RSK Insertion for Set Partitions and Diagram Algebras

Tom Halverson* and Tim Lewandowski*

Department of Mathematics and Computer Science
Macalester College, Saint Paul, MN 55105 USA
halverson@macalester.edu   tim.lewandowski@gmail.com

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In honor of Richard Stanley on his 60th birthday.

Abstract

We give combinatorial proofs of two identities from the representation theory of the partition algebra $\mathbb{C}A_k(n), n \geq 2k$. The first is $n^k = \sum_\lambda f^\lambda m_k^\lambda$, where the sum is over partitions λ of n, f^λ is the number of standard tableaux of shape λ, and m_k^λ is the number of "vacillating tableaux" of shape λ and length $2k$. Our proof uses a combination of Robinson-Schensted-Knuth insertion and jeu de taquin. The second identity is $B(2k) = \sum_\lambda (m_k^\lambda)^2$, where $B(2k)$ is the number of set partitions of $\{1, \dots, 2k\}$. We show that this insertion restricts to work for the diagram algebras which appear as subalgebras of the partition algebra: the Brauer, Temperley-Lieb, planar partition, rook monoid, planar rook monoid, and symmetric group algebras.

1 Introduction

Two fundamental identities in the representation theory of the symmetric group S_k are

$$\text{(a)} \qquad n^k = \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} f^\lambda d_\lambda, \qquad \text{and} \qquad \text{(b)} \qquad k! = \sum_{\lambda \vdash k} (f^\lambda)^2, \tag{1.1}$$

where λ varies over partitions of the integer k of length $\ell(\lambda) \leq n$, f^λ is the number of standard Young tableaux of shape λ, and d_λ is the number of column strict tableaux of shape λ with entries from $\{1, \dots, n\}$. The Robinson-Schensted-Knuth (RSK) insertion algorithm provides a a bijection between sequences $(i_1, \dots, i_k), 1 \leq i_j \leq n$, and pairs (P_λ, Q_λ) consisting of a standard Young tableau P_λ of shape λ and a column strict tableau

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Q_λ of shape λ, thus providing a combinatorial proof of (1.1.a). If we restrict $i_1, \ldots, i_k$ to be a permutation of $1, \ldots, k$, then Q_λ is a standard tableau and we have a proof of (1.1.b).

Identity (1.1.b) comes from the decomposition of the group algebra $\mathbb{C}[S_k]$ into irreducible S_k-modules $V^\lambda, \lambda \vdash k$, where $\dim(V^\lambda) = f^\lambda$ and the multiplicity of V^λ in $\mathbb{C}[S_k]$ is also f^λ. Identity (1.1.a) comes from the Schur-Weyl duality between S_k and the general linear group $GL_n(\mathbb{C})$ on the k-fold tensor product $V^{\otimes k}$ of the fundamental representation V of $GL_n(\mathbb{C})$. There is an action of S_k on $V^{\otimes k}$ by tensor place permutations, and via this action, $\mathbb{C}[S_k]$ is isomorphic to the centralizer algebra $End_{GL_n(\mathbb{C})}(V^{\otimes k})$. As a bimodule for $S_k \times GL_n(\mathbb{C})$,

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} S^\lambda \otimes V^\lambda, \tag{1.2}$$

where S^λ is an irreducible S_k-module of dimension f^λ and V^λ is an irreducible $GL_r(\mathbb{C})$-module of dimension d_λ. We get (1.1.a) by computing dimensions on each side of (1.2).

Brauer [Br] defined an algebra $\mathbb{C}B_k(n)$, which is isomorphic to the centralizer algebra of the orthogonal group $O_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$, when $n \geq 2k$, i.e., $\mathbb{C}B_k(n) \cong End_{O_n(\mathbb{C})}(V^{\otimes k})$. The dimension of the Brauer algebra is $(2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$. Since $O_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$, their centralizers satisfy $\mathbb{C}B_k(n) \supseteq \mathbb{C}[S_k]$. Berele [Be] generalized the RSK correspondence to give combinatorial proof of the $\mathbb{C}B_k(n)$-analog of (1.1.a). Sundaram [Sun] (see also [Ter]) gave a combinatorial proof of the $\mathbb{C}B_k(n)$-analog of (1.1.a).

We now take this restriction further to $S_{n-1} \subseteq S_n \subseteq O_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$, where S_n is viewed as the subgroup of permutation matrices in $GL_n(\mathbb{C})$ and $S_{n-1} \subseteq S_n$ corresponds to the permutations that fix n. Under this restriction, V is the permutation representation of S_n, and when $n \geq 2k$, the centralizer algebras are the partition algebras,

$$\mathbb{C}A_k(n) \cong End_{S_n}(V^{\otimes k}) \quad \text{and} \quad \mathbb{C}A_{k+\frac{1}{2}}(n) \cong End_{S_{n-1}}(V^{\otimes k}).$$

The partition algebra $\mathbb{C}A_k(n)$ first appeared independently in the work of Martin [Mar1, Mar2, Mar3] and Jones [Jo] arising from applications in statistical mechanics. See [HR2] for a survey paper on partition algebras.

For $k \in \mathbb{Z}_{>0}$ and $n \geq 2k$, the $\mathbb{C}$-algebras $\mathbb{C}A_k(n)$ and $\mathbb{C}A_{k+\frac{1}{2}}(n)$ are semisimple with bases indexed by set partitions of $\{1, \ldots, 2k\}$ and $\{1, \ldots, 2k+1\}$, respectively. Thus, $\dim(\mathbb{C}A_k(n))) = B(2k)$ and $\dim(\mathbb{C}A_{k+\frac{1}{2}}(n))) = B(2k+1)$, where $B(\ell)$ is the ℓth Bell number. Define,

$$\Lambda_n^k = \left\{ \lambda \vdash n \,\middle|\, |\lambda| - \lambda_1 \leq k \right\}, \tag{1.3}$$

(these are partitions of n with at most k boxes below the first row of their Young diagram). Then the irreducible representations of $\mathbb{C}A_k(n)$ are indexed by partitions in the set Λ_n^k, and the irreducible representations of $\mathbb{C}A_{k+\frac{1}{2}}(n)$ are indexed by partitions in the set Λ_{n-1}^k.

Using the Schur-Weyl duality between S_n and $\mathbb{C}A_k(n)$ we get the identity

$$n^k = \sum_{\lambda \in \Lambda_n^k} f^\lambda m_k^\lambda, \tag{1.4}$$

where m_k^λ is the number of vacillating tableaux of shape λ and length $2k$ (defined in Section 2.2), which are sequences of integer partitions in the Bratteli diagram of $\mathbb{C}A_k(n)$. Identity (1.4) is the partition algebra analog of (1.1.a). In Section 3, we prove (1.4) using RSK column insertion and jeu de taquin. Decomposing $\mathbb{C}A_k(n)$ as a bimodule for $\mathbb{C}A_k(n) \otimes \mathbb{C}A_k(n)$ gives

$$B(2k) = \sum_\lambda (m_k^\lambda)^2, \tag{1.5}$$

which is the $\mathbb{C}A_k(n)$-analog of (1.1.b). In Section 4, we give a bijective proof of (1.5) that contains as a special cases the RSK algorithms for $\mathbb{C}S_k$ and $\mathbb{C}B_k(n)$.

Martin and Rollet [MR] have given a *different* combinatorial proof of the second identity (1.5). Their bijection has the elegant property that pairs of paths in the difference between the Bratteli diagrams of $\mathbb{C}A_k(\ell)$ and $\mathbb{C}A_k(\ell-1)$ are in exact correspondence with the set partitions of $\{1,\ldots,2k\}$ into ℓ parts. The advantages of the correspondence in this paper are:

1. Our algorithm for (1.5) contains, as special cases, the known RSK correspondences for a number of diagram algebras which appear as subalgebras of $\mathbb{C}A_k(n)$:

   (a) The group algebra of the symmetric group $\mathbb{C}S_k$,
   (b) The Brauer algebra $\mathbb{C}B_k(n)$,
   (c) The Temperley-Lieb algebra $\mathbb{C}T_k(n)$,
   (d) The planar partition algebra $\mathbb{C}P_k(n)$.
   (e) The rook monoid algebra $\mathbb{C}R_n$ and the planar rook monoid algebra $\mathbb{C}PR_n$.

   Thus we obtain combinatorial proofs of the analog of (1.5) for each of these algebras (see equations (5.1), (5.2), (5.3), and (5.4)).

2. Using Fomin growth diagrams we show that our algorithm for (1.5) is symmetric in the sense that if $d \to (P,Q)$ then $\mathrm{flip}(d) \mapsto (Q,P)$ where $\mathrm{flip}(d)$ is the diagram d flipped over its horizontal axis. This is the generalization of the property for the symmetric group that if $\pi \to (P,Q)$ then $\pi^{-1} \to (Q,P)$. As a consequence, we show that the number of symmetric diagrams equals the sum of the dimensions of the irreducible representations (for each of the diagram algebras mentioned in item 1 above).

3. Our algorithms for the bijections in (1.4) and (1.5) each use iterations of RSK insertion and jeu de taquin (see equations (3.4) and (4.4)).

The main idea for the algorithm in this paper came from a bijection of R. Stanley between fixed point free involutions in the symmetric group S_{2k} and Brauer diagrams. This led to the insertion scheme of Sundaram in [Sun] for the Brauer algebra. After we distributed a preliminary version of this paper, R. Stanley and colleagues independently came out with the paper [CDDSY], which studies the crossing and nesting properties of an extended version of the insertion used in this paper for the bijection in (1.5). We have adopted the term "vacillating tableaux" from [CDDSY], and we use the crossing property of [CDDSY] to show that our algorithm restricts appropriately to the planar partition algebra.

In Section 2, we give two equivalent notations for vacillating tableaux. We show that when $n \geq 2k$, there is a bijection between Λ_n^k and $\Gamma_k = \{\lambda \vdash t \mid 0 \leq t \leq k\}$ given by removing the first part of $\lambda \in \Lambda_n^k$ to produce $\lambda^* \in \Gamma_k$. We use two notations because the Λ_n^k notation is best suited to state and prove (1.4) and the Γ_k notation is best suited to state and prove (1.5). The vacillating tableaux in [CDDSY] are the same as those in this paper under the Γ_k-notation, however we use the term "vacillating tableaux" for both notations.

2 The Partition Algebra and Vacillating Tableaux

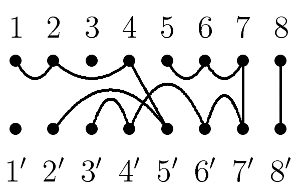
For $k \in \mathbb{Z}_{>0}$, let

$$A_k = \left\{\text{set partitions of } \{1, 2, \ldots, k, 1', 2', \ldots, k'\}\right\}, \qquad \text{and} \tag{2.1}$$
$$A_{k+\frac{1}{2}} = \left\{d \in \Pi_{k+1} \;\middle|\; (k+1) \text{ and } (k+1)' \text{ are in the same block}\right\}. \tag{2.2}$$

The propagating number of $d \in A_k$ is

$$\mathrm{pn}(d) = \begin{pmatrix} \text{the number of blocks in } d \text{ that contain both an element} \\ \text{of } \{1, 2, \ldots, k\} \text{ and an element of } \{1', 2', \ldots, k'\} \end{pmatrix}. \tag{2.3}$$

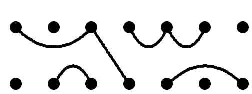
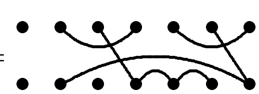
For convenience, represent a set partition $d \in A_k$ by a graph with k vertices in the top row, labeled $1, \ldots, k$, and k vertices in the bottom row, labeled $1', \ldots, k'$, with vertex i and vertex j connected by a path if i and j are in the same block of the set partition d. For example,



represents $\{\{1, 2, 4, 2', 5'\}, \{3\}, \{5, 6, 7, 3', 4', 6', 7'\}, \{8, 8'\}, \{1'\}\}$,

and has propagating number 3. The graph representing d is not unique.

We compose two partition diagrams d_1 and d_2 as follows. Place d_1 above d_2 and identify each vertex j' in the bottom row of d_1 with the corresponding vertex j in the top row of d_2. This new super diagram is a graph g on 3 rows of vertices. Remove any connected components of g that live entirely in the middle row. Then make a partition diagram $d_3 \in A_k$ so that two vertices in $d_3 \in A_k$ are in the same connected component if and only if the corresponding vertices in the top or bottom rows of g are in the same connected components. Then let $d_1 \circ d_2 = d_3$. For example, if

$d_1 =$  and $d_2 =$ 

then

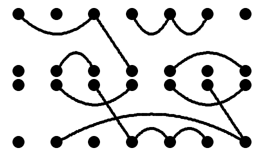
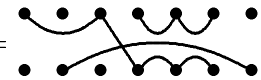
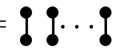
$$d_1 \circ d_2 =$$  $$=$$  .

Diagram multiplication makes A_k into an associative monoid with identity, $1 =$ , and the propagating number satisfies $\mathrm{pn}(d_1 \circ d_2) \leq \min(\mathrm{pn}(d_1), \mathrm{pn}(d_2))$.

For $k \in \frac{1}{2}\mathbb{Z}_{>0}$ and $n \in \mathbb{C}$, the partition algebra $\mathbb{C}A_k(n) = \mathbb{C}\text{span-}\{d \in A_k\}$ is an associative algebra over $\mathbb{C}$ with basis A_k. Multiplication in $\mathbb{C}A_k(n)$ is defined by

$$d_1 d_2 = n^\ell (d_1 \circ d_2),$$

where ℓ is the number of blocks removed from the the middle row when constructing the composition $d_1 \circ d_2$. In the example above $d_1 d_2 = n^2 d_1 \circ d_2$.

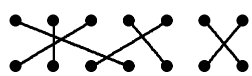
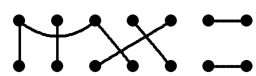
For each $k \in \mathbb{Z}_{>0}$, the following are submonoids of the partition monoid A_k:

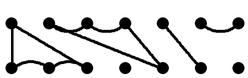
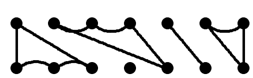
$$\begin{aligned}
S_k &= \{d \in A_k \mid \mathrm{pn}(d) = k\}, \qquad\qquad I_t = \{d \in A_k \mid \mathrm{pn}(d) \leq t\}, \quad 0 < t \leq k, \\
B_k &= \{d \in A_k \mid \text{all blocks of } d \text{ have size } 2\}, \\
R_k &= \left\{ d \in A_k \;\middle|\; \begin{array}{l} \text{all blocks of } d \text{ have at most one vertex in } \{1, \ldots k\} \\ \text{and at most one vertex in } \{1', \ldots k'\} \end{array} \right\}.
\end{aligned}$$

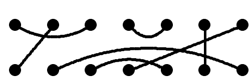
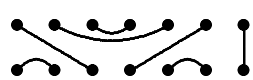
Here, S_k is the symmetric group, B_k is the Brauer monoid, and R_k is the rook monoid. A set partition is *planar* [Jo] if it can be represented as a graph without edge crossings inside of the rectangle formed by its vertices. The following are planar submonoids,

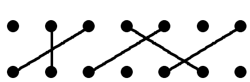
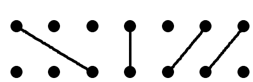
$$P_k = \{d \in A_k \mid d \text{ is planar}\}, \quad T_k = B_k \cap P_k, \quad PR_k = R_k \cap P_k, \quad 1 = S_k \cap P_k.$$

Here, T_k is the Temperley-Lieb monoid. Examples of diagrams in the various submonoids are:

 $\in S_7$.  $\in I_4$,

 $\in P_7$,  $\in P_{6+\frac{1}{2}}$,

 $\in B_7$,  $\in T_7$,

 $\in R_7$.  $\in PR_7$.

For each monoid, we make an associative algebra in the same way that we construct the partition algebra $\mathbb{C}A_k(n)$ from the partition monoid $A_k(n)$. For example, we obtain the the Brauer algebra $\mathbb{C}B_k(n)$, the Temperley-Lieb algebra $\mathbb{C}T_k(n)$, the group algebra of the symmetric group $\mathbb{C}S_k$ in this way. Multiplication in the rook monoid algebra $\mathbb{C}R_k$ is done without the coefficient n^ℓ (see [Ha]).

For $\ell \in \mathbb{Z}_{>0}$, the Bell number $B(\ell)$ is the number of set partitions of $\{1, 2, \ldots, \ell\}$, the Catalan number is $C(\ell) = \frac{1}{\ell+1}\binom{2\ell}{\ell} = \binom{2\ell}{\ell} - \binom{2\ell}{\ell+1}$, and $(2\ell)!! = (2\ell-1)\cdot(2\ell-3)\cdots 5\cdot 3\cdot 1$. These have generating functions (see [Sta, 1.24f, and 6.2]),

$$\sum_{\ell\geq 0} B(\ell)\frac{z^\ell}{\ell!} = \exp(e^z - 1), \qquad \sum_{\ell\geq 0} C(\ell-1)z^\ell = \frac{1-\sqrt{1-4z}}{2z},$$

and

$$\sum_{\ell\geq 0}(2(\ell-1))!!\frac{z^\ell}{\ell!} = \frac{1-\sqrt{1-2z}}{z}.$$

For $k \in \frac{1}{2}\mathbb{Z}_{>0}$, the monoids have cardinality

$$\mathrm{Card}(A_k) = B(2k), \qquad \mathrm{Card}(P_k) = \mathrm{Card}(T_{2k}) = C(2k), \qquad \text{for } k \in \tfrac{1}{2}\mathbb{Z}_{>0}$$

and

$$\begin{array}{lll} \mathrm{Card}(S_k) = k!, & \mathrm{Card}(B_k) = (2k)!!, & \\ \mathrm{Card}(R_k) = \displaystyle\sum_{\ell=0}^{k}\binom{k}{\ell}^2 \ell! & \mathrm{Card}(PR_k) = \dbinom{2k}{k}, & \text{for } k \in \mathbb{Z}_{>0}. \end{array}$$

2.1 Schur-Weyl Duality Between S_n and $\mathbb{C}A_k(n)$

The irreducible representations of S_n are indexed by integer partitions of n. If $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in (\mathbb{Z}_{\geq 0})^\ell$ with $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \cdots + \lambda_\ell = n$, then λ is a partition of n, denoted $\lambda \vdash n$. If $\lambda \vdash n$, then we write $|\lambda| = n$. If $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ are partitions such that $\mu_i \leq \lambda_i$ for each i, then we say that $\mu \subseteq \lambda$, and λ/μ is the skew shape given by deleting the boxes of μ from the Young diagram of λ.

Let V be the n-dimensional permutation representation of the symmetric group S_n. If we view $S_{n-1} \subseteq S_n$ as the subgroup of permutations that fix n, then V is isomorphic to the left coset representation $\mathbb{C}[S_n/S_{n-1}]$. Let $V^{\otimes k}$ be the k-fold tensor product representation of V, and let $V^{\otimes 0} = \mathbb{C}$. From the "tensor identity" (see for example [HR2]), we have the following restriction-induction rule rule for $i \geq 0$,

$$V^{\otimes(i+1)} \cong V^{\otimes i} \otimes \mathbb{C}[S_n/S_{n-1}] \cong \mathrm{Ind}_{S_{n-1}}^{S_n}(\mathrm{Res}_{S_{n-1}}^{S_n}(V^{\otimes i})).$$

Thus, $V^{\otimes k}$ is obtained from k iterations of restricting to S_{n-1} and inducing back to S_n.

Let V^λ denote the irreducible representations of S_n indexed by $\lambda \vdash n$. The restriction and induction rules for $S_{n-1} \subseteq S_n$ are given by

$$\mathrm{Res}_{S_{n-1}}^{S_n}(V^\lambda) \cong \bigoplus_{\mu\vdash(n-1),\, \mu\subseteq\lambda} V^\mu, \qquad \text{for } \lambda \vdash n \tag{2.4}$$

$$\mathrm{Ind}_{S_{n-1}}^{S_n}(V^\mu) \cong \bigoplus_{\lambda\vdash n,\, \mu\subseteq\lambda,} V^\lambda, \qquad \text{for } \mu \vdash (n-1). \tag{2.5}$$

In each case λ/μ consists of a single box. Starting with the trivial representation $V^{(n)} \cong \mathbb{C}$ and iterating the restriction (2.4) and induction (2.5) rules, we see that the irreducible S_n-representations that appear in $V^{\otimes k}$ are labeled by the partitions in

$$\Lambda_n^k = \left\{ \lambda \vdash n \,\middle|\, |\lambda| - \lambda_1 \leq k \right\}, \tag{2.6}$$

and the irreducible S_{n-1}-representations that appear in $V^{\otimes k}$ are labeled by the partitions in Λ_{n-1}^k.

There is an action of $\mathbb{C}A_k(n)$ on $V^{\otimes k}$ (see [Jo, MR, HR2]) that commutes with S_n and maps $\mathbb{C}A_k(n)$ surjectively onto the centralizer $End_{S_n}(V^{\otimes k})$. This generalizes the actions of $\mathbb{C}[S_k]$, by place permutations, and $B_k(n)$ on $V^{\otimes k}$. Furthermore, when $n \geq 2k$ we have

$$\mathbb{C}A_k(n) \cong End_{S_n}(V^{\otimes k}) \qquad \text{and} \qquad \mathbb{C}A_{k+\frac{1}{2}}(n) \cong End_{S_{n-1}}(V^{\otimes k}). \tag{2.7}$$

By convention, we let $\mathbb{C}A_0(n) = \mathbb{C}A_{\frac{1}{2}}(n) = \mathbb{C}$. Since anything that commutes with S_n on $V^{\otimes k}$ will also commute with S_{n-1}, we have $\mathbb{C}A_k(n) \subseteq \mathbb{C}A_{k+\frac{1}{2}}(n)$, and thus

$$\mathbb{C}A_0(n) \subseteq \mathbb{C}A_{\frac{1}{2}}(n) \subseteq \mathbb{C}A_1(n) \subseteq \mathbb{C}A_{1\frac{1}{2}}(n) \subseteq \cdots \subseteq \mathbb{C}A_{(k-\frac{1}{2}}(n) \subseteq \mathbb{C}A_k(n). \tag{2.8}$$

The *Bratteli diagram* for $\mathbb{C}A_k(n)$ consists of rows of vertices, with the rows labeled by $0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, k$, such that the vertices in row i are Λ_n^i and the vertices in row $i + \frac{1}{2}$ are Λ_{n-1}^i. Two vertices are connected by an edge if they are in consecutive rows and they differ by exactly one box. Figure 1 shows the Bratteli diagram for $\mathbb{C}A_k(6)$. Let $n \geq 2k$. By double centralizer theory (see, for example, [HR2]), we know that

(1) The irreducible representations of $\mathbb{C}A_k(n)$ can be indexed by Λ_n^k, so we let M_k^λ denote the irreducible $\mathbb{C}A_k(n)$ representation indexed by $\lambda \in \Lambda_n^k$.

(2) The decomposition of $V^{\otimes k}$ as an $S_n \times \mathbb{C}A_k(n)$-bimodule is given by

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_n^k} V^\lambda \otimes M_k^\lambda. \tag{2.9}$$

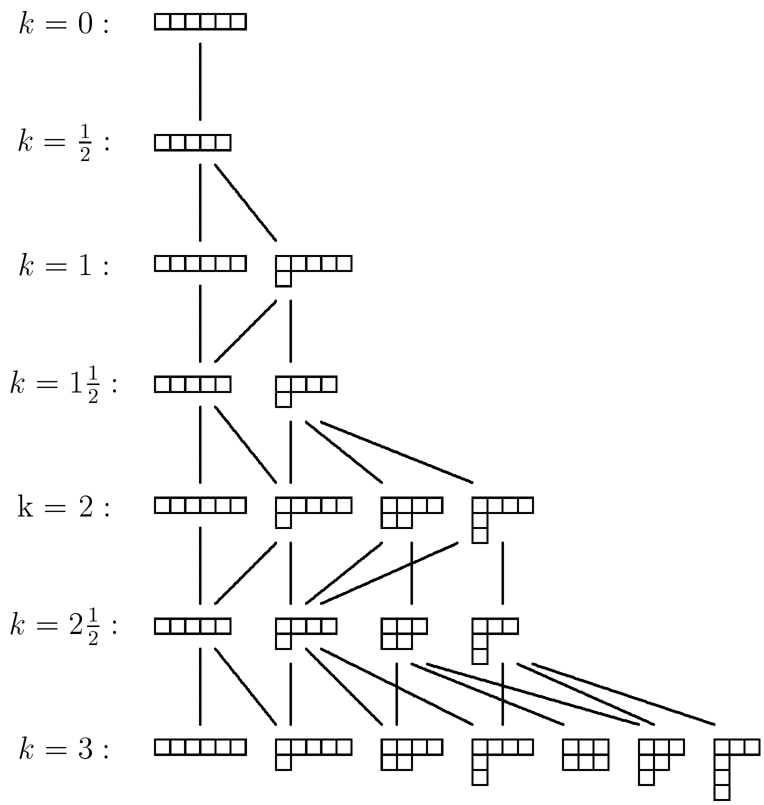
(3) The dimension of M_k^λ equals the multiplicity of V^λ in $V^{\otimes k}$. The edges in the Bratteli diagram exactly follow the restriction and induction rules in (2.4), and (2.5). and so

$$m_k^\lambda = \dim(M_k^\lambda) = \left\{ \begin{array}{l} \text{the number of paths from the top} \\ \text{of the Bratteli diagram to } \lambda \end{array} \right\}.$$

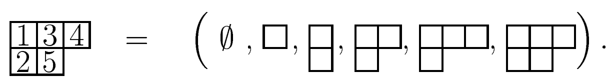
2.2 Vacillating Tableaux

The dimension of the irreducible S_n-module V^λ equals the number f^λ of standard tableaux of shape λ. A standard tableau of shape λ is a filling of the Young diagram of λ with the numbers $1, 2, \ldots, n$ in such a way that each number appears exactly once, the rows

Figure 1: Bratteli Diagram for $\mathbb{C}A_k(6)$



increase from left to right, and the columns increase from top to bottom. We can identify a standard tableaux T_λ of shape λ with a sequence $(\emptyset = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda)$ such that $|\lambda^{(i)}| = i$, $\lambda^{(i)} \subseteq \lambda^{(i+1)}$, and such that $\lambda^{(i)}/\lambda^{(i-1)}$ is the box containing i in T_λ. For example,



The sequence $(\emptyset = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda)$ is a path in Young's lattice, which is the Bratteli diagram for S_n. The number of standard tableaux f^λ can be computed using the hook formula (see [Sag, §3.10]). We let $\mathcal{SYT}(\lambda)$ denote the set of standard tableaux of shape λ.

Let $\lambda \in \Lambda_n^k$. A *vacillating tableaux of shape* λ *and length* $2k$ is a sequence of partitions,

$$\left((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \ldots, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)} = \lambda\right),$$

satisfying, for each i,

(1) $\lambda^{(i)} \in \Lambda_n^i$ and $\lambda^{(i+\frac{1}{2})} \in \Lambda_{n-1}^i$,

(2) $\lambda^{(i)} \supseteq \lambda^{(i+\frac{1}{2})}$ and $|\lambda^{(i)}/\lambda^{(i+\frac{1}{2})}| = 1$,

(3) $\lambda^{(i+\frac{1}{2})} \subseteq \lambda^{(i+1)}$ and $|\lambda^{(i+1)}/\lambda^{(i+\frac{1}{2})}| = 1$.

The vacillating tableaux of shape λ correspond exactly with paths from the top of the Bratteli diagram to λ. Thus, if we let $\mathcal{VT}_k(\lambda)$ denote the set of vacillating tableaux of shape λ and length k, then

$$m_k^\lambda = \dim(M_k^\lambda) = |\mathcal{VT}_k(\lambda)|. \tag{2.10}$$

Let $n \geq 2k$, and for a partition λ, define λ^* and $\bar{\lambda}$ as follows,

$$\begin{array}{llll} \text{if} & \lambda = (\lambda_1, \ldots, \lambda_\ell) \ \vdash \ n, & \text{then} & \lambda^* = (\lambda_2, \ldots, \lambda_\ell) \ \vdash \ (n - \lambda_1), \\ \text{if} & \lambda = (\lambda_1, \ldots, \lambda_\ell) \ \vdash \ s \leq k, & \text{then} & \bar{\lambda} = (n - s, \lambda_1, \ldots, \lambda_\ell) \ \vdash \ n. \end{array}$$

Since $n \geq 2k$ we are guaranteed that $\bar{\lambda}$ is a partition and that $0 \leq |\lambda^*| \leq k$. The sets

$$\Lambda_n^k = \left\{ \lambda \vdash n \;\middle|\; |\lambda^*| \leq k \right\} \qquad \text{and} \qquad \Gamma_k = \left\{ \lambda \vdash t \;\middle|\; 0 \leq t \leq k \right\} \tag{2.11}$$

are in bijection with one another using the maps,

$$\begin{array}{ccc} \Lambda_n^k & \to & \Gamma_k \\ \lambda & \mapsto & \lambda^* \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Gamma_k & \to & \Lambda_n^k \\ \lambda & \mapsto & \bar{\lambda} \end{array}. \tag{2.12}$$

Via these bijections, we can use either Γ_k or Λ_n^k to index the irreducible representations of $\mathbb{C}A_k(n)$. For example, the following sequences represent the same vacillating tableau P_λ, the first using diagrams from Λ_n^k and the second from Γ_k,

$$\begin{aligned} P_\lambda &= \left(\ydiagram{6}, \ydiagram{5}, \ydiagram{5,1}, \ydiagram{4,1}, \ydiagram{4,2}, \ydiagram{3,2}, \ydiagram{4,2} \right), \\ &= \left(\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2}, \right). \end{aligned}$$

For our bijection in Section 3 we will use Λ_n^k, and for our bijection in Section 4 we will use Γ_k. The Bratteli diagram for $\mathbb{C}A_k(n), n \geq 2k$, is given in Figure 2, using labels from Γ_k, along with the number of vacillating tableaux for each shape λ.

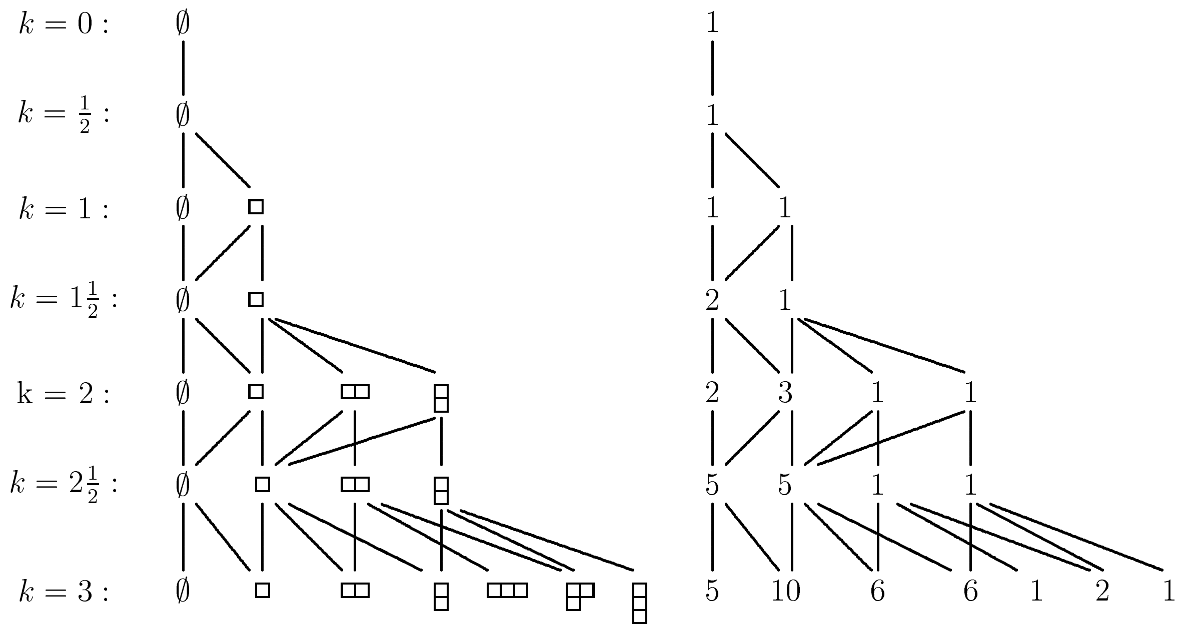
3 A Bijective Proof of $n^k = \sum_{\lambda \in \Lambda_n^k} f^\lambda \, m_k^\lambda$

Comparing dimensions on both sides of the identity (2.9) gives

$$n^k = \sum_{\lambda \in \Lambda_n^k} f^\lambda \, m_k^\lambda. \tag{3.1}$$

For example, when $n = 6$ and $k = 3$, the f^λ in the bottom row of Figure 1 (see also Figure 2) are 1, 5, 9, 10, 5, 16, 10, the corresponding m_3^λ are 5, 10, 6, 6, 1, 2, 1, and we have $6^3 = 216 = 1{\cdot}5 + 5{\cdot}10 + 9{\cdot}6 + 10{\cdot}6 + 5{\cdot}1 + 16{\cdot}2 + 10{\cdot}1$.

Figure 2: Bratteli Diagram for $\mathbb{C}A_k(n), n \geq 2k$.



To give a combinatorial proof of (3.1), we need to find a bijection of the form

$$\left\{ (i_1, \ldots, i_k) \,\middle|\, 1 \leq i_j \leq n \right\} \longleftrightarrow \bigsqcup_{\lambda \in \Lambda_n^k} \mathcal{SYT}(\lambda) \times \mathcal{VT}_k(\lambda). \tag{3.2}$$

To do so, we construct an invertible function that turns a sequence $(i_1, \ldots, i_k)$ of numbers in the range $1 \leq i_j \leq n$ into a pair (T_λ, P_λ) consisting of a standard tableaux T_λ of shape λ and vacillating tableaux P_λ of shape λ and length $2k$ for some $\lambda \in \Lambda_n^k$. Our bijection uses jeu de taquin and RSK column insertion.

If T is a standard tableau of shape $\lambda \vdash n$, then Schützenberger's [Scü] *jeu de taquin* provides an algorithm for removing the box containing x from T and producing a standard tableau S of shape $\mu \vdash (n-1)$ with $\mu \subseteq \lambda$ and entries $\{1, \ldots, n\} \setminus \{x\}$. We only need a special case of jeu de taquin for our purposes. See [Sag, §3.7] or [Sta, §A1.2] for the full-strength version and its applications.

If S is a standard tableau, let $S_{i,j}$ denote the entry of S in row i (numbered left-to-right) and column j (numbered top-to-bottom). We say that a *corner* of S is a box whose removal leaves the Young diagram of a partition. Thus the corners of S are the boxes that are both at the end of a row and the end of a column. The following algorithm will delete x from T leaving a standard tableau S with x removed. We denote this process by $x \xleftarrow{\text{jdt}} T$.

1. Let $c = S_{i,j}$ be the box containing x.
2. While c is not a corner, do

A. Let c' be the box containing $\min\{S_{i+1,j}, S_{i,j+1}\}$;
B. Exchange the positions of c and c'.
3. Delete c.

If only one of $S_{i+1,j}, S_{i,j+1}$ exists at step 2.A, then the minimum is taken to be that single value. Below is an example of $2 \xleftarrow{\text{jdt}} T$,

$$\begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & 8 & 12 \\ 7 & 10 & 11 & \\ 9 & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 4 & 5 & 6 \\ 3 & 2 & 8 & 12 \\ 7 & 10 & 11 & \\ 9 & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 4 & 5 & 6 \\ 3 & 8 & 2 & 12 \\ 7 & 10 & 11 & \\ 9 & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 4 & 5 & 6 \\ 3 & 8 & 11 & 12 \\ 7 & 10 & 2 & \\ 9 & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 4 & 5 & 6 \\ 3 & 8 & 11 & 12 \\ 7 & 10 & & \\ 9 & & & \end{array}.$$

The jeu de taquin algorithm is invertible in the following sense. Suppose that T is a tableaux that is standard but missing an element x, and suppose that we are given the position on the border of T from which x was deleted. Then place x in that border position and move it to a standard position in T using reverse jeu de taquin. That is, iteratively swap x with the larger of the numbers just above it or just to its left. Reading the above example backwards is an example of reverse jeu de taquin.

A bijective proof of the S_n identity $k! = \sum_{\lambda \vdash k} (f^\lambda)^2$ was originally found by Robinson [Rob] and later found, independently and in the form we present here, by Schensted [Sch]. Knuth [Kn] analyzed this algorithm and extended it to prove the identity (1.1.a). See [Sta, §7 Notes] for a nice history of the RSK algorithm. Let S be a tableau of partition shape μ, with $|\mu| < n$, with increasing rows and columns, and with distinct entries from $\{1, \ldots, n\}$. Let x be a positive integer that is not in S. The following algorithm inserts x into S producing a standard tableau T of shape λ with $\mu \subseteq \lambda$, $|\lambda/\mu| = 1$, whose entries are the union of those from S and $\{x\}$. We denote this process by $x \xrightarrow{\text{RSK}} S$.

1. Let R be the first row of S.
2. While x is less than some element in R, do
   A. Let y be the smallest element of R greater than x;
   B. Replace $y \in R$ with x;
   C. Let $x := y$ and let R be the next row.
3. Place x at the end of R (which is possibly empty).

For example, here is the insertion of 2 into the output of the jeu de taquin example above

$$\begin{array}{lllll} 1 & 4 & 5 & 6 & \leftarrow 2 \\ 3 & 8 & 11 & 12 & \\ 7 & 10 & & & \\ 9 & & & & \end{array} \Rightarrow \begin{array}{lllll} 1 & 2 & 5 & 6 & \\ 3 & 8 & 11 & 12 & \leftarrow 4 \\ 7 & 10 & & & \\ 9 & & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & 11 & 12 \\ 7 & 10 & \leftarrow 8 & \\ 9 & & & \end{array} \Rightarrow \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & 11 & 12 \\ 7 & 8 & & \\ 9 & \leftarrow 10 & & \end{array} \Rightarrow \begin{array}{llll} 1 & 2 & 5 & 6 \\ 3 & 4 & 11 & 12 \\ 7 & 8 & & \\ 9 & 10 & & \end{array}$$

It is possible to invert the process of row insertion using row *uninsertion*. See [Sag] for details. In the example above, the number 2 and the leftmost tableau are the result of uninserting 10 from the rightmost tableau.

Given $i_1, \ldots, i_k$, with $1 \leq i_j \leq n$, we will produce a pair (T_λ, P_λ), $\lambda \in \Lambda_n^k$, consisting of a standard tableau T_λ and a vacillating tableau P_λ. First, initialize the 0th tableau to be the standard tableaux of shape (n), namely,

$$T^{(0)} = \boxed{1 \mid 2 \mid \quad \cdots \quad \mid n} \,. \tag{3.3}$$

Then, recursively define standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ by

$$\begin{array}{rcl} T^{(j+\frac{1}{2})} & = & \left(i_{j+1} \xleftarrow{\text{jdt}} T^{(j)} \right), \\ T^{(j+1)} & = & \left(i_{j+1} \xrightarrow{\text{RSK}} T^{(j+\frac{1}{2})} \right), \end{array} \qquad 0 \leq j \leq k-1. \tag{3.4}$$

Let $\lambda^{(j)} \in \Lambda_n^j$ be the shape of $T^{(j)}$, and let $\lambda^{(j+\frac{1}{2})} \in \Lambda_{n-1}^j$ be the shape of $T^{(j+\frac{1}{2})}$. Then let

$$P_\lambda = \left(\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \ldots, \lambda^{(k)} \right) \qquad \text{and} \qquad T_\lambda = T^{(k)}, \tag{3.5}$$

so that P_λ is a vacillating tableau of shape $\lambda = \lambda^{(k)} \in \Lambda_n^k$, and T_λ is a standard tableaux of the same shape λ. We denote this iterative "delete-insert" process that associates the pair (T_λ, P_λ) to the sequence $(i_1, \ldots, i_k)$ by

$$(i_1, \ldots, i_k) \xrightarrow{\text{DI}} (T_\lambda, P_\lambda). \tag{3.6}$$

Figure 3 shows the calculations which give

$$(2,4,3) \xrightarrow{\text{DI}} \left(\begin{ytableau} 1 & 2 & 3 & 6 \\ 4 \\ 5 \end{ytableau}, \left(\ydiagram{6}, \ydiagram{5}, \ydiagram{5,1}, \ydiagram{4,1}, \ydiagram{4,2}, \ydiagram{4,1}, \ydiagram{4,1,1} \right) \right).$$

Theorem 3.1. *The function* $(i_1, \ldots, i_k) \xrightarrow{\text{DI}} (T_\lambda, P_\lambda)$ *provides a bijection between sequences in* $\left\{ (i_1, \ldots, i_k) \,\middle|\, 1 \leq i_j \leq n \right\}$ *and* $\bigsqcup_{\lambda \in \Lambda_n^k} \mathcal{SYT}(\lambda) \times \mathcal{VT}_k(\lambda)$ *and thus gives a combinatorial proof of (3.1).*

Proof. We prove the theorem by constructing the inverse of $\xrightarrow{\text{DI}}$.

Let $\lambda^{(j+\frac{1}{2})} \subseteq \lambda^{(j+1)}$ with $\lambda^{(j+1)} \in \Lambda_n^{(j+1)}$ and $\lambda^{(j+\frac{1}{2})} \in \Lambda_{n-1}^j$, and let $T^{(j+1)}$ be a standard tableau of shape $\lambda^{(j+1)}$. We can uniquely determine i_{j+1} and a tableau $T^{(j+\frac{1}{2})}$ of shape $\lambda^{(j+\frac{1}{2})}$ such that $T^{(j+1)} = (i_{j+1} \xrightarrow{\text{RSK}} T^{(j+\frac{1}{2})})$. To do this let b be the box in $\lambda^{(j+1)}/\lambda^{(j+\frac{1}{2})}$. Then let i_{j+1} and $T^{(j+\frac{1}{2})}$ be the result of uninserting the number in box b of $T^{(j+1)}$ (using the fact that RSK insertion is invertible).

Now let $T^{(j+\frac{1}{2})}$ be a tableau of shape $\lambda^{(j+\frac{1}{2})} \in \Lambda_{n-1}^{(j)}$ with increasing rows and columns and entries $\{1, \ldots, n\} \setminus \{i_{j+1}\}$ and let $\lambda^{(j)} \subseteq \lambda^{(j+\frac{1}{2})}$ with $\lambda^{(j)} \in \Lambda_n^j$. We can uniquely produce a standard tableau $T^{(j)}$ such that $T^{(j+\frac{1}{2})} = (i_{j+1} \xleftarrow{\text{jdt}} T^{(j)})$. To do this, let b be the box in $\lambda^{(j)}/\lambda^{(j+\frac{1}{2})}$, put i_{j+1} in position b of $T^{(j+\frac{1}{2})}$, and perform the inverse of jeu de taquin to produce $T^{(j)}$. That is, move i_{j+1} into a standard position by iteratively swapping it with the larger of the numbers just above it or just to its left.

Given $\lambda \in \Lambda_n^k$ and $(P_\lambda, T_\lambda) \in \mathcal{SYT}(\lambda) \times \mathcal{VT}_k(\lambda)$, we apply the process above to $\lambda^{(k-\frac{1}{2})} \subseteq \lambda^{(k)}$ and $T^{(k)} = T_\lambda$ producing i_k and $T^{(k-1)}$. Continuing this way, we can produce $i_k, i_{k-1}, \ldots, i_1$ and $T^{(k)}, T^{(k-1)}, \ldots, T^{(1)}$ such that $(i_1, \ldots, i_k) \xrightarrow{\text{DI}} (T_\lambda, P_\lambda)$. $\square$

Figure 3: Delete-Insertion of $(2, 4, 3)$

j	i_j	$T^{(j)}$
0		$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline 1&2&3&4&5&6\\\hline\end{array}$
$\frac{1}{2}$	$2 \xleftarrow{\text{jdt}}$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&3&4&5&6\\\hline\end{array}$
1	$2 \xrightarrow{\text{RSK}}$	$\begin{matrix}1&2&4&5&6\\3\end{matrix}$
$1\frac{1}{2}$	$4 \xleftarrow{\text{jdt}}$	$\begin{matrix}1&2&5&6\\3\end{matrix}$
2	$4 \xrightarrow{\text{RSK}}$	$\begin{matrix}1&2&4&6\\3&5\end{matrix}$
$2\frac{1}{2}$	$3 \xleftarrow{\text{jdt}}$	$\begin{matrix}1&2&4&6\\5\end{matrix}$
3	$3 \xrightarrow{\text{RSK}}$	$\begin{matrix}1&2&3&6\\4\\5\end{matrix}$

4 A Bijective Proof of $B(2k) = \sum_{\lambda\in\Gamma_k} (m_k^\lambda)^2$

If we view $\mathbb{C}A_k(n)$ as a bimodule for $\mathbb{C}A_k(n) \otimes \mathbb{C}A_k(n)$, with the first tensor factor acting by left multiplication on $\mathbb{C}A_k(n)$ and the second tensor factor acting by right multiplication on $\mathbb{C}A_k(n)$, then the decomposition into irreducibles for $\mathbb{C}A_k(n) \otimes \mathbb{C}A_k(n)$ is $\mathbb{C}A_k(n) \cong \bigoplus_{\lambda\in\Gamma_k} M_k^\lambda \otimes \bar{M}_k^\lambda$, where $\bar{M}_k^\lambda$ is the irreducible right $\mathbb{C}A_k(n)$-module indexed by λ (so $M_k^\lambda \cong \bar{M}_k^\lambda$). This is a generalization of the decomposition of the group algebra of a finite group. Comparing dimensions on both sides gives,

$$B(2k) = \sum_{\lambda\in\Gamma_k} (m_k^\lambda)^2. \tag{4.1}$$

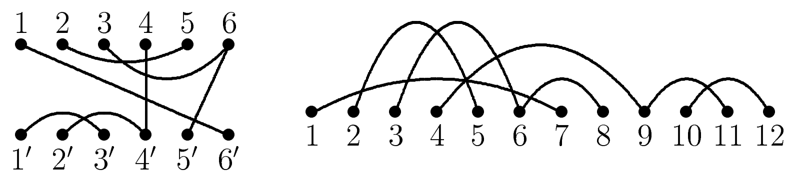
For example using the numbers m_3^λ from the bottom row of Figure 2, we have $B(6) = 203 = 5^2 + 10^2 + 6^2 + 6^2 + 1^2 + 2^2 + 1^2$.

To give a combinatorial proof of (4.1), we find a bijection of the form

$$A_k \longleftrightarrow \bigsqcup_{\lambda\in\Gamma_k} \mathcal{VT}_k(\lambda) \times \mathcal{VT}_k(\lambda), \tag{4.2}$$

by constructing a function that takes a set partition $d \in A_k$ and produces a pair (P_λ, Q_λ) of vacillating tableaux. In Section 5 we show that our bijection restricts to work for all the diagram subalgebras described in Section 2. In particular, if d corresponds to a permutation in S_k then our bijection is the usual RSK algorithm described in Section 3.2.

We will draw diagrams $d \in A_k$ using a *standard representation* as single row with the vertices in order $1, \ldots, 2k$, where we relabel vertex j' with the label $2k - j + 1$. We draw the edges of the standard representation of $d \in A_k$ in a specific way: connect vertices i and j, with $i \leq j$, if and only if i and j are related in d and there does not exist k related to i and j with $i < k < j$. In this way, each vertex is connected only to its nearest neighbors in its block. For example,

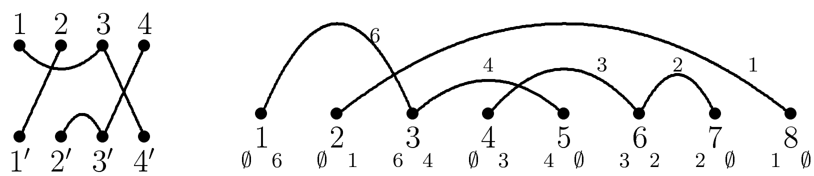


We label each edge e of the diagram d with $2k + 1 - \ell$, where ℓ is the right vertex of e. Define the *insertion sequence* of a diagram to be the sequence $E = (E_j)$ indexed by j in the sequence $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots, 2k-1, 2k-\frac{1}{2}, 2k$, where

$$E_j = \begin{cases} a, & \text{if vertex } j \text{ is the left endpoint of edge } a, \\ \emptyset, & \text{if vertex } j \text{ is not a left endpoint,} \end{cases}$$

$$E_{j-\frac{1}{2}} = \begin{cases} a, & \text{if vertex } j \text{ is the right endpoint of edge } a, \\ \emptyset, & \text{if vertex } j \text{ is not a right endpoint.} \end{cases}$$

Example 4.1. The edge labeling for the diagram of the set partition $\{\{1,3,4'\},\{2,1'\},\{4,3',2'\}\}$ is given by



and the insertion sequence is

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
E_j	$\emptyset$	6	$\emptyset$	1	6	4	$\emptyset$	3	4	$\emptyset$	3	2	2	$\emptyset$	1	$\emptyset$

The insertion sequence of a standard diagram completely determines the edges, and thus the connected components, of the diagram, so the following proposition follows immediately.

Proposition 4.2. *$d \in A_k$ is completely determined by its insertion sequence.*

For $d \in A_k$ with insertion sequence $E = (E_j)$, we will produce a pair (P_λ, Q_λ) of vacillating tableaux. Begin with the empty tableaux,

$$T^{(0)} = \emptyset. \tag{4.3}$$

Then recursively define standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ by

$$\begin{array}{rcl} T^{(j+\frac{1}{2})} & = & \begin{cases} E_{j+\frac{1}{2}} \stackrel{\text{jdt}}{\longleftarrow} T^{(j)}, & \text{if } E_{j+\frac{1}{2}} \neq \emptyset, \\ T^{(j)}, & \text{if } E_{j+\frac{1}{2}} = \emptyset, \end{cases} \\ T^{(j+1)} & = & \begin{cases} E_{j+1} \stackrel{\text{RSK}}{\longrightarrow} T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} \neq \emptyset, \\ T^{(j+\frac{1}{2})}, & \text{if } E_{j+1} = \emptyset, \end{cases} \end{array} \qquad 0 \leq j \leq 2k-1. \tag{4.4}$$

Let $\lambda^{(i)}$ be the shape of $T^{(i)}$, let $\lambda^{(i+\frac{1}{2})}$ be the shape of $T^{(i+\frac{1}{2})}$, and let $\lambda = \lambda^{(k)}$. Define

$$\begin{array}{rcl} Q_\lambda & = & \left(\emptyset, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \ldots, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)}\right), \\ P_\lambda & = & \left(\lambda^{(2k)}, \lambda^{(2k-\frac{1}{2})}, \ldots, \lambda^{(k+\frac{1}{2})}, \lambda^{(k)}\right). \end{array}$$

In this insertion process, every edge of the partition diagram is inserted when we come to its left endpoint and deleted when we come to its right endpoint, so the final shape is $\lambda^{(2k)} = \emptyset$. This means that $P_\lambda \in \mathcal{VT}(\lambda)$ and $Q_\lambda \in \mathcal{VT}(\lambda)$, so we have associated a pair of vacillating tableaux (P_λ, Q_λ) to a set partition $d \in A_k$. We denote this we process by

$$d \to (P_\lambda, Q_\lambda).$$

In Figure 4, we see the insertion of the diagram in Example 4.1. The result of this is to pair the diagram in Example 4.1 with the following pair of vacillating tableaux,

$$\begin{array}{rcl} Q_\lambda & = & \left(\emptyset,\ \emptyset, \square, \square, \begin{array}{l}\square\\[-3pt]\square\end{array}, \square, \square\square, \square\square, \begin{array}{l}\square\square\\[-3pt]\square\end{array}\right), \\ P_\lambda & = & \left(\emptyset,\ \emptyset, \square, \square, \square\square, \square, \square\square, \square\square, \begin{array}{l}\square\square\\[-3pt]\square\end{array}\right). \end{array}$$

Remark 4.3. We have numbered the edges of the standard diagram of d in increasing order from right to left, so if $E_{j+\frac{1}{2}} \neq \emptyset$, then $E_{j+\frac{1}{2}}$ will be the largest element of $T^{(j)}$. Thus, in $T^{(j+\frac{1}{2})} = (E_{j+\frac{1}{2}} \stackrel{\text{jdt}}{\longleftarrow} T^{(j)})$ we know that $E_{j+\frac{1}{2}}$ is in a corner box, and jeu de taquin simply deletes that box.

Theorem 4.4. *The function $d \to (P_\lambda, Q_\lambda)$ provides a bijection between the set partitions A_k and pairs of vacillating tableaux $\bigsqcup_{\lambda \in \Gamma_k} \mathcal{VT}(\lambda) \times \mathcal{VT}_k(\lambda)$ and thus gives a combinatorial proof of (4.1).*

Proof. We prove the theorem by constructing the inverse of $d \to (P_\lambda, Q_\lambda)$. First, we use Q_λ followed by P_λ in reverse order to construct the sequence $\lambda^{(\frac{1}{2})}, \lambda^{(1)}, \ldots, \lambda^{(2k-\frac{1}{2})}, \lambda^{(2k)}$. We initialize $T^{(2k)} = \emptyset$.

Figure 4: Insertion of the Set Partition in Example 4.1.

j	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
E_j	$\emptyset$	6	$\emptyset$	1	6	4	$\emptyset$	3	4	$\emptyset$	3	2	2	$\emptyset$	1	$\emptyset$

j	E_j		$T^{(j)}$
0			$\emptyset$
$\frac{1}{2}$	$\emptyset$	$\xleftarrow{\text{jdt}}$	$\emptyset$
1	6	$\xrightarrow{\text{RSK}}$	$\boxed{6}$
$1\frac{1}{2}$	$\emptyset$	$\xleftarrow{\text{jdt}}$	$\boxed{6}$
2	1	$\xrightarrow{\text{RSK}}$	$\begin{array}{\|c\|}\hline 1\\\hline 6\\\hline\end{array}$
$2\frac{1}{2}$	6	$\xleftarrow{\text{jdt}}$	$\boxed{1}$
3	4	$\xrightarrow{\text{RSK}}$	$\begin{array}{\|c\|c\|}\hline 1&4\\\hline\end{array}$
$3\frac{1}{2}$	$\emptyset$	$\xleftarrow{\text{jdt}}$	$\begin{array}{\|c\|c\|}\hline 1&4\\\hline\end{array}$
4	3	$\xrightarrow{\text{RSK}}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array}$

j	E_j		$T^{(j)}$
8	$\emptyset$	$\xrightarrow{\text{RSK}}$	$\emptyset$
$7\frac{1}{2}$	1	$\xleftarrow{\text{jdt}}$	$\emptyset$
7	$\emptyset$	$\xrightarrow{\text{RSK}}$	$\boxed{1}$
$6\frac{1}{2}$	2	$\xleftarrow{\text{jdt}}$	$\boxed{1}$
6	2	$\xrightarrow{\text{RSK}}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}$
$5\frac{1}{2}$	3	$\xleftarrow{\text{jdt}}$	$\boxed{1}$
5	$\emptyset$	$\xrightarrow{\text{RSK}}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline\end{array}$
$4\frac{1}{2}$	4	$\xleftarrow{\text{jdt}}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline\end{array}$
4			$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array}$

We now show how to construct $T^{(i+\frac{1}{2})}$ and E_{i+1} so that $T^{(i+1)} = (E_{i+1} \xrightarrow{\text{RSK}} T^{(i+\frac{1}{2})})$. If $\lambda^{(i+\frac{1}{2})} = \lambda^{(i+1)}$, then let $T^{(i+\frac{1}{2})} = T^{(i+1)}$ and $E_{i+1} = \emptyset$. Otherwise, $\lambda^{(i+1)}/\lambda^{(i+\frac{1}{2})}$ is a box b, and we use uninsertion on the value in b to produce $T^{(i+\frac{1}{2})}$ and E_{i+1} such that $T^{(i+1)} = (E_{i+1} \xrightarrow{\text{RSK}} T^{(i+\frac{1}{2})})$. Since we uninserted the value in position b, we know that $T^{(i+\frac{1}{2})}$ has shape $\lambda^{(i+\frac{1}{2})}$.

We then show how to construct $T^{(i)}$ and $E_{i+\frac{1}{2}}$ so that $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{\text{jdt}} T^{(i)})$. If $\lambda^{(i)} = \lambda^{(i+\frac{1}{2})}$, then let $T^{(i)} = T^{(i+\frac{1}{2})}$ and $E_{i+\frac{1}{2}} = \emptyset$. Otherwise, $\lambda^{(i)}/\lambda^{(i+\frac{1}{2})}$ is a box b. Let $T^{(i)}$ be the tableau of shape $\lambda^{(i)}$ with the same entries as $T^{(i+1)}$ and having the entry $2k-i$ in box b. Let $E_{i+\frac{1}{2}} = 2k-i$. At any given step i (as we work our way down from step $2k$ to step $\frac{1}{2}$), $2k-i$ is the largest value added to the tableau thus far, so $T^{(i)}$ is standard. Furthermore, $T^{(i+\frac{1}{2})} = (E_{i+\frac{1}{2}} \xleftarrow{\text{jdt}} T^{(i)})$, since $E_{i+\frac{1}{2}} = 2k-i$ is already in a corner and thus jeu de taquin simply deletes it.

This iterative process will produce $E_{2k}, E_{2k-1}, \ldots, E_{\frac{1}{2}}$ which completely determines the set partition d. By the way we have constructed d, we have $d \to (P_\lambda, Q_\lambda)$. $\square$

5 Properties of the Insertion Algorithm

In this section we investigate a number of the nice properties of the insertion algorithm from Section 4. First, we note that our algorithm restricts to the insertion algorithms for the subalgebras discussed in Section 1:

Remark 5.1. (a) If $d \in S_k$ is a permutation diagram, then the standard representation of d pairs vertices $1, \ldots k$ with vertices $k+1, \ldots, 2k$. Our algorithm inserts on steps j, with $1 \leq j \leq k$, and deletes on steps $j + \frac{1}{2}$, with $k \leq j \leq 2k-1$. The algorithm does nothing on all other steps. If we ignore these "do-nothing" steps, then the algorithm becomes the usual RSK insertion and P_λ and Q_λ are the insertion and recording tableaux, respectively.

(b) If $d \in B_k$ is a Brauer diagram, then each vertex in the standard representation of d is incident to exactly one edge, so exactly one of $E_{(j-\frac{1}{2})}$ or E_j will be nonempty for each $1 \leq j \leq 2k$. If we ignore the steps where we do nothing, then for each vertex we either insert or delete the edge number associated to that vertex. This is precisely the insertion scheme of Sundaram [Sun] (see also [Ter]). This algorithm produces "oscillating tableaux," which either increase or decrease by one box on each step. If we insert all the Brauer diagrams, we produce the Bratteli diagram for $\mathbb{C}B_k(n)$ (see for example [HR1]).

(c) If $d \in R_k$ is a rook monoid diagram, then our algorithm has us insert or do nothing on steps $0 < j \leq k$ and delete or do nothing on steps $j + \frac{1}{2}$ with $k \leq j \leq 2k-1$. If we insert all rook monoid diagrams, we obtain the Bratteli diagram for $\mathbb{C}R_k$ (see [Ha]) .

(d) If $d \in A_{k-\frac{1}{2}}$ then the standard representation of d has an edge, labeled k, connecting vertices k and $k+1$. At step k in the insertion process, we will insert this edge into $T^{(k-\frac{1}{2})}$. The label k is larger than all the entries in $T^{(k-\frac{1}{2})}$, since the entries that are in $T^{(k-\frac{1}{2})}$ correspond to edges whose right endpoint is larger than $k+1$. Thus we insert this entry at the end of the first row and in the next step, $k+\frac{1}{2}$, we immediately delete it, so that $\lambda^{(k-\frac{1}{2})} = \lambda^{(k+\frac{1}{2})}$. If we remove $\lambda^{(k)}$ from our pair of paths, we obtain a pair of vacillating tableaux $(P_{\lambda^{(k-\frac{1}{2})}}, Q_{\lambda^{(k-\frac{1}{2})}})$ of length $2k-1$, and we get a combinatorial formula for the following dimension identity for $\mathbb{C}A_{k-\frac{1}{2}}(n)$ involving the odd Bell numbers,

$$B(2k-1) = \sum_{\lambda \in \Gamma_{k-\frac{1}{2}}} (m^\lambda_{k-\frac{1}{2}})^2. \tag{5.1}$$

Proposition 5.2. *If $d \in A_k$ such that $d \to (P_\lambda, Q_\lambda)$, then $|\lambda| = \mathrm{pn}(d)$, where $\mathrm{pn}(d)$ is the propagating number defined in (2.3).*

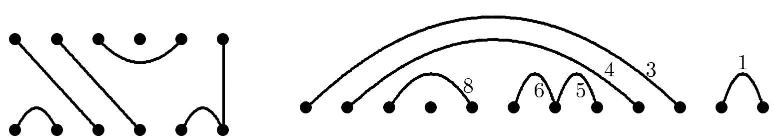
Proof. If we draw a vertical line between vertex k and $k+1$ in the standard representation of d, then $\mathrm{pn}(d)$ is the number of edges that cross this line. The labels of these edges are exactly the numbers which have been inserted but not deleted at step k of our algorithm. Thus the tableau $T^{(k)}$ has exactly these numbers in it and so $\lambda^{(k)}$ consists of $\mathrm{pn}(d)$ boxes. $\square$

Recall that I_t is spanned by the diagrams whose propagating number satisfies $\mathrm{pn}(d) \leq t$. The chain of ideals $\mathbb{C}I_0(n) \subseteq \mathbb{C}I_1(n) \subseteq \cdots \subseteq \mathbb{C}I_k(n) = \mathbb{C}A_k(n)$ is a key feature of the "basic construction" of $\mathbb{C}A_k(n)$ (see [HR2, Mar3]). The irreducible components of $\mathbb{C}I_t(n)/(\mathbb{C}I_{t-1}(n))$ are those indexed by $\lambda \in \Gamma_k$ with $|\lambda| = t$. The previous proposition tells us that our insertion algorithm respects this basic construction and gives a combinatorial proof of the following dimension identity for $\mathbb{C}I_t(n)/(\mathbb{C}I_{t-1}(n))$,

$$\mathrm{Card}\left(\{d \in A_k \mid \mathrm{pn}(d) = t\}\right) = \sum_{\substack{\lambda \in \Gamma_k \\ |\lambda| = t}} (m_k^\lambda)^2. \tag{5.2}$$

Proposition 5.3. *Let $d \to (P, Q) = (\lambda^{(0)}, \ldots, \lambda^{(2k)})$. Then d is planar if and only if the length of every partition in the sequence satisfies $\ell(\lambda^{(i)}) \leq 1$.*

Proof. This is a special case of the crossing and nesting theorem in [CDDSY]. If d is planar, then the standard representation of d is also planar. For example,



Let $a < b$ be the labels of two edges in a planar diagram d and suppose that b is inserted before a in the insertion sequence of d. Since $a < b$ the right vertex of b is left of the right vertex of a. Since b is inserted before a, the left vertex of b is left of the left vertex of a. Since d is planar, the right vertex of b must then be left of the left vertex of a. Thus, b is inserted and deleted before a is inserted. This means that when we are inserting an edge label a into $T^{(i)}$, we know that a is larger than all the entries of $T^{(i)}$, and so a placed at the end of the first row. Thus at each step of the insertion, we only ever obtain the empty tableau or a tableau consisting of 1 row. Conversely, to obtain a partition that has more than one column, there must be a pair $a < b$ such that a is inserted after b is inserted but before b is deleted. This is necessary for a to bump b into the next row. By the reasoning above, this forces an edge crossing in the partition diagram. $\square$

Remark 5.4. (a) The planar partition diagrams $d \in P_k$ are exactly paired with the paths (P, Q) whose vertices contain only length 0 or 1 partitions. The planar partition algebra $\mathbb{C}P_k(n)$ is isomorphic to the Temperley-Lieb algebra $\mathbb{C}T_{2k}(n)$, $k \in \frac{1}{2}\mathbb{Z}_{>0}$, (see [Jo] or [HR2]) and has the Bratteli diagram shown in Figure 5. Our algorithm gives a combinatorial proof of the following dimension identity for $\mathbb{C}T_{2k}(n)$,

$$C(2k) = \sum_{\substack{\lambda \in \Lambda_k \\ \ell(\lambda) \leq 1}} \left(\binom{2k}{\lfloor k \rfloor - |\lambda|} - \binom{2k}{\lfloor k \rfloor - |\lambda| - 1} \right)^2, \qquad k \in \frac{1}{2}\mathbb{Z}_{>0}, \tag{5.3}$$

where $\lfloor\ \rfloor$ is the floor function and $\binom{2k}{\lfloor k \rfloor - |\lambda|} - \binom{2k}{\lfloor k \rfloor - |\lambda| - 1}$ is the dimension of the irreducible $\mathbb{C}T_{2k}(n)$-module labeled by λ (this is also the number of paths to λ in the Bratteli diagram of $\mathbb{C}T_{2k}(n)$).

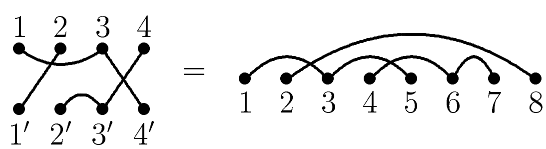
(b) The planar Brauer diagrams T_k span another copy of the Temperley-Lieb algebra $\mathbb{C}T_k(n)$ inside $\mathbb{C}A_k(n)$, and our insertion algorithm when applied to these diagrams gives a different bijective proof of the identity in (a).

(c) In the insertion for planar rook monoid diagrams in PR_k, we insert or do nothing for steps j with $0 < j \leq k$, we delete or do nothing for steps $j + \frac{1}{2}$ with $k \leq j < 2k$, and we do nothing on all other steps. Thus, the Bratteli diagram, which is shown in Figure 5, has levels j and $j + \frac{1}{2}$ merged and is isomorphic to Pascal's triangle. The irreducible representation associated to the partition of shape $\lambda = (\ell)$ has dimension $\binom{k}{\ell}$ (see [HH] for the representation theory of PR_k). Our algorithm gives an RSK insertion proof of following the well-known binomial identity,

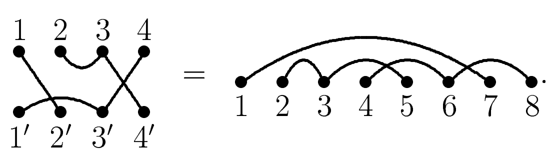
$$\binom{2k}{k} = \sum_{\ell=0}^{k} \binom{k}{\ell}^2. \tag{5.4}$$

Finally, we show that the algorithm has the symmetry property of the usual RSK algorithm for the symmetric group. Namely, if $\pi \to (P, Q)$ for $\pi \in S_k$, then $\pi^{-1} \to (Q, P)$. Using diagrams, the inverse of π is simply the diagram of π reflected over its horizontal axis. We define the flip of any partition diagram to be this reflection, so for example,

d =



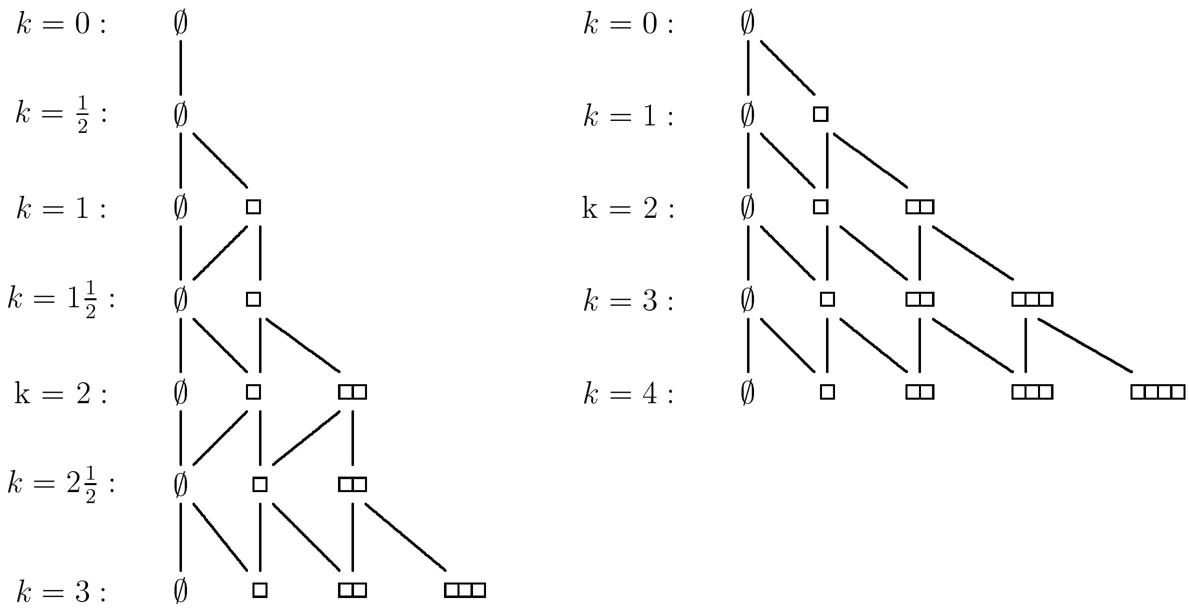
flip(d) =



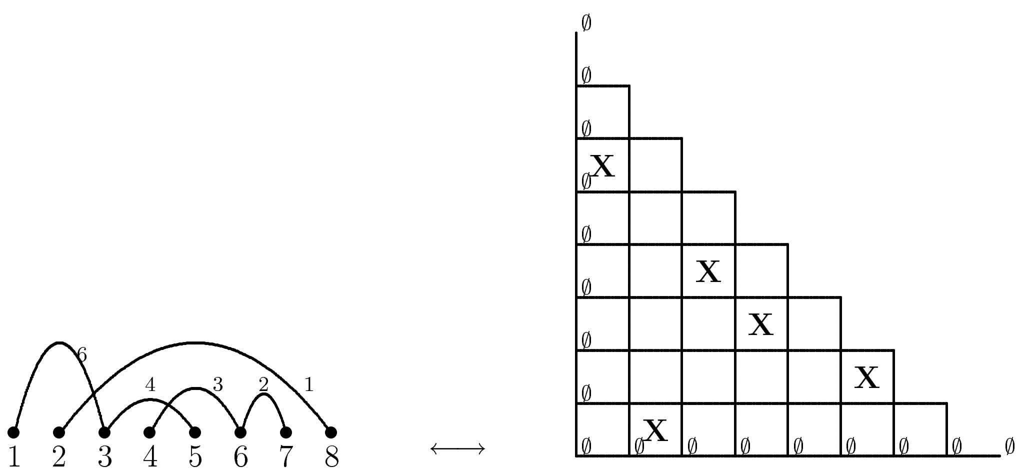
Notice that in the standard representation of d, a flip corresponds to a reflection over the vertical line between vertices k and $k+1$. Our goal is to show that if $d \to (P, Q)$, then flip$(d) \to (Q, P)$.

Given a partition $d \in A_k$, construct a triangular grid in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that contains the points in the triangle whose vertices are $(0, 0), (2k, 0)$ and $(0, 2k)$. Number

Figure 5: Bratteli Diagrams for the Planar Partition Algebra $\mathbb{C}P_k(n)$ and the Planar Rook Monoid PR_k

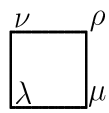


the columns $1, \ldots, 2k$ from left to right and the rows $1, \ldots, 2k$ from bottom to top. Place an $\mathbf{X}$ in the box in column i and row j if and only if, in the one-row diagram of d, vertex i is the left endpoint of edge j. We then label the vertices of the diagram on the bottom row and left column with the empty partition $\emptyset$. For example, in our diagram d from Example 4.1 we have



Note that the triangular array completely determines the partition diagram and vice versa.

Now we inductively label the remaining vertices using the local rules of S. Fomin (see [Ry1, Ry2] or [Sta, 7.13] and the references there). If a box is labeled with μ, ν, and λ as shown below. Then we add the label ρ according the rules that follow:

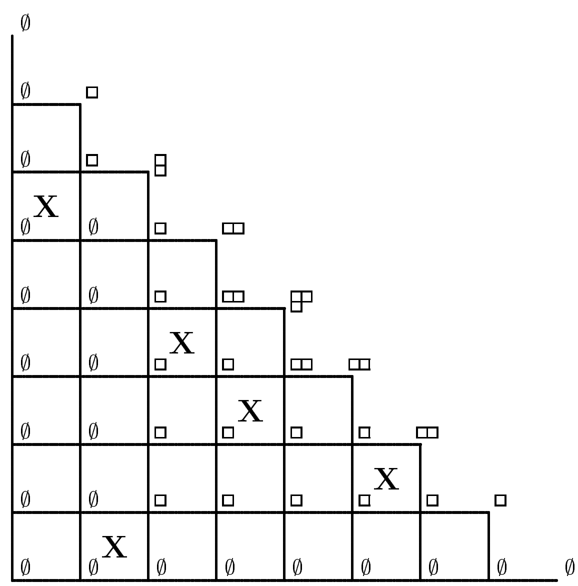


(L1) If $\mu \neq \nu$, let $\rho = \mu \cup \nu$, i.e., $\rho_i = \max(\mu_i, \nu_i)$.

(L2) If $\mu = \nu$, $\lambda \subset \mu$, and $\lambda \neq \mu$, then this will automatically imply that μ can be obtained from λ by adding a box to λ_i. Let ρ be obtained from μ by adding a box to μ_{i+1}.

(L3) If $\mu = \nu = \lambda$, then if the square does not contain an **X**, let $\rho = \lambda$, and if the square does contain an **X**, let ρ be obtained from λ by adding 1 to λ_1.

Using these rules, we can uniquely label every corner, one sep at a time. The resulting diagram is called the *growth diagram* G_d for d. The diagram corresponding to the above example is

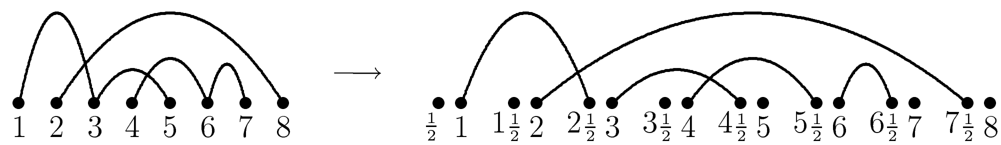


Let P_d denote the chain of partitions that follow the staircase path on the diagonal of G_d from $(0, 2k)$ to (k, k) and let Q_d denote the chain of partitions that follow the staircase path on the diagonal of G_d from $(2k, 0)$ to (k, k). The pair (P_d, Q_d) represents a pair of vacillating tableaux whose shape is the partition at (k, k). The staircase path in our growth diagram above is the same as the path we get from insertion in Figure 4.

Theorem 5.5. *Let $d \in A_k$ with $d \to (P, Q)$. Then $P_d = P$ and $Q_d = Q$.*

Proof. Turn the diagram $d \in A_k$ into a diagram d' on $4k$ vertices by splitting each vertex i into two vertices labeled by $i - \frac{1}{2}$ and i. If there is an edge from vertex j to vertex i in d, with $j < i$, let j be adjacent to $i - \frac{1}{2}$ in d'. If there is an edge from vertex j to vertex i

in d, with $j > i$, let i be adjacent to $j - \frac{1}{2}$ in d'. Thus, in our example we have,

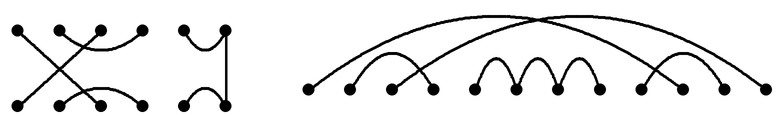


In this way we turn each diagram in A_k into a partial matching in A_{2k}. In his dissertation, Roby [Ry1] shows that the up-down tableaux coming from growth diagrams are equivalent to those from RSK insertion of Brauer diagrams (matchings). Dulucq and Sagan [DS] extend this insertion to work for partial matchings (and other generalizations for skew oscillating tableaux) and Roby [Ry2] shows that the growth diagrams also work for the Duluc-Sagan insertion of partial matchings. Furthermore, the insertion sequences that come from our diagram d are the same as the insertion sequences that come from d' using the methods of [Sun, DS]. $\square$

A key advantage of the use of growth diagrams is that the symmetry of the algorithm is nearly obvious. We have that i is the left endpoint of the edge labeled j in d if and only if j is the left endpoint of the edge labeled i in $\mathrm{flip}(d)$. Thus the growth diagram of G_d is the reflection over the line $y = x$ of the growth diagram of $G_{\mathrm{flip}(d)}$, and so $P_d = Q_{\mathrm{flip}(d)}$ and $Q_d = P_{\mathrm{flip}(d)}$. Combining this point with the Theorem 5.5 gives

Corollary 5.6. *If* $d \to (P, Q)$ *then* $\mathrm{flip}(d) \to (Q, P)$.

We say that $d \in A_k$ is symmetric if $d = \mathrm{flip}(d)$. The following is an example of a symmetric diagram in A_6,



In S_k the symmetric diagrams are involutions.

Corollary 5.7. *A diagram* $d \in A_k$ *is symmetric if and only if* $d \to (P, P)$.

Proof. If d is symmetric, then by Corollary 5.6 we must have $P = Q$. To prove the converse, let $P = Q$ and place these vacillating tableaux on the staircase border of a growth diagram. The local rules above are invertible: given μ, ν and ρ one can the rules backwards to uniquely find λ and determine whether there is an $\mathbf{X}$ in the box. Thus, the interior of the growth diagram is uniquely determined. By the symmetry of having $P = Q$ along the staircase, the growth diagram must have a symmetric interior and a symmetric placement of the $\mathbf{X}$s. This forces d to be symmetric. $\square$

This corollary tells us that the number of symmetric diagrams in A_k is equal to the number of vacillating tableaux of length $2k$, or the number of paths to level k in the Bratteli diagram of $\mathbb{C}A_k(n)$. Thus,

$$\mathrm{Card}(\{d \in A_k \mid d \text{ is symmetric } \}) = \sum_{\lambda \in \Lambda_k} m_k^{\lambda}. \tag{5.5}$$

Furthermore, since our insertion restricts to the diagram subalgebras, (5.5) is true if we replace A_k with any of the monoids $S_k, B_k, T_k, I_t, R_k, PR_k$ and replace m_k^λ with the dimension of the appropriate irreducible representation. In the case of the symmetric group, (5.5) reduces to the fact that the number of involutions in S_k equals $\sum_{\lambda \vdash k} f^\lambda$.

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