{2(1-q^2)}(1-t^2q^{2n})q(1-q)(1-Z).\end{aligned}$$

Because of the presence of the factors of the form $(1-t^2q^{2n})$ we are led to introduce the coefficients $C_n(t,q,X,Y,Z):=b_n(t)(t^2;q^2)_{n+1}$ $(n\ge 0)$. By multiplying the latter equation by $(t^2;q^2)_n$ we get for $n\ge 2$

$$\begin{aligned}(3.5)\ (1-q^2)&C_n(t,q,X,Y,Z)\\=&\Bigl(X(1-q^2)+(Zq+q^2)(1-t^2q^{2n-2})+t^2q^{2n-1}(1-q^2)ZY\Bigr)C_{n-1}(t,q,X,Y,Z)\\&\qquad-\frac12(1-t^2)q(1+q)(1+Z)C_{n-1}(tq,q,X,Y,Z)\\&\qquad+\frac12(1-t^2)q(1-q)(1-Z)C_{n-1}(-tq,q,X,Y,Z),\end{aligned}$$

while $C_0(t,q,X,Y,Z)=1+t$, $\quad C_1(t,q,X,Y,Z)=(1+t)(X+tqYZ)$.

Finally, with $C_n(t,q,X,Y,Z):=(1+t)B_n(t,q,X,Y,Z)$ $(n\ge 0)$ we get the recurrence formula (3.3), knowing that the factorial generating function for the polynomials $C_n(t,q,X,Y,Z)=(1+t)B_n(t,q,X,Y,Z)$ is given by (3.1). As all the steps are perfectly reversible, the equivalence holds. $\square$

4. The MacMahon Verfahren

Now having three equivalent definitions for the record-signed-Euler-Mahonian polynomial $B_n(t,q,X,Y,Z)$, our next task is to prove the identity

$$(4.1)\qquad {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)=B_n(t,q,X,Y,Z).$$

Let $\mathbb{N}^n$ (resp. $\mathrm{NIW}(n)$) be the set of all the words (resp. all the nonincreasing words) of length n, whose letters are nonnegative integers. As we have seen in section 2 (Theorem 2.2), we know how to calculate the generating function for words by a certain four-variable statistic. The next step is to map each pair $(b,w)\in\mathrm{NIW}(n)\times B_n$ onto $c\in\mathbb{N}^n$ in such a way that the geometry on w can be derived from the latter statistic on c.

For the construction we proceed as follows. Write the signed permutation w as the linear word $w=x_1x_2\dots x_n$, where x_k is the image of the integer k $(1\le k\le n)$. For each $k=1,2,\dots,n$ let z_k be the number of descents in the right factor $x_kx_{k+1}\dots x_n$ and ϵ_k be equal to 0 or 1 depending on whether x_k is positive or negative. Next, form the words $z=z_1z_2\dots z_n$ and $\epsilon=\epsilon_1\epsilon_2\dots\epsilon_n$.

Now, take a nonincreasing word $b = b_1 b_2 \ldots b_n$ and define $a_k := b_k + z_k$, $c'_k := 2a_k + \epsilon_k$ $(1 \le k \le n)$, then $a := a_1 a_2 \ldots a_n$ and $c' := c'_1 c'_2 \ldots c'_n$. Finally, form the two-matrix $\begin{pmatrix} c'_1 & c'_2 & \ldots & c'_n \\ |x_1| & |x_2| & \ldots & |x_n| \end{pmatrix}$. Its bottom row is a permutation of $1\,2\,\ldots\,n$; rearrange the columns in such a way that the bottom row is precisely $1\,2\,\ldots\,n$. Then the word $c = c_1 c_2 \ldots c_n$ which corresponds to the pair (b, w) is defined to be the top row in the resulting matrix.

Example. Start with the pair (b, w) below and calculate all the necessary ingredients:

$$\begin{aligned} b &= 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ w &= 6\ \bar{5}\ \bar{4}\ 1\ 7\ \bar{3}\ \bar{2} \\ z &= 2\ 1\ 1\ 1\ 1\ 0\ 0 \\ \epsilon &= 0\ 1\ 1\ 0\ 0\ 1\ 1 \\ a &= 3\ 2\ 2\ 1\ 1\ 0\ 0 \\ c' &= 6\ 5\ 5\ 2\ 2\ 1\ 1 \\ c &= 2\ 1\ 1\ 5\ 5\ 6\ 2 \end{aligned}$$

Theorem 4.1. *For each nonnegative integer r the above mapping is a bijection of the set of all the pairs $(b, w) = (b_1 b_2 \ldots b_n,\ x_1 x_2 \ldots x_n) \in \mathrm{NIW}(n) \times B_n$ such that $2b_1 + \mathrm{fdes}\, w = r$ onto the set of the words $c = c_1 c_2 \ldots c_n \in \mathbb{N}^n$ such that $\max c = r$. Moreover,*

$$2b_1 + \mathrm{fdes}\, w = \max c; \quad 2\,\mathrm{tot}\, b + \mathrm{fmaj}\, w = \mathrm{tot}\, c; \tag{4.2}$$

$$\mathrm{lowerp}\, w = \mathrm{evenlower}\, c; \quad \mathrm{lowern}\, w = \mathrm{oddlower}\, c; \quad \mathrm{neg}\, w = \mathrm{odd}\, c. \tag{4.3}$$

Before giving the proof of Theorem 4.1 we derive its analytic consequences. First, it is q-routine to prove the three identities, where b_1 is the first letter of b,

$$\frac{1}{(u;q)_N} = \sum_{n \ge 0} \begin{bmatrix} N+n-1 \\ n \end{bmatrix}_q u^n; \tag{4.4}$$

$$\begin{bmatrix} N+n \\ n \end{bmatrix}_q = \sum_{b \in \mathrm{NIW}(N),\, b_1 \le n} q^{\mathrm{tot}\, b}; \tag{4.5}$$

$$\frac{1}{(u;q)_{N+1}} = \sum_{n \ge 0} u^n \sum_{b \in \mathrm{NIW}(N),\, b_1 \le n} q^{\mathrm{tot}\, b}. \tag{4.6}$$

We then consider

$$\frac{1+t}{(t^2;q^2)_{n+1}}\, {}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z),$$

where

$${}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) := \sum_{w \in B_n} t^{\mathrm{fdes}\, w} q^{\mathrm{fmaj}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} Z^{\mathrm{neg}\, w},$$

which we rewrite as

$$\begin{aligned}
&\sum_{r'\geq 0}(t^{2r'}+t^{2r'+1})\begin{bmatrix}n+r'\\ r'\end{bmatrix}_{q^2}{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) && \text{[by (4.4)]}\\
&=\sum_{r\geq 0}t^r\begin{bmatrix}n+\lfloor r/2\rfloor\\ \lfloor r/2\rfloor\end{bmatrix}_{q^2}{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z) && \text{[and by (4.5)]}\\
&=\sum_{r\geq 0}t^r\sum_{b\in \mathrm{NIW}(n),\,2b_1\leq r}q^{2\,\mathrm{tot}\,b}\sum_{w\in B_n}t^{\mathrm{fdes}\,w}q^{\mathrm{fmaj}\,w}X^{\mathrm{lowerp}\,w}Y^{\mathrm{lowern}\,w}Z^{\mathrm{neg}\,w}\\
&=\sum_{s\geq 0}t^s\sum_{\substack{b\in \mathrm{NIW}(n),\,w\in B_n\\ 2b_1+\mathrm{fdes}\,w\leq s}}q^{2\,\mathrm{tot}\,b+\mathrm{fmaj}\,w}X^{\mathrm{lowerp}\,w}Y^{\mathrm{lowern}\,w}Z^{\mathrm{neg}\,w}.
\end{aligned}$$

By Theorem 4.1 the set $\{(b,w): b\in \mathrm{NIW}(n),\, w\in B_n,\, 2b_1+\mathrm{fdes}\,w\leq s\}$ is in bijection with the set $\{0,1,\ldots,s\}^n$ and (4.2) and (4.3) hold. Hence,

$$\frac{1+t}{(t^2;q^2)_{n+1}}\,{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)=\sum_{s\geq 0}t^s\sum_{c\in\{0,\ldots,s\}^n}q^{\mathrm{tot}\,c}X^{\mathrm{evenlower}\,c}Y^{\mathrm{oddlower}\,c}Z^{\mathrm{odd}\,c}.$$

Now use Theorem 2.2:

$$\begin{aligned}
\sum_{s\geq 0}t^sH_s(u)&=\sum_{s\geq 0}t^s\sum_{c\in\{0,1,\ldots,s\}^*}u^{|c|}q^{\mathrm{tot}\,c}X^{\mathrm{evenlower}\,c}Y^{\mathrm{oddlower}\,c}Z^{\mathrm{odd}\,c}\\
&=\sum_{n\geq 0}u^n\sum_{s\geq 0}t^s\sum_{c\in\{0,1,\ldots,s\}^n}q^{\mathrm{tot}\,c}X^{\mathrm{evenlower}\,c}Y^{\mathrm{oddlower}\,c}Z^{\mathrm{odd}\,c}\\
&=\sum_{n\geq 0}(1+t)\,{}^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)\frac{u^n}{(t^2;q^2)_{n+1}}.
\end{aligned}$$

Thus, identity (4.1) is proved, as well as Theorem 1.1.

Let us now complete the proof of Theorem 4.1. First, $\max c=c_1'=2a_1+\epsilon_1=2b_1+2z_1+\epsilon_1=2b_1+\mathrm{fdes}\,w$; also $\mathrm{tot}\,c=\mathrm{tot}\,c'=2\,\mathrm{tot}\,a+\mathrm{tot}\,\epsilon=2\,\mathrm{tot}\,b+2\,\mathrm{tot}\,z+\mathrm{tot}\,\epsilon=2\,\mathrm{tot}\,b+\mathrm{fmaj}\,w$. Hence (4.2) holds.

Next, prove that $(b,w)\mapsto c$ is bijective. As both sequences b and z are nonincreasing, $a=b+z$ is also nonincreasing. If $x_k>x_{k+1}$, then $z_k=z_{k+1}+1$ and $a_k\geq a_{k+1}+1$, then $c_k'=2a_k+\epsilon_k\geq 2a_{k+1}+2+\epsilon_k\geq 2a_{k+1}+2>2a_{k+1}+\epsilon_{k+1}=c_{k+1}'$. Thus,

$$x_k>x_{k+1}\Rightarrow c_k'>c_{k+1}'. \tag{4.7}$$

To construct the reverse bijection we proceed as follows. Start with a sequence $c=c_1c_2\ldots c_n$; form the word $\delta=\delta_1\delta_2\ldots\delta_n$, where $\delta_i:=\chi(c_i\text{ even})-\chi(c_i\text{ odd})$ $(1\leq i\leq n)$ and the two-row matrix

$$\begin{pmatrix}c_1 & c_2 & \ldots & c_n\\ 1\delta_1 & 2\delta_2 & \ldots & n\delta_n\end{pmatrix}.$$

Rearrange the columns in such a way that $\binom{c_i}{i}$ occurs to the left of $\binom{c_j}{j}$, if either $c_i > c_j$, or $c_i = c_j$, $i < j$. The bottom of the new matrix is a *signed* permutation w. After those two transformations the new matrix reads

$$\begin{pmatrix} c' \\ w \end{pmatrix} = \begin{pmatrix} c'_1 & c'_2 & \dots & c'_n \\ x_1 & x_2 & \dots & x_n \end{pmatrix}.$$

The sequences ϵ and z are defined as above, as well as the sequence $a := (c' - \epsilon)/2$. As the sequence c' is nonincreasing, the inequality $c'_k - \epsilon_k \geq c'_{k+1} - \epsilon_{k+1}$ holds if $\epsilon_k = 0$ or if $\epsilon_k = \epsilon_{k+1} = 1$. It also holds when $\epsilon_k = 1$ and $\epsilon_{k+1} = 0$, because in such a case c'_k is odd and c'_{k+1} even and then $c'_k \geq 1 + c'_{k+1}$. Hence, a is also nonincreasing.

Finally, define $b := a - z$. Because of (4.7) we have $z_k = z_{k+1} + 1 \Rightarrow c'_k > c'_{k+1}$. If c'_k and c'_{k+1} are of the same parity, then $c'_k > c'_{k+1} \Rightarrow a_k > a_{k+1}$. If $c'_k > c'_{k+1}$ holds and the two terms are of different parity, then c'_k is even and c'_{k+1} odd. Hence, $a_k = c'_k/2 > (c'_{k+1} - \epsilon_{k+1})/2 = a_{k+1}$. Thus $z_k = z_{k+1} + 1 \Rightarrow a_k > a_{k+1}$. As $z_n = 0$, we conclude by a decreasing induction that $b_k = a_k - z_k \geq a_{k+1} - z_{k+1} = b_{k+1}$, so that b is a nonincreasing sequence of nonnegative integers.

There remains to prove (4.3). First, the letter c_j of c is odd, if and only if j occurs with the minus sign in w, so that $\operatorname{neg} w = \operatorname{odd} c$. Next, suppose that x_j is a positive lower record of $w = x_1 x_2 \dots x_n$. For $j < i$ we have $x_j < |x_i|$. Hence, the following implications hold: $|x_i| < x_j \Rightarrow i < j \Rightarrow c'_i \geq c'_j \Rightarrow c_{|x_j|} \leq c_{|x_i|}$ and c_{x_i} is an even lower record of c. If x_j is a negative lower record of w, then $j < i \Rightarrow -x_j < |x_i|$, so that $|x_i| < -x_j \Rightarrow i < j \Rightarrow c'_i > c'_j \Rightarrow c_{|x_j|} < c_{|x_i|}$ and $c_{|x_i|}$ is an odd lower record of c. □

5. The Insertion Method

Another method for proving identity (4.1) is to make use of the insertion method. Each signed permutation $w' = x'_1 \dots x'_{n-1}$ of order $(n-1)$ gives rise to $2n$ signed permutations of order n when n or $-n$ is inserted to the left or to the right w', or between two letters of w'. Assuming that w' has a flag descent number equal to $\operatorname{fdes} w' = k$, our duty is then to watch how the statistics "fmaj", "lowerp", "lowern", "neg" are modified after the insertion of n or $-n$ into the possible n slots. Such a method has already been used by Adin et al. [ABR01], Chow and Gessel [ChGe04], Haglund et al. [HLR04], for "fmaj" only. They all have observed that for each $j = 0, 1, \dots, 2n-1$ there is one and only one signed permutation of order n derived by the insertion of n or $-n$ whose flag-major index is increased by j.

In our case we observe that the number of positive (resp. negative) lower records remains alike, except when n (resp. $-n$) is inserted to the right of w', where it increases by 1. The number of negative letters increases only when $-n$ is inserted. For controlling "fdes" we make a distinction between the signed permutations having an even flag descent number and those having an odd one. For the former ones the first letter is positive. When n (resp. $-n$) is inserted to the left of w', the flag descent number increases by 2 (resp. by 1). For the latter ones the first letter is negative. When n (resp. $-n$) is inserted to the left of w', the flag descent number increases by 1 (resp. remains invariant).

For $n\ge 2$, $0\le k\le 2n-1$ let

$$^{\mathrm{fmaj}}B_{n,k} := \sum_{w\in B_n,\ \mathrm{fdes}\, w=k} q^{\mathrm{fmaj}\, w} X^{\mathrm{lowerp}\, w} Y^{\mathrm{lowern}\, w} Z^{\mathrm{neg}\, w},$$

and $^{\mathrm{fmaj}}B_{0,0}:=1$, $^{\mathrm{fmaj}}B_{0,k}:=0$ when $k\ne 0$; $^{\mathrm{fmaj}}B_{1,0}=X$, $^{\mathrm{fmaj}}B_{1,1}=qYZ$. Making use of the observations above we easily see that the polynomials $^{\mathrm{fmaj}}B_{n,k}$ satisfy the same recurrence relation, displayed in (3.4), as the $B_{n,k}(q,Y,Y,Z)$.

Remark. How to compare the MacMahon Verfahren and the insertion method? Recurrence relations may be difficult to be truly verified. The first procedure has the advantage of giving both a new identity on ordinary words (Theorem 2.1) and a closed expression for the factorial generating function for the polynomials $^{\mathrm{fmaj}}B_n(t,q,X,Y,Z)$ (formula (1.8)).

6. Specializations

When a variable has been deleted in the following specializations of the polynomials $B_n(t,q,X,Y,Z)$, this means that the variable has been given the value 1. For instance, $B_n(q,X,Y,Z):=B_n(1,q,X,Y,Z)$.

When s tends to infinity, then $H_s(u)$ tends to $H_\infty(u)$ given in (1.1). Hence, when identity (3.1) is multiplied by $(1-t)$ and t is replaced by 1, identity (3.1) specializes into

$$\sum_{n\ge 0} B_n(q,X,Y,Z)\frac{u^n}{(q^2;q^2)_n} = H_\infty(u) = \frac{\Bigl(uq\Bigl(\dfrac{Z+q}{1-q^2}-ZY\Bigr);q^2\Bigr)_\infty}{\Bigl(u\Bigl(\dfrac{q(Z+q)}{1-q^2}+X\Bigr);q^2\Bigr)_\infty}. \tag{6.1}$$

Now expand $H_\infty(u)$ by means of the q-binomial theorem [GaRa90, chap. 1]. We get:

$$H_\infty(u) = \sum_{n\ge 0}\left(\frac{q\Bigl(\dfrac{Z+q}{1-q^2}-YZ\Bigr)}{\dfrac{q(Z+q)}{1-q^2}+X};q^2\right)_n \Bigl(u\Bigl(\frac{q(Z+q)}{1-q^2}+X\Bigr)\Bigr)^n \Big/ (q^2;q^2)_n.$$

By identification

$$B_n(q,X,Y,Z) = \left(\frac{q\Bigl(\dfrac{Z+q}{1-q^2}-YZ\Bigr)}{\dfrac{q(Z+q)}{1-q^2}+X};q^2\right)_n \Bigl(\frac{q(Z+q)}{1-q^2}+X\Bigr)^n,$$

which can also be written as

$$\begin{aligned} B_n(q,X,Y,Z) &= (X+qZ+q^2+q^3Z+\cdots+q^{2n-2}+YZq^{2n-1}) \\ &\quad \cdots\times(X+qZ+q^2+q^3Z+q^4+q^5YZ)(X+qZ+q^2+q^3YZ)(X+qYZ), \end{aligned} \tag{6.2}$$

or, by induction,

$$B_n(q,X,Y,Z) = (X+qZ+q^2+q^3Z+\cdots+q^{2n-2}+YZq^{2n-1})B_{n-1}(q,X,Y,Z) \tag{6.3}$$

for $n\ge 2$ and $\quad B_1(q,X,Y,Z)=X+qYZ$.

In particular, the polynomial ${}^{\mathrm{fmaj}}B_n(q,X,Y)$ defined in the introduction is equal to

$$\begin{aligned}(6.4)\quad B_n(q,X,Y) &= (X+q+q^2+q^3+\cdots+q^{2n-2}+Yq^{2n-1})\\ &\cdots\times(X+q+q^2+q^3+q^4+q^5Y)(X+q+q^2+q^3Y)(X+qY).\end{aligned}$$

Finally, when $X=Y=Z:=1$, identity (6.4) reads

$$(6.5)\qquad \sum_{n\ge0}(1+t)B_n(t,q)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\ge0}t^s\frac{1}{1-u(1+q+q^2+\cdots+q^s)},$$

an identity derived by several authors (Adin et al. [ABR01], Chow & Gessel [ChGe04], Haglund et al. [HLR04]) with *ad hoc* methods. Also notice that for $X=Y=Z:=1$ formula (6.2) yields $B_n(q)=(q^2;q^2)_n/(1-q)^n$.

For $Z=0$ we get

$${}^{\mathrm{fmaj}}B_n(t,q,X,Y,0)=\sum_{w\in\mathfrak{S}_n}t^{\mathrm{fdes}\,w}q^{\mathrm{fmaj}\,w}X^{\mathrm{lowerp}\,w},$$

since the monomials corresponding to signed permutations having negative letters vanish. The summation is then over the *ordinary permutations*. Also notice that ${}^{\mathrm{fmaj}}B_n(t,q,X,Y,0)={}^{\mathrm{fmaj}}B_n(t,q,X,0,0)$. Moreover, for each ordinary permutation w we have: $\mathrm{fdes}\,w=2\,\mathrm{des}\,w$ and $\mathrm{fmaj}\,w=2\,\mathrm{maj}\,w$. When $Y=Z=0$ we also have

$$H_{2+1}(u)=H_{2s}(u)=\frac{\left(\dfrac{uq^2}{1-q^2+uq^{2(s+1)}};q^2\right)_{s+1}}{\left(u\dfrac{X(1-q^2)+q^2}{1-q^2+uq^{2(s+1)}};q^2\right)_{s+1}}$$

and

$$\sum_{n\ge0}(1+t)\ {}^{\mathrm{fmaj}}B_n(t,q,X,0,0)\frac{u^n}{(t^2;q^2)_{n+1}}=\sum_{s\ge0}(t^{2s}+t^{2s+1})H_{2s}(u).$$

With $A_n(t,q,X):={}^{\mathrm{fmaj}}B_n(t^{1/2},q^{1/2},X,0,0)=\sum\limits_{w\in\mathfrak{S}_n}t^{\mathrm{des}\,w}q^{\mathrm{maj}\,w}X^{\mathrm{lowerp}\,w}$ we obtain the identity

$$(6.7)\qquad \sum_{n\ge0}A_n(t,q,X)\frac{u^n}{(t;q)_{n+1}}=\sum_{s\ge0}t^s\frac{\left(\dfrac{uq}{1-q+uq^{s+1}};q\right)_{s+1}}{\left(u\dfrac{X(1-q)+q}{1-q+uq^{s+1}};q\right)_{s+1}},$$

apparently a new identity for the generating function for the symmetric groups $\mathfrak{S}_n$ by the three-variable statistic (des, maj, lowerp), as well as the following one obtained by multiplying the identity by $(1-t)$ and letting t go to infinity:

$$(6.8)\qquad \sum_{n\ge0}A_n(q,X)\frac{u^n}{(q;q)_n}=\frac{\left(\dfrac{uq}{1-q};q\right)_\infty}{\left(uX+\dfrac{uq}{1-q};q\right)_\infty}.$$

7. Lower Records and Flag-inversion Number

The purpose of this section is prove Theorem 1.2. We proceed as follows. For each (ordinary) permutation $\sigma=\sigma_1\sigma_2\dots\sigma_n$ of order n and for each $i=1,2,\dots,n$ let $b_i(\sigma)$ denote the number of letters σ_j *to the left of* σ_i such that $\sigma_j<\sigma_i$. On the other hand, for each *signed* permutation $w=x_1x_2\dots x_n$ of order n let $\operatorname{abs} w$ denote the (ordinary) permutation $|x_1|\,|x_2|\dots|x_n|$. It is straightforward to see that another expression for the flag-inversion number (or the length function), $\operatorname{finv} w$, of w is the following

$$\operatorname{finv} w=\operatorname{inv}\operatorname{abs} w+\sum_{1\le i\le n}(2b_i(\operatorname{abs} w)+1)\chi(x_i<0). \tag{7.1}$$

The generating polynomial ${}^{\operatorname{finv}}B_n(q,X,Y)$ can be derived as follows. Let $\operatorname{Lower}\sigma$ denote the set of the (necessarily positive) records of the ordinary permutation σ. By (7.1) we have

$$\begin{aligned}
{}^{\operatorname{finv}}B_n(q,X,Y,Z)&=\sum_{w\in B_n} q^{\operatorname{finv} w}X^{\operatorname{lowerp} w}Y^{\operatorname{lowern} w}Z^{\operatorname{neg} w}\\
&=\sum_{\sigma\in\mathfrak{S}_n}\sum_{\operatorname{abs} w=\sigma} q^{\operatorname{finv} w}X^{\operatorname{lowerp} w}Y^{\operatorname{lowern} w}Z^{\operatorname{neg} w}\\
&=\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{inv}\sigma}\prod_{\sigma_i\in\operatorname{Lower}\sigma}(X+YZq^{2b_i(\sigma)+1})\prod_{\sigma_i\notin\operatorname{Lower}\sigma}(1+Zq^{2b_i(\sigma)+1})\\
&:=\sum_{\sigma\in\mathfrak{S}_n} f(\sigma),
\end{aligned}$$

showing that ${}^{\operatorname{finv}}B_n(q,X,Y,Z)$ is simply a generating polynomial for the group $\mathfrak{S}_n$ itself.

But $\mathfrak{S}_n$ can be generated from $\mathfrak{S}_{n-1}$ by inserting the letter n into the possible n slots of each permutation of order $n-1$. Let $\sigma'=\sigma'_1\sigma'_2\dots\sigma'_{n-1}$ be such a permutation. Then

$$\begin{aligned}
f(\sigma'_1\sigma'_2\dots\sigma'_{n-2}\sigma'_{n-1}n)&=f(\sigma')(X+YZq^{2n-1});\\
f(\sigma'_1\sigma'_2\dots\sigma'_{n-2}n\sigma'_{n-1})&=f(\sigma')q(1+Zq^{2n-3});\\
\cdots\qquad&\qquad\cdots\\
f(\sigma'_1n\sigma'_2\dots\sigma'_{n-2}\sigma'_{n-1})&=f(\sigma')q^{n-2}(1+Zq^3);\\
f(n\sigma'_1\sigma'_2\dots\sigma'_{n-2}\sigma'_{n-1})&=f(\sigma')q^{n-1}(1+Zq).
\end{aligned}$$

Hence, ${}^{\operatorname{finv}}B_n(q,X,Y,Z)={}^{\operatorname{finv}}B_{n-1}(q,X,Y,Z)(X+q+\dots+q^{n-1}+q^nZ+\dots+q^{2n-2}Z+q^{2n-1}YZ)$ $(n\ge2)$. As ${}^{\operatorname{finv}}B_1(q,X,Y,Z)=X+qYZ$, we get the expression displayed in (1.9). □

Comparing (6.2) with (1.9) we see that the two polynomials ${}^{\operatorname{finv}}B_n(q,Z)$ and $B_n(q,Z)$ are different as soon as $n\ge2$, while ${}^{\operatorname{finv}}B_n(q,X,Y)=B_n(q,X,Y)$ for all n. This means that any bijection Ψ of the group B_n onto itself having the property that

$$\operatorname{fmaj} w=\operatorname{finv}\Psi(w)$$

does not leave the number of negative letters "neg" invariant. It was, in particular, the case for the bijection constructed in our previous paper [FoHa05].

Remark 1. Instead of considering lower records from right to left we can introduce lower records *from left to right*. Let $\operatorname{lowerp}' w$ and $\operatorname{lowern}' w$ denote the numbers of such records, positive and negative, respectively and introduce

$$^{\mathrm{finv}}B_n(q,X,Y,X',Y') := \sum_{w\in B_n} q^{\operatorname{finv} w} X^{\operatorname{lowerp} w} Y^{\operatorname{lowern} w} X'^{\operatorname{lowerp}' w} Y'^{\operatorname{lowern}' w}.$$

Using the method developed in the proof of the previous theorem we can calculate this polynomial in the form

$$\begin{aligned}
&^{\mathrm{finv}}B_n(q,X,Y,X',Y')\\
&\quad = (X+q+\cdots+q^{n-2}+q^{n-1}X'+q^nY'+q^{n+1}+\cdots+q^{2n-2}+q^{2n-1}Y)\\
&\qquad \cdots\times(X+q+q^2X'+q^3Y'+q^4+q^5Y)(X+qX'+q^2Y'+q^3Y)(XX'+qYY').
\end{aligned}$$

Remark 2. The same method may be used for ordinary permutations. For each permutation σ let $\operatorname{upper}\sigma$ denote the number of its *upper* records *from right to left* and define:

$$^{\mathrm{inv}}A_n(q,X,V) := \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{inv}\sigma} X^{\operatorname{lowerp}\sigma} V^{\operatorname{upper}\sigma}.$$

Then

$$^{\mathrm{inv}}A_n(q,X,V) = XV(X+qV)(X+q+q^2V)\cdots(X+q+q^2+\cdots+q^{n-1}V),$$

an identity which can be put into the form

$$\sum_{n\ge0} {}^{\mathrm{inv}}A_n(q,X,V)\frac{u^n}{(q;q)_n} = 1-\frac{XV}{X+V-1}+\frac{XV}{X+V-1}\frac{\Bigl(\dfrac{u}{1-q}-ux;q\Bigr)_\infty}{\Bigl(uX+\dfrac{uq}{1-q};q\Bigr)_\infty}, \tag{7.2}$$

which specializes into

$$\sum_{n\ge0} {}^{\mathrm{inv}}A_n(q,X)\frac{u^n}{(q;q)_n} = \frac{\Bigl(\dfrac{uq}{1-q};q\Bigr)_\infty}{\Bigl(uX+\dfrac{uq}{1-q};q\Bigr)_\infty}; \tag{7.3}$$

$$\sum_{n\ge0} {}^{\mathrm{inv}}A_n(q,V)\frac{u^n}{(q;q)_n} = \frac{\Bigl(\dfrac{u}{1-q}-uV;q\Bigr)_\infty}{\Bigl(\dfrac{u}{1-q};q\Bigr)_\infty}. \tag{7.4}$$

Comparing (7.3) with (6.8) we then see that $A_n(q,X) = {}^{\mathrm{inv}}A_n(q,X)$. In other words, the generating polynomial for $\mathfrak{S}_n$ by $(\operatorname{maj},\operatorname{lowerp})$ is the same as the generating polynomial by $(\operatorname{inv},\operatorname{lowerp})$. A combinatorial proof of this result is due to Björner and Wachs [BjW88], who have made use of the transformation constructed in [Fo68]. Finally, the expression of $^{\mathrm{inv}}A_n(q,X,V)$ for $q=1$ was derived by David and Barton ([DaBa62], chap. 10).

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